Münster J. of Math. **6** (2013), 117–166
urn:nbn:de:hbz:6-25319483725
Münster Journal of Mathematics

The extender algebra and vagaries of Σ^2_1 absoluteness

Philipp Doebler and Ralf Schindler

(Communicated by Katrin Tent)

Abstract. We review the construction of the extender algebra, a Boolean algebra which is due to Woodin, with δ-many generators. The resulting genericity iteration is applied to prove a new Σ^2_1-absoluteness theorem for c.c.c. forcings with ordinal parameters. We also introduce and discuss sets of ordinals that extend to a class with unique condensation.

This paper mostly deals with the extender algebra, a Boolean algebra which was discovered by W. H. Woodin and the absoluteness results one can obtain using the extender algebra. Given any model $\mathcal{M}$ that contains a Woodin cardinal δ, one can construct the extender algebra W_δ in $\mathcal{M}$. If $\mathcal{M}$ is countable and sufficiently iterable, then given *any* $x \subset \omega$, one can find an iteration map $j : \mathcal{M} \to \mathcal{M}^*$ such that x is generic over $\mathcal{M}^*$ for $j(W_\delta)$. This iteration is known as a *genericity iteration*. More generally, if $\mathbb{P}$ is any notion of forcing and if τ is a name for a real in $V^{\mathbb{P}}$, then there is in V an iterate $\mathcal{M}^*$ of $\mathcal{M}$, such that regardless of the choice of a $G \subset \mathbb{P}$ which is generic over V we have that τ^G is generic over $\mathcal{M}^*$. These statements are due to Woodin. They are a key technical tool for producing some of the deepest insights made by contemporary pure set theory.

This paper is meant to be mostly a survey article which also presents a collection of new results.

We will carry out the construction of the extender algebra for a model with a Woodin cardinal δ in Section 2. There are two variants of this construction: the one with ω-many generators (intended for making subsets of ω generic) and the one with δ-many generators (intended for making subsets of ω_1 generic). The former one is presented in [13], and the latter one is also written up in

Both authors gratefully acknowledge support by DFG grant no. SCHI 484/3-1. This article contains material from the first author's PhD thesis. We thank Ilijas Farah and the referee for their very helpful comments.

[3].[1] These genericity iterations are applied to prove Woodin's Σ^2_1-absoluteness Theorem 4.1 (see also the discussion in the following section) and a new Σ^2_1-absoluteness theorem for c.c.c. forcings with ordinal parameters, Theorem 5.4.

In Section 6 we introduce and discuss sets that extend to a class with unique condensation: $A \subset \omega_1$ *extends to* A^* *with unique condensation* if A^* is class of ordinals such that $A^* \cap \omega_1 = A$ and for all uncountable cardinals κ

> if $\lambda > \kappa$ is a sufficiently large regular cardinal, then there is a club $C(A^*, \kappa, \lambda)$ of countable substructures $X \prec H_\lambda$ such that $A^* \cap \kappa \in X$ and
> $$A \cap \bar{\kappa} = \overline{A^* \cap \kappa},$$
> where π is the inverse of the collapse of X and $\pi(\bar{\kappa}, \overline{A^* \cap \kappa}) = \kappa, A^*$.

We analyze the sets that extend to classes with unique condensation in detail and construct nontrivial examples. We show that these sets can trivialize in the following sense: granted large cardinals in V and an iterability assumption, we show that every set with a uniquely condensing extension is constructible from a real. See Theorem 6.18 for the precise statement.

We will also show that given a sufficiently iterable model $\mathcal{M}$ with a measurable Woodin cardinal and a name τ for a subset of ω_1 that extends to a class with unique condensation, a genericity iteration of $\mathcal{M}$ of length ω_1 can be constructed such that all interpretations of τ are generic over the final model. We show that if τ is the name related to a reasonable forcing extension, then such a genericity iteration behaves like genericity iterations for reals, see Lemma 6.21. We apply this Lemma to show two absoluteness results, Theorem 6.31 and Theorem 6.30.

1. Introduction to absoluteness for $L(\mathbb{R})$

In this section we will review some results about forcing absoluteness for $L(\mathbb{R})$. Woodin has shown the following Σ^2_1 absoluteness with respect to forcing extensions of V, Theorem 1.1.

In the statement of Theorem 1.1, $M^\sharp_{\mathsf{mw}}$ is a fine structural premouse that contains a measurable Woodin cardinal. The paper [13] discusses such fine structural premice and their iterations. However, fine structure will not play any role in the current paper. The reader may think of $M^\sharp_{\mathsf{mw}}$ as a countable transitive model $(M; \in, U)$ such that

(a) M is a model of ZFC^- with a largest cardinal, say κ,[2]
(b) $(M; U) \models$ "U is a measure on κ witnessing that κ is a measurable cardinal," and
(c) $M \models$ "κ is a Woodin cardinal."

[1]The first version of [3] already existed a few years before the first version of our paper came into existence, and our exposition in §2 follows §1 of [3] very closely, but a certain part of these two papers were written in parallel.

[2]Here, ZFC^- is ZFC without the power set axiom

Models with measurable cardinals may be iterated. A model $(M;\in,U)$ as above is ω_1+1-iterable if and only if there is a strategy Σ such that if $\mathcal{T}$ is an iteration tree of limit length $\lambda\le\omega_1$ which is according to Σ (i.e., for all limit ordinals $\bar\lambda<\lambda$, Σ was used to choose the branch in $\mathcal{T}$ through $\mathcal{T}\restriction\bar\lambda$), then Σ yields a cofinal well-founded branch through $\mathcal{T}$. For details, see [13].

Theorem 1.1 (Woodin). *Suppose $M^\sharp_{\mathsf{mw}}$ exists and is (ω,ω_1+1)-iterable in all set forcing extensions. Assume* CH *holds. Let $\mathbb{P}$ be a notion of forcing and let $G\subset\mathbb{P}$ be V-generic. Let z be a real in V. If in $V[G]$*

$$\exists A\subset\mathbb{R}^{V[G]}L(\mathbb{R}^{V[G]},A)\models\phi(A,z),$$

then in V

$$\exists A\subset\mathbb{R}^{V}L(\mathbb{R}^{V},A)\models\phi(A,z).$$

Furthermore if CH *holds in $V^{\mathbb{P}}$, then the converse is true.*

The existence of $M^\sharp_{\mathsf{mw}}$ was not the original hypothesis of the Σ^2_1 absoluteness theorem 1.1. Woodin's first proof used class many measurable Woodins and the stationary tower forcing. We will give a full proof of the above theorem, see Theorem 4.1.

It is natural to ask if one can add ordinal parameters to the statement of the above theorem. Woodin studied a class of forcings larger than the reasonable forcings: A poset $\mathbb{P}$ is called *weakly proper* if and only if for all ordinals α, $([\alpha]^\omega)^V$ is cofinal in $([\alpha]^\omega)^{V^{\mathbb{P}}}$. The conclusion of the following theorem is more general than the conclusion of the Embedding Theorem from [12]. A set of reals A is weakly homogeneously Suslin if and only if A admits a tree representation via a homogeneity system of measures.

Theorem 1.2 ([15, 10.63]). *Let $\mathbb{P}\in V_\delta$ be a* weakly proper *notion of forcing. Assume $A\subset\mathbb{R}$, $L(A,\mathbb{R})\models$* AD[3] *and every set in $\mathcal{P}(\mathbb{R})\cap L(A,\mathbb{R})$ is δ-weakly homogeneously Suslin. Let $G\subset\mathbb{P}$ be generic over V. There is then an elementary embedding*

$$j_G: L(A,\mathbb{R})\to L(A_G,\mathbb{R}_G)$$

such that $j_G(\alpha)=\alpha$ for all ordinals α.

In the statement of Theorem 1.2, A is $<\delta$-universally Baire, so that it makes sense to consider the natural reinterpretations A_G, $\mathbb{R}_G$ in the forcing extension $V[G]$. Notice also that the embedding j_G from Theorem 1.2 is canonically defined.

Woodin has shown relative to large cardinals that there is a semiproper forcing extension $V[G]$ of V such that there is some elementary embedding

$$j_G: L(\mathbb{R})\to L(\mathbb{R}_G)$$

which is *not* the identity on the ordinals. So one cannot hope to generalize the above theorem to a larger class of forcings.

[3] AD is the Axiom of Determinacy.

Another result in this direction due to Woodin is the following theorem published in [16]. We state it with the reduced large cardinal assumption Larson obtained in [8, Thm. 3.4.17].

Theorem 1.3 (Woodin with stronger hypothesis, Larson)**.** *Let Γ_{uB} denote the class of universally Baire sets. Suppose there is a proper class of Woodin cardinals. Suppose δ is supercompact and $V_{\delta+1}$ is countable in $V[G]$, G set generic over V. Let $V[G][g]$ be any set generic extension of $V[G]$. Then*

(1) $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}, \Gamma_{uB})^{V[G]} = \Gamma_{uB}^{V[G]}$,
(2) $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}, \Gamma_{uB})^{V[G][g]} = \Gamma_{uB}^{V[G][g]}$,
(3) $(\Gamma_{uB}^{V[G]})^\sharp \subset (\Gamma_{uB}^{V[G][g]})^\sharp$, *where each set in* $\Gamma_{uB}^{V[G]}$ *is identified with its reinterpretation in* $V[G][g]$.

The above theorem says that the theory of $L(\mathbb{R}, \Gamma_{uB})$ is sealed in $V[G]$ with respect to set forcing and hence generalizes the Σ^2_1 absoluteness for $L(\mathbb{R})$. The referee points out to us that to the best of her/his knowledge it is still open whether large cardinals imply that the theory of $L(\mathbb{R}, \Gamma_{uB})$ is sealed in V.

All the above theorems are shown with modern set theoretic methods. Stationary tower forcing is one way to show Σ^2_1 absoluteness for $L(\mathbb{R})$ and is also used to show the above theorem. The second way to show Σ^2_1 absoluteness for $L(\mathbb{R})$ is the extender algebra; also the Embedding Theorem is shown using the extender algebra. Besides stationary tower forcing and the extender algebra there is yet another method to show Σ^2_1 absoluteness for $L(\mathbb{R})$: Todorčević imitated the stationary tower proof by Levy collapsing a measurable Woodin cardinal to ω_2. In such a Levy collapse a ω_2-saturated ideal on ω_1 exists and one can force with this ideal. A detailed proof of Σ^2_1 absoluteness has been published by Farah in [4].

In the literature there are other variants and extensions of Σ^2_1 absoluteness. For example one can enrich the language and add predicates for universally Baire sets of reals; see [6] and [5] for such a result and other extensions of Σ^2_1 absoluteness.

In this paper, we shall prove:

Theorem (Theorem 5.4)**.** Suppose $M^\sharp_{\mathsf{mw}}$ exists and is $(\omega, \kappa+1)$-iterable. Assume CH holds. Furthermore assume $\mathbb{P}$ is a c.c.c. forcing of size κ. Let $G \subset \mathbb{P}$ be V-generic. Then

$$V \models \exists A \subset \mathbb{R} : L(\mathbb{R}, A) \models \phi(A, z, \vec{\alpha})$$

if and only if

$$V[G] \models \exists A \subset \mathbb{R}^{V[G]} : L(\mathbb{R}^{V[G]}, A) \models \phi(A, z, \vec{\alpha}).$$

Here z is a real parameter and $\vec{\alpha}$ are finitely many ordinal parameters.

Note that it is not possible to substitute c.c.c. by ω-closed in the statement of the above theorem: let $G \subset \mathbb{P} = \mathrm{Col}(\omega_1, \omega_2)$ be V generic. Then the following statement in parameters ω_1^V and ω_2^V is true in $V[G]$ but absurd in V:

$$\exists A \subset \mathbb{R}^{V[G]} : L(\mathbb{R}^{V[G]}, A) \models A \text{ codes a surjection from } \omega_1^V \text{ onto } \omega_2^V.$$

So we turn to more restrictive subsets of ω_1: the sets $A \subset \omega_1$ that extend to a class with unique condensation. We develop a genericity iteration for $x \subset \omega_1$ in (reasonable) forcing extensions that extend to a class with unique condensation and use this genericity iteration to show a weak absoluteness result.

These subsets of ω_1 can trivialize: granted a large cardinal hypothesis and an iterability hypothesis, then every A that extends to a class with unique condensation is constructible from a real, see Theorem 6.18.

The referee pointed out to us that it is a direct consequence of [6, Thm. 4.1] that, granted the the existence of a proper class of measurable Woodin cardinals, the conclusion of Theorem 5.4 holds true for *any* poset $\mathbb{P}$ and *any* formula with predicates for universally Baire sets and the nonstationary ideal on ω_1 but without parameters for (uncountable) ordinals.

2. The extender algebra

We begin by recalling the Lindenbaum algebra and some basic facts. Then we will construct the extender algebra.

Definition 2.1. For a cardinal δ and an ordinal $\beta \leq \delta$ let $L_{\beta,\delta,0}$ be the propositional logic with β many propositional variables a_ξ, $\xi < \beta$, allowing conjunctions $\bigwedge_{\xi<\kappa} \phi_\xi$ for all $\kappa < \delta$. In addition to the axioms and rules for finitary propositional logic we have for all $\eta < \kappa < \delta$ and all $\langle \phi_\xi ; \xi < \kappa \rangle$ the abbreviation

$$\bigvee_{\xi<\kappa} \phi_\xi \equiv \neg \bigwedge_{\xi<\kappa} \neg\phi_\xi,$$

the axiom

$$\bigwedge_{\xi<\kappa} \phi_\xi \to \phi_\eta$$

and an infinitary rule of inference for each $\kappa < \delta$

$$\text{from} \vdash \phi_\xi \text{ for all } \xi < \kappa \text{ infer} \vdash \bigwedge_{\xi<\kappa} \phi_\xi.$$

Every $x \subset \beta$ naturally defines a valuation ν_x for $L_{\beta,\delta,0}$ via $\nu_x(a_\xi) = \mathsf{true}$ if and only if $\xi \in x$. For $\phi \in L_{\beta,\delta,0}$ let

$$A_\phi = \{x \subset \beta\,;\, x \models \phi\}.$$

If $T \subset L_{\beta,\delta,0}$ is a theory, we set $A_T = \{x \subset \beta\,;\, x \models T\}$.

Note that $x \models \phi$ is absolute between transitive models of ZFC containing x and ϕ; in particular collapsing δ to ω makes no difference.

Lemma 2.2. *For every $\phi \in L_{\beta,\delta,0}$ the following are equivalent.*

(1) $\vdash \phi$.
(2) $A_\phi = \mathcal{P}(\beta)$ *in all generic extensions.*
(3) $A_\phi = \mathcal{P}(\beta)$ *in all generic extensions by* $\mathrm{Col}(\omega, \delta)$.

Furthermore: for every theory $T \subset L_{\beta,\delta,0}$ and every $\phi \in L_{\beta,\delta,0}$ the following are equivalent:

(1) $T \vdash \phi$.
(2) $A_{T\cup\{\phi\}} = A_T$ *in all generic extensions.*
(3) $A_{T\cup\{\phi\}} = A_T$ *in all generic extensions by* $\mathrm{Col}(\omega, \delta)$.

Proof. We only show the first part since the characterization of $T \vdash \phi$ has almost the same proof. It suffices to show (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (1). Let us suppose that $\vdash \phi$. Since $\vdash \phi$ is upwards absolute, it holds in all generic extensions. So we need to verify the correctness of $L_{\beta,\delta,0}$, i.e. $\vdash \phi \implies x \models \phi$ for all $x \subset \beta$. We omit this argument since it is an easy induction on the rank of the proof for ϕ.

So let us suppose that $A_\phi = \mathcal{P}(\beta)$ holds in all generic extensions by $\mathrm{Col}(\omega, \delta)$. We assume that $\vdash \phi$ fails and construct a forcing of size δ that adds an $x \subset \beta$ such that $x \not\models \phi$. Since the forcing we are going to construct completely embeds into $\mathrm{ro}(\mathrm{Col}(\omega, \delta))$, this will suffice.

Let $\mathbb{P} = \{p \subset L_{\beta,\delta,0}\,;\, p \not\vdash \phi \wedge \mathrm{Card}(p) < \delta\}$ ordered by reverse inclusion. For $p \in \mathbb{P}$ and $\psi \in L_{\beta,\delta,0}$ we claim that either $p \cup \{\psi\}$ or $p \cup \{\neg\psi\}$ belongs to $\mathbb{P}$. Otherwise we would have $p \vdash \psi \to \phi$ and $p \vdash \neg\psi \to \phi$. Hence, by elementary inference rules, we have $p \vdash \phi$, contradiction to $p \in \mathbb{P}$! So the set $D_\psi = \{p \in \mathbb{P}\,;\, \psi \in p \vee \neg\psi \in p\}$ is dense in $\mathbb{P}$, and hence every generic $\Gamma \subset \mathbb{P}$ is forced to be a complete theory such that $\Gamma \not\vdash \phi$. In $V[\Gamma]$ define $x_\Gamma \subset \beta$ by $\xi \in x_\Gamma$ if and only if $a_\xi \in \Gamma$. Then $x_\Gamma \not\models \phi$. $\square$

Let $B_{\beta,\delta,0}$ be the Lindenbaum algebra of $L_{\beta,\delta,0}$, i.e. we set

$$\phi \sim \psi \text{ if and only if } \vdash \phi \leftrightarrow \psi$$

and let $[\phi]$ denote the $\sim$-equivalence class of ϕ. Let

$$\phi \leq \psi \text{ if and only if } \vdash \phi \to \psi,$$

we then set $B_{\beta,\delta,0} = \langle \{[\phi]\,;\, \phi \in L_{\beta,\delta,0}\}, \leq / \sim \rangle$.

For a theory T we define the quotient Lindenbaum algebra $B_{\beta,\delta,0}/T$ as follows:

$$\phi \sim_T \psi \text{ if and only if } T \vdash \phi \leftrightarrow \psi$$

and let $[\phi]_T$ denote the $\sim_T$-equivalence class of ϕ. Let

$$\phi \leq_T \psi \text{ if and only if } T \vdash \phi \to \psi;$$

then $B_{\beta,\delta,0}/T = \langle \{[\phi]_T\,;\, \phi \in L_{\beta,\delta,0}\}, \leq_T / \sim_T \rangle$.

Lemma 2.3. *For every theory T if $B_{\beta,\delta,0}/T$ has the δ-chain condition, then $B_{\beta,\delta,0}/T$ is a complete Boolean algebra.*

Proof. $B_{\beta,\delta,0}$ is δ-complete, since for any $\kappa < \delta$

$$\Sigma_{\xi<\kappa}[\phi_\xi] = \left[\bigvee_{\xi<\kappa} \phi_\xi\right];$$

the same clearly holds for $B_{\beta,\delta,0}/T$. Let $X \subset B_{\beta,\delta,0}/T$. We have to show that ΣX exists. Fix an antichain Y that is maximal with respect to the following property: if $x \in X$, then there is some $y \in Y$ such that $y \leq x$. By the δ-chain condition, Y has cardinality $< \delta$, hence ΣY exists. It is easy to verify that $\Sigma Y = \Sigma X$. □

For $x \subset \beta$ such that $x \models T$ define an ultrafilter $\Gamma_x \subset B_{\beta,\delta,0}/T$ by

$$\Gamma_x = \{[\phi]_T\,;\, x \models \phi\}.$$

Note that Γ_x is well-defined on the $\sim_T$-equivalence classes since $x \models T$. For a generic $\Gamma \subset B_{\beta,\delta,0}/T$ we also set $x_\Gamma = \{\xi < \delta\,;\, [a_\xi]_T \in \Gamma\}$. Then $\Gamma_{x_\Gamma} = \Gamma$ and it is also not difficult to check that $x_{\Gamma_x} = x$ for any x such that $x \models T$.

Lemma 2.4. *Let δ be an ordinal. Assume M is a transitive model of* $\mathsf{ZFC} - \mathsf{Powerset}$ + *"$\mathcal{P}(\delta)$ exists" such that for some $T \in M$ the Boolean algebra $B_{\beta,\delta,0}/T$ has the δ-chain condition. Then for every $x \subset \beta$ such that $x \models T$ the filter $\Gamma_x \subset B_{\beta,\delta,0}/T$ is generic over M. In particular, since Γ_x and x are interdefinable, x is generic over M.*

Proof. Fix $x \subset \beta$, $x \models T$. Assume $\{[\phi_\xi]\,;\, \xi < \kappa\}$ is a maximal antichain of $B_{\beta,\delta,0}/T$ that belongs to M. By Lemma 2.2 it suffices to verify $x \in A_{T\cup\{\phi_\xi\}}$ for some $\xi < \kappa$. Assume otherwise. Let $G \subset \mathrm{Col}(\omega,\delta)$ be V-generic. Note that G is also M-generic. Let $\{\psi_n\,;\, n \in \omega\}$ be an enumeration of $\{\phi_\xi\,;\, \xi < \kappa\}$ in order type ω in $M[G] \subset V[G]$. Since the statement "there is an $x \subset \beta$, $x \models T$ such that $x \not\models \psi_n$ for all $n < \omega$" is a $\underset{\sim}{\Sigma}^1_1$ statement true in $V[G]$, it is also true in $M[G]$. Therefore $\mathrm{Col}(\omega,\delta)$ forces over M that there is an $x \subset \beta$, $x \models T$ such that $x \not\models \bigvee_{\xi<\kappa} \phi_\xi$. Hence by Lemma 2.2 the sentence $\neg \bigvee_{\xi<\kappa} \phi_\xi$ is consistent with T. This statement is absolute and holds in M, contradicting the maximality of the antichain. □

We now define the extender algebra relative to a sequence of extenders $\vec{E}$. For details regarding extender sequences, premice and other concepts of inner model theory we refer the conscientious reader to [13]. Given a premouse $\mathcal{M}$ one can define an iteration game for $\mathcal{M}$. Such a game exists for all ordinals α. In an iteration game of length α two players construct an iteration tree $\mathcal{T}$ of length α on $\mathcal{M}^{\mathcal{T}}_0 = \mathcal{M}$. For each node β in the tree there is a model $\mathcal{M}^{\mathcal{T}}_\beta$, and the models at each direct $<_{\mathcal{T}}$-successor of a node β are obtained by forming an ultrapower of (an initial segment of) $\mathcal{M}^{\mathcal{T}}_\beta$ with some extender. These extenders are chosen by player I at successor stages of the game. Player II only plays at limit stages. It is player II's responsibility to pick well-founded branches through the tree at limit stages; i.e. the direct limit of the models along the branch is well-founded. An (ω,α)-iteration strategy Σ for $\mathcal{M}$ is a winning strategy for player II in the iteration game of length α. So Σ tells us what branches to pick. We will then choose player I's moves to obtain an iteration. A premouse is (ω,α)-iterable if there is an (ω,α)-iteration strategy.

Note that the extender algebra can also be defined for "coarse" iterable models with Woodin cardinals, see [3]; in this case iterability refers to the concept in [9].

Definition 2.5. Let $\mathcal{M} = \langle J_\rho[\vec{E}]; \in, \vec{E}, E_\rho \rangle$ be a premouse such that $\mathcal{M} \models \delta$ is Woodin, let $\beta \le \delta$ and let $\zeta < \rho$. Then $T(\vec{E} \restriction \zeta, \beta) \subset L_{\beta,\delta,0}$ is the theory containing the axioms

$$\bigvee_{\alpha<\kappa} \phi_\alpha \leftrightarrow \bigvee_{\alpha<\lambda} i_E(\langle \phi_\xi ; \xi < \kappa \rangle)_\alpha$$

for E on the sequence $\vec{E} \restriction \zeta$ such that $\mathrm{crit}(E) = \kappa \le \lambda$, and $\nu(E)$ is a $\mathcal{M}$-cardinal such that $i_E(\langle \phi_\xi ; \xi < \kappa \rangle) \restriction \lambda \in \mathcal{J}^{\mathcal{M}}_{\nu(E)}$.

If $\delta = \zeta$, we will simply write $T(\vec{E}, \beta)$ for $T(\vec{E} \restriction \delta, \beta)$. We will call $W_\delta(\vec{E}, \beta) := B_{\beta,\delta,0}/T(\vec{E}, \beta)$ the *extender algebra of* $\vec{E}$ *with* β*-many generators*. If $\beta = \delta = \zeta$, then we will write $W_\delta(\vec{E})$ and $T(\vec{E})$ respectively.

Note that the extender algebra of $\vec{E}$ with β many generators exists in $\mathcal{M}$. If β and $\vec{E}$ are clear from the context, we will omit them. Also note that the extender algebra only depends on $\vec{E} \restriction \delta$ and not on the whole sequence $\vec{E}$.

For us the most interesting case is $\beta = \delta$. The extender algebra with δ-many generators is used to make subsets of ω_1 generic. Sometimes it is convenient to use the extender algebra with less than δ many generators; we will especially need the case with ω-many generators to make reals generic over iterates.

Another well known trick is the following: one can restrict the extender sequence $\vec{E}$ such that only extenders with critical point $> \kappa$ for some $\kappa < \delta$ appear on $\vec{E}$. It is not difficult to see that it is possible to restrict in such a way, that the restricted sequence still witnesses that δ is Woodin. We cannot hope that the restriction of $\vec{E}$ is a fine extender sequence in the sense of [13].

Note that the extender algebra has atoms: for less than δ many generators this is easy to see. In the case of δ-many generators, look at the $L_{\delta,\delta,0}$ statement $\phi :\equiv \bigwedge_{\xi<\kappa} a_\xi$, where κ is a cardinal strong up to δ such that this strongness is witnessed by $\vec{E}$. For all $\kappa < \lambda < \delta$ we have, using the axioms induced by extenders with critical point κ,

$$T(\vec{E}) \vdash \bigvee_{\xi<\kappa} \phi \leftrightarrow \bigvee_{\xi<\lambda} \Big(\bigwedge_{\xi<\lambda} a_\xi \Big),$$

so

$$T(\vec{E}) \vdash \phi \leftrightarrow \bigwedge_{\xi<\lambda} a_\xi.$$

Hence the condition $[\bigwedge_{\xi<\kappa} a_\xi]_{T(\vec{E})}$ is an atom, since $\bigwedge_{\xi<\lambda} a_\xi \in [\bigwedge_{\xi<\kappa} a_\xi]_{T(\vec{E})}$ for $\kappa < \lambda < \delta$.

Let us recall some notation for iteration trees. Let $\mathcal{T} = \langle \alpha, <_\mathcal{T} \rangle$ be an iteration tree of length α in the sense of [13], then $\mathcal{M}^\mathcal{T}_\beta$ denotes the βth model of this tree. The set $[\beta, \gamma]_\mathcal{T}$ is the branch through $\mathcal{T}$ from β to γ. If γ is a $\mathcal{T}$-successor of β, then there is an iteration map $j^\mathcal{T}_{\beta,\gamma} : \mathcal{M}^\mathcal{T}_\beta \to \mathcal{M}^\mathcal{T}_\gamma$. If an extender was picked to continue the iteration at stage β, then we denote this extender by $E^\mathcal{T}_\beta$. Any notions left undefined are to be found in [13].

3. The genericity iteration

The proof of the following theorem produces some of the key technical tools for our later applications, in particular for Theorem 4.1 and its variants. Theorem 3.1 and Corollary 3.5 are also shown in [3]. The paper [6] also contains results which will be shown below.

Theorem 3.1 (Woodin and Steel independently). *Let $\mathfrak{M} = \langle J_\rho[\vec{E}]; \in, \vec{E}, E_\rho\rangle$ be a sound premouse that is active and has a $(\omega, \omega_1 + 1)$-iteration strategy Σ such that $\vec{E}$ witnesses the measurability and Woodiness of δ in $\mathfrak{M}$. Let $W_\delta := W_\delta(\vec{E}, \delta)$ denote the extender algebra of $\vec{E}$ with δ many generators. Let $x \subset \omega_1$. Then there is an iteration tree $\mathcal{T}$ on $\mathfrak{M}$ of height $\omega_1 + 1$ with $i^{\mathcal{T}}_{0,\omega_1} : \mathfrak{M} \to \mathcal{M}^{\mathcal{T}}_{\omega_1}$ such that x is $i^{\mathcal{T}}_{0,\omega_1}(W_\delta)$-generic over $\mathcal{M}^{\mathcal{T}}_{\omega_1}$.*

Note that if $x \subset \omega$, i.e. the situation when the extender algebra is only constructed with ω-many generators, then the measurability of δ is not required, see [13, 7.14]. The proof we are about to give mainly follows the proof of [13, 7.14]; the notes [3] were also very helpful.

Proof. The extender algebra W_δ is built using extenders witnessing that δ is Woodin; we will make use of this fact in the following claim:
Claim 1. W_δ is δ-c.c. in $\mathfrak{M}$.
Proof of Claim 1. Working in $\mathfrak{M}$ we pick a set $A = \{[\phi_\xi]_{T(\vec{E})}\,;\, \xi < \delta\}$. We have to show that A is not an antichain. Let $\kappa < \delta$ be $\langle \phi_\xi; \xi < \delta\rangle$-reflecting and let this fact be witnessed by $\vec{E}$. Let ν be a cardinal such that $\langle \phi_\xi; \xi < \kappa+1\rangle \in \mathcal{J}^{\mathfrak{M}}_\nu$ and let F on $\vec{E}$ witness the reflection of κ at this ν. Let E be the trivial completion of $F \restriction \nu$. Then

$$i_E\Bigg(\bigvee_{\xi<\kappa} \phi_\xi\Bigg) \restriction (\kappa + 1) = \bigvee_{\xi\leq\kappa} \phi_\xi.$$

Hence

$$T(\vec{E}) \vdash \bigvee_{\xi<\kappa} \phi_\xi \leftrightarrow \bigvee_{\xi\leq\kappa} \phi_\xi$$

and hence also

$$T(\vec{E}) \vdash \phi_\kappa \to \bigvee_{\xi<\kappa} \phi_\xi$$

Reformulating this fact gives $[\phi_\kappa]_{T(\vec{E})} \leq [\bigvee_{\xi<\kappa} \phi_\xi]_{T(\vec{E})}$. So A is not an antichain. □(Claim 1)

By Lemma 2.3 W_δ is a complete Boolean algebra. In general an arbitrary $y \subset \delta$ will not satisfy $T(\vec{E})$. We will produce a normal iteration tree $\mathcal{T}$ of height $\omega_1 + 1$ such that for $i^{\mathcal{T}}_{0,\omega_1} : \mathfrak{M} \to \mathcal{M}^{\mathcal{T}}_{\omega_1}$

$$i^{\mathcal{T}}_{0,\omega_1}(\delta) = \omega_1 \text{ and } x \models i^{\mathcal{T}}_{0,\omega_1}(T(\vec{E})),$$

for a fixed $x \subset \omega_1$. If we achieve this, then by Lemma 2.4 the set x will be $j(W_\delta)$-generic over $\mathcal{M}^{\mathcal{T}}_{\omega_1}$.

There is a normal measure U on δ such that U's trivial completion appears on $\vec{E}$. Let us assume that U's index is minimal; i.e. the trivial completion of U is E_{ζ_0}, where ζ_0 is minimal among all ordinals ζ' such $E_{\zeta'}$ is the trivial completion of a normal measure on δ.

Let $i_U : \mathfrak{M} \to \mathfrak{M}' \simeq \mathrm{Ult}(\mathfrak{M}, U)$ and let $\vec{F}$ denote the extender sequence of $\mathfrak{M}'$. The model $\mathfrak{M}$ can see a part of $i_U(T(\vec{E}))$: the coherence of $\mathfrak{M}$'s fine extender sequence implies

$$\vec{F} \restriction \zeta_0 = \vec{E} \restriction \zeta_0.$$

So if E_α is an extender on $\vec{E}$ with $\alpha < \zeta_0$ and $\mathrm{crit}(E) = \kappa$, then $E_\alpha = F_\alpha$; hence every axiom of the form

$$\bigvee_{\alpha<\kappa} \phi_\alpha \leftrightarrow \bigvee_{\alpha<\lambda} i_E(\langle \phi_\xi ; \xi < \kappa \rangle)_\alpha,$$

with $i_{E_\alpha}(\langle \phi_\xi ; \xi < \kappa \rangle) \restriction \lambda \in \mathcal{J}^{\mathfrak{M}'}_{\nu(E_\alpha)}$ is in $\mathfrak{M}$. We introduce a notation for this expanded theory: set $T(\vec{E})^+ = T(\vec{E} \restriction \zeta_0, \zeta_0)$.

We now recursively construct the iteration tree $\mathcal{T}$ for a fixed $x \subset \omega_1$. Before giving more details we outline our plan: we will show that for club many $\gamma < \omega_1$ we have

$$x \cap i^{\mathcal{T}}_{0,\gamma}(\delta) \models i^{\mathcal{T}}_{0,\gamma}(T(\vec{E})).$$

We call such a γ a *weak closure point*. At a weak closure point γ we would like to use the trivial completion of $i^{\mathcal{T}}_{0,\gamma}(U)$ to continue the iteration, but we need to ensure that the resulting iteration is normal. For this we define: γ is a *closure point*, if there is no extender with index $< i^{\mathcal{T}}_{0,\gamma}(\zeta_0)$ that induces an axiom not satisfied by $x \cap i^{\mathcal{T}}_{0,\gamma}(\zeta_0)$, or equivalently:

$$x \cap i^{\mathcal{T}}_{0,\gamma}(\zeta_0) \models i^{\mathcal{T}}_{0,\gamma}(T(\vec{E})^+).$$

Clearly every closure point is a weak closure point. Moreover we will show in the end that there are also club many closure points. Note that this is not trivial: using the agreement of models in an iteration tree, it is not difficult to see that limits of closure points are weak closure points, but in general such limits are not closure points.

We now give more details how to iterate away the least extender which induces an axiom not satisfied by x. Set $\mathcal{M}^{\mathcal{T}}_0 = \mathfrak{M}$ and suppose $\mathcal{T}$ on $\mathcal{M}$ has been constructed up to some countable stage β; furthermore suppose that $D^{\mathcal{T}} = \varnothing$, i.e. the tree has not dropped. If β is a limit ordinal we use the strategy Σ to continue the iteration. If β is a successor there are two cases: if β is a closure point, then we continue the construction of $\mathcal{T}^*$ by picking (the trivial completion of) the least normal measure witnessing that $i^{\mathcal{T}}_{0,\beta}(\delta)$ is measurable.

The second case is: β is not a closure point. Let E be on the $\mathcal{M}_\beta$-sequence such that E induces an axiom of $i_{0,\beta}(T(\vec{E})^+)$ not satisfied by x, and such that $\mathrm{lh}(E)$ is minimal among all extenders on the $\mathcal{M}_\beta$ sequence with this property.

We set $E^{\mathcal{T}}_\beta = E$ and use E according to the rules for ω-maximal iteration trees to extend $\mathcal{T}$ one more step. Note that $\mathrm{lh}(E) < i^{\mathcal{T}}_{0,\beta}(\zeta_0)$ in this case.

The following is easily verified: if an extender E induces an axiom ϕ false of x, then $i_E(\phi)$ is true of x, where i_E is the ultrapower formed with E. In this sense we iterate away false axioms.

We check that all moves are valid in the iteration game. For this we must check that $\gamma < \beta \implies \mathrm{lh}(E^{\mathcal{T}}_\gamma) < \mathrm{lh}(E^{\mathcal{T}}_\beta)$ to see that E is a valid move of player I in the iteration game. There are four cases:

(1) If γ and β are closure points, then $\mathrm{lh}(E^{\mathcal{T}}_\gamma) = \mathrm{lh}(i^{\mathcal{T}}_{0,\gamma}(U)) = i^{\mathcal{T}}_{0,\gamma}(\zeta_0) < i^{\mathcal{T}}_{0,\beta}(\zeta_0) = \mathrm{lh}(i^{\mathcal{T}}_{0,\beta}(U)) = \mathrm{lh}(E^{\mathcal{T}}_\beta)$.

(2) If β is a closure point, then an easy induction, using the definition of closure point, yields that $\mathrm{lh}(E^{\mathcal{T}}_\beta) = i^{\mathcal{T}}_{0,\beta}(\zeta_0)$ is an upper bound for the length of all extenders used at stages $< \beta$ (note that (1) is a special case of (2)).

(3) Now suppose neither γ nor β is a closure point. Suppose the implication does not hold for $\gamma < \beta$. The agreement of models in an ω-maximal iteration tree implies that E_β is on the sequence of $\mathcal{M}^{\mathcal{T}}_\gamma$. We show that $\nu(E^{\mathcal{T}}_\beta)$ is a cardinal of $\mathcal{M}^{\mathcal{T}}_\gamma$: $\nu(E^{\mathcal{T}}_\gamma)$ is a cardinal of $\mathcal{M}^{\mathcal{T}}_\gamma$ and any cardinal $\leq \nu(E^{\mathcal{T}}_\gamma)$ of $\mathcal{M}_\beta$ is a cardinal of $\mathcal{M}^{\mathcal{T}}_\gamma$. By our assumption $\nu(E^{\mathcal{T}}_\beta) < \mathrm{lh}(E^{\mathcal{T}}_\beta) \leq \mathrm{lh}(E^{\mathcal{T}}_\gamma)$ and there are no cardinals in the open interval $]\nu(E^{\mathcal{T}}_\gamma), \mathrm{lh}(E^{\mathcal{T}}_\gamma)[$, so $\nu(E^{\mathcal{T}}_\beta) \leq \nu(E^{\mathcal{T}}_\gamma)$ is a cardinal in $\mathcal{M}^{\mathcal{T}}_\gamma$. So clearly the false axiom induced by $E^{\mathcal{T}}_\beta$ is also induced in $\mathcal{M}^{\mathcal{T}}_\gamma$. But this contradicts our choice of $E^{\mathcal{T}}_\gamma$, since $\mathrm{lh}(E^{\mathcal{T}}_\gamma)$ was not minimal.

(4) Now suppose γ is a closure point and at stage $\beta > \gamma$ we used the extender $E^{\mathcal{T}}_\beta$ to iterate away a false axiom. Like in (3) we suppose towards a contradiction $\mathrm{lh}(E^{\mathcal{T}}_\beta) \leq \mathrm{lh}(E^{\mathcal{T}}_\gamma)$. Then the argument for (3) yields that $E^{\mathcal{T}}_\beta$ is in $\mathcal{M}^{\mathcal{T}}_\gamma$, so in fact $\mathrm{lh}(E^{\mathcal{T}}_\beta) < \mathrm{lh}(E^{\mathcal{T}}_\gamma)$. Moreover $E^{\mathcal{T}}_\beta$ also induces in $\mathcal{M}^{\mathcal{T}}_\gamma$ an axiom false of x, but then γ is not a closure point! Contradiction.

We must check that $[0, \beta+1]_{\mathcal{T}}$ does not drop; that is $E^{\mathcal{T}}_\beta$ measures all subsets of its critical point κ in the model $\mathcal{M}^{\mathcal{T}}_\gamma$ to which it is applied. In the closure point case this is clear. In the other case this is true because $\kappa < \nu(E^{\mathcal{T}}_\gamma)$, $\nu(E^{\mathcal{T}}_\gamma)$ is a cardinal of $\mathcal{M}^{\mathcal{T}}_\gamma$, and $\mathcal{M}^{\mathcal{T}}_\beta$ agrees with $\mathcal{M}^{\mathcal{T}}_\gamma$ below $\nu(E^{\mathcal{T}}_\gamma)$. This finishes the successor step of the construction in both cases.

Set $\mathcal{M}^* = \mathcal{M}^{\mathcal{T}}_{\omega_1}$ and let $b = [0, \omega_1]_{\mathcal{T}}$ denote the branch that yields $\mathcal{M}^*$. We now show that b contains ω_1 many closure points.

So suppose not and aim for a contradiction, say the closure points are bounded by some ζ. Let H_η be large enough such that $x, \mathcal{T}, \mathfrak{M}, \Sigma, \zeta \in H_\eta$ and pick some countable, elementary

$$\pi : H \to H_\eta,$$

such that H is transitive and $\zeta < \gamma := \mathrm{crit}(\pi) = \omega_1^H$ and all the objects mentioned are in the range of π. Let $\pi(\bar{\mathcal{T}}) = \mathcal{T}$ and set $\gamma = \mathrm{crit}(\pi) = \omega_1^H$. Set $\delta^* = i^{\mathcal{T}}_{0,\gamma}(\delta)$ and $\zeta^* = i^{\mathcal{T}}_{0,\gamma}(\zeta_0)$. Like in the proof that the comparison process terminates we get the following claim.

Claim 2. We have
$$V_\gamma^{\mathcal{M}_\gamma^{\bar{\mathcal{T}}}} = V_\gamma^{\mathcal{M}_\gamma^{\mathcal{T}}}$$
and
$$\pi \restriction V_\gamma^{\mathcal{M}_\gamma^{\bar{\mathcal{T}}}} = i_{\gamma,\omega_1}^{\mathcal{T}} \restriction V_\gamma^{\mathcal{M}_\gamma^{\mathcal{T}}}.$$
□

Let $\beta + 1 \in b$ be the $\mathcal{T}$-successor of γ. Because the critical points of the extenders used along b are increasing, we have $\mathrm{crit}(E_\beta^{\mathcal{T}}) = \mathrm{crit}(i_{\gamma,\omega_1}^{\mathcal{T}}) = \gamma$. Also we have an axiom
$$\bigvee_{\xi<\gamma} \phi_\xi \leftrightarrow i_{E_\beta^{\mathcal{T}}}\Big(\bigvee_{\xi<\gamma} \phi_\xi\Big) \restriction \lambda$$
of $i_{0,\beta}^{\mathcal{T}}(T(\vec{E})^+)$ induced by $E_\beta^{\mathcal{T}}$ that does not hold for $x \cap \zeta^*$. The falsity of this axiom means that the right hand side is true of $x \cap \zeta^*$, but the left hand side is not. But now $\bigvee_{\xi<\gamma} \phi_\xi$ is essentially a subset of γ, and therefore, by the agreement of the models of the iteration, contained as an element in $\mathcal{M}_\gamma^{\mathcal{T}}$. Recall that $\lambda < \nu(E_\beta^{\mathcal{T}})$; since generators are not moved on $\mathcal{T}$
$$i_{E_\beta^{\mathcal{T}}}\Big(\bigvee_{\xi<\gamma} \phi_\xi\Big) \restriction \lambda = i_{\gamma,\omega_1}^{\mathcal{T}}\Big(\bigvee_{\xi<\gamma} \phi_\xi\Big) \restriction \lambda = \pi\Big(\bigvee_{\xi<\gamma} \phi_\xi\Big) \restriction \lambda.$$
Now $\gamma \leq \zeta^*$ and $x \cap \zeta^* \not\models \bigvee_{\xi<\gamma} \phi_\xi$ implies that $x \cap \gamma \not\models \bigvee_{\xi<\gamma} \phi_\xi$. Since $x \cap \gamma \in H$ and $\pi(x \cap \gamma) = x$, we have $x \not\models \pi(\bigvee_{\xi<\gamma} \phi_\xi)$. This contradicts the fact that $x \cap \zeta^*$ satisfies the initial segment $i_{E_\beta^{\mathcal{T}}}(\bigvee_{\xi<\gamma} \phi_\xi) \restriction \lambda$ of this disjunction. In other words γ is a closure point: contradiction!

So b contains uncountably many closure points (in fact club many, but we have no use for this fact here), hence the least normal measure on δ witnessing the measurability of δ (resp. its image) was used ω_1-many times. For a closure point γ, note that $x \cap i_{0,\gamma}^{\mathcal{T}}(\delta) \models i_{0,\gamma}^{\mathcal{T}}(T(\vec{E}))$, so $x \models i_{0,\gamma}^{\mathcal{T}}(T(\vec{E}))$. Also note that the existence of unboundedly many closure points in b implies $i_{0,\omega_1}^{\mathcal{T}}(\delta) = \omega_1$. We need to show $x \models i_{0,\omega_1}^{\mathcal{T}}(T(\vec{E}))$, i.e. ω_1 is a weak closure point. For this fix some $\psi \in i_{0,\omega_1}^{\mathcal{T}}(T(\vec{E}))$. Clearly there is some $\bar{\psi}$ and some closure point $\gamma \in b$ such that $i_{\gamma,\omega_1}^{\mathcal{T}}(\bar{\psi}) = \psi$. But since $\bar{\psi} \in i_{0,\gamma}^{\mathcal{T}}(T(\vec{E}))$, it is basically a bounded subset of $i_{0,\gamma}^{\mathcal{T}}(\delta) = \mathrm{crit}(i_{\gamma,\omega_1}^{\mathcal{T}})$, hence $\bar{\psi} = \psi$. Since $x \cap i_{0,\gamma}^{\mathcal{T}}(\delta) \models \bar{\psi}$ clearly $x \models \psi$. □

We will call an iteration as above a *genericity iteration*. In the following we will refine the concept of genericity iteration. Note that the argument above for $x \models i_{0,\omega_1}^{\mathcal{T}}(T(\vec{E}))$ also yields that limits of closure points are weak closure points and moreover that the weak closure points are club in ω_1.

3.2. **First applications of genericity iterations.** We use genericity iterations to present Corollary 3.5, an absoluteness argument due to Steel and Woodin independently; this is not the most general result though, but the proof is quite easy to grasp. We will refine the argument later to add more parameters and to obtain Σ^2_1 absoluteness, see Theorem 4.1.

Definition 3.3. Let $x \subset \mathsf{OR}$ and let $\vec{E}$ be a fine extender sequence over x. We let $M^\sharp_{\mathsf{mw}}(x) = \langle J_\beta(x)^{\vec{E}}; \in, x, \vec{E} \restriction \beta, E_\beta \rangle$ denote the minimal sound x-premouse that satisfies the following properties:

(1) $M^\sharp_{\mathsf{mw}}(x)$ is active, i.e. $E_\beta \neq \varnothing$, and $\mathrm{crit}(E_\beta) > \delta$,
(2) $M^\sharp_{\mathsf{mw}}(x)$ has a $(\omega, \omega_1 + 1)$-iteration strategy Σ,
(3) $\vec{E}$ witnesses the measurability and Woodiness of δ.

Note that we demand that the witnesses for the measurability and Woodiness of δ are on $\vec{E}$. We can describe the top measure of $M^\sharp_{\mathsf{mw}}$; we do not prove the following fact, it follows from the minimality of $M^\sharp_{\mathsf{mw}}$. If $M^\sharp_{\mathsf{mw}}(x) = \langle J_\beta(x)^{\vec{E}}; \in, x, \vec{E} \restriction \beta, E_\beta \rangle$, then on $\vec{E} \restriction \beta$ there is no extender witnessing the measurability of $M^\sharp_{\mathsf{mw}}(x)$'s measurable Woodin; E_β is in fact the only extender so that E_β is the trivial completion of a normal measure on δ. Furthermore, since we use the indexing of [13], $\beta = \delta^{++\mathrm{Ult}(M^\sharp_{\mathsf{mw}}(x), E_\beta)}$. Without a proof we state the following fact that we will make use of without further notice:

Remark 3.4. If $M^\sharp_{\mathsf{mw}} := M^\sharp_{\mathsf{mw}}(\varnothing)$ exists, then $M^\sharp_{\mathsf{mw}}(x)$ exists for every $x \subset \omega$.

The key ingredients to show the above are the following: first one observes that even without the least Woodin cardinal η of $M^\sharp_{\mathsf{mw}}$ there are measure one many Woodins in $M^\sharp_{\mathsf{mw}}$, say the thinned out sequence of extenders with critical point $> \eta$ is called $\vec{F}$. Then one performs a genericity iteration to make x generic over some iterate of $M^\sharp_{\mathsf{mw}}$ for W_η, where W_η is constructed with ω-many generators. So in the generic extension containing x (the image of) $\vec{F}$ witnesses that there is still an iterable system of extenders.

Contemporary inner model theory has not yet been able to construct $M^\sharp_{\mathsf{mw}}$ under any hypothesis.

Notice that the next theorem goes beyond Shoenfield's Absoluteness Theorem due to the requirement that $\sup(X) = \omega_1$ (i.e., that X be an unbounded subset of ω_1).

Corollary 3.5 (Woodin and Steel independently)**.** *Suppose $M^\sharp_{\mathsf{mw}}$ exists and is $(\omega, \omega_1 + 2)$-iterable. Let ϕ be a statement in the language of set theory with one free variable. There is a statement ϕ^*, definable from ϕ in a uniform way, such that*

$$\exists X \subset \omega_1 : (\sup(X) = \omega_1 \wedge L[X] \models \phi(X))$$

if and only if

$$M^\sharp_{\mathsf{mw}} \models \phi^*.$$

Proof. Let δ be the measurable Woodin cardinal of $M^\sharp_{\mathsf{mw}}$. We will define ϕ^* in a moment. Suppose $L[X] \models \phi(X)$ for some unbounded $X \subset \omega_1$. If necessary we modify ϕ and X a little so that $L[X] \models \omega_1 = \omega_1^V$. By Theorem 3.1 there is an elementary map $j : M^\sharp_{\mathsf{mw}} \to \mathfrak{M}^*$ such that $j(\delta) = \omega_1$ and X is generic over $\mathfrak{M}^*$. Then $\mathfrak{M}^*$ has a top measure U and the critical point of this top measure is $j(\delta) = \omega_1$. Let $h : \mathfrak{M}^{**} = \mathrm{Ult}(\mathfrak{M}^*, U)$ and note that the extender U must be applied to $\mathfrak{M}^*$ by the rules of the iteration game. Since $V_{\omega_1}^{\mathfrak{M}^{**}} = V_{\omega_1}^{\mathfrak{M}^*}$, we have that X is also generic over $\mathfrak{M}^{**}$. Because $h(\omega_1)$ is still measurable in $\mathfrak{M}^{**}[X]$, we have that $h(\omega_1)$ is an X-indiscernible. So

$$L_{h(\omega_1)}[X] \models \phi(X).$$

In $\mathfrak{M}^*$, the existence of an X, such that X is generic for the extender-algebra W_{ω_1} and $L_{h(\omega_1)}[X] \models \phi(X)$ in the ultrapower with U is a first order statement in the parameters ω_1 and U, call it $\phi^*(\omega_1, U)$. By elementarity $\phi^*(\delta, \bar{U})$ holds in $M_{\mathsf{mw}}^\sharp$, where $\bar{U}$ is $M_{\mathsf{mw}}^\sharp$'s top-measure.

For the other direction pick some $G \subset W_\delta$, $G \in V$ that is generic over $M_{\mathsf{mw}}^\sharp$ such that for some $Y \subset \omega_1^{M_{\mathsf{mw}}^\sharp[G]}$ unbounded in $\omega_1^{M_{\mathsf{mw}}^\sharp[G]}$, $Y \in M_{\mathsf{mw}}^\sharp[G]$ and Y is a witness for ϕ^*, say $p \in G$ is a condition that forces Y to be a witness for ϕ^*. Then we iterate $M_{\mathsf{mw}}^\sharp$ linearly ω_1-many times using only its top measure on δ. We need to apply the technique we call "piecing together end-extending generics" from the proof of Theorem 4.1; since we give a very detailed and far more general version of this technique there, we omit the details of this construction and just sum up the result. Set $G_0 = G$. For each countable iterate $\mathfrak{M}_i$ of $M_{\mathsf{mw}}^\sharp$, $i < \omega_1$, obtained by linearly iterating, we have a generic $G_i \subset j_{0,i}(W_\delta)$, where $j_{0,i} : M_{\mathsf{mw}}^\sharp \to \mathfrak{M}_i$ is the iteration map. For $i < j$ the generics G_i, G_j end-extend each other, i.e.: $G_i \subset G_j$. Then $G_{\omega_1} = \bigcup\{G_i\,;\, i < \omega_1\}$ is generic over $\mathfrak{M}_{\omega_1}$. We have $p \in G_{\omega_1}$, so Y_{ω_1} has the desired properties, where Y_{ω_1} is calculated from G_{ω_1} in the same way as Y was calculated from G. □

We then have the following obvious corollary which looks like a bounded forcing axiom, except that it lacks interesting parameters. We refer the reader to [1] on bounded forcing axioms.

Corollary 3.6. *Suppose $M_{\mathsf{mw}}^\sharp$ exists and furthermore suppose that the $(\omega, \omega_1 + 1)$-iterability of $M_{\mathsf{mw}}^\sharp$ is preserved in all generic extensions. Then for every forcing $\mathbb{P}$ and every Δ_0 statement ϕ with one free variable*

$$H_{\omega_2}^{V^{\mathbb{P}}} \models \exists X \subset \omega_1(\sup(X) = \omega_1 \wedge \phi(X)) \implies$$
$$H_{\omega_2} \models \exists X \subset \omega_1(\sup(X) = \omega_1 \wedge \phi(X)).$$

□

The above corollary is suboptimal. With different methods one can show far more than the above corollary using a weaker large cardinal hypothesis: in [6, Thm. 5.2], assuming the existence of two Woodin cardinals but not the existence of $M_{\mathsf{mw}}^\sharp$, a similar absoluteness result is shown using a more expressive language as in the corollary above. The language in [6, Thm. 5.2] in addition contains a predicate for NS_{ω_1} and predicates for all universally Baire sets of reals, as well as constants for every member of H_{ω_1}.

3.7. **Adding parameters.** We now explore what parameters one can reasonably hope to add to the statement of the above corollaries. The arguments to follow are blueprints which can be applied for example to add parameters to the statement of Theorem 4.1. Let us first consider a real z: if we demand that $M^\sharp_{\mathsf{mw}}$ exists then we have remarked that $M^\sharp_{\mathsf{mw}}(z)$ exists.

Corollary 3.8. *Suppose $M^\sharp_{\mathsf{mw}}$ exists. Let z be a real and suppose that in all generic extensions $M^\sharp_{\mathsf{mw}}(z)$ is (ω, ω_1+1)-iterable. Then for every forcing $\mathbb{P}$ and every Δ_0 statement ϕ with two free variables*

$$H^{V^{\mathbb{P}}}_{\omega_2} \models \exists X \subset \omega_1(\sup(X) = \omega_1 \wedge \phi(X, z)) \implies$$
$$H_{\omega_2} \models \exists X \subset \omega_1(\sup(X) = \omega_1 \wedge \phi(X, z)).$$

□

We now study parameters for which forcing names exist in some generic extension of $M^\sharp_{\mathsf{mw}}(z)$. We need some notation first.

Definition 3.9. For $S \subset \omega_1$ let $\mathrm{code}(S) = \{x \in \mathsf{WO}\,;\, \|x\| \in S\}$. A set $A \subset \mathbb{R}$ is *closed under ordertypes* if $x \in A \cap \mathsf{WO}$ and $\|x\| = \|y\|$ for some y implies $y \in A$. Let $A \subset \mathbb{R}$, we then set $\mathrm{decode}(A) = \{\alpha < \omega_1\,;\, \exists x \in A \cap \mathsf{WO} : \|x\| = \alpha\}$. Let $\mathfrak{M}$ be a (ω, ω_1+1)-iterable premouse that contains a Woodin cardinal δ and let $A \subset \mathbb{R}$. We say a term for a set of reals $\tau \in \mathfrak{M}^{\mathrm{Col}(\omega,\delta)}$ *captures* A if for all countable iterations $\pi : \mathfrak{M} \to \mathfrak{M}^*$ and all $g \subset \pi(\mathrm{Col}(\omega, \delta))$ generic over $\mathfrak{M}^*$, $g \in V$,

$$\pi(\tau)^g = A \cap \mathfrak{M}^*.$$

We will say that $S \subset \omega_1$ is *captured by* $\tau \in \mathfrak{M}^{\mathrm{Col}(\omega,<\delta)}$ *over* $\mathfrak{M}$ if for all iterations $\pi : \mathfrak{M} \to \mathfrak{M}^*$ such that $\pi(\delta) = \omega_1$ and for all $g \subset \pi(\mathrm{Col}(\omega, <\delta))$ generic over $\mathfrak{M}^*$, $g \in V$,

$$\pi(\tau)^g \cap \mathsf{WO} = \mathrm{code}(S) \cap \mathfrak{M}^*[g].$$

Note that equivalently we could say

$$\mathrm{decode}(\pi(\tau)^g) = S \cap \omega_1^{\mathfrak{M}^*[g]}$$

in the last part of the definitions above. Moreover note that in the presence of large cardinals lots of definable sets can be captured.

Lemma 3.10. *Let $\mathfrak{M} = \langle J_\beta[\vec{E}]; \in, \vec{E}, E_\beta\rangle$ be a sound premouse that is active and has a (ω, ω_1+1)-iteration strategy Σ such that $\vec{E}$ witnesses the measurability and Woodiness of δ. Furthermore assume that $S \subset \omega_1$ is captured by τ over $\mathfrak{M}$. Let ϕ be a statement in the language of set theory with two free variables, then*

$$\exists X \subset \omega_1 : L[X, S] \models \phi(X, S)$$

if and only if

$$\exists p \in W^{\mathfrak{M}}_\delta : p \Vdash^{\mathfrak{M}} \exists X \subset \check{\delta} : L_\kappa[X, \mathrm{decode}(\tau)] \models \phi(X, \mathrm{decode}(\tau)),$$

where κ is the critical point of the top measure of $\mathfrak{M}$.

Proof. The proof is similar to the proof of 3.5. First assume $L[X,S] \models \phi(X,S)$ for some $X \subset \omega_1$. Then produce a genericity iteration $\pi : \mathfrak{M} \to \mathfrak{M}^*$ such that $\pi(\delta) = \omega_1$ and X is $\pi(W_\delta)$-generic over $\mathfrak{M}^*$. So $\pi(\tau)^X \cap \mathsf{WO} = \mathrm{code}(S) \cap \mathfrak{M}^*[X]$ by our hypothesis. So $S \in \mathfrak{M}^*[X]$. Then by (X,S)-indiscernibility of $\pi(\kappa)$

$$L[X,S]_{\pi(\kappa)} \models \phi(X,S),$$

where κ is the critical point of the top measure of $\mathfrak{M}$. It remains to appeal to the elementarity of π.

The presence of τ does change the proof of the other direction. We need to piece together end-extending local generic objects for the other direction. Since a more complex argument of this type is given in the proof for 4.1, we omit it here. □

Unfortunately there are serious restrictions on the complexity of a parameter S such that code(S) is captured.

Lemma 3.11. *Let $S \subset \omega_1$ be such that* code(S) *is captured by some τ over some countable sound premouse $\mathfrak{M}$ that is active and (ω, ω_1+1)-iterable. Then*

(1) there is a $\boldsymbol{\Delta}^1_2$-set A such that $S = \mathrm{decode}(A)$;
(2) if furthermore sharps for all reals exist, then either S or $\omega_1 \setminus S$ contains a club.

Proof. We show how to calculate $A \subset \mathsf{WO}$ with the desired properties. For this let us fix a a cardinal $\delta \in \mathfrak{M}$ such that there is a total measure U on δ. Let $x \in \mathsf{WO}$, say $\|x\| = \alpha$. Pick a countable linear iteration $\pi : \mathfrak{M} \to \mathfrak{M}^*$ that is obtained using only U and its images such that $\pi(\delta) > \alpha$. In V pick a $g \subset \mathrm{Col}(\omega, <\pi(\delta))$ generic over $\mathfrak{M}^*$. If $x \in \mathfrak{M}^*[g]$ then by the choice of τ

$$x \in \pi(\tau)^g \iff \alpha \in S.$$

So we define A such that $x \in A$ if and only if

$$\forall\pi\forall g[\phi_0(\pi, U, \mathfrak{M}) \wedge \pi(\delta) > \|x\| \wedge \phi_1(\pi, g, \delta) \to \\ \exists y \in \mathsf{WO} \cap \mathfrak{M}^* : \|y\| = \|x\| \wedge y \in \pi(\tau)^g],$$

here $\phi_0(\pi, U, \mathfrak{M})$ expresses that π is a linear iteration of $\mathfrak{M}$ using only U and its images and $\phi_1(\pi, g, \delta)$ expresses that g is $\mathrm{Col}(\omega, \pi(\delta))$ generic over the last model of the iteration π and $\mathfrak{M}^*$ denotes π's last model.

We can also calculate A in the following fashion: $x \in A$ if and only if

$$\exists\pi\exists g[\phi_0(\pi, U, \mathfrak{M}) \wedge \pi(\delta) > \|x\| \wedge \phi_1(\pi, g, \delta) \wedge \\ (\exists y \in \mathsf{WO} \cap \mathfrak{M}^* : \|y\| = \|x\| \wedge y \in \pi(\tau)^g)].$$

By choosing a nice coding we see that the first formula defining A is $\Pi^1_2(z)$ where z is a real coding $(\mathfrak{M}, \delta, U)$ and the second is $\Sigma^1_2(z)$. Hence A is $\Delta^1_2(z)$.

This clearly implies that S is constructible from the real z. If $z^\sharp$ exists, then there is either a z-indiscernible in S or in $\omega_1 \setminus S$, hence there are either club many z-indiscernibles in S or in $\omega_1 \setminus S$. □

This shows that we cannot hope to capture (a code for) a stationary and costationary set if we have sharps for reals. Also we cannot capture a ladder system for ω_1, since such a system would allow to partition ω_1 into ω_1-many stationary sets (a ladder system is in fact the amount of choice one needs to calculate such a partition).

4. Σ^2_1 ABSOLUTENESS

We now work a little harder to obtain Σ^2_1 absoluteness which was first shown by Woodin using the stationary tower forcing. Our proof differs substantially from Woodin's first proof and uses genericity iterations instead of the stationary tower. The proof we are going to present is due to Steel and Woodin independently. The paper [3] also contains a write-up of this result.

Theorem 4.1 (Woodin)**.** *Suppose $M^\sharp_{\mathsf{mw}}$ exists and is (ω, ω_1+1)-iterable in all set forcing extensions. Assume* CH *holds. Let $\mathbb{P}$ be a notion of forcing and let $G \subset \mathbb{P}$ be V-generic. Let z be a real in V. If in $V[G]$*

$$\exists A \subset \mathbb{R}^{V[G]} L(\mathbb{R}^{V[G]}, A) \models \phi(A, z),$$

then in V

$$\exists A \subset \mathbb{R}^{V} L(\mathbb{R}^{V}, A) \models \phi(A, z).$$

Furthermore if CH *holds in $V^{\mathbb{P}}$, then the converse is true.*

Before we give proof, we want to state three Lemmata. The first one is part of the folklore; for a more general result see (for example) [7, 10.10].

Lemma 4.2. *Let $\mathbb{P}$ and $\mathbb{Q}$ be notions of forcings in V such that in $V^{\mathbb{P}}$ for all $q \in \mathbb{Q}$ a $\mathbb{Q}$-generic containing q exists. Then a $\mathbb{Q}$ name $\dot{R}$ exists such that $V^{\mathbb{Q}*\dot{R}} = V^{\mathbb{P}}$.* □

The above Lemma is shown using Boolean algebras. If $\mathbb{P}$ and $\mathbb{Q}$ are complete Boolean algebras, then the conclusion of the above Lemma reads: $\mathbb{Q}$ is a regular subalgebra of $\mathbb{P}$.

The second lemma is also part of the folklore; we do not explicitly state it for fine-structural models since it clearly also holds in the coarse case.

Lemma 4.3. *Let $\mathbb{P}$ be a complete Boolean algebra that satisfies the δ-c.c. and let $j : V \to M$ be an elementary embedding with critical point δ. Then j"$\mathbb{P}$ is a regular subalgebra of $j(\mathbb{P})$. Furthermore if δ is Woodin as witnessed by the extender-sequence $\vec{E}$, $\omega \leq \beta \leq \delta$ and $\mathbb{P} = W_\delta = W_\delta(\beta, \vec{E})$, then the embedding*

$$[\phi]_{T(\vec{E})} \mapsto [\phi]_{j(T(\vec{E}))}$$

witnesses that W_δ is a regular subalgebra of $j(W_\delta)$.

Proof. As to the first part, let A be a maximal antichain of $\mathbb{P}$. Then $\mathrm{Card}(A) < \delta$, so $j(A) = j$"A is a maximal antichain of $j(\mathbb{P})$. Hence j"$\mathbb{P}$ is a regular subalgebra of $j(\mathbb{P})$.

The second part is immediate. □

The third lemma discusses the relationship of the extender algebra with ω-many generators and small forcing. It is a slight generalization of the genericity iteration to make a real generic.

Lemma 4.4 (Woodin)**.** *Let $\mathfrak{M} = \langle J_\rho[\vec{E}]; \in, \vec{E}, E_\rho \rangle$ be a sound premouse that is active and has a $(\omega, \omega_1 + 1)$-iteration strategy Σ such that $\vec{E}$ witnesses the Woodiness of δ in $\mathfrak{M}$. Let $\mathbb{P} \in V_\kappa^{\mathfrak{M}}$, $\kappa < \delta$, be a notion of forcing. Let $\vec{F}$ denote the total extenders of $\vec{E}$ with critical point $> \kappa$ and index $< \delta$. Let $x \subset \omega$. Then the following hold true:*

(1) If $g \subset \mathbb{P}$ is generic over $\mathfrak{M}$ and $\alpha < \delta$ such that $F_\alpha \neq \varnothing$, then there is a total extender $\tilde{F}_\alpha \in \mathfrak{M}[g]$ such that $\tilde{F}_\alpha \cap \mathfrak{M} = F_\alpha$ (here, F_α is the extender of $\vec{F}$ with index α). We will say F_α induces *$\tilde{F}_\alpha$.*

(2) For $g \subset \mathbb{P}$ is generic over $\mathfrak{M}$, let $W_\delta^g := W_\delta^g(\vec{F}, \omega)$ denote the extender algebra with ω many generators calculated from the set of induced extenders $\{\tilde{F}_\alpha\,;\, \alpha < \delta\}$ in $\mathfrak{M}[g]$ and let $W_\delta^{\dot{g}}$ denote a name for that forcing. If $g \subset \mathbb{P}$ is generic over $\mathfrak{M}$, then there is an iteration tree $\mathcal{T}$ on $\mathfrak{M}$ of some height $\alpha + 1 < \omega_1$ such that:

(a) if E is the extender we apply at stage β of the construction of $\mathcal{T}$, then E is on the sequence $i_{0,\beta}^{\mathcal{T}}(\vec{F})$;

(b) $\mathrm{crit}(i_{0,\alpha}^{\mathcal{T}}) > \kappa$;

(c) if $g \subset \mathbb{P}$ is generic over $\mathfrak{M}$, then g is generic over $\mathcal{M}_\alpha^{\mathcal{T}}$, and moreover x is generic for $i_{0,\alpha}^{\mathcal{T}}(W_\delta^{\dot{g}})^g$ over $\mathcal{M}_\alpha^{\mathcal{T}}[g]$.

(3) Moreover there is an iteration tree $\mathcal{T}$ on $\mathfrak{M}$ of some height $\alpha + 1 < \omega_1$ such that for all $g \subset \mathbb{P}$ generic over $\mathfrak{M}$ the real x is generic for $i_{0,\alpha}^{\mathcal{T}}(W_\delta^{\dot{g}})^g$ over $\mathcal{M}_\alpha^{\mathcal{T}}[g]$.

We will give the key ideas for this Lemma only. For (1) one needs to run the argument that shows that the measurability of some cardinal is preserved under small forcing. Note that (2) of the above lemma is identical to [13, 7.16] and (3) has almost the same proof: one performs a genericity iteration for x using only the extenders from $\vec{F}$ and their images. We indicate how to pick extenders to obtain a tree like in (3). At stage β of the tree construction do the following: if there is a condition $p \in \mathbb{P}$ and an extender $E \in i_{0,\beta}^{\mathcal{T}}(\vec{F})$ such that p forces that $\tilde{E}$ induces an axiom false of x, then pick the minimal such E to continue the construction of $\mathcal{T}$. The rest runs similar to the proof of [13, 7.14]. It is routine to check that the extenders on $\vec{F}$ witness that δ is Woodin and that the extenders induced from $\vec{F}$ continue to do so in $\mathfrak{M}^{\mathbb{P}}$. So W_δ^g is well-defined and δ-c.c. We shall give no more details.

We now prove 4.1.

Proof. We fix $G \subset \mathbb{P}$ generic over V and some $A \in \mathcal{P}(\mathbb{R})^{V[G]}$ such that

$$\psi(A) :\equiv L(\mathbb{R}^{V[G]}, A) \models \phi(A, z),$$

where $z \in \mathbb{R}^V$. We force CH over $V[G]$ using $\mathrm{Col}(\omega_1, 2^\omega)^{V[G]}$ and call the resulting extension W. For a while we will work in W. We code A and $\mathbb{R}^{V[G]}$

by a set $B \subset \omega_1$. Clearly there is a formula ψ' such that $L[B] \models \psi'(B)$ if and only if $\psi(A)$ holds. By our hypothesis, we have that $\mathfrak{M} := M^\sharp_{\mathsf{mw}}(z)$ has a $(\omega, \omega_1 + 1)$-iteration strategy Σ in W, so by Corollary 3.5 there is an iteration $j : \mathfrak{M} \to \mathfrak{M}^*$ such that B is generic over $\mathfrak{M}$ for the extender algebra. Let δ denote $\mathfrak{M}$'s measurable Woodin and let W_δ be the extender algebra calculated in $\mathfrak{M}$ relative to $\mathfrak{M}$'s extender sequence $\vec{E}$. Hence by elementarity of j there is a condition $p \in W_\delta$, say $p = [\phi]_{T(\vec{E})}$, such that

$$p \Vdash_{\mathfrak{M}} \check{\delta} = \omega_1 \wedge \exists \dot{B} : L[\dot{B}] \models \psi'(\dot{B}).$$

Our plan is as follows: we will construct in V an iteration tree $\mathcal{T}$ of length $\omega_1 + 1$ and Γ generic over the last model $\mathfrak{M}^*$ of $\mathcal{T}$ such that $p \in \Gamma$, $\mathbb{R}^V \subset \mathfrak{M}^*[\Gamma]$ and p is not moved by $j^{\mathcal{T}}_{0,\omega_1}$. The tree $\mathcal{T}$ will be constructed in ω_1 many rounds; for each round i there is an ordinal α_i, and in round i we will construct the map

$$j^{\mathcal{T}}_{\alpha_i,\alpha_{i+1}} : \mathcal{M}^{\mathcal{T}}_{\alpha_i} \to \mathcal{M}^{\mathcal{T}}_{\alpha_{i+1}}.$$

Before we can go into details we need to care for a minor technical thing. Recall that the members of W_δ are of the form $[\phi]_{T(\vec{E})}$; alternatively we could have constructed W_δ using the $<_{M^\sharp_{\mathsf{mw}}}$-least formula in an equivalence class. So for the rest of the proof we assume without loss of generality that W_δ contains formulae and so the maps of the form

$$[\phi]_{j^{\mathcal{T}}_{0,\alpha_i}(T(\vec{E}))} \mapsto [\phi]_{j^{\mathcal{T}}_{0,\alpha_{i+1}}(T(\vec{E}))}$$

are the identity on formulae. This identification eases the reasoning considerably, since $L_{\delta,\delta,0}$ formulae are not moved by maps with critical point δ. One consequence we will need later is that nice names for reals are not moved by such maps; another consequence of this and Lemma 4.3 is the following: if $j : \mathfrak{M} \to \mathfrak{M}'$ has critical point δ, then $W_\delta = j"W_\delta$ is a regular subalgebra of $j(W_\delta)$.

For book-keeping pick an enumeration $\{x_i\,;\, 0 < i < \omega_1 \text{ is not a limit ordinal}\}$ of the reals in V. We call what follows *piecing together end extending generics.* We now construct in V an iteration tree $\mathcal{T}$ of length $\omega_1 + 1$, a sequence of ordinals $\langle \alpha_i ; i < \omega_1 + 1\rangle$ and a sequence of generics $\langle \Gamma_i ; i < \omega_1 + 1\rangle$ such that

(1) $\langle \alpha_i ; i < \omega_1\rangle$ is a normal sequence, i.e. $\{\alpha_i\,;\, i < \omega_1\}$ is closed unbounded in ω_1 and $\alpha_{\omega_1} = \omega_1$,
(2) $p \in \Gamma_0$,
(3) p is not moved by $j^{\mathcal{T}}_{0,\omega_1}$,
(4) $\mathrm{crit}(j^{\mathcal{T}}_{\alpha_i,\omega_1}) = j^{\mathcal{T}}_{0,\alpha_i}(\delta)$,
(5) $\Gamma_i \subset j^{\mathcal{T}}_{0,\alpha_i}(W_\delta)$ is generic over $\mathcal{M}^{\mathcal{T}}_{\alpha_i}[\Gamma_j]$ for $j < i$,
(6) if $i > 0$ is not a limit ordinal, then $x_i \in \mathcal{M}^{\mathcal{T}}_{\alpha_i}[\Gamma_i]$ and
(7) if $i \leq j$, then $\Gamma_i \subset \Gamma_j$.

Let U denote (the trivial completion of) the least normal measure on δ that is on $\vec{E}$. Set $\mathcal{M}^{\mathcal{T}}_0 = \mathfrak{M}$ and set $\alpha_0 = 0$. In V we can pick Γ_0 such that $p \in \Gamma_0$. This finishes the construction of α_0 and Γ_0.

At all stages α_i of the iteration we use the trivial completion of $i^{\mathcal{T}}_{0,\alpha_i}(U)$ to continue the iteration. At limit stages $\lambda \leq \omega_1$ we set $\alpha_\lambda = \sup\{\alpha_i\,;\, i < \lambda\}$ and we use the iteration strategy Σ to continue the iteration. We set

$$\Gamma_\lambda := \bigcup\{\Gamma_i\,;\, i < \lambda\} \subset j^{\mathcal{T}}_{0,\alpha_\lambda}(W_\delta).$$

All antichains of the extender algebra are small and $\mathrm{crit}(j^{\mathcal{T}}_{\alpha_i,\omega_1}) = j^{\mathcal{T}}_{0,\alpha_i}(\delta)$ for $i < \lambda$, so we have that Γ_λ is generic over $\mathcal{M}^{\mathcal{T}}_{\alpha_\lambda}$.

We now discuss the successor case. Fix $i < \omega_1$ and let $\gamma = \alpha_i$. We continue the iteration by picking $j^{\mathcal{T}}_{0,\gamma}(U)$ as the next extender. At stage $\gamma + 1$ let η_γ be the least Woodin cardinal in $\mathcal{M}^{\mathcal{T}}_{\gamma+1}$ in the open interval $]j^{\mathcal{T}}_{0,\gamma}(\delta), j^{\mathcal{T}}_{0,\gamma+1}(\delta)[$. Let $\vec{F}$ consist of the extenders on $\mathcal{M}^{\mathcal{T}}_{\gamma+1}$'s extender sequence with critical point $> j^{\mathcal{T}}_{0,\gamma}(\delta)$ and index $< \eta_\gamma$ that witness that η_γ is Woodin. As in Lemma 4.4 we define from $\vec{F}$ an extender algebra $W^{\Gamma_\gamma}_{\eta_\gamma} \in \mathcal{M}^{\mathcal{T}}_{\gamma+1}[\Gamma_\gamma]$ with ω-many generators. We now apply (2) of Lemma 4.4: we continue the iteration tree $\mathcal{T}$ by performing a genericity iteration to make x_{i+1} generic for $i^{\mathcal{T}}_{\gamma+1,\beta}(W^{\Gamma_\gamma}_{\eta_\gamma})$ over $\mathcal{M}^{\mathcal{T}}_\beta[\Gamma_\gamma]$, where $\mathcal{M}^{\mathcal{T}}_\beta$ is some iterate of $\mathcal{M}^{\mathcal{T}}_{\gamma+1}$, such that $\mathrm{crit}(i^{\mathcal{T}}_{\gamma+1,\beta}) > j^{\mathcal{T}}_{0,\gamma}(\delta)$.

A model of the form $\mathcal{M}^{\mathcal{T}}_\beta[\Gamma_\gamma]$ is well-defined since Γ_γ is small forcing over $\mathcal{M}^{\mathcal{T}}_{\gamma+1}$ and $\mathrm{crit}(j^{\mathcal{T}}_{\gamma+1,\beta}) > j^{\mathcal{T}}_{0,\gamma}(\delta)$ by Lemma 4.4. Also the genericity iteration to make x_{i+1} generic over a small forcing extension of an iterate terminates after countably many steps. Note that we never apply extenders to models with index $< \gamma + 1$ (every extender used in the construction of $W^{\Gamma_\gamma}_{\eta_\gamma}$, i.e. every extender on $\vec{F}$, has critical point $> j^{\mathcal{T}}_{0,\gamma}(\delta)$; since $\nu(E_\zeta) < j^{\mathcal{T}}_{0,\gamma}(\delta)$ for all $\zeta < \gamma$ we see that the extenders are never applied to models with index $< \gamma + 1$). So we have that x_{i+1} is generic over $\mathcal{M}^{\mathcal{T}}_\beta[\Gamma_\gamma]$. We now want to apply Lemma 4.2 to find Γ_β. Let D denote the collection of all dense sets of $j^{\mathcal{T}}_{\gamma+1,\beta}(W^{\Gamma_\gamma}_{\eta_\gamma})$ computed in $\mathcal{M}^{\mathcal{T}}_\beta[\Gamma_\gamma]$. Recall that $p \Vdash \omega_1 = \check{\delta}$, hence we have for all $q \in j^{\mathcal{T}}_{\gamma+1,\beta}(W^{\Gamma_\gamma}_{\eta_\gamma})$

$$p \Vdash^{\mathcal{M}^{\mathcal{T}}_\beta[\Gamma_\gamma]} \exists g \subset j^{\mathcal{T}}_{\gamma+1,\beta}(W^{\Gamma_\gamma}_{\eta_\gamma})[\check{q} \in g \wedge g \text{ meets every } d \in \check{D}].$$

So by Lemma 4.3 and Lemma 4.2 we find a generic filter Γ_β extending Γ_γ such that $x_{i+1} \in \mathcal{M}^{\mathcal{T}}_\beta[\Gamma_\beta]$. This finishes the construction of $\mathcal{T}$ and $\langle \Gamma_i; i \leq \omega_1\rangle$.

Let $b = [0, \omega_1]_{\mathcal{T}}$ be the uncountable branch through $\mathcal{T}$. By construction, b contains every α_i; hence $j^{\mathcal{T}}_{0,\omega_1}(\delta) = \omega_1^V$.

So $\Gamma := \Gamma_{\omega_1} \subset j^{\mathcal{T}}_{0,\omega_1}(W_\delta)$ is generic over $\mathcal{M}^{\mathcal{T}}_{\omega_1}$, and $j^{\mathcal{T}}_{0,\omega_1}(p) \in \Gamma$. We have to check $\mathbb{R}^V \in \mathcal{M}^{\mathcal{T}}_{\omega_1}[\Gamma]$. Consider some $x_i \in \mathbb{R}^V$. By construction $x_i \in \mathcal{M}^{\mathcal{T}}_{\alpha_i}[\Gamma_i]$, so there is a nice name σ such that $x_i = \sigma^\Gamma_i$. By the δ-c.c. of W_δ, σ is not moved by $j^{\mathcal{T}}_{\alpha_i,\omega_1}$ and since $\Gamma_i = \Gamma \cap W_{\delta_i}$, we have $x_i = \sigma^\Gamma_i \in \mathcal{M}^{\mathcal{T}}_{\omega_1}$.

Recall that $p \in \Gamma$ and that p was not moved by $j^{\mathcal{T}}_{0,\omega_1}$. By elementarity it now suffices to iterate the top-extender of $\mathcal{M}^{\mathcal{T}}_{\omega_1}[\Gamma]$ out of the universe to obtain

$$V \models \exists A' \subset \mathbb{R}^V L(\mathbb{R}^V, A') \models \phi(A', z).$$

The same method yields a proof for the converse direction: basically one changes the roles of $V[G]$ and V; i.e. in $V[G]$ replace $\langle x_i; i < \omega_1\rangle$ by an enumeration of the reals of $V[G]$, run the according tree construction in $V[G]$. □

It is possible to add parameters besides reals to the formulae above, using for example Lemma 3.10. Also one can add a subset of the reals captured by a term for example. Nevertheless the same restrictions to the complexity of such parameters as before apply, see Lemma 3.11.

5. Subsets of ω_1 in forcing extensions

The classic genericity iteration to make a fixed real generic has a generalization for reals living in forcing extensions. It is possible to produce a long iteration such that all interpretations of a name for a real are generic:

Theorem 5.1 (Woodin). *Let $\mathbb{P}$ be a forcing of size κ and suppose the sound premouse $\mathfrak{M} = \langle J_\beta[\vec{E}]; \in, \vec{E}, E_\beta\rangle$ is active and has a $(\omega, \kappa^+ + 1)$-iteration strategy Σ such that $\vec{E}$ witnesses the Woodiness of δ. Let W denote the extender algebra with ω many generators relative to $\vec{E}$. Let $\dot{x} \in V^{\mathbb{P}}$ be a name for a real. Then there exists an iteration $j : \mathfrak{M} \to \mathfrak{M}^*$ in V of length $< \kappa^+$ such that for all $G \subset \mathbb{P}$ generic over V the real $\dot{x}^G$ is $j(W)$-generic over $\mathfrak{M}$.*

We do not give a proof of the above theorem but refer the reader to the appendix [11] of [12]; we will give a proof of a more general result, Lemma 6.21, with a similar proof. We aim to generalize the above theorem to subsets of ω_1. The first generalization is the following theorem which allows us to make subsets of ω_1 in c.c.c. forcing extensions generic over an iterate living in V; clearly the following theorem also generalizes Theorem 3.1. The second generalization is Lemma 6.21, which allows us to make certain subsets of ω_1 living in reasonable extensions generic over an iterate in V.

Theorem 5.2. *Let $\mathbb{P}$ be any c.c.c. forcing. Let $\dot{A}$ be a $\mathbb{P}$-name such that*

$$\mathbf{1}_{\mathbb{P}} \Vdash \dot{A} \subset \check{\omega}_1.$$

Let $\mathfrak{M} = \langle J_\beta[\vec{E}]; \in, \vec{E}, E_\beta\rangle$ be a sound premouse that is active and has a $(\omega, \omega_1 + 1)$-iteration strategy Σ such that $\vec{E}$ witnesses the Woodiness and measurability of δ. Then there exists an iteration $j : \mathfrak{M} \to \mathfrak{M}^*$ of length ω_1 in V such that for all $G \subset \mathbb{P}$ generic over V the set $\dot{A}^G$ is $j(W_\delta)$-generic over $\mathfrak{M}$.

The proof we are about to give is very similar to the one for Theorem 3.1; we will omit some details that we gave in the proof for Theorem 3.1.

Proof. Let U on $\vec{E}$ be the extender with the least index witnessing the measurability of δ, i.e. U is (the trivial completion of) a normal total measure on δ. Let ζ_0 be the index of U. We construct an iteration tree $\mathcal{T}$ of length $\omega_1 + 1$ on $\mathcal{M}^{\mathcal{T}}_0 = \mathfrak{M}$. We will call an $\alpha \leq \omega_1$ a $\mathbb{P}$-*weak closure point for* $\dot{A}$ if for all $p \in \mathbb{P}$

$$p \Vdash \dot{A} \cap j^{\mathcal{T}}_{0,\alpha}(\delta) \models j^{\mathcal{T}}_{0,\alpha}(T(\vec{E})).$$

To ensure normality of the resulting iteration we need a more technical definition: $\alpha \leq \omega_1$ is a $\mathbb{P}$-*closure point for* $\dot{A}$ if for all $p \in \mathbb{P}$ and all $\zeta < j^{\mathcal{T}}_{0,\alpha}(\zeta_0)$ and $\vec{F}$ on $\mathcal{M}^{\mathcal{T}}_\alpha$'s extender sequence

$$p \Vdash \dot{A} \cap j^{\mathcal{T}}_{0,\alpha}(\zeta_0) \text{ does not contradict any axiom induced by } \check{F}_\zeta.$$

Clearly any $\mathbb{P}$-closure point for $\dot{A}$ is a $\mathbb{P}$-weak closure point for $\dot{A}$ and limits of $\mathbb{P}$-closure points for $\dot{A}$ are $\mathbb{P}$-weak closure points for $\dot{A}$.

We define the iteration as follows: in the limit case we use Σ to continue the iteration. In the successor case there are two subcases: if $\alpha < \omega_1$ is a $\mathbb{P}$-closure point for $\dot{A}$, then we use $j^{\mathcal{T}}_{0,\alpha}(U)$ to continue the iteration. If α is not a closure point, then there is a least "bad" extender E on the extender sequence of $\mathcal{M}^{\mathcal{T}}_\alpha$ and some $p \in \mathbb{P}$ such that

$$p \Vdash \dot{A} \cap j^{\mathcal{T}}_{0,\alpha}(\zeta_0) \not\models \phi,$$

where ϕ is some axiom induced by E. We then use E to continue the iteration. This finishes the construction of $\mathcal{T}$. The arguments we have given before make sure $\mathcal{T}$ is a normal tree. Let $b = [0, \omega_1]_{\mathcal{T}}$ and let $j = j^{\mathcal{T}}_{0,\omega_1} : \mathfrak{M} \to \mathcal{M}^{\mathcal{T}}_{\omega_1}$. We set $\mathfrak{M}^* = \mathcal{M}^{\mathcal{T}}_{\omega_1}$. Let us now check that there are unboundedly many (in fact club many) $\mathbb{P}$-closure points for $\dot{A}$ in b; so suppose towards a contradiction that the set of closure points is bounded in ω_1 say by $\eta < \omega_1$. Pick a countable $X \prec V_\lambda$ for some large enough λ such that $\omega_1 \cap X > \eta$ and $\dot{A}, \mathcal{T}, \mathbb{P} \in X$. Let $\pi : H \to X$ denote the inverse of the transitive collapse of X and let $\alpha = \omega_1 \cap X$. Then $\pi \restriction \mathcal{M}^{\mathcal{T}}_\alpha = j^{\mathcal{T}}_{\alpha,\omega_1}$. Since $\alpha \in b$ there is a direct $\mathcal{T}$-successor of α, say $\gamma + 1$. Then there is a $p \in \mathbb{P}$ that forces that the extender $E^{\mathcal{M}^{\mathcal{T}}_\gamma}_\zeta$ on the $\mathcal{M}^{\mathcal{T}}_\gamma$-sequence is the minimal extender that induces a bad axiom. Let $G \subset \mathbb{P}$ be V-generic such that $p \in G$. We show that G is generic over X: Let $D \in X$ be an antichain of $\mathbb{P}$; then $q \in G \cap D$ for a unique q. Since D can be enumerated in ordertype ω, we have $q \in X \cap G$. Moreover we show: $X[G] \cap V = X$, this follows from the following claim:

Claim 1. Let $\tau \in X$ be a $\mathbb{P}$-name, let $\mathbb{B} = \mathrm{ro}(\mathbb{P})$ and let $q := [[\tau \in \check{V}]]_{\mathbb{B}}$. Then $q \in X$ and there is a countable set $y \in X$ such that $q \Vdash \tau \in \check{y}$.

Proof of Claim 1. Clearly $q \in X$ by elementarity. Let

$$\mathcal{A} = \{q' \leq q\,;\, 0 \neq q' = [[\tau = \check{x}]] \text{ for some } x \in V\}.$$

Since $\mathbb{P}$ is c.c.c. $\mathcal{A}$ is countable. By elementarity $\mathcal{A} \in X$. Since $\mathcal{A}$ is countable we have $y \in X$. □(Claim 1)

Let $\hat{\pi} : \hat{H} \to X[G]$ denote the inverse of the transitive collapse of $X[G]$. Since $X[G] \cap V = X$, we have that $H \subset \hat{H}$ and $\hat{\pi} \restriction H = \pi$. Let $\bar{A}, \bar{G}$ be such that $\hat{\pi}(\bar{A}, \bar{G}) = \dot{A}, G$. So

$$\hat{H} \models \bar{A}^{\bar{G}} \cap j^{\mathcal{T}}_{0,\alpha}(\delta) \not\models j^{\mathcal{T}}_{0,\alpha}(T(\vec{E})).$$

As before we get the following claim:

Claim 2. We have
$$\hat\pi \restriction V_{\delta^*}^{\mathcal{M}_\alpha^{\bar{\mathcal{T}}}} = i^{\mathcal{T}}_{\alpha,\omega_1} \restriction V_{\delta^*}^{\mathcal{M}_\alpha^{\mathcal{T}}}.$$

□(Claim 2)

We have now reproduced the situation in the proof of Theorem 3.1 and can proceed like in that proof. Hence α is a $\mathbb{P}$-closure point for $\dot{A}$. By the argument at the end of the proof of Theorem 3.1 we have that ω_1 is a $\mathbb{P}$-weak closure point for $\dot{A}$. This suffices to show. □

We now refine the previous argument to show more Σ^2_1 absoluteness for the class of c.c.c. forcings; we allow not only real parameters but also ordinals.

5.3. $\mathbf{\Sigma^2_1}$ absoluteness and c.c.c. forcing axtensions.

Theorem 5.4. *Suppose $M^\sharp_{\mathsf{mw}}$ exists and is $(\omega, \kappa+1)$-iterable. Assume* CH *holds. Furthermore assume $\mathbb{P}$ is a c.c.c. forcing.*
$$V \models \exists A \subset \mathbb{R} : L(\mathbb{R}, A) \models \phi(A, z, \vec{\alpha})$$
if and only if
$$V[G] \models \exists A \subset \mathbb{R}^{V[G]} : L(\mathbb{R}^{V[G]}, A) \models \phi(A, z, \vec{\alpha}).$$
Here z is a real parameter and $\vec{\alpha}$ are finitely many ordinal parameters.

The proof will use ideas from the previous proof and from the proof of Theorem 4.1. It is convenient to introduce some notation: we will code two subsets of ω_1 into one. For this purpose we define the useful $\oplus$-operation and its reverse operations:

Definition 5.5. Given (possibly set-sized) classes $A, B \subset \mathsf{OR}$ we define the class $A \oplus B$ by $\gamma \in A \oplus B$ if and only if
$$(\exists \alpha \in A : \exists \alpha' \in \mathrm{Lim} : \exists n \in \omega : \alpha = \alpha' + n \wedge \gamma = \alpha' + 2n) \vee$$
$$(\exists \alpha \in B : \exists \alpha' \in \mathrm{Lim} : \exists n \in \omega : \alpha = \alpha' + n \wedge \gamma = \alpha' + 2n + 1).$$
Furthermore we implicitly define operations $(\cdot)_{\mathsf{even}}$ and $(\cdot)_{\mathsf{odd}}$ acting on classes of ordinals by demanding:
$$(A \oplus B)_{\mathsf{even}} = A$$
and
$$(A \oplus B)_{\mathsf{odd}} = B.$$

The intuition in the above definition is that A is mapped to the "even" ordinals and B to the "odd" ordinals. In the following we will make us of the following fact: if $\dot{A}$ and $\dot{B}$ are forcing names for sets of ordinals, then we can compute a forcing name $\dot{C}$ such that it is forced that $\dot{C} = \dot{A} \oplus \dot{B}$. In an abuse of notation, we will denote a name $\dot{C}$ as above by $\dot{A} \oplus \dot{B}$. We now show Theorem 5.4.

Proof. We will first give a detailed proof of the downwards direction of the absoluteness, i.e. we assume

$$L(\mathbb{R}^{V[G]}, \dot{B}^G) \models \phi(\dot{B}^G, z, \vec{\alpha})$$

for some $\dot{B}$, and we want to show

$$L(\mathbb{R}^V, B) \models \phi(B, z, \vec{\alpha})$$

for some $B \in V$. The converse direction of this absoluteness is a variant of what we are going to show now; we will mention some details for the upwards direction at the end of the proof.

Notice that as $\mathbb{P}$ has the c.c.c., any canonical name for A has size $\aleph_1$, so that by $V \models \mathsf{CH}$ we may assume without loss of generality that $V^{\mathbb{P}} \models \mathsf{CH}$.

We denote the measurable Woodin cardinal in $\mathfrak{M} = M^\sharp_{\mathsf{mw}}$ by δ and we let Σ denote $\mathfrak{M}$'s $(\omega, \omega_1 + 1)$-iteration strategy. Let us fix a $\mathbb{P}$-name $\dot{B}$ such that for all $G \subset \mathbb{P}$ generic over V

$$L(\mathbb{R}^{V[G]}, \dot{B}^G) \models \phi(\dot{B}^G, z, \vec{\alpha}) \wedge \dot{B}^G \subset \check{\omega}_1.$$

In the following we will suppress z and work with $\mathfrak{M} = M^\sharp_{\mathsf{mw}}$. We will construe $\mathfrak{M}$ and its iterates as class sized models if convenient (i.e. we will confuse $\mathfrak{M}$ with the class sized model one obtains when iterating $\mathfrak{M}$'s top measure out of the universe); we will need this fact to allow for arbitrarily large ordinal parameters at the end of this proof.

Set $\dot{A} = \dot{B} \oplus \dot{R}$ for some name $\dot{R}$ such that

$$\mathbf{1}_{\mathbb{P}} \Vdash \dot{R} \subset \check{\omega}_1 \wedge \dot{R} \text{ codes a well-ordering of } \mathbb{R}.$$

Our aim is to produce an iteration tree $\mathcal{T} \in V$, p and $\Gamma \in V$ such that

- $\mathcal{T}$ on $\mathcal{M}_0^{\mathcal{T}} = \mathfrak{M}$ is of length $\omega_1 + 1$,
- for all $G \subset \mathbb{P}$ generic over V the set $\dot{A}^G$ is generic for $j^{\mathcal{T}}_{0,\omega_1}(W_\delta)$ over $\mathcal{M}^{\mathcal{T}}_{\omega_1}$,
- $p \in j^{\mathcal{T}}_{0,\omega_1}(W_\delta)$ is such that

$$p \Vdash L(\mathbb{R}, (\dot{\Gamma})_{\mathsf{even}}) \models \phi((\dot{\Gamma})_{\mathsf{even}}, \vec{\alpha}),$$

  where $\dot{\Gamma}$ is the canonical name for a $j^{\mathcal{T}}_{0,\omega_1}(W_\delta)$-generic,
- $\Gamma \subset j^{\mathcal{T}}_{0,\omega_1}(W_\delta)$ contains p and is generic over $\mathcal{M}^{\mathcal{T}}_{\omega_1}$,
- $\mathbb{R}^V \subset \mathcal{M}^{\mathcal{T}}_{\omega_1}[\Gamma]$.

For this our strategy is as follows: like in the proof for Theorem 4.1 we have to piece together end-extending generics. Again it is helpful to assume that the conditions of the extender algebra are not equivalence classes of formulae, but take the form of (minimal) formulae. In the proof of 4.1, we knew p from the beginning, in this proof we will have to consider all possible p; also we have ordinal parameters present which are moved in general by iterating, so we will have to arrange that in $V^{\mathbb{P}}$ the set $\dot{A}$ is generic over $\mathcal{M}^{\mathcal{T}}_{\omega_1}$.

We will drop the superscript $\mathcal{T}$ in the rest of this proof; i.e. $\mathcal{T}$ has models $\mathcal{M}_\alpha$ and maps $j_{\alpha,\beta}$. We prepare a book-keeping device: let $\langle y_i; i < \omega_1\rangle$ be an enumeration of the reals of V and for $i < \omega_1$ let $x_i \in V$ be such that

$\langle y_j ; j \leq i\rangle \in L[x_i]$. Let U on $\vec{E}$ be the extender with the least index witnessing the measurability of δ, i.e. U is (the trivial completion of) a normal total measure on δ, and let ζ_0 be the index of U (so that in fact ζ_0 is the height of $M^\sharp_{\mathsf{mw}}$), and U is $M^\sharp_{\mathsf{mw}}$'s top-measure. An ordinal α is a $\mathbb{P}$-*closure point for* $\dot{A}$ if for all $q \in \mathbb{P}$ and all $\zeta < j_{0,\alpha}(\zeta_0)$

$$q \Vdash \dot{A} \cap j_{0,\alpha}(\zeta_0) \text{ does not contradict any axiom induced by } \check{F}_\zeta,$$

where $\vec{F}$ denotes $\mathcal{M}_\alpha$'s extender sequence. In this case we clearly have that for all $q \in \mathbb{P}$

$$q \Vdash \dot{A} \cap j_{0,\alpha}(\delta) \models j_{0,\alpha}(T(\vec{E}));$$

we will call α a $\mathbb{P}$-*weak closure point for* $\dot{A}$ if it only satisfies this weaker property (cp. the proof of Theorem 5.2). As before we have that a limit of $\mathbb{P}$-weak closure points for $\dot{A}$ is also a $\mathbb{P}$-weak closure point for $\dot{A}$; in general a limit of $\mathbb{P}$-closure points for $\dot{A}$ is just a $\mathbb{P}$-weak closure point for $\dot{A}$.

We now formally define the iteration tree $\mathcal{T}$ in ω_1-many rounds; each round i starts at a stage α_i of $\mathcal{T}$. Set $\mathcal{M}_0 = \mathfrak{M}$. In each round α_i we have $\mathcal{T}\restriction(\alpha_i+1)$ defined and so $\mathcal{M}_{\alpha_i}$ exists. The tree $\mathcal{T}$ and the ordinals $\langle \alpha_i ; i < \omega_1\rangle$ will have the following properties:

(1) $\mathcal{T} \in V$ is an iteration tree on $\mathfrak{M}$ of length $\omega_1 + 1$,

(2) the set $\{\alpha_i\,;\, i < \omega_1\}$ is a club of $\mathbb{P}$-weak closure points for $\dot{A}$.

Additionally, for $i < \omega_1$ and $p \in j_{0,\alpha_i}(W_\delta)$ such that

$$p \Vdash j_{0,\alpha_i}(\check{\delta}) = \dot{\omega}_1$$

we will pick a generic Γ^p_i, with $p \in \Gamma^p_i$. For this it is convenient to introduce some objects: we will define a partial regressive function j that maps α_i to the maximal $\alpha_j < \alpha_i$ such that the generic Γ^p_j can be extended to a generic Γ^p_i. We now define j formally: for $\gamma < \omega_1$ we inductively define

$$J(\gamma) := \{j\,;\, \alpha_j \in [0,\gamma[_\mathcal{T} \wedge \operatorname{crit}(j_{\alpha_j,\gamma}) = j_{0,\alpha_j}(\delta)\},$$

So if $j \in J(\gamma)$ we have $j_{\alpha_j,\gamma}\restriction j_{0,\alpha_j}(\delta) = \mathrm{id}$. It is not difficult to check that $J(\gamma)$ is closed in γ. We set

$$j(\gamma) = \max(J(\gamma)),$$

if $\max(J(\gamma)) < \gamma$ exists, and let $j(\gamma)$ be undefined else (i.e., if $J(\gamma)$ is empty or unbounded in γ). For $j \leq i$ look at the map $j_{\alpha_j,\alpha_i} : \mathcal{M}_{\alpha_j} \to \mathcal{M}_{\alpha_i}$, if it exists. If this map does not exist, then the following definition trivializes, i.e. $P_{j,i} = \varnothing$. Let

$$P_{j,i} := \{p \in j_{0,\alpha_j}(W_\delta)\,;\, j_{\alpha_j,\alpha_i}\restriction j_{0,\alpha_j}(\delta) = \mathrm{id} \wedge p \Vdash j_{0,\alpha_j}(\check{\delta}) = \dot{\omega}_1\},$$

so that $P_{j,i}$ is empty if $j_{\alpha_j,\alpha_i}\restriction j_{0,\alpha_j}(\delta)$ is not the identity. Let

$$P_i := P_{j(\alpha_i),i}$$

if $j(\alpha_i)$ is defined and empty otherwise. Finally let

$$P^i := \{p \in j_{0,\alpha_i}(W_\delta)\,;\, p \Vdash j_{0,\alpha_i}(\check{\delta}) = \dot{\omega}_1 \wedge p \notin P_i\}.$$

The generic Γ^p_i will satisfy the following conditions:

(3) if $p \in P_i \cup P^i$, then $p \in \Gamma_i^p \subset j_{0,\alpha_i}(W_\delta)$ and Γ_i^p is generic over $\mathcal{M}_{\alpha_i}$,
(4) if $j(\alpha_i)$ is defined and if $p \in P_i$, then Γ_i^p end-extends $\Gamma_{j(\alpha_i)}^p$,
(5) if λ is a limit ordinal and $J(\alpha_\lambda)$ is unbounded in λ, then

$$\Gamma_\lambda^p = \bigcup\{\Gamma_j^p\, ;\, j \in J(\alpha_\lambda)\}$$

for all $p \in P^\lambda$,
(6) if $p \in P_i$ and $j(\alpha_i)$ is defined, then the real $x_{j(\alpha_i)}$ is generic over $\mathcal{M}_{\alpha_i}[\Gamma_{j(\alpha_i)}^p]$ for a forcing of cardinality $< j_{0,\alpha_i}(\delta)$,
(7) if $j(\alpha_i)$ is defined and $p \in P_i$, then $x_{j(\alpha_i)} \in \mathcal{M}_{\alpha_i}[\Gamma_i^p]$,
(8) each Γ_i^p is generic over all models with index $\gamma \geq \alpha_i$.

Once we state how we construct the iteration in each round, the last item above will follow easily by the agreement of models of an iteration tree. At each limit ordinal $\leq \omega_1$ we use Σ to continue the iteration tree $\mathcal{T}$. Suppose we have already constructed the iteration with the above properties up to a stage α_i, i.e. we have produced $\mathcal{T} \restriction (\alpha_i + 1)$. We now describe a tree $\mathcal{U}$ of length $\omega_1 + 1$ that continues $\mathcal{T} \restriction (\alpha_i + 1)$. After having done so, we will decide which countable β is α_{i+1}, i.e. $\mathcal{T} \restriction (\alpha_{i+1} + 1) = \mathcal{U} \restriction \beta$ for some countable β.

Say the construction of $\mathcal{U}$ has reached a countable stage $\beta \geq \alpha_i$. There are three rules, (P1), (P2) and (P3) that define the iteration at a stage β. These rules tell us which extender we use; (P1) in fact gives rise to countably many rules. In the formulation of (P1), $\tilde{E}$ is the induced extender in the sense of Lemma 4.4. We use the minimal extender E with

(P1) $j_{0,\alpha_i}(\zeta_0) < \mathrm{crit}(E) < j_{0,\beta}(\delta)$ and there is some $j \leq i$, some $k \leq i$ and some $p \in P^j \cup P_j$ such that in $\mathcal{M}_\beta[\Gamma_j^p]$ the extender $\tilde{E}$ induces an axiom false of x_k, or
(P2) there is some $q \in \mathbb{P}$ such that

$$q \Vdash \dot{A} \cap j_{0,\beta}(\check{\zeta}_0) \text{ does contradict an axiom induced by } \check{E}.$$

We explicitly do not fix a system of extenders and a Woodin cardinal. One can define axioms induced by extenders independently of a Woodin cardinal. Of course later we will specify a system. Also note that $\mathcal{M}_\beta[\Gamma_j^p]$ in the definition of (P1) is well-defined by condition (8). For $\beta = \alpha_i$ the rule (P1) is trivial.

(P3) If neither (P1) nor (P2) implies that we use an extender, use the top-measure $j_{0,\beta}(U)$ of $\mathcal{M}_\beta$, i.e. the measure witnessing that $j_{0,\beta}(\delta)$ is measurable.

So if we use $j_{0,\beta}(U)$, then especially β is a $\mathbb{P}$-closure point for $\dot{A}$. This fact and the fact that we always picked the minimal extenders at all stages yield that the extenders we used are of increasing length, i.e. the resulting iteration is normal.

We now show that in the construction of $\mathcal{U}$ we reach a stage where neither (P1) nor (P2) implies that we use an extender, so that the top-measure is actually used. Assume that this was not the case and work towards a contradiction. So in the ith round we produce a tree $\mathcal{U}$ of length $\omega_1 + 1$ such that rule

(P3) was not used at a stage $\beta \geq \alpha_i$. This implies that (P1) was unboundedly often the reason why we had to apply an extender, otherwise by the argument from 5.2 we reach a $\mathbb{P}$-closure point for $\dot{A}$ after countably many steps.

Claim 1. Club often (P1) was the reason why we had to continue the iteration.

Proof of Claim 1. If (P1) was at stationary many stages not the reason why we had to continue the iteration, then there is a stationary $S \subset \omega_1$ of points such that (P1) was not the reason, and hence (P2) was. Now pick an elementary substructure $X \prec V_\lambda$ for a large enough λ such that $\gamma = X \cap \omega_1 \in S$ and $\mathcal{T} \in X$. Making use of $\mathbb{P}$'s c.c.c., like in the proof for Theorem 5.2, we can now look at $X[G] \prec V_\lambda[G]$ for a $G \subset \mathbb{P}$ generic over V. But then an argument like in the proof for Theorem 5.2 shows that γ is a $\mathbb{P}$-closure point for $\dot{A}$. Contradiction! □(Claim 1)

By an application of Fodor's Theorem there is a stationary S, a k and a p such that for all $\beta \in S$: some extender $\tilde{E}$ induces an axiom false of x_k, in the sense of (P1). Pick an elementary substructure $X \prec V_\lambda$ for a large enough λ such that $\alpha_i < \gamma = X \cap \omega_1 \in S$ and $\mathcal{U} \in X$. But then an argument like in the proof for Theorem 3.1 shows that x_k is generic over $\mathcal{M}_\gamma[\Gamma^p_i]$ (here one has to keep in mind that $\mathcal{M}_\gamma[\Gamma^p_i]$ is a small forcing extension of $\mathcal{M}_\gamma$, see Lemma 4.4). This contradicts $\gamma \in S$!

This shows that unboundedly often during the construction of $\mathcal{U}$ rule (P3) implied that we had to use the top-measure at a stage $\beta \geq \alpha_i$. An easier version of this argument shows that we reach unboundedly many stages β, such that β is a $\mathbb{P}$-weak closure point for $\dot{A}$ and (P1) is not the reason why we have to use an extender at stage β, call such a stage *extraordinary.* Then we let α_{i+1} be the least extraordinary stage such that we have used the top-measure at some stage γ, $\alpha_i \leq \gamma < \alpha_{i+1}$. We set $\mathcal{T} \restriction (\alpha_{i+1} + 1) := \mathcal{U} \restriction (\alpha_{i+1} + 1)$. Clearly α_{i+1} is a $\mathbb{P}$-weak closure point, since α_{i+1} is extraordinary. We have to show how to pick the generics of the form Γ^p_{i+1} such that conditions (3) through (8) are satisfied. We first show:

Claim 2. For $j \leq i$, $k \leq i$ and $p \in P^j \cup P_j$ the real x_k is generic over $\mathcal{M}_{\alpha_{i+1}}[\Gamma^p_j]$ for a forcing of cardinality $< j_{0,\alpha_{i+1}(\delta)}$.

Proof of Claim 2. Let $\eta < j_{0,\alpha_{i+1}(\delta)}$, $\eta > j_{0,\alpha_i}(\zeta_0)$ be a Woodin cardinal such that there are extenders $\vec{F}$ on $\mathcal{M}_{\alpha_{i+1}}$'s extender sequence that witness that η is Woodin. In the construction of $\mathcal{U}$ we picked α_{i+1} as an extraordinary stage, so none of the extenders from $\vec{F}$ induce an extender $\tilde{F}$ in $\mathcal{M}_{\alpha_{i+1}}[\Gamma^p_j]$ that induces an axiom false of x_k. So x_k is generic over $\mathcal{M}_{\alpha_{i+1}}[\Gamma^p_j]$ for the extender algebra with ω-many generators calculated from $\{\tilde{F} \in \mathcal{M}_{\alpha_{i+1}}[\Gamma^p_j];\, F \text{ on } \vec{F}\}$. □(Claim 2)

This claim clearly shows more than what we demand in condition (6) Recall for any $j < \omega_1$ and any $p \in P^j \cup P_j$ we have $p \Vdash j_{0,\alpha_j}(\check{\delta}) = \dot{\omega}_1$, so if $j_{\alpha_j,\alpha_{i+1}}$ exists and p is not moved by $j_{\alpha_j,\alpha_{i+1}}$, we have by elementarity $p \Vdash j_{0,\alpha_{i+1}}(\check{\delta}) = \dot{\omega}_1$. So the powerset of any forcing of cardinality $< j_{0,\alpha_{i+1}}(\delta)$ is forced to be countable. We recall this fact because we are about to apply Lemma 4.2. Now pick a p in P_{i+1} and let $j = j(\alpha_{i+1})$. Inductively we already picked Γ^p_j. An argument like in the proof for Theorem 4.1, using $\mathrm{crit}(j_{\alpha_j,\alpha_{i+1}}) = j_{0,\alpha_j}(\delta)$, Lemmata 4.2 and 4.3, shows that we find Γ^p_{i+1}, an end-extension of Γ^p_j, such that $x_j \in \mathcal{M}_{\alpha_{i+1}}[\Gamma^p_{i+1}]$. This choice satisfies condition (7). For $p \in P^{i+1}$ we pick $\Gamma^p_{i+1} \subset j_{0,\alpha_{i+1}}(W_\delta)$ generic over $\mathcal{M}_{\alpha+1}$, so (4) is satisfied. Note that since α_{i+1} is extraordinary, we will not pick an extender with length $< j_{0,\alpha_{i+1}}(\delta)$ when we continue the iteration at stage α_{i+1}, this shows that condition (8) holds.

For a limit λ of rounds we set $\alpha_\lambda = \sup\{\alpha_i\,;\, i < \lambda\}$. It is not difficult to see, using the agreement of models along the iteration, that α_λ is a $\mathbb{P}$-weak closure point for $\dot{A}$. We now have to pick the generics of the form Γ^p_λ. There are three cases. The first case is: $J(\alpha_\lambda)$ is unbounded in λ. Then we pick the generics according to condition (5): for all $p \in P^\lambda$ set

$$\Gamma^p_\lambda = \bigcup\{\Gamma^p_j\,;\, j \in J(\alpha_\lambda)\}.$$

The second case is: $j(\alpha_\lambda)$ is undefined but the first case does not hold, then for $p \in P^{i+1}$ we pick $\Gamma^p_\lambda \subset j_{0,\alpha_\lambda}(W_\delta)$ generic over $\mathcal{M}_\lambda$. The third case is: $j := j(\alpha_\lambda)$ exists, then inductively we picked Γ^p_j. Recall that $j < \lambda$ and hence $\alpha_{j+1} < \alpha_\lambda$. The above claim shows: in the jth round we produced an iteration such that x_j was generic over $\mathcal{M}_{\alpha_{j+1}}$ for a forcing of cardinality $< j_{0,\alpha_{j+1}}(\delta)$. Since α_{j+1} is extraordinary, we used an extender with length $\geq j_{0,\alpha_{i+1}}(\delta)$ to continue the iteration, so by the agreement of models of $\mathcal{T}$

$$V^{\mathcal{M}_{\alpha_{j+1}}}_{j_{0,\alpha_{j+1}}(\delta)} = V^{\mathcal{M}_{\alpha_\lambda}}_{j_{0,\alpha_{j+1}}(\delta)}.$$

This implies that x_j is also generic over $\mathcal{M}_{\alpha_\lambda}$ for small forcing. This shows condition (6). By the argument from the successor case we find Γ^p_λ, an end-extension of Γ^p_j, such that $x_j \in \mathcal{M}_{\alpha_\lambda}[\Gamma^p_\lambda]$, making (7) true.

This finishes the construction of $\mathcal{T}$ and the family $(\Gamma^p_i)_i$.

Let $b = [0, \omega_1]_\mathcal{T}$ and let $j^* = j_{0,\omega_1} : \mathfrak{M} \to \mathcal{M}_{\omega_1}$. We set $\mathfrak{M}^* = \mathcal{M}_{\omega_1}$. We have shown that in each round we produce a countable iteration that terminates at an extraordinary stage. For this we showed that we actually use the top-measure. Another Skolem-hull argument of this type shows that there are unboundedly many $\beta \in b$ where we use the top-measure, so $j^*(\delta) = \omega_1$. It is easy to see that there are club many $\mathbb{P}$-weak closure points α_i for $\dot{A}$ in b such that $\mathrm{crit}(j_{\alpha_i,\omega_1}) = \alpha_i$. Since ω_1 is a limit of $\mathbb{P}$-weak closure points for $\dot{A}$, we have that ω_1 is also a $\mathbb{P}$-weak closure point for $\dot{A}$. Hence: if $G \subset \mathbb{P}$ is V-generic, then in $V[G]$ the set $\dot{A}^G$ is generic over $\mathfrak{M}^*$. Let $C \subset b$ denote a club of $\mathbb{P}$-weak closure points β for $\dot{A}$ such that $\mathrm{crit}(j_{\beta,\omega_1}) = \alpha_\beta = \beta = j_{0,\beta}(\delta)$.

We now iterate the top-measure of $\mathfrak{M}^*$ linearly and write $\mathfrak{M}^{**}$ for the resulting class sized model; we do this to make sure $\vec{\alpha} \in \mathfrak{M}^{**}$. Note that all generics of the form Γ^p_i are still generic over $\mathfrak{M}^{**}$ and $V^{\mathfrak{M}^*}_{\omega_1} = V^{\mathfrak{M}^{**}}_{\omega_1}$. Moreover if $G \subset \mathbb{P}$ is V-generic, then in $V[G]$ the set $\dot{A}^G$ is generic over $\mathfrak{M}^{**}$. So clearly there is a condition $p \in j(W_\delta)$ such that

$$p \Vdash^{j(W_\delta)}_{\mathfrak{M}^{**}} L(\dot{\mathbb{R}}, (\dot{\Gamma})_{\mathsf{even}}) \models \phi((\dot{\Gamma})_{\mathsf{even}}, \check{\vec{\alpha}}) \wedge j^*(\check{\delta}) = \dot{\omega}_1,$$

where $\dot{\Gamma}$ is the canonical name for the $j(W_\delta)$-generic. (Notice that it will be forced that $(\dot{\Gamma})_{\mathsf{even}} = \dot{B}$.) Then $p \in P_{j_0,\omega_1}$ for some countable $\alpha_{j_0} \in C$. Set

$$\Gamma = \bigcup\{\Gamma^p_i\,;\, j_0 < i \wedge i \in C\}.$$

We show that Γ is well defined: by the choice of C, we have that $C \cap \lambda \subset J(\alpha_\lambda)$ for every limit point $\lambda = \alpha_\lambda$ of C, so $J(\lambda)$ is unbounded in λ in this case. Then conditions (4) and (5) imply that the generics of the form Γ^p_i, $i \in C$, extend each other. Using that the antichains of $j^*(W_\delta)$ are of cardinality $< j^*(\delta)$, we see that $\Gamma \in V$ is generic over $\mathfrak{M}^*$ and hence over $\mathfrak{M}^{**}$.

We have to check $\mathbb{R}^V \subset \mathfrak{M}^*[\Gamma]$. If y is a real in V, then $y \in L[x_i]$ for all large enough i. Let $\alpha_i \in C$ and let α_j denote the least $\alpha_k > \alpha_i$ in b. We have $j(\alpha_j) = i$ and so $x_j \in \mathcal{M}_{\alpha_j}[\Gamma^p_j]$ by condition (7). Hence $x_i \in \mathfrak{M}^*[\Gamma]$, because a nice name for x_i is not moved by j_{α_i,ω_1}. Since i can be arbitrary large, we have $\mathbb{R}^V \in \mathfrak{M}^*[\Gamma]$. So $\mathbb{R}^V \in \mathfrak{M}^{**}[\Gamma]$.

By the choice of p

$$\mathfrak{M}^{**}[\Gamma] \models L(\mathbb{R}, (\Gamma)_{\mathsf{even}}) \models \phi((\Gamma)_{\mathsf{even}}, \vec{\alpha}).$$

This is what we needed to show for the downwards direction of the absoluteness.

For the upwards direction of the absoluteness, one runs a similar argument: we reverse the roles of $V[G]$ and V, still the construction takes place in V. Note for this we have to replace the sequence $\langle y_i; i < \omega_1\rangle$ by a sequence of names for reals. If $\dot{y} \in V^{\mathbb{P}}$ is a name for a real, then a genericity iteration for $\dot{y}$ still terminates after countably many steps, see [12, Lemma 3]. This is the key fact one additionally needs in the converse direction. We replace $\dot{B}$ with a $B \in V$ such that $L(\mathbb{R}^V, B) \models \phi(B, z, \vec{\alpha})$ and so we replace $\mathbb{P}$-(weak) closure points for $\dot{A}$ with (weak)-closure points (for an appropriate A) in the sense of Theorem 3.1. We shall give no more details. □

6. Sets that extend to a class with unique condensation

We mentioned that we cannot hope to generalize Theorem 5.4 to all proper forcings. One problem is that the witnesses for Σ^2_1 absoluteness are very general. If we restrict the choice of witnesses, then we can generalize Theorem 5.4. Sets that extend to a class with unique condensation, which we are about to define, are well-suited witnesses as we will see in Theorem 6.31.

We will systematically study the sets that extend to classes with unique condensation. Besides constructing examples, we will also show that a set that extends to a class with unique condensation, granted a large cardinal hypothesis, is constructible from a real.

Definition 6.1. Let $A \subset \omega_1$ and let $\kappa > \omega_1$ be a cardinal. We will say A *extends to* A^* *with unique condensation up to* κ if $A^* \subset \kappa$ is a set such that

(1) $A^* \cap \omega_1 = A$,

(2) if $\lambda > \kappa$ is a sufficiently large regular cardinal, then there is a club $C(A^*, \kappa, \lambda)$ of countable substructures $X \prec H_\lambda$ such that $A^* \in X$ and $A \cap \bar{\kappa} = \bar{A^*}$, where π is the inverse of the collapse of X and $\pi(\bar{\kappa}, \bar{A^*}) = \kappa, A^*$.

We will say A *extends to a class* A^* *with unique condensation* if $A^* \subset \mathsf{OR}$ is a class such that

(1) $A^* \cap \omega_1 = A$,
(2) for all cardinals $\kappa > \omega_1$ A extends to $A^* \cap \kappa$ with unique condensation.

In the above definition sufficiently large means that for every $\kappa > \omega_1$ as above there is some λ_0 such that for all $\lambda \geq \lambda_0$ the above property holds. If A extends to a class A^* with unique condensation, then $\omega_1 \setminus A$ also extends to $\mathsf{OR} \setminus A^*$ with unique condensation. In Lemma 6.8 we will show that all $A \subset \omega_1$ that live in a model of AD extend to a class with unique condensation. We now show that the term "unique" is justified in the above definition.

Lemma 6.2. *If $A \subset \omega_1$ extends to A^* with unique condensation, then A^* is unique with this property.*

Proof. Suppose A^* and A^{**} are both classes to which A extends with unique condensation. It suffices to show that all ordinals β are in A^* if and only if $\beta \in A^{**}$. Fix an ordinal β. Let $\kappa > \omega_1$ be a regular cardinal $> \beta$ and let $X \in C(A^*, \kappa, \lambda) \cap C(A^{**}, \kappa, \lambda)$ be a countable substructure such that $A^* \cap \kappa, A^{**} \cap \kappa, \beta \in X$; here λ is sufficiently large for A^* and A^{**}. Let $\pi : M \to X$ denote the inverse of the collapsing map of X and let $\pi(\bar{A^*}, \overline{A^{**}}, \bar{\beta}, \bar{\kappa}) = A^* \cap \kappa, A^{**} \cap \kappa, \beta, \kappa$. Then by our hypothesis

$$\bar{A^*} = A \cap \bar{\kappa} = \overline{A^{**}}.$$

So by elementarity of π

$$\beta \in A^* \iff \bar{\beta} \in \bar{A^*} \iff \bar{\beta} \in \overline{A^{**}} \iff \beta \in A^{**}. \qquad \square$$

We will call an A^* as above a *uniquely condensing extension of* A.

Lemma 6.3. *Let $A, B \subset \omega_1$.*

(1) If A extends to a class A^* with unique condensation, then A contains a club if $\omega_1 \in A^*$ and A is nonstationary if $\omega_1 \notin A^*$.
(2) If A is bounded in ω_1, then A extends to a class with unique condensation.
(3) If A and B both extend to a class with unique condensation, then $A \oplus B$ extends to a class with unique condensation, too.
(4) If A extends to a class with unique condensation and A' is obtained from A by replacing a countable initial segment of A with another countable set, then A' extends to a class with unique condensation.

Proof. For (1) we will show that if $\omega_1 \in A^*$, then A contains a club; the other implication then follows from the previous lemma and the fact that $\omega_1 \setminus A$ extends to $\mathsf{OR} \setminus A^*$ with unique condensation. The following set contains a club $C := \{\alpha < \omega_1\,;\, \exists X \in C(A^*, \omega_2, \theta) : \alpha = X \cap \omega_1\}$, for some sufficiently large θ. By the unique condensation of A^* and $\omega_1 \in A^*$, we have that $C \subset A$.

The rest of the lemma is straightforward to verify: we list the appropriate witnesses for each case: If A is bounded, then $A^* = A$ is a uniquely condensing extension of A. If A and B both extend to a class with unique condensation, then there are classes A^*, B^* that witness this fact; it is not difficult to see that $A^* \oplus B^*$ witnesses that $A \oplus B$ extends to a class with unique condensation. If A^* is a uniquely condensing extension of A and $A' = (A \setminus \alpha) \cup a$ for some $a \subset \alpha < \omega_1$, then $A'^* := (A^* \setminus \alpha) \cup a$ is a uniquely condensing extension of A'. □

If A^* is a uniquely condensing extension of some A, then A^* satisfies even better condensation properties, as the following lemma shows:

Lemma 6.4. *Let $\kappa > \omega_1$ be a cardinal and let A extend to a class A^* with unique condensation. Let $F : [H_\theta]^{<\omega} \to H_\theta$ be such that the club $C_F := \{X \in [H_\theta]^\omega\,;\, X$ is closed under $F\} \subset C(A^*, \kappa, \theta)$. Let $X \subset H_\theta$ of cardinality $< \kappa$ such that X is closed under F and let $\pi : M \to X$ denote the inverse of the transitive collapse of X. Then*

$$A^* \cap \bar{\kappa} = \overline{A^* \cap \kappa},$$

where $\pi(\bar{\kappa}, \overline{A^* \cap \kappa}) = \kappa, A^* \cap \kappa$.

Proof. Let $\theta' > \theta$ be a regular cardinal such that $H_\theta \in H_{\theta'}$. Let Y be a countable substructure of $H_{\theta'}$ such that $X, F \in Y$. Then $Z := X \cap Y$ is a countable substructure of H_θ and by elementarity of Y, Z is closed under F. So $Z \in C(A^*, \kappa, \theta)$. Let $\sigma : N \to Y$ denote the inverse of the transitive collapse of Y, so

$$\rho := \sigma^{-1}(\pi) : \sigma^{-1}(M) \to Z$$

is the inverse of the transitive collapse of Z. Then

$$\rho^{-1}(A^* \cap \kappa) = A \cap \rho^{-1}(\kappa).$$

By elementarity of σ we have that π also has the above property, i.e.

$$\pi^{-1}(A^* \cap \kappa) = A \cap \pi^{-1}(\kappa).$$

□

Lemma 6.5. *Let $\mathsf{OR} \subset M \subset N$ denote two transitive models of set theory such that $\omega_1^M = \omega_1^N$. If $A \subset \omega_1$ extends to a class A^* with unique condensation in M, then A also extends to a class with unique condensation in N.*

Proof. We suppose that A^* did not witness that A extends to class with unique condensation in N and work towards a contradiction. So given an uncountable N-cardinal $\lambda > \omega_1$, there are unboundedly many θ such that the second part of Definition 6.1 fails for λ and θ in N; i.e. the set of countable $X \prec H_\theta^N$ such that $A, A^* \cap \lambda \in X$ and $\pi^{-1}(A \cap \lambda) \neq A \cap \pi^{-1}(\lambda)$ is stationary. On the other hand, there is a club $C(A^*, \theta, \lambda)$ in M that witnesses that A^* has unique condensation. Say all countable structures closed under $F : [H_\theta]^{<\omega} \to H_\theta$ are in $C(A^*, \theta, \lambda)$ for some $F \in M$.

Pick $X \prec \langle H_\theta^N; \in, H_\theta^M \rangle$ countable with $A, A^* \cap \lambda, p \in X$ and $\pi^{-1}(A \cap \lambda) \neq A \cap \pi^{-1}(\lambda)$, where $\pi : \langle \bar{X}; \in, \bar{H} \rangle \to X$ denote the inverse of the transitive collapse of X. Since $\omega_1^M = \omega_1^N$, we can assume without loss of generality that $Y = X \cap H_\theta^M$ is closed under F. Then $\bar{H}$ is the transitive collapse of Y and $\pi \restriction \bar{H}$ is its uncollapsing map. So

$$(\pi \restriction \bar{H})^{-1}(A \cap \lambda) = \pi^{-1}(A \cap \lambda) \neq A \cap \pi^{-1}(\lambda) = A \cap (\pi \restriction \bar{H})^{-1}(\lambda).$$

We now look at the tree T of height ω searching for a countable substructure Z of H_θ^M, Z closed under F such that $\sigma^{-1}(A \cap \lambda) \neq A \cap \sigma^{-1}(\lambda)$, where σ is the inverse of the transitive collapse of Z. Then $T \in M$ and Y witnesses that T is ill-founded in N. By absoluteness of well-foundedness, we have a branch Z through T, $Z \in M$. But since Z is closed under F, we have $Z \in C(A^*, \theta, \lambda)$, a contradiction! □

6.6. **Constructing sets with uniquely condensing extensions.** We now show that any $A \subset \omega_1$ coded by a universally Baire sets of reals extends to a class with unique condensation. For this let us fix a recursive function $\{(\cdot)^i \, ; \, i < \omega\}$ that maps a real y to a countable set of reals $\{y^i \, ; \, i < \omega\}$. The proof to follow will show that we need to assume a bit more than that $A = \{\|x\| \, ; \, x \in B\}$ for some universally Baire set B.

Lemma 6.7. *Let $A \subset \omega_1$ be unbounded in ω_1 and let $B \subset \omega^\omega$ be a set of reals with the following properties:*

(1) B is universally Baire;
(2) if $y \in B$ and $\{y^i \, ; \, i < \omega\} \subset \mathsf{WO}$, then $\{\|y^i\| \, ; \, i < \omega\} = A \cap \alpha$ for some $\alpha < \omega_1$;
(3) for every $\beta < \omega_1$ there is some $y \in B$ such that $\{y^i \, ; \, i < \omega\} \subset \mathsf{WO}$ and $\{\|y^i\| \, ; \, i < \omega\} = A \cap \alpha$ for some $\beta < \alpha < \omega_1$;
(4) if $y \in B$ and $\{y^i \, ; \, i < \omega\} \subset \mathsf{WO}$ and $z \in \omega^\omega$ is such that $\{z^i \, ; \, i < \omega\} \subset \mathsf{WO}$ and $\{\|y^i\| \, ; \, i < \omega\}$ end-extends $\{\|z^i\| \, ; \, i < \omega\}$, then $z \in B$.

Then A extends to a class with unique condensation.

Proof. For every cardinal κ we fix trees T_κ, S_κ such that $B = p[T_\kappa]$ and $p[T_\kappa] = \omega^\omega \setminus p[S_\kappa]$ in all forcing extensions by forcings of cardinality $\leq \kappa$. We can now define the uniquely condensing extension A^* of A. We set $\alpha \in A^*$ if and only if

$$V^{\mathrm{Col}(\omega,\alpha)} \models \exists y \in p[\check{T}_\kappa] : \{y^i \, ; \, i < \omega\} \subset \mathsf{WO} \wedge \check{\alpha} \in \{\|y^i\| \, ; \, i < \omega\},$$

where κ is the least cardinal $> \alpha$. Note that by the homogeneity of $\mathrm{Col}(\omega, \alpha)$ the above statement is decided by $\mathbf{1}_{\mathrm{Col}(\omega,\alpha)}$.

Claim 1. A^* is definable from B and furthermore A^* does not depend on the choice of the family of trees $(T_\kappa, S_\kappa)_\kappa$.

Proof of Claim 1. It will suffice to show that the set A^* does not depend on the choice of the trees T_κ, S_κ. If we can show this, then A^* is definable from any class of trees witnessing the universal Baireness of B.

We fix another pair of trees T'_κ, S'_κ witnessing that B is κ-universally Baire. Assume for some $\alpha < \kappa$ there is some real $\dot{y} \in V^{\mathrm{Col}(\omega,\alpha)}$ such that $\dot{y} \in p[\check{T}_\kappa]$

and $\dot{y} \in p[\check{S}'_\kappa]$, then the tree U searching for a branch through T_κ and S'_κ is ill-founded in $V^{\mathrm{Col}(\omega,\alpha)}$; note that U is without loss of generality in V. By the absoluteness of well-foundedness, this tree is ill-founded in V. So there is some $z \in V$ such that $z \in p[T_\kappa] \cap p[S'_\kappa]$. This contradicts the fact that in V

$$p[T_\kappa] = \omega^\omega \setminus p[S_\kappa] = \omega^\omega \setminus p[S'_\kappa].$$

□(Claim 1)

We now have to show that $A^* \cap \omega_1 = A$. By the choice of B it is not difficult to see that $A \subset A^*$. So let $\alpha \in A^* \cap \omega_1$, we have to show $\alpha \in A$. Let $\dot{y} \in V^{\mathrm{Col}(\omega,\alpha)}$ be such that

$$V^{\mathrm{Col}(\omega,\alpha)} \models \dot{y} \in p[\check{T}_\kappa] \wedge \{\dot{y}^i\,;\, i < \omega\} \subset \mathsf{WO} \wedge \check{\alpha} \in \{\|\dot{y}^i\|\,;\, i < \omega\}.$$

Let λ be regular and large enough such that $\dot{y}, T_{\omega_1}, S_{\omega_1}, \mathbb{R} \in H_\lambda$ and let $X \prec H_\lambda$ be countable such that $\dot{y}, T_{\omega_1}, S_{\omega_1} \in X$ and $\alpha < X \cap \omega_1$. Let $\pi : \bar{H} \to X$ be the inverse of the transitive collapse of X and let $\pi(\bar{y}, \bar{T}, \bar{S}) = \dot{y}, T_{\omega_1}, S_{\omega_1}$. Let $g \in V$, $g \subset \mathrm{Col}(\omega,\alpha)$ be an arbitrary generic over $\bar{H}$. Then $\bar{y}^g \in p[\bar{T}]$, so for some f with domain ω we have $(\bar{y}^g, f) \in [\bar{T}]$. Hence back in V we have $(\bar{y}^g, \cup\{\pi(f \restriction n\,;\, n \in \omega\}) \in [T_\kappa]$, so $\bar{y}^g \in B$. This implies that $\alpha \in A$ by our hypothesis 2.

We have to show the second item in Definition 6.1; the argument for this will be similar to the argument we have just given for $A^* \cap \omega_1 = A$, but we also need to exploit hypotheses (3) and (4). So let us fix a cardinal $\kappa > \omega_1$ and let λ be regular and large enough such that $B, A^* \cap \kappa, T_\kappa, S_\kappa \in H_\lambda$. Pick $X \prec H_\lambda$ such that $A^* \cap \kappa, B \in X$ and let $\pi : \bar{H} \to X$ denote the inverse of the transitive collapse of X. Since $B \in X$, there are two trees $T, S \in X$ that witness that B is κ-universally Baire. Let $\pi(\bar{\kappa}, \bar{A}^*, \bar{T}, \bar{S}) = \kappa, A^* \cap \kappa, T, S$. We have to show $A \cap \bar{\kappa} = \bar{A}^*$. Fix $\alpha \in \bar{A}^*$. Then by elementarity there is some $\dot{y} \in \bar{H}^{\mathrm{Col}(\omega,\alpha)}$ such that

$$\bar{H}^{\mathrm{Col}(\omega,\alpha)} \models \dot{y} \in p[\check{\bar{T}}] \wedge \{\dot{y}^i\,;\, i < \omega\} \subset \mathsf{WO} \wedge \check{\alpha} \in \{\|\dot{y}^i\|\,;\, i < \omega\}.$$

Let $g \in V$, $g \subset \mathrm{Col}(\omega,\alpha)$ be an arbitrary $\bar{H}$ generic. Then $\alpha \in \{\|\dot{y}^i\|\,;\, i < \omega\}$ and $\dot{y}^g \in p[\bar{T}]$. By the same reasoning as before $\dot{y}^g \in p[T]$ in V. Hence $\dot{y}^g \in B$ and $\alpha \in A \cap \bar{\kappa}$ by hypothesis 2.

Let us assume the other inclusion fails and work towards a contradiction. Let $\alpha < \bar{\kappa}$ be minimal such that $\alpha \in A$ but $\alpha \notin \bar{A}^*$. Hence there is a condition $p \in \mathrm{Col}(\omega,\alpha)$ such that

$$\bar{H}^{\mathrm{Col}(\omega,\alpha)} \models p \Vdash \forall y : \{y^i\,;\, i < \omega\} \subset \mathsf{WO} \wedge \check{\alpha} \in \{\|y^i\|\,;\, i < \omega\} \implies y \in p[\check{\bar{S}}].$$

Let $p \in g \in V$, $g \subset \mathrm{Col}(\omega,\alpha)$ generic over $\bar{H}$. Note that $\bar{H}^{\mathrm{Col}(\omega,\alpha)}$ can calculate $A \cap \alpha = \bar{A}^* \cap \alpha$. In $\bar{H}^{\mathrm{Col}(\omega,\alpha)}$ we find a real y such that $\{\|y^i\|\,;\, i < \omega\} = A \cap (\alpha+1)$. Since $p \in g$, we have that $y \in p[\bar{S}]$. By the same argument as before $y \in p[S]$ and hence $y \notin B$. Hence by hypothesis (4) there is no $z \in B$ such that $\{\|y^i\|\,;\, i < \omega\}$ is end-extended by $\{\|z^i\|\,;\, i < \omega\}$. Hence $\alpha \notin A$ by hypothesis (3). This is a contradiction to our choice of α. This finishes the proof of the lemma. □

As a consequence to the previous lemma we can show that subsets of ω_1 living in determinacy models extend to uniquely condensing classes.

Lemma 6.8. *Let M be a transitive class sized model such that $\mathbb{R} \subset M \models \mathsf{ZF} + \mathsf{AD}$. Let $A \in M$ be a subset of $\omega_1^M = \omega_1^V$. Then A extends to a class with unique condensation.*

Proof. Let $A \in M \models \mathsf{AD}$, $A \subset \omega_1$. We aim to show that there is a universally Baire B that satisfies the properties in the statement of the previous lemma. For this we study the following well-known Solovay Game

$$G(A): \begin{array}{c|ccccc} I & x_0 & & x_1 & & \\ II & & y_0 & & y_1 & \cdots \end{array}$$

Here player I is obliged to play some $x = \langle x_i; i < \omega\rangle \in \mathsf{WO}$, else II wins, and Player II has to respond by playing a real $y = \langle y_i; i < \omega\rangle$ such that y codes (in some fixed recursive way) a countable set $\{y^i\,;\, i \in \omega\} \subset \mathsf{WO}$, else I wins. Player II wins $G(A)$ if $\{\|y^i\|\,;\, i < \omega\} = A \cap \alpha$ for some $\alpha > \|x\|$.

We show that player I cannot have a winning strategy: let σ be a strategy (not necessarily winning) for I, then the set $\{\sigma * y\,;\, y \in \omega^\omega\}$ is a $\boldsymbol{\Sigma}^1_1$ subset of WO. Hence by boundedness there is a countable ordinal α such that $\alpha > \|\sigma * y\|$ for all $y \in \omega^\omega$. So player II can play a y such that $\{\|y^i\|; i < \omega\} = A \cap \alpha$ and win against the strategy σ, hence σ is not winning.

By the determinacy hypothesis a winning strategy τ for player II exists. With the help of τ we will define B. Set $x \in B$ if and only if

$$\begin{aligned} \phi_0(x) :\equiv &\{x^i\,;\, i < \omega\} \subset \mathsf{WO} \wedge \\ &\exists y(y \in \mathsf{WO} \wedge \{\|(y * \tau)^i\|\,;\, i < \omega\} \text{ end-extends } \{\|x^i\|\,;\, i < \omega\}). \end{aligned}$$

A straightforward calculation shows that ϕ_0 is Σ^1_2 in a code for τ. We promise that the next claim shows that we can also define B as follows: $x \in B$ if and only if

$$\begin{aligned} \phi_1(x) :\equiv \{x^i\,;\, i < \omega\} \subset \mathsf{WO} \wedge \forall y[(y \in \mathsf{WO} \wedge \|y\| > \sup\{\|x^i\|\,;\, i < \omega\}) \implies \\ \{\|(y * \tau)^i\|\,;\, i < \omega\} \text{ end-extends } \{\|x^i\|\,;\, i < \omega\}]. \end{aligned}$$

Another straightforward calculation shows that ϕ_1 is Π^1_2 in a code for τ. The following statement is true in V by the fact that τ is a winning strategy for player II in $G(A)$:

$$\begin{aligned} \forall y, z[y, z \in \mathsf{WO} \implies (&\{\|(y * \tau)^i\|\,;\, i < \omega\} \text{ end-extends } \{\|(z * \tau)^i\|\,;\, i < \omega\} \vee \\ &\{\|(z * \tau)^i\|\,;\, i < \omega\} \text{ end-extends } \{\|(y * \tau)^i\|\,;\, i < \omega\})]. \end{aligned}$$

It is not difficult to see that that ψ is a Π^1_2 statement in a code for τ. Hence by Shoenfield Absoluteness ψ holds in all forcing extensions of V.

Claim 1. If a transitive model of set theory containing τ satisfies ψ, then

$$\forall x : \phi_0(x) \iff \phi_1(x).$$

Proof of Claim 1. Clearly $\phi_1(x)$ implies $\phi_0(x)$. So suppose $\phi_0(x)$ and let $y \in \mathsf{WO}$ be such that Let $y \in \mathsf{WO}$ be such that $\{\|(y*\tau)^i\|\,;\, i < \omega\}$ end-extends $\{\|x^i\|\,;\, i < \omega\}$. Now let $z \in \mathsf{WO}$ be arbitrary such that $\|z\| > \sup\{\|x^i\|\,;\, i < \omega\}$. Since ψ holds, we have that $\{\|(z*\tau)^i\|\,;\, i < \omega\}$ end-extends $\{\|(y*\tau)^i\|\,;\, i < \omega\}$ or $\{\|(y*\tau)^i\|\,;\, i < \omega\}$ end-extends $\{\|(z*\tau)^i\|\,;\, i < \omega\}$. In either case $\{\|(z*\tau)^i\|\,;\, i < \omega\}$ end-extends $\{\|x^i\|\,;\, i < \omega\}$. □(Claim 1)

So B is (in a weak sense) provably $\mathbf{\Delta^1_2}$, hence B is universally Baire.

We have to check that B satisfies the properties stated in the previous lemma; for all nonobvious properties this is verified by using the fact that τ is a winning strategy for player I. Hence by the previous lemma A extends to a class with unique condensation. □

Note that if $A \subset \omega_1$ is in a model of AD, then A is constructible from a real; in fact $A \in L[\sigma]$ where σ is a winning strategy for player II in $G(A)$. In this sense, the set A trivializes. Nevertheless nontrivial examples of sets with uniquely condensing extensions exist if the universe has a uniform shape:

Example 6.9. Suppose sharps for all sets exist. Let $V = L^\sharp$, the smallest inner model that is closed under the $\sharp$ operation. Using the Gödel pairing function and the well order $<$ of $L^\sharp$, we can uniformly code initial segments of $L^\sharp$ in the following way: if $\alpha < \beta$ are limit ordinals, then the code A_α for $L^\sharp_\alpha$ is a subset of α and the code A_β for $L^\sharp_\beta$ end-extends A_α, i.e. $A_\beta \cap \alpha = A_\alpha$. By A^* we denote the class coding $L^\sharp$. Set $A = A_{\omega_1}$. We claim that A^* is a uniquely condensing extension of A. For $\theta > \kappa$ both regular uncountable cardinals consider a countable substructure $X \prec L^\sharp \| \theta$ such that $A_\kappa = A^* \cap \kappa \in X$. Let $\pi : M \to X$ denote the transitive collapse of M and let $\pi(\bar\kappa, \bar A) = \kappa, A_\kappa$. By elementarity of π

$$M \models V = L^\sharp,$$

and for all $x \in M$ the set $(x^\sharp)^M$ is embedded into $(\pi(x))^\sharp$, hence $(x^\sharp)^M = x^\sharp$. Thus M is an initial segment of $L^\sharp$. So, since we defined the sets of the form A_α uniformly, we have $\bar A = A_{\bar\kappa}$.

We claim that A is not constructible from a real. Suppose otherwise that $A \in L[z]$ for some real $z \in V = L^\sharp$. Then $z^\sharp$ exists and is clearly not in $L[z]$. On the other hand, in $L^\sharp$, every hereditarily countable set is in $L_{\omega_1}[A]$, so that in particular $z^\sharp \in L_{\omega_1}[A] \subset L[z]$. Contradiction!

Note that given any mouse operator J, the same construction works for L^J, the smallest inner model that is closed under J.

6.10. **Sets with uniquely condensing extensions, precipitous ideals and CC*.** We analyze how sets with uniquely condensing extensions behave in the presence of ideals and the combinatorial principle CC*.

Lemma 6.11. *Let I be a precipitous ideal on ω_1 with the following property: if $G \subset \mathbb{P} := \mathcal{P}(\omega_1) \setminus I$ is generic, then $j(\omega_1^V) = \omega_2^V$, where j is the generic ultrapower induced by G. Let A be a set that extends to a class A^* with unique condensation. If j is a generic ultrapower induced by some generic $G \subset \mathcal{P}(\omega_1) \setminus I$, then*

$$j(A) = A^* \cap \omega_2 = A \cup \tilde{A},$$

where $\tilde{A}$ is the set of $\omega_1 \leq \alpha < \omega_2$ such that there is a club C and a canonical function f_α such that $f_\alpha(\beta) \in A$ for every $\beta \in C$, i.e. the Tilde operation applied to A.

Proof. Fix some generic G and j as above. We first show $j(A) \subset A^* \cap \omega_2$. Let $\alpha \in j(A)$. So there is some I-positive $S \in G$ and some canonical function f_α such that $f_\alpha(\beta) \in A$ for all $\beta \in S$. The set

$$C := \{\beta < \omega_1\, ;\, \beta = X \cap \omega_1 \text{ for some } X \in C(A^*, \omega_2, \theta) \text{ with } \alpha, f_\alpha \in X\}$$

is club, where θ is sufficiently large. Since S is stationary we find some $\beta \in S \cap C$, say $X \prec H_\theta$ witnesses $\beta \in C$. Let $\pi : M \to X$ be the inverse of the transitive collapse of X and let $\pi(\bar{\alpha}, \overline{A^* \cap \omega_2}) = \alpha, A^* \cap \omega_2$. Then $f_\alpha(\beta) = \text{otp}(X \cap \alpha) = \bar{\alpha} \in A$ and since A extends we have that $\bar{\alpha} \in \overline{A^* \cap \omega_2}$. Applying π yields: $\alpha \in A^* \cap \omega_2$.

We show $A^* \cap \omega_2 \subset A \cup \tilde{A}$. Let $\alpha \in A^* \cap \omega_2$, $\alpha \geq \omega_1$. Fix a surjection $g : \omega_1 \to \alpha$ and let $f : \omega_1 \to \omega_1$ be the canonical function induced by g. Consider the club

$$C := \{\beta < \omega_1\, ;\, \beta = X \cap \omega_1 \text{ for some } X \in C(A^*, \omega_2, \theta) \text{ with } \alpha, g, f \in X\}$$

for some sufficiently large θ. Let $\beta \in C$ and $X \prec H_\theta$ be a witnesses for this, let $\pi : M \to X$ be the inverse of the transitive collapse of X and let $\pi(\bar{\alpha}) = \alpha$. Since $\alpha \in A^* \cap \omega_2$ we have $\bar{\alpha} = \text{otp}(X \cap \alpha) = \text{otp}(g''\beta) = f(\beta) \in A$. Hence C, f witness that $\alpha \in \tilde{A}$.

Trivially $A \cup \tilde{A} \subset j(A)$. This finishes the proof. □

We conjecture that the existence of a strong enough ideal on ω_1 implies that every set with a uniquely condensing extension is constructible from a real.

Using the combinatorial principle CC^* we can show that only countably many reals can be constructed from a set with a uniquely condensing extension.

Definition 6.12 (Todorčević [14])**.** We say CC^* holds if there are arbitrarily large regular cardinals θ such that for all well-orderings $<$ of H_θ and for all countable $X \prec \langle H_\theta; \in, < \rangle$ there is a countable $Y \prec \langle H_\theta; \in, < \rangle$ such that[4] $X \sqsubset Y$ and $X \cap \omega_2 \neq Y \cap \omega_2$.

Lemma 6.13. *If CC^* holds and A extends to a class A^* with unique condensation, then $L[A]$ only contains countably many reals.*

[4] $X \sqsubset Y$ means that $X \subset Y$ and $X \cap \omega_1 = Y \cap \omega_1$.

Proof. Suppose otherwise and work towards a contradiction. Then $L[A]$ contains uncountably many reals and any real of $L[A]$ is in some $L_\alpha[A]$ for a countable α. Let $f : \omega_1 \to \omega_1$ be such that $f(\alpha)$ is the least ordinal such that the $<_{L[A]}$-least real $a \notin L_\alpha[A]$ is in $L_{f(\alpha)}[A]$. Clearly $f \in L[A]$. Let $C = C(A^*, \omega_2, \theta)$ for some sufficiently large θ and suppose there is a function $F : [H_\theta]^{<\omega} \to H_\theta$ such that C contains exactly the $X \prec H_\theta$ closed under F. Let $X \prec H_{\theta^+}$ be countable such that $F, A, A^* \cap \omega_2 \in X$. Let $\alpha = X \cap \omega_1$ and let $\pi : M \to X$ be the inverse of the transitive collapse of X. In general we can not compute $f(\alpha)$ in M, we apply CC* $f(\alpha) + 1$-many times to find a countable $Y \supset X$, $Y \cap \omega_1 = \alpha$, $Y \prec H_{\theta^+}$ such that $\mathrm{otp}(Y \cap \omega_2) > f(\alpha)$. Let $\sigma : N \to Y$ denote the inverse of the transitive collapse of Y and note that by elementarity $Y \cap H_\theta$ is closed under F. Let $\sigma(\bar{A}, \overline{A^* \cap \omega_2}, \beta) = A, A^* \cap \omega_2, \omega_2$. Then, since $Y \cap H_\theta \in C$, we have $\overline{A^* \cap \omega_2} = A \cap \beta$ and $\beta > f(\alpha)$. Hence $L_\beta[A] \in N$ and we can compute $f(\alpha)$ in N. By $F \in Y$ we have that $H_\theta \in Y$. In Y we find a countable $Y'' \prec H_\theta$ that contains $A, A^* \cap \omega_2, f(\alpha)$ and is closed under F, so $Y'' \in C$. Let $\pi(Y') = Y''$ and let $\rho : N' \to Y'$ be the inverse of the transitive collapse of Y'. Note that $N' \in N$ and $\mathrm{crit}(\rho) < \alpha$. Then in N the set $\overline{A^* \cap \omega_2}$ witnesses that $\bar{A}$ has a countable extension that condenses uniquely up to β. Let $\rho(\overline{\overline{A^* \cap \omega_2}}, \bar{\beta}) = \overline{A^* \cap \omega_2}, \beta$. Hence $\overline{\overline{A^* \cap \omega_2}} = \bar{A} \cap \bar{\beta}$ and so $\overline{\overline{A^* \cap \omega_2}} = A \cap \bar{\beta}$. In N' compute $L_{\bar{\beta}}[\overline{\overline{A^* \cap \omega_2}}] = L_{\bar{\beta}}[A]$. By elementarity $f(\alpha) \in L_{\bar{\beta}}[A]$, a contradiction to the fact that $\bar{\beta} < \alpha$. □

6.14. **Sets with uniquely condensing extensions and term-capturing.** If V contains ω-many Woodin cardinals and a measurable above and satisfies an iterability hypothesis, we can show that sets with uniquely condensing extensions are constructible from a real. The key idea is the following: if $A \subset \omega_1$ is in a model of determinacy, then it is constructible from a real. So we aim to show $L(\mathbb{R}) = L(\mathbb{R}, A) \models \mathsf{AD}$, this is Theorem 6.18. This is of course similar to a proof of $\mathsf{AD}^{L(\mathbb{R})}$ from ω-many Woodin cardinals. There are various ways to show determinacy from large cardinals. We will use the technique of capturing sets of reals over sufficiently iterable premice. In contrast to the rest of this chapter, we will work with coarse premice in the sense of Martin and Steel [9], since we will apply [10].

Definition 6.15. Let $B \subset \omega^\omega$. Let $\mathfrak{M}$ be a premouse with an iteration strategy Σ that contains ω-many Woodin cardinals $(\delta_i)_{i\in\omega}$. Let τ be a $\mathrm{Col}(\omega, \delta_0)$ term in $\mathfrak{M}$. We say τ *captures* B *with respect to* Σ if and only if for all countable iteration maps $i : \mathfrak{M} \to \mathfrak{M}^*$, $i \in V$, obtained by using Σ and for all $g \subset \mathrm{Col}(\omega, i(\delta_0))$, $g \in V$,

$$i(\tau)^g = B \cap \mathfrak{M}^*[g].$$

Definition 6.16. Let λ be an infinite ordinal. Let $G \subset \mathrm{Col}(\omega, < \lambda)$ be $\mathfrak{M}$-generic for some suitable $\mathfrak{M}$. Then we set

$$\mathbb{R}^*_G = \bigcup\{\mathbb{R} \cap \mathfrak{M}[G \cap \mathrm{Col}(\omega, < \alpha)]\,;\, \alpha < \lambda\}.$$

If $\mathfrak{M}$ is sufficiently iterable and contains ω-many Woodin cardinals, it is possible to use $\mathfrak{M}$ to verify parts of the theory of $L(\mathbb{R}^V)$; for this recall the following result:

Theorem 6.17 ([13, 7.15]). *Suppose that $\mathfrak{M} \models \lambda$ is a limit of Woodin cardinals, where λ is countable in V, and that Σ is an ω_1+1-iteration strategy for $\mathfrak{M}$. Let H be $\mathrm{Col}(\omega,\mathbb{R})$-generic over V; then in $V[H]$ there is an iteration map $i : \mathfrak{M} \to \mathfrak{M}^*$ coming from an iteration tree all of whose proper initial segments are played by Σ, and a G which is $\mathrm{Col}(\omega, < i(\lambda))$-generic over $\mathfrak{M}^*$, such that*

$$\mathbb{R}^*_G = \mathbb{R}^V.$$

Moreover, given a $g \subset \mathrm{Col}(\omega,\alpha)$ for an $\alpha < \lambda$, we can construct i, G such that $\mathrm{crit}(i) > \alpha^{++\mathfrak{M}^*}$ and G is generic over $\mathfrak{M}^*[g]$. □

First note that [13, 7.15] deals with fine-structural premice, nevertheless the proof of [13, 7.15] works in the coarse case, too. As we stated earlier the extender algebra, which is the main tool in the proof of [13, 7.15], can also be used to construct genericity iterations with coarse premice. Also note that the moreover part of the above theorem is only implicit in [13]; it follows by a minor modification of the proof using [13, 7.16]. We apply the previous theorem to obtain:

Theorem 6.18. *Suppose*

$V \models \lambda'$ is the limit of ω-many Woodin cardinals and $\kappa' > \lambda'$ is measurable

Suppose A extends to $A^* \subset \lambda'$ with unique condensation up to λ'. Let $\theta > \theta' > (2^{\kappa'})^+$ be large enough such that

(1) the club $C = C(A^*, \lambda', \theta')$ of countable substructures of $V_{\theta'}$ witnesses that A extends to A^* with unique condensation,
(2) θ is large enough so that $\langle V_\theta; \in, \lambda'\rangle$ is a premouse in the sense of [9],
(3) if $X \prec V_\theta$, $A^*, \lambda, C \in X$ is a countable elementary substructure with $\pi : \mathfrak{M} \to X$ the inverse of the transitive collapse, then
   (a) $\mathfrak{M}$ has a ω_1+1-iteration strategy Σ, and
   (b) (Re-embedding) if $i : \mathfrak{M} \to \mathfrak{M}^*$ is a countable iteration map obtained by using Σ, then there is an elementary $\pi_{\mathfrak{M}^*} : \mathfrak{M}^* \to V_\theta$ satisfying $\pi_{\mathfrak{M}^*} \circ i = \pi$.

Then

(1) $L(\mathbb{R}, A) \models \mathsf{AD}$,
(2) A is constructible from a real, and
(3) $L(\mathbb{R}, A) = L(\mathbb{R})$.

Proof. We first discuss the conclusions: Clearly (2) implies (3). If A is contained in a model of AD, then it is a well-known fact that the determinacy of the Solovay-Game $G(A)$ implies that A is constructible from a real that codes a winning strategy for player II in $G(A)$. Hence (1) implies (2). So it suffices to show (1).

For this assume $L(\mathbb{R}, A) \models \neg\mathsf{AD}$, hence there is a set of reals B that is not determined. By minimizing the ordinal parameters in the definition of B we can assume without loss of generality that B is definable without ordinal parameters in $L(\mathbb{R}, A)$. Let z be the only real parameter used in the definition of B. If we can show that B is captured by a term over some countable sufficiently iterable model then by [10, Lemma 1.7] the set is determined, contradicting our assumption. So we aim to show that B is captured.

Assume $x \in B$ if and only if

$$L(\mathbb{R}, A) \models \phi(x, z, A).$$

Pick $X \prec V_\theta$ with $z, C, A^*, \lambda \in X$. Let $\pi : \mathfrak{M} \to X$ denote the inverse of the transitive collapse and let $\pi(\bar{A}, \lambda, \kappa) = A^*, \lambda', \kappa'$ and let $(\delta_i)_{i\in\omega}$ denote the countably many Woodin cardinals in $\mathfrak{M}$. In $\mathfrak{M}$ we define a $\mathrm{Col}(\omega, \delta_0)$-term τ as follows: if $g \subset \mathrm{Col}(\omega, \delta_0)$ is generic over $\mathfrak{M}$, then $x \in \tau^g$ if and only if

$$\psi(x, z, A) \equiv: \mathbf{1}_{\mathrm{Col}(\omega,<\lambda)} \Vdash J_{\check{\kappa}}(\mathbb{R}^*_{\dot{G}}, \check{\bar{A}}) \models \phi(\check{x}, \check{z}, \check{A}),$$

here $\dot{G}$ is a canonical name for a $\mathrm{Col}(\omega, <\lambda)$ generic and κ is the measurable $> \lambda$ in $\mathfrak{M}$. We need to verify that τ captures B. Assume $i : \mathfrak{M} \to \mathfrak{M}^*$ is a countable iteration according to Σ and let $g \subset \mathrm{Col}(\omega, i(\delta_0))$ be generic over $\mathfrak{M}^*$. Let $x \in \mathbb{R} \cap \mathfrak{M}^*[g]$. We have to show

$$(L(\mathbb{R}^V, A) \models \phi(x, z, A)) \iff (\mathfrak{M}^*[g] \models \psi(x, z, A)).$$

By the previous theorem, we find an iteration map $j : \mathfrak{M}^* \to \mathfrak{M}^{**}$, $j \in V^{\mathrm{Col}(\omega,\mathbb{R}^V)}$, with $\mathrm{crit}(j) > i(\delta_0)^{++\mathfrak{M}^*}$ coming from an iteration tree $\mathcal{T}$ of length $\omega_1 + 1$ on $\mathfrak{M}^*$ all of whose proper initial segments are played by Σ, and a G which is $\mathrm{Col}(\omega, < j(i(\lambda)))$-generic over $\mathfrak{M}^{**}[g]$, such that

$$\mathbb{R}^*_G = \mathbb{R}^V.$$

For this note, that since $\mathrm{crit}(j)$ is large enough, g is a $\mathfrak{M}^{**}$-generic; moreover j lifts to

$$\hat{j} : \mathfrak{M}^*[g] \to \mathfrak{M}^{**}[g],$$

where $\hat{j}(\sigma^g) = j(\sigma)^g$. In an abuse of notation we shall write j for $\hat{j}$. By our re-embedding hypothesis, we have for $\alpha < \omega_1$ an elementary embedding

$$\pi_\alpha : \mathcal{M}^{\mathcal{T}}_\alpha \to V_\theta,$$

such that $\pi_\alpha(j^{\mathcal{T}}_{0,\alpha}(i(\bar{A}))) = A^*$. The model $\mathfrak{M}^{**}$ is the direct limit of the models $\mathcal{M}^{\mathcal{T}}_\alpha$, and hence there is a map

$$\pi^{**} : \mathfrak{M}^{**} \to V_\theta.$$

Since for every $\gamma < \omega_1^V$ a real $x_\gamma \in \mathsf{WO}$ with $\|x\| = \gamma$ is in $\mathfrak{M}^{**}[g, G \cap \mathrm{Col}(\omega, < \alpha)]$ for some $\alpha < j(i(\lambda))$, we have that $j(i(\lambda)) \geq \omega_1^V$ and by a standard homogeneity argument and the symmetry of the name $\mathbb{R}^*_{\dot{G}}$ we have $\mathbb{R} \cap L(\mathbb{R}^*_G, A) = \mathbb{R}^*_G$ and hence $j(i(\lambda)) \leq \omega_1^V$. So $j(i(\lambda)) = \omega_1^V$. We now apply that A extends to A^* with unique condensation: for all $\alpha < \omega_1$ we have that $\mathrm{ran}(\pi_\alpha) \cap V_{\theta'} \in C$. Hence

$$\pi_\alpha^{-1}(A^*) = A \cap \pi_\alpha^{-1}(\lambda').$$

From $\sup\{\pi_\alpha^{-1}(\lambda') ; \alpha < \omega_1\} = j(i(\lambda)) = \omega_1^V$ it follows $(\pi^{**})^{-1}(A^*) = A$. We can now calculate

$$J_{j(i(\kappa))}(\mathbb{R}^*_G, j(i(\bar{A})))^{\mathfrak{M}^{**}[G]} = J_{j(i(\kappa))}(\mathbb{R}^V, A).$$

Note that $\pi^{**} \circ j \circ i(\kappa) = \pi(\kappa) = \kappa'$. By elementarity of j and the fact that we can iterate the measure on $j(i(\kappa))$ out of the universe

$$\begin{aligned} &\mathfrak{M}^*[g] \models \psi(x, z, A) \\ &\iff J_{j(i(\kappa))}(\mathbb{R}^*_G, j(i(\bar{A}))) \models \phi(x, z, A) \\ &\iff J_{j(i(\kappa))}(\mathbb{R}^V, A) \models \phi(x, z, A) \\ &\iff L(\mathbb{R}^V, A) \models \phi(x, z, A). \end{aligned}$$

This shows that B is captured by $\mathfrak{M}$. This is what we needed to show. □

6.19. **Sets with uniquely condensing extensions in forcing extensions.** We work with fine-structural premice again. Given a forcing name $\dot{x} \in V^{\mathbb{P}}$ for a real and granted that $M^\sharp_{\mathsf{mw}}$ exists and is sufficiently iterable, one can construct an iteration tree $\mathcal{T}$ of length $< \mathrm{Card}(\mathbb{P})^+ + 1$ such that for every $G \subset \mathbb{P}$ generic over V the real $\dot{x}^G$ is generic over $\mathcal{T}$'s last model, see Theorem 5.1. In general such an iteration is uncountable. Neeman and Zapletal showed that, given one generic $G \subset \mathbb{P}$ for a reasonable forcing $\mathbb{P}$, one finds $\alpha < \omega_1$ such that $\dot{x}^G$ is generic over $\mathcal{M}^{\mathcal{T}}_\alpha$, see [12, Lemma 3]. We generalize this to names for subset of ω_1 with uniquely condensing extensions; before we can state the lemma we need a definition:

Definition 6.20. Let κ be an ordinal and let $\mathfrak{M} = \langle J_\beta[\vec{E}]; \in, \vec{E}, E_\beta \rangle$ be a countable sound premouse that has a $(\omega, \kappa+1)$-iteration strategy Σ. We will say Σ *condenses to fragments* if it satisfies the following property: if λ is a regular cardinal such that $\mathfrak{M}, \Sigma \in H_\lambda$, and if $X \prec H_\lambda$ is countable with uncollapsing map $\pi : \bar{H} \to X$ and $\pi(\bar{\Sigma}) = \Sigma$, then $\bar{\Sigma} = \Sigma \restriction \mathrm{dom}(\bar{\Sigma})$.

Here we do not want to construct iteration strategies that condense to fragments; nevertheless let us note that there are (at least) two ways to see that they exist: in the large cardinal area below one Woodin cardinal one is always in the situation that there is at most one well-founded branch through an iteration tree, hence there is only at most one (highly absolute) iteration strategy, this is one of the main results of [9]; for a fine-structural version see [13, Thm. 6.10]. Beyond that one uses $\mathcal{Q}$-structures in the construction of iteration strategies. Under the assumption that the ultimate projectum drops below the least Woodin cardinal of a tame premouse $\mathfrak{M}$, there is a unique branch b through an iteration tree on $\mathfrak{M}$ such that b comes with a $\mathcal{Q}$-structure (if there is any). Again this gives rise to an absolute iteration strategy. For more details on $\mathcal{Q}$-structures and iteration trees see, for example, the introduction of [2].

Lemma 6.21. *Let $\mathfrak{M} = \langle J_\beta[\vec{E}]; \in, \vec{E}, E_\beta\rangle$ be a sound premouse that is active and has a $(\omega, \kappa^+ + 1)$-iteration strategy Σ such that $\vec{E}$ witnesses the Woodiness and measurability of $\delta < \beta$. Let $\mathbb{P}$ be a forcing of size $\leq \kappa$. Let $\dot{A} \in V^{\mathbb{P}}$ be a name such that*

$$\mathbf{1}_{\mathbb{P}} \Vdash \dot{A} \subset \check{\omega}_1 \textit{ extends to } \dot{A}^* \textit{ with unique condensation;}$$

here we see $\dot{A}^*$ as a $\mathbb{P}$-name for a class definable from some set in $V^{\mathbb{P}}$.

It is possible to construct an iteration tree $\mathcal{T}$ of height $\kappa^+ + 1$ with the following properties:

(1) There are arbitrary large ordinals $\beta < \kappa^+$ such that for any $G \subset \mathbb{P}$ that is generic over V, the set $\dot{A}^{*G} \cap j^{\mathcal{T}}_{0,\beta}(\delta)$ is $j^{\mathcal{T}}_{0,\beta}(W_\delta)$-generic over $\mathcal{M}^{\mathcal{T}}_\beta$, where W_δ is the extender algebra with δ many generators calculated in $\mathfrak{M}$. We call such a β a weak closure point*.*

(2) If $\mathbb{P}$ is a reasonable forcing and additionally Σ condenses to fragments, then in $V^{\mathbb{P}}$ there are club many weak closure points *$\beta \in [0, \omega_1]_{\mathcal{T}}$; i.e. for any $G \subset \mathbb{P}$ that is generic over V there are club many β such that set $\dot{A}^G \cap j^{\mathcal{T}}_{0,\beta}(\delta)$ is $j^{\mathcal{T}}_{0,\beta}(W_\delta)$-generic over $\mathcal{M}^{\mathcal{T}}_\beta$.*

(3) Especially: if $\mathbb{P}$ is a reasonable forcing and additionally Σ condenses to fragments, then for any $G \subset \mathbb{P}$ that is generic over V, the set $\dot{A}^G \subset \omega_1$ is $j^{\mathcal{T}}_{0,\omega_1}(W_\delta)$-generic over $\mathcal{M}^{\mathcal{T}}_{\omega_1}$, where W_δ is the extender algebra with δ many generators calculated in $\mathfrak{M}$. So ω_1 is a weak closure point.

Note that Theorem 5.1 is a special case of conclusion (1) above: if $\dot{A} \subset \omega$, then $\dot{A}$ extends to $\dot{A}$ with unique condensation.

Before we begin the proof of the above lemma, we need a suitable notation. Following Neeman and Zapletal, we extend our notation for the axioms that arise in the construction of the extender-algebra.

Definition 6.22. Let $\mathcal{M} = \langle J_\beta[\vec{E}]; \in, \vec{E}, E_\beta\rangle$ be a premouse such that $\mathcal{M} \models \delta$ is Woodin. Let $\vec{\phi} = \langle \phi_\xi; \xi < \kappa\rangle$ be a sequence of $L_{\delta,\delta,0}$-sentences and let $E = E_\rho$ be a extender on $\vec{E}$. Let λ such that $\mathrm{crit}(E) = \kappa \leq \lambda < \rho$, and suppose $\nu(E)$ is a $\mathcal{M}$-cardinal such that $i_E(\langle \phi_\xi; \xi < \kappa\rangle) \restriction \lambda \in \mathcal{J}^{\mathcal{M}}_{\nu(E)}$. We set

$$a_{\kappa,\lambda,\rho,\vec{\phi}} :\equiv \bigvee_{\alpha<\kappa} \phi_\alpha \leftrightarrow \bigvee_{\alpha<\lambda} i_E(\langle \phi_\xi; \xi < \kappa\rangle)_\alpha.$$

Now we are ready to prove the lemma.

Proof. We construct $\mathcal{T}$ using the strategy we have used many times: iterate up to some closure point and then hit the measure (or its image respectively) witnessing the measurability of δ. This time the tree will be of height $\kappa^+ + 1$. Let us fix $\mathbb{P}$ not necessarily reasonable. Let U be the least extender on the extender sequence of $\mathfrak{M}$ that witnesses that δ is measurable, and let ζ_0 be the ordinal where U is indexed. First we define what a *weak closure point* is in the context of this proof: We will call an $\alpha \leq \kappa^+$ a *weak closure point* if for all $p \in \mathbb{P}$

$$p \Vdash \dot{A}^* \cap j^{\mathcal{T}}_{0,\alpha}(\delta) \models j^{\mathcal{T}}_{0,\alpha}(T(\vec{E})).$$

Note the following subtlety: we are talking about $\dot{A}^*$ above, the uniquely condensing extension of $\dot{A}$; this allows us to discuss the case $\alpha > \omega_1$ in contrast to the previous proofs. Now $\alpha < \kappa^+$ is a *closure point for* if for all $p \in \mathbb{P}$ and all $\zeta < i^{\mathcal{T}}_{0,\alpha}(\zeta_0)$

$$p \Vdash \check{F}_\zeta \text{ does not induce an axiom false of } \dot{A}^* \cap j^{\mathcal{T}}_{0,\alpha}(\zeta_0),$$

where $\vec{F}$ is $\mathcal{M}^{\mathcal{T}}_\alpha$'s extender sequence. Clearly every closure point is a weak closure point. We will show something a little stronger than what we state in conclusion (1); we actually show that there is a closure point.

We construct an iteration tree $\mathcal{T}$ of length $\kappa^+ + 1$ on $\mathcal{M}^{\mathcal{T}}_0 = \mathfrak{M}$. We will refer to this construction as a *genericity iteration for* $\dot{A}^*$. We define the iteration as follows: in the limit case we use Σ to continue the iteration. In the successor case there are subcases: if $\alpha < \omega_1$ is a closure point, then we use $j^{\mathcal{T}}_{0,\alpha}(U)$ to continue the iteration.

If α is not a closure point, then there is a least "bad" extender $E^{\mathcal{M}^{\mathcal{T}}_\alpha}_\rho$ on the extender sequence of $\mathcal{M}^{\mathcal{T}}_\alpha$ and some $p_\alpha \in \mathbb{P}$ such that

$$p_\alpha \Vdash \dot{A}^* \cap j^{\mathcal{T}}_{0,\alpha}(\zeta_0) \not\models a,$$

where a is some axiom in $\mathcal{M}^{\mathcal{T}}_\alpha$ of the form $a_{\kappa_\alpha,\lambda_\alpha,\rho,\vec{\phi}}$ for some $(\lambda_\alpha, \vec{\phi}) \in \mathcal{M}^{\mathcal{T}}_\alpha$. Furthermore we pick p_α so that it decides the value of a and minimizes λ_α, i.e. there is some $\vec{\phi}^\alpha \in \mathcal{M}^{\mathcal{T}}_\alpha$ such that

$$p_\alpha \Vdash \dot{A}^* \cap j^{\mathcal{T}}_{0,\alpha}(\zeta_0) \not\models a_{\kappa_\alpha,\lambda_\alpha,\rho,\vec{\phi}^\alpha},$$

and λ_α is minimal among all λ with

$$p_\alpha \Vdash \dot{A}^* \cap j^{\mathcal{T}}_{0,\alpha}(\zeta_0) \not\models a_{\kappa_\alpha,\lambda,\rho,\vec{\phi}'^\alpha}$$

for some ϕ'^α. We then use $E^{\mathcal{M}^{\mathcal{T}}_\alpha}_\rho$ to continue the iteration. This finishes the construction of $\mathcal{T}$. The arguments we have given before make sure $\mathcal{T}$ is a normal tree. Let $b = [0, \kappa^+]_{\mathcal{T}}$ and let $j = j^{\mathcal{T}}_{0,\kappa^+} : \mathfrak{M} \to \mathcal{M}^{\mathcal{T}}_\kappa$. Note that b is club in κ^+. We set $\mathfrak{M}^* = \mathcal{M}^{\mathcal{T}}_\kappa$. We aim to show the first part of the theorem, i.e. that there is a closure point $< \kappa^+$.

For every $\alpha \in b$ let α^+_b be the least ordinal such that $\alpha \mathcal{T} \alpha^+_b + 1$. So there is an extender $E^{\mathcal{T}}_{\alpha^+_b}$ which we used to continue the iteration at stage α^+_b; let $\kappa_{\alpha^+_b} = \operatorname{crit}(E^{\mathcal{T}}_{\alpha^+_b})$. Since $\mathcal{M}^{\mathcal{T}}_\alpha$ and $\mathcal{M}^{\mathcal{T}}_{\alpha^+_b+1}$ agree on subsets of $\kappa_{\alpha^+_b}$, it follows that $\vec{\phi}^{\alpha^+_b} \in \mathcal{M}^{\mathcal{T}}_\alpha$. Let us denote $\vec{\phi}^{\alpha^+_b}$ by $\vec{\psi}^\alpha$.

Let $S_1 = b \cap \mathrm{Lim}$. For $\alpha \in S_1$ the model $\mathcal{M}^{\mathcal{T}}_\alpha$ is a direct limit and contains $\vec{\psi}^\alpha$, so there is some $h(\alpha) < \alpha$ such that $\vec{\psi}^\alpha \in \operatorname{ran}(j^{\mathcal{T}}_{h(\alpha),\alpha})$. So Fodor's Theorem yields a stationary $S_2 \subset S_1$ such that $h(\alpha) = \beta$ for all $\alpha \in S_2$. Since $\mathcal{M}^{\mathcal{T}}_\beta$ has at most cardinality κ, further thinning of S_2 produces a stationary $S_3 \subset S_2$ and a fixed $\vec{\psi} \in \mathcal{M}^{\mathcal{T}}_\beta$ such that $\vec{\psi}^\alpha = j^{\mathcal{T}}_{\beta,\alpha}(\vec{\psi})$ for all $\alpha \in S_3$. Since $\mathbb{P}$ has cardinality $\leq \kappa$, we can also assume that there is a fixed $p \in \mathbb{P}$ such that $p_{\alpha^+_b} = p$ for all $\alpha \in S_3$.

Let α be any element of S_3 and set $\gamma = \alpha_b^+$ (hence $\gamma+1 \in b$). So

$$p \Vdash \dot{A}^* \cap j^{\mathcal{T}}_{0,\gamma}(\zeta_0) \not\models a_{\kappa,\lambda_\gamma,\rho,j^{\mathcal{T}}_{\beta,\alpha}(\vec{\psi})},$$

where ρ satisfies our minimality assumption and $\kappa_\gamma = \mathrm{crit}(E^{\mathcal{T}}_\gamma)$ and $a_{\kappa,\lambda_\gamma,\rho,j^{\mathcal{T}}_{\beta,\alpha}(\vec{\psi})}$ is calculated in $\mathcal{M}^{\mathcal{T}}_\gamma$. Hence

$$p \Vdash \dot{A}^* \cap j^{\mathcal{T}}_{0,\gamma}(\zeta_0) \not\models \bigvee_{\xi<\kappa_\gamma} j^{\mathcal{T}}_{\beta,\alpha}(\vec{\psi})_\xi \text{ and } \dot{A}^* \cap j^{\mathcal{T}}_{0,\gamma}(\zeta_0) \models \bigvee_{\xi<\nu(E^{\mathcal{T}}_\gamma)} i^{\mathcal{M}^{\mathcal{T}}_\gamma}_{E^{\mathcal{T}}_\gamma}(j^{\mathcal{T}}_{\beta,\alpha}(\vec{\psi}))_\xi.$$

Note that $i^{\mathcal{M}^{\mathcal{T}}_\gamma}_{E^{\mathcal{T}}_\gamma}(j^{\mathcal{T}}_{\beta,\alpha}(\vec{\psi})) = i^{\mathcal{M}^{\mathcal{T}}_\alpha}_{E^{\mathcal{T}}_\gamma}(j^{\mathcal{T}}_{\beta,\alpha}(\vec{\psi}))$, so we will drop the superscript. Since $i_{E^{\mathcal{T}}_\gamma}(j^{\mathcal{T}}_{\beta,\alpha}(\vec{\psi}))$ is $j^{\mathcal{T}}_{\alpha,\gamma+1}(j^{\mathcal{T}}_{\beta,\alpha}(\vec{\psi}))$, we can rewrite the above statement as

$$(*)\quad p \Vdash \dot{A}^* \cap j^{\mathcal{T}}_{0,\gamma}(\zeta_0) \not\models \bigvee_{\xi<\kappa_\gamma} j^{\mathcal{T}}_{\beta,\alpha}(\vec{\psi})_\xi \text{ and } \dot{A}^* \cap j^{\mathcal{T}}_{0,\gamma}(\zeta_0) \models \bigvee_{\xi<\nu(E^{\mathcal{T}}_\gamma)} j^{\mathcal{T}}_{\beta,\gamma+1}(\vec{\psi})_\xi.$$

Let $\alpha' \in S_3$ such that $\alpha' > \gamma+1$. Then $\mathrm{crit}(j^{\mathcal{T}}_{\gamma+1,\alpha'}) \geq \nu(E^{\mathcal{T}}_\gamma)$ and so for $\xi < \nu(E^{\mathcal{T}}_\gamma)$, $j^{\mathcal{T}}_{\beta,\gamma+1}(\vec{\psi})_\xi$ is not moved by $j^{\mathcal{T}}_{\gamma+1,\alpha'}$. Thus

$$p \Vdash \bigvee_{\xi<\nu(E^{\mathcal{T}}_\gamma)} j^{\mathcal{T}}_{\beta,\alpha'}(\vec{\psi}).$$

But then clearly

$$p \Vdash \bigvee_{\xi<\kappa'} j^{\mathcal{T}}_{\beta,\alpha'}(\vec{\psi}),$$

where $\kappa' = \mathrm{crit}(E^{\mathcal{T}}_{\alpha'^+_b})$. This clearly contradicts $(*)$. Hence we have shown that there is a closure point $< \kappa^+$; in fact we did not need that $\dot{A}^*$ is (a name for) a uniquely condensing extension but our argument works for any subset of κ^+. Also it is obvious that there are arbitrarily large closure points $< \kappa^+$, since we could run the same argument starting with $S_1 \setminus \eta$ instead of S_1 for an arbitrary $\eta < \kappa^+$.

We now additionally assume that $\mathbb{P}$ is reasonable and that Σ condenses to fragments. We show the second part of the theorem; the third easily follows from the second. We fix a countable ordinal η and some $G \subset \mathbb{P}$ generic over V. We aim to find a *weak closure point*, i.e. some $\beta > \eta$, $\beta < \omega_1$ such that for some $q \in G$

$$q \Vdash \dot{A}^* \cap j^{\mathcal{T}}_{0,\beta}(\delta) \models j^{\mathcal{T}}_{0,\beta}(T(\vec{E})).$$

In V pick a countable $X \prec H_\lambda$ for some large enough regular λ such that $\omega_1 \cap X > \eta$ and $\dot{A}, \dot{A}^* \cap \check{\kappa}^+, \Sigma, \mathcal{T}, \mathbb{P} \in X$. Let $\pi : H \to X$ denote the inverse of the transitive collapse of X and let $\alpha = \omega_1 \cap X$. By the definition of reasonable forcings we can assume without loss of generality that $G \cap X$ is $\mathbb{P}$-generic over X and $X[G \cap X] \prec H_\lambda[G]$; this implies that π lifts, i.e.

$$\hat{\pi} : H[\bar{G}] \to H_\lambda[G],$$
$$\tau^{\bar{G}} \mapsto \pi(\tau)^G$$

is an elementary embedding, where $\bar{G} := \pi^{-1}``G$. We write π again for $\hat{\pi}$. Let $\pi(\bar{\kappa}, \bar{A}, \bar{A}^*, \bar{\Sigma}, \bar{\mathcal{T}}) = \kappa, \dot{A}^G, (\dot{A}^* \cap \check{\kappa}^+)^G, \Sigma, \mathcal{T}$. Since $(\dot{A}^*)^G$ is the uniquely condensing extension of $\dot{A}^G$ we have $\dot{A}^G \cap \bar{\kappa} = \bar{A}^*$. By the first part of the theorem there is an uncountable closure point $\beta \in \kappa^+ \cap X$. Let $\pi(\bar{\beta}) = \beta$, then by elementarity $\dot{A}^G \cap j^{\bar{\mathcal{T}}}_{0,\bar{\beta}}(\delta) = \bar{A}^* \cap j^{\bar{\mathcal{T}}}_{0,\bar{\beta}}(\delta)$ is $j^{\bar{\mathcal{T}}}_{0,\bar{\beta}}(W_\delta)$-generic over $\mathcal{M}^{\bar{\mathcal{T}}}_{\bar{\beta}}$. Also by elementarity the tree $\bar{\mathcal{T}}$ in H is built using the same rules as in the construction of $\mathcal{T}$, but at limit stages one uses the strategy $\bar{\Sigma}$ to pick branches. By our hypothesis we have that $\bar{\Sigma}$ is a fragment of Σ, hence we know that the branches picked by $\bar{\Sigma}$ in $\bar{H}$ are the same branches that Σ picked in V. This implies that the iteration tree $\bar{\mathcal{T}}$ is an initial segment of $\mathcal{T}$. So in $V[G]$ we have

$$V[G] \models \dot{A}^G \cap j^{\mathcal{T}}_{0,\bar{\beta}}(\delta) \models j^{\mathcal{T}}_{0,\bar{\beta}}(T(\vec{E})).$$

So there is a condition $q \in G$ that forces that $\bar{\beta}$ is a weak closue point. $\square$

Note that if $\mathbb{P}$ is reasonable but not c.c.c. we can not hope to show that in the previous tree construction there are countable stages α such that

$$\mathbf{1}_{\mathbb{P}} \Vdash j^{\mathcal{T}}_{0,\alpha}(\delta) \cap \dot{A} \models j^{\mathcal{T}}_{0,\alpha}(T(\vec{E})).$$

To see this pick a maximal antichain $\mathcal{A} \subset \mathbb{P}$ of cardinality ω_1 (we can do so if we without loss of generality suppose that $\mathbb{P}$ is a Boolean algebra). Assume CH and let $f : \mathcal{A} \to \mathcal{P}(\omega)$ be a surjection. Choose a name $\dot{A}$ such that $a \Vdash \dot{A} = f\check{(a)}$ for every $a \in \mathcal{A}$. Then clearly $\dot{A}$ is a name for a bounded subset of ω_1 and hence $\dot{A}$ extends to class with unique condensation. Let $\mathcal{T}$ be the iteration tree given by the previous lemma. Now assume towards a contradiction that for a countable α

$$\mathbf{1}_{\mathbb{P}} \Vdash j^{\mathcal{T}}_{0,\alpha}(\delta) \cap \dot{A} \models j^{\mathcal{T}}_{0,\alpha}(T(\vec{E})).$$

Let a be such that $f(a)$ codes an ordertype $> \mathcal{M}^{\mathcal{T}}_\alpha \cap \mathsf{OR}$. Then

$$a \Vdash f\check{(a)} \text{ is generic over } \mathcal{M}^{\mathcal{T}}_\alpha,$$

a contradiction.

Remark 6.23. Lemma 6.21 can be easily seen to generalize. Using the notation of Lemma 6.21 : if $\mathbb{P}$ is reasonable and Σ condenses to fragments and we have an arbitrary iteration tree $\mathcal{T}$ on $\mathfrak{M}$ of height $< \omega_2$, then one can continue $\mathcal{T}$ by performing a genericity iteration for A^* as in the proof of Lemma 6.21 of length $\kappa^+ + 1$. Note that in general one will have to apply extenders to models of $\mathcal{T}$ in this process. By the same reflection argument this genericity iteration terminates after $< \omega_2$-many steps. So we reach a closure point in the sense of Lemma 6.21 at some stage $< \omega_2$.

6.24. **Applications of sets with uniquely condensing extensions.** The following lemma is part of the folklore surrounding measurable cardinals and not too difficult to prove. A detailed proof can be found in [8, 1.1.20, 1.1.19]

Lemma 6.25 (folklore). *Let κ be measurable and assume $\theta \geq (2^\kappa)^+$ is a regular cardinal. Let $Z \prec M$ be a substructure such that $\mu \in Z$ and $Z \cap \mathcal{P}(\kappa)$ has cardinality $< \kappa$ and suppose* $\mathrm{Card}(Z) < \theta$. *Then for all $\gamma < \kappa$*

(1) $Z[\gamma] = \{f(\gamma)\,;\, f \in Z, f : \kappa \to M\} \prec M$,
(2) $\bigcap\{A\,;\, A \in Z \cap \mu\} \neq \varnothing$,
(3) if $\gamma \in \cap\{A\,;\, A \in Z \cap \mu\} \neq \varnothing$, *then* $Z[\gamma] \cap \gamma = Z \cap \gamma$. □

Recall that Chang's Conjecture is equivalent to the statement: the set $\{X \subset \omega_2\,;\, \mathrm{otp}(X) = \omega_1\}$ intersects all strongly closed unbounded subsets of $[\omega_2]^{\omega_1}$, where $C \subset [\kappa]^{\omega_1}$ is called strongly closed if and only if $C = C_F = \{X \in [\kappa]^{\omega_1}\,;\, F\text{"}[X]^{<\omega} \subset X\}$ for some $F\colon [\kappa]^{<\omega} \to \kappa$. In the light of this, the following lemma can be seen as a generalization of Chang's Conjecture.

Lemma 6.26 (folklore)**.** *Let κ be measurable. Let $M = H_\theta$ for some regular $\theta > 2^\kappa$. Then*

$$\{X \subset \kappa\,;\, \mathrm{otp}(X) = \omega_1 \wedge \exists Z \prec M : X = Z \cap \kappa\}$$

intersects all strongly closed unbounded subsets of $[\kappa]^{\omega_1}$. Especially

$$\{X \subset \kappa\,;\, \mathrm{otp}(X) = \omega_1\}$$

intersects all strongly closed unbounded subsets of $[\kappa]^{\omega_1}$.

Proof. Fix a function $F : [\kappa]^{<\omega} \to \kappa$. We have to show that

$$\{X \subset \kappa\,;\, \mathrm{otp}(X) = \omega_1 \wedge \exists Z \prec M : X = Z \cap \kappa\}$$

intersects C_F. We build a chain of length ω_1 of elementary substructures $\langle Z_\alpha; \alpha < \omega_1\rangle$ of M such that $Z_\alpha \subset Z_{\alpha+1}$ for $\alpha < \omega_1$ and $Z_\alpha \cap \kappa = Z_{\alpha+1} \cap \sup(Z_\alpha \cap \kappa)$. Let $Z_0 \prec M$ be a countable substructure with $\kappa, F \in Z_0$. At limit stages form unions. If Z_α is already constructed, then an application of Lemma 6.25 yields a $Z_{\alpha+1} \prec M$, $Z_\alpha \subset Z_{\alpha+1}$ with $Z_\alpha \cap \kappa = Z_{\alpha+1} \cap \sup(Z_\alpha \cap \kappa)$. Set $Z = \bigcup\{Z_\alpha\,;\, \alpha < \omega_1\}$. Then Z is closed under F and $\mathrm{otp}(Z \cap \kappa) = \omega_1$, the latter holds because the sets of the form $Z_\alpha \cap \kappa$, $\alpha < \kappa$, end-extend each other. This finishes the proof of the lemma. □

We show that the conclusion of the previous lemma is preserved under c.c.c. forcing.

Lemma 6.27. *Let $\mathbb{P}$ be notion of forcing that satisfies the c.c.c. Let $\kappa > \omega_1$ be a cardinal and let $S \subset [\kappa]^{\omega_1}$ be such that S intersect all strongly closed unbounded subsets of $[\kappa]^{\omega_1}$. Then in $V^{\mathbb{P}}$ the set S also intersects all strongly closed unbounded subsets of $[\kappa]^{\omega_1}$.*

Proof. Fix a name $\dot{F} \in V^{\mathbb{P}}$ such that

$$\mathbf{1}_{\mathbb{P}} \Vdash \dot{F} : [\check{\kappa}]^{<\check{\omega}} \to \kappa.$$

Let θ be large enough such that all the dense sets of $\mathbb{P}$, $\dot{F}$, $S \in V_\theta$. The set of C' of $Y \prec V_\theta$ of cardinality ω_1 such that $\mathbb{P}, \dot{F}, S \in Y$ is a strongly closed unbounded subsets of $[V_\theta]^{\omega_1}$, so the set $C = \{Y \cap \kappa\,;\, Y \in C\}$ is strongly closed unbounded in $[\kappa]^{\omega_1}$. Pick $Y \in C'$ such that $Y \cap \kappa \in C \cap S$. Then, by the argument for Claim 1 in the proof of Lemma 5.2, we have for all V-generics $G \subset \mathbb{P}$

$$Y[G] \prec V_\theta[G] \text{ and } Y[G] \cap V = Y.$$

By elementarity $Y \cap \kappa = Y[G] \cap \kappa$ is closed under $\dot{F}^G$. This suffices to show. □

Using the characterization of CC which was mentioned above we obtain:

Corollary 6.28 (folklore)**.** *Chang's Conjecture is preserved by c.c.c. forcing.*

□

The following concept is not standard; we introduce it to state the following theorems.

Definition 6.29. Let $\kappa > \omega_1$ be a cardinal and let $S \subset [\kappa]^{\omega_1}$ such that S intersects all strongly closed unbounded subsets of $[\kappa]^{\omega_1}$. A notion of forcing $\mathbb{P}$ *preserves* S, if in $V^{\mathbb{P}}$ the set S intersects all strongly closed unbounded subsets of $[\kappa]^{\omega_1}$.

Theorem 6.30. *Let κ be measurable and let $\mathbb{P}$ be a forcing of cardinality $< \kappa$. Let $\dot{A}$ be a name such that*

$$\mathbf{1}_{\mathbb{P}} \Vdash L[\dot{A}] \models \phi(\dot{A}, \check{\vec{\alpha}}),$$

where $\vec{\alpha}$ are finitely many ordinal parameters, and $\dot{A}^*$ be a name for subset of κ such that

$$\mathbf{1}_{\mathbb{P}} \Vdash \dot{A} \textit{ extends to } \dot{A}^* \textit{ with unique condensation.}$$

Let $\theta \geq (2^\kappa)^+$ be large enough such that $\dot{A}^* \in H_\theta$ and set $M = \langle H_\theta; \in, \mathbb{P}, \dot{A}, \dot{A}^* \rangle$. Suppose $M^\sharp_{\mathsf{mw}}$ exists and has an $(\omega, \kappa+1)$-iteration strategy Σ that condenses to fragments. Suppose that

$$S := \{X \subset \kappa\, ; \operatorname{otp}(X) = \omega_1 \wedge \exists Y \prec M : Y \cap \kappa = X\}^V$$

intersects all strongly closed unbounded subsets of $[\kappa]^{\omega_1}$ and $\mathbb{P}$ preserves S.

Then there is some $A \subset \omega_1$, $A \in V$ such that

$$L[A] \models \phi(A, \vec{\alpha}).$$

Proof. Set $\mathfrak{M} = M^\sharp_{\mathsf{mw}}$ and let δ denote $\mathfrak{M}$'s measurable Woodin. Let $U \in M^\sharp_{\mathsf{mw}}$ denote the (trivial completion of the) least normal measure on δ and let ζ_0 denote the index of U on $M^\sharp_{\mathsf{mw}}$'s extender sequence. Our strategy is as follows: using the ideas of Lemma 6.21 we build an iteration tree $\mathcal{T} \in V$ on $\mathfrak{M}$ of height $\kappa+1$ such that for all $G \subset \mathbb{P}$ the set $\dot{A}^{*G}$ is generic over $\mathcal{M}^{\mathcal{T}}_\kappa$. Once $\mathcal{T}$ is constructed it will follow from the hypothesis on S that we find some elementary substructure $X \prec M$, $X \in V$, $\operatorname{otp}(X \cap \kappa) = \omega_1$ such that $\dot{A}^G$ is generic over this substructure.

We construct $\mathcal{T}$; we will omit superscripts $\mathcal{T}$ where possible, so $\mathcal{T}$ has iteration maps $j_{\alpha,\beta}$ and models $\mathcal{M}_\alpha$. We will call $\alpha < \kappa$ a *closure point* if

$$\mathbf{1}_{\mathbb{P}} \Vdash \dot{A}^* \cap j_{0,\alpha}(\zeta_0) \models j_{0,\alpha}(W_\delta)$$

where W_δ is the extender algebra with δ-many generators calculated in $\mathfrak{M}$. If α is a closure point we use $j_{0,\alpha}(U)$ to continue the iteration. If α is not a closure point we perform a genericity iteration for $\dot{A}^*$ in the sense of Lemma 6.21. Since $\mathbb{P}$ has size $< \kappa$ we can apply conclusion (1) of Lemma 6.21; so we reach the next closure point after $<\kappa$-many stages. Literally Lemma 6.21 only tells us where the first weak closure point is; nevertheless from the proof we also obtain a closure point. Moreover it is easy to see the following: after we apply (the image of) U at a closure point α, we can perform another genericity iteration for $\dot{A}^*$ in the sense of Lemma 6.21. This completes the definition of $\mathcal{T}$.

Clearly $\mathcal{T} \in M$. Let $b = [0,\kappa]_{\mathcal{T}}$. By the argument we have given before b contains unboundedly many (and hence club many) closure points. So club often we have used (the images of) U to continue the iteration. Moreover $j_{0,\kappa}(\delta) = \kappa$ and κ is a weak closure point. Let $\mathfrak{M}^* = \mathcal{M}_\kappa$.

Now pick an arbitrary $G \subset \mathbb{P}$ generic over V. In $V[G]$ we consider the structure $M^+ := \langle H^V_\theta; \in, \mathbb{P}, \dot{A}, \dot{A}^*, \dot{A}^{*G}\rangle$. Trivially all substructures of M^+ contain $\mathcal{T}$. The set

$$C = \{X \subset \kappa\,;\, \mathrm{otp}(X) = \omega_1 \wedge \exists Y \prec M^+ : Y \cap \kappa = X\}^{V[G]}$$

is strongly closed unbounded. By our hypothesis on S, we find some $Y \prec M$, $Y \in V$ such that $\mathrm{otp}(Y \cap \kappa) = \omega_1$ and there is some $Y^+ \in C$ such that $Y^+ \cap \kappa = Y \cap \kappa$. Since κ is a closure point $Y^+ \models \dot{A}^{*G} \cap \kappa$ is generic over $\mathfrak{M}^*$. Let $\pi^+ : N^+ \to Y^+$ denote the inverse of the transitive collapse of Y^+. By Lemma 6.4 and $\mathrm{otp}(Y^+ \cap \kappa) = \omega_1$ we have $(\pi^+)^{-1}(\dot{A}^{*G} \cap \kappa) = \dot{A}^G$. Set $\bar{\mathcal{T}} = (\pi^+)^{-1}(\mathcal{T})$ and let $\pi : N \to Y$ denote the inverse of the transitive collapse of Y. Since $Y^+ \cap \kappa = Y \cap \kappa$ we have that $\bar{\mathcal{T}} = \pi^{-1}(\mathcal{T})$. Since Σ condenses to fragments, $\bar{\mathcal{T}}$ is build according to Σ. So by elementarity of π^+, $\bar{\mathcal{T}}$ contains ω_1-many closure points for $\dot{A}^G$ and hence $\dot{A}^G$ is generic over $\bar{\mathfrak{M}}^* = \mathcal{M}^{\bar{\mathcal{T}}}_{\omega_1}$. Summing up we have that $\dot{A}^G$ is generic over $\bar{\mathfrak{M}}^* \in V$. If necessary we iterate $\bar{\mathfrak{M}}^*$'s top-measure to make sure $\vec{\alpha} \in \bar{\mathfrak{M}}^*$. Pick a condition $q \in j^{\bar{\mathcal{T}}}_{0,\omega_1}(W_\delta)$ such that

$$q \Vdash \omega_1 = j^{\bar{\mathcal{T}}}_{0,\omega_1}(\delta)\check{} \wedge \exists A : L[A] \models \phi(A, \check{\vec{\alpha}}).$$

We want to find $\Gamma \in V$, $q \in \Gamma$, $\Gamma \subset j^{\bar{\mathcal{T}}}_{0,\omega_1}(W_\delta)$ generic over $\bar{\mathfrak{M}}^*$. For this we need to piece together end-extending generics; since we do not need to make sure that $\mathbb{R}^V \subset \bar{\mathfrak{M}}^*[\Gamma]$ the argument is much simpler than in the proof for Theorem 4.1. Especially we do not have to pick generics while iterating this time. Let $C \subset [0,\omega_1]_{\bar{\mathcal{T}}}$ denote the club of points α where we used $j_{0,\alpha}(U)$ to continue the iteration. Let $\alpha_0 \in C$ be such that $q \in \mathcal{M}^{\bar{\mathcal{T}}}_{\alpha_0}$ and q is not moved by $j^{\bar{\mathcal{T}}}_{\alpha_0,\omega_1}$. In V pick $\Gamma_0 \subset j^{\bar{\mathcal{T}}}_{0,\alpha_0}(W_\delta)$ generic over $\mathcal{M}^{\bar{\mathcal{T}}}_{\alpha_0}$. Let $\alpha_1 = \min(C \setminus (\alpha_0 + 1))$. Using Lemma 4.3, we can end-extend Γ_0 to some Γ_1 generic over $\mathcal{M}^{\bar{\mathcal{T}}}_{\alpha_1}$. In this fashion we continue all the way up through C: at successor stages repeat the argument we have just given, at limit stages $\lambda \in C$ form the union $\Gamma_\lambda = \bigcup\{\Gamma_i\,;\, i < \lambda\}$. Using the fact that anitchains are small, it is not difficult to see that Γ_λ is generic over $\mathcal{M}^{\bar{\mathcal{T}}}_\lambda$. Finally we set $\Gamma = \bigcup\{\Gamma_i\,;\, i < \omega_1\}$. Then Γ is as desired.

There is some $A \subset \omega_1$, $A \in \bar{\mathfrak{M}}^*[\Gamma]$ such that

$$L_\rho[A] \models \phi(A, \vec{\alpha}),$$

where ρ is the critical point of $\bar{\mathfrak{M}}^*$'s top measure. Iterating this top measure out of the universe we obtain

$$L[A] \models \phi(A, \vec{\alpha}).$$

This finishes the proof. $\square$

The previous theorem has a variant:

Theorem 6.31. *Let $\mathbb{P}$ be a reasonable forcing of cardinality $< \kappa$, κ regular. Let $\dot{A}$ be a name such that*

$$\mathbf{1}_{\mathbb{P}} \Vdash L[\dot{A}] \models \phi(\dot{A}, \check{\vec{\alpha}}),$$

where $\vec{\alpha}$ are finitely many ordinal parameters, and $\dot{A}^*$ be a name for subset of κ such that

$$\mathbf{1}_{\mathbb{P}} \Vdash \dot{A} \text{ extends to } \dot{A}^* \text{ with unique condensation.}$$

Let $\theta \geq (2^\kappa)^+$ be large enough such that $\dot{A}^* \in H_\theta$ and set $M = \langle H_\theta; \in, \mathbb{P}, \dot{A}, \dot{A}^* \rangle$. Suppose $M^\sharp_{\mathsf{mw}}$ exists and has an $(\omega, \kappa+1)$-iteration strategy Σ that condenses to fragments. Suppose that

$$S := \{X \subset \omega_2\,;\, \mathrm{otp}(X) = \omega_1 \wedge \exists Y \prec M : Y \cap \omega_2 = X\}^V$$

intersects all strongly closed unbounded subsets of $[\omega_2]^{\omega_1}$ and $\mathbb{P}$ preserves S.

Then there is some $A \subset \omega_1$, $A \in V$ such that

$$L[A] \models \phi(A, \vec{\alpha}).$$

Note that S intersects all strongly club sets in V is equivalent to Chang's Conjecture. The proof of the above theorem is similar to the proof of the previous theorem. We nevertheless give some details, especially at the point where the reasonability of $\mathbb{P}$ is applied.

Proof. Set $\mathfrak{M} = M^\sharp_{\mathsf{mw}}$ and let δ denote $\mathfrak{M}$'s measurable Woodin. Let $U \in M^\sharp_{\mathsf{mw}}$ denote the (trivial completion of the) least normal measure on δ and let ζ_0 denote the index of U on $M^\sharp_{\mathsf{mw}}$'s extender sequence. Using Lemma 6.21 we are going to build an iteration tree $\mathcal{T} \in V$ on $\mathfrak{M}$ of height $\omega_2 + 1$ such that for all $G \subset \mathbb{P}$ the set $\dot{A}^{*G} \cap \omega_2$ is generic over $\mathcal{M}^{\mathcal{T}}_{\omega_2}$. Once $\mathcal{T}$ is constructed it will follow from the hypothesis on S that we find some elementary substructure $X \prec M$, $X \in V$, $\mathrm{otp}(X \cap \omega_2) = \omega_1$ such that $\dot{A}^G$ is generic over this substructure.

We construct $\mathcal{T}$; we will omit superscripts $\mathcal{T}$ where possible, so $\mathcal{T}$ has iteration maps $j_{\alpha,\beta}$ and model $\mathcal{M}_\alpha$. We will call $\alpha < \omega_2$ a *closure point* if

$$\mathbf{1}_{\mathbb{P}} \Vdash \dot{A}^* \cap j_{0,\alpha}(\zeta_0) \models j_{0,\alpha}(W_\delta)$$

where W_δ is the extender algebra with δ-many generators calculated in $\mathfrak{M}$. If α is a closure point we use $j_{0,\alpha}(U)$ to continue the iteration. If α is not a closure point we perform a genericity iteration for $\dot{A}^*$ in the sense of Lemma 6.21. Since $\mathbb{P}$ is reasonable and Σ condenses to fragments we can apply Lemma 6.21, so, taking note of Remark 6.23, we reach the next closure point after $< \omega_2$-many stages. This completes the definition of $\mathcal{T}$. Clearly $\mathcal{T} \in M$. Let

$b = [0, \omega_2]_\mathcal{T}$. Clearly b contains unboundedly many (and hence club many) closure points. So club often we have used (the images of) U to continue the iteration. Moreover $j_{0,\omega_2}(\delta) = \omega_2$ and ω_2 is a weak closure point.

The rest follows like in the previous proof if one replaces κ with ω_2. □

References

[1] J. Bagaria, Bounded forcing axioms as principles of generic absoluteness, Arch. Math. Logic **39** (2000), no. 6, 393–401. MR1773776 (2001i:03103)

[2] D. Busche and R. Schindler, The strength of choiceless patterns of singular and weakly compact cardinals, Ann. Pure Appl. Logic **159** (2009), no. 1-2, 198–248. MR2523718 (2010j:03060)

[3] I. Farah, Completely additive liftings, Bull. Symbolic Logic **4** (1998), no. 1, 37–54. MR1609187 (99h:03025)

[4] I. Farah, A proof of the Σ^2_1-absoluteness theorem, in *Advances in logic*, 9–22, Contemp. Math., 425 Amer. Math. Soc., Providence, RI. MR2322360 (2008b:03072)

[5] I. Farah, R. Ketchersid, P. B. Larson, and M. Magidor, Absoluteness for universally Baire sets and the uncountable. II, in *Computational prospects of infinity. Part II. Presented talks*, 163–191, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., 15 World Sci. Publ., Hackensack, NJ. MR2449464 (2009i:03048)

[6] I. Farah and P. B. Larson, Absoluteness for universally Baire sets and the uncountable. *Set theory: recent trends and applications*, 47–92, Quad. Mat., 17 Dept. Math., Seconda Univ. Napoli, Caserta. MR2374762 (2008j:03076)

[7] A. Kanamori, *The higher infinite*, second edition, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. MR1994835 (2004f:03092)

[8] P. B. Larson, *The stationary tower*, University Lecture Series, 32, Amer. Math. Soc., Providence, RI, 2004. MR2069032 (2005e:03001)

[9] D. A. Martin and J. R. Steel, Iteration trees, J. Amer. Math. Soc. **7** (1994), no. 1, 1–73. MR1224594 (94f:03062)

[10] I. Neeman, Optimal proofs of determinacy, Bull. Symbolic Logic **1** (1995), no. 3, 327–339. MR1349683 (96m:03032)

[11] I. Neeman and J. Zapletal, Proper forcing and $L(\mathbb{R})$, arXiv:math/0003027 (2000).

[12] I. Neeman and J. Zapletal, Proper forcing and $L(\mathbb{R})$, J. Symbolic Logic **66** (2001), no. 2, 801–810. MR1833479 (2002m:03078)

[13] J. R. Steel, An outline of inner model theory, *Handbook of set theory.* Vols. 1, 2, 3, 15951684, Springer, Dordrecht, 2010. MR2768698

[14] S. Todorčević, Conjectures of Rado and Chang and cardinal arithmetic, *Finite and infinite combinatorics in sets and logic (Banff, AB, 1991)*, 385–398, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 411, Kluwer Acad. Publ., Dordrecht, 1993. MR1261218

[15] W. H. Woodin, *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, de Gruyter Series in Logic and its Applications, 1, de Gruyter, Berlin, 1999. MR1713438 (2001e:03001)

[16] W. H. Woodin, The continuum hypothesis, in *Logic Colloquium 2000*, 143–197, Lect. Notes Log., 19 Assoc. Symbol. Logic, Urbana, IL. MR2143878 (2006m:03079)

Received June 4, 2010; accepted September 2, 2012

Philipp Doebler
Westfälische Wilhelms-Universität Münster, Institut für Psychologie,
Fliednerstr. 21, D-48149 Münster, Germany
E-mail: doebler@uni-muenster.de
URL: http://wwwpsy.uni-muenster.de/Psychologie.inst4/AEHolling/personen/P_Doebler.html

Ralf Schindler
Westfälische Wilhelms-Universität Münster, Institut für Mathematische Logik und Grundlagenforschung,
Einsteinstr. 62, D-48149 Münster, Germany
E-mail: rds@uni-muenster.de
URL: http://www.math.uni-muenster.de/u/rds/