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C*-Algebras Associated to Irreversible Semigroup Dynamical Systems

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Abstract

In the recent past, Cuntz and Vershik introduced a notion of independence for pairs of commuting group endomorphisms of a discrete abelian group with finite cokernel. Here we generalize their concept to the case of arbitrary commuting group endomorphisms of a discrete group. We show that the characterizations of independence established by Cuntz and Vershik do not carry over in general. Therefore we will differentiate between independence and strong independence. In addition, we establish a close connection between the latter notion and the concept of $*$-commutativity introduced by Arzumanian and Renault. We then define irreversible algebraic dynamical systems and irreversible $*$-commutative dynamical systems to mirror both algebraic and topological aspects of dynamical systems like $\times 2, \times 3 : \mathbb{T} \longrightarrow \mathbb{T}$.

To both kinds of dynamical systems, we associate C*-algebras by means of generators and relations. In the case of irreversible algebraic dynamical systems, this C*-algebra is a natural generalization of the one that has been studied by Hirshberg, Cuntz and Vershik, and Vieira. We prove that, under mild assumptions, this C*-algebra is a UCT Kirchberg algebra. Moreover, we analyse its diagonal subalgebra, relate its core subalgebra to generalized Bunce-Deddens algebras in the sense of Orfanos and establish crossed product pictures. For irreversible $*$-commutative dynamical systems, the C*-algebra takes into account a reconstruction formula reminiscent of Parseval frames. Given that an irreversible algebraic dynamical system corresponds to an irreversible $*$-commutative dynamical system via Pontryagin duality, we prove that the two C*-algebras we obtain are canonically isomorphic.

In a different direction, we associate a discrete product system of Hilbert bimodules to either of the two types of dynamical systems. For irreversible algebraic dynamical systems, these product systems turn out to have coherent systems of orthonormal bases on the fibres. In the case of irreversible $*$-commutative dynamical systems, we only obtain coherent systems of finite Parseval frames. Nevertheless, this enables us to show that, for both kinds of dynamical systems, the C*-algebra we constructed via an explicit presentation coincides with the Cuntz-Nica-Pimsner algebra associated to the product systems of Hilbert bimodules obtained from the dynamical system.

For irreversible $*$-commutative dynamical systems, we use this identification to characterize topological freeness of the dynamical system by properties of the associated C*-algebra. This extends the corresponding result of Meier Carlsen and Silvestrov for a single surjective local homeomorphism of a compact Hausdorff space. An almost immediate consequence is a necessary and sufficient simplicity criterion for C*-algebras associated to irreversible $*$-commutative dynamical systems. As an application, we show that the conditions imposed on irreversible algebraic dynamical systems to obtain UCT Kirchberg algebras are in fact necessary in the case where the involved group is abelian and the group endomorphisms have finite cokernel.

Contents

Introduction

Let G be a countable discrete group and $(\xi_g)_{g\in G}$ denote the standard orthonormal basis of the Hilbert space $\ell^2(G)$. Suppose φ is an injective group endomorphism of G. Then $S_\varphi\xi_g = \xi_{\varphi(g)}$ defines an isometry on $\ell^2(G)$. For $g \in G$, let U_g denote the canonical unitary on $\ell^2(G)$ given by left translation. Then $S_\varphi U_g = U_{\varphi(g)}S_\varphi$ holds for all $g \in G$. This leads to the C*-algebra $\mathcal{O}_r[\varphi]$ generated by the isometry S_φ and the unitaries $(U_g)_{g\in G}$. A natural object to study within this context is a universal model for $\mathcal{O}_r[\varphi]$, which is a C*-algebra $\mathcal{O}[\varphi] = C^*(\{s_\varphi, (u_g)_{g\in G} \mid \mathcal{R}\})$ generated by an isometry s_φ and unitaries u_g satisfying a suitable set of relations $\mathcal{R}$.

In the case where φ is a group automorphism of G, the C*-algebra $C^*(S_\varphi, (U_g)_{g\in G})$ is related to the crossed product $C^*_r(G) \rtimes_\alpha \mathbb{Z}$, where $\alpha(u_g) = u_{\varphi(g)}$. It is well-known that this crossed product is canonically isomorphic to the reduced group C*-algebra of the semidirect product $G \rtimes_\varphi \mathbb{Z}$. Hence, the full group C*-algebra of $G \rtimes_\varphi \mathbb{Z}$ can be regarded as a universal model for $\mathcal{O}_r[\varphi]$, even though the latter tends to be a proper quotient of the former. The structure of these C*-algebras has already been studied extensively, see [Wil07]. In contrast, the situation for an injective, but non-surjective group endomorphism φ has started to receive more attention in the recent past. Let us remark that G has to be infinite in this case. The most elementary examples of such endomorphisms are $\times 2 : \mathbb{Z} \longrightarrow \mathbb{Z}$ and the one-sided shift on $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$ for $n \geq 2$.

Restricting to the case where G is amenable and $G/\varphi(G)$ is finite, Ilan Hirshberg introduced a universal C*-algebraic model $\mathcal{O}[\varphi]$ for $\mathcal{O}_r[\varphi]$ in 2002, see [Hir02]. He showed that the core $\mathcal{F} \subset \mathcal{O}[\varphi]$, which is the fixed point algebra under the canonical gauge action, is simple if $(\varphi^n(G))_{n\in\mathbb{N}}$ separates the points in G, that is, $\bigcap_{n\in\mathbb{N}} \varphi^n(G) = \{1_G\}$. Using simplicity of $\mathcal{F}$, he concluded that $\mathcal{F}$ is the crossed product of a natural commutative subalgebra $\mathcal{D}$, called the diagonal, by G. Assuming that the family of subgroups $(\varphi^n(G))_{n\in\mathbb{N}}$ separates the points in G and consists of normal subgroups of G, Hirshberg established that $\mathcal{O}[\varphi]$ is simple and therefore isomorphic to $\mathcal{O}_r[\varphi]$. Additionally, he computed the K-theory of $\mathcal{O}[\varphi]$ based on the K-theory of $\mathcal{F}$ and the Pimsner-Voiculescu six-term exact sequence for $\times n : \mathbb{Z} \longrightarrow \mathbb{Z}, n \geq 2$, the shift on $\bigoplus_{\mathbb{N}} H$, where H is a finite group, and $\varphi : \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}, a \mapsto bab, b \mapsto aba$, where a, b denote the standard generators of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

A decade later, Felipe Vieira extended Hirshberg's results to the case where G is amenable and $(\varphi^n(G))_{n\in\mathbb{N}}$ separates the points in G, see [Vie13]. His approach used techniques for semigroup crossed products as well as partial group crossed products. One remarkable outcome of his work is the connection to semigroup C*-algebras for left cancellative semigroups as introduced by Xin Li in [Li12, Li13]: If G is amenable, $(\varphi^n(G))_{n\in\mathbb{N}}$ separates the points in G, and $G/\varphi(G)$ is infinite, then $\mathcal{O}[\varphi]$ is canonically isomorphic to the full semigroup C*-algebra of $G \rtimes_\varphi \mathbb{N}$. Furthermore, Vieira showed that this is the same as the reduced semigroup C*-algebra for this semidirect product.

At about the same time, Joachim Cuntz and Anatoly Vershik examined the case where G is abelian, $G/\varphi(G)$ is finite, and $(\varphi^n(G))_{n\in\mathbb{N}}$ separates the points in G, see [CV13]. They proved that $\mathcal{O}[\varphi]$ is a UCT Kirchberg algebra and provided a general method to compute the K-theory of $\mathcal{O}[\varphi]$. In addition, they found that the spectrum of the diagonal $\mathcal{D}$ is a compact abelian group G_φ, which can be interpreted as a completion of G with respect to φ. Another interesting outcome of [CV13] is the fact that $\mathcal{F} \cong C(G_\varphi) \rtimes G$ is also isomorphic to $C(\hat{G}) \rtimes \hat{G}_\varphi$.

Summarizing the current status, it is fair to say that a lot is known about the C*-algebras $\mathcal{O}[\varphi]$, $\mathcal{F}$ and $\mathcal{D}$ associated to a single injective, non-surjective group endomorphism φ of a countably infinite, discrete group G. Indeed, in many cases we are able, at least in principle, to compute the K-theory for $\mathcal{O}[\varphi]$, which is known to be a complete invariant due to the celebrated classification theorem by Eberhard Kirchberg and Christopher N. Phillips, see [Kir, Phi00]. Thus, by computing the K-theory of $\mathcal{O}[\varphi]$, we can recover the information on the dynamical system (G,φ) that is encoded in $\mathcal{O}[\varphi]$. It is therefore natural to ask whether analogous results hold for similar dynamical systems involving more than one transformation.

To motivate this question, let us mention an important example which showcases some interesting phenomena for such dynamical systems. In 1967, Hillel Furstenberg proved the following result, which applies for instance to $\times 2, \times 3 : \mathbb{T} \longrightarrow \mathbb{T}$, the Pontryagin dual of $\times 2, \times 3 : \mathbb{Z} \longrightarrow \mathbb{Z}$, see [Fur67, Part IV]: Every closed subset of $\mathbb{T}$, which is invariant under the action of a non-lacunary subsemigroup of $\mathbb{Z}^\times$, is either finite or equals $\mathbb{T}$. This led him to conjecture that a stronger form of rigidity might be true: Any invariant ergodic Borel probability measure on $\mathbb{T}$ is either atomic or the Lebesgue measure on $\mathbb{T}$. In its general form, this conjecture is still open. An important reduction step has been achieved by Daniel J. Rudolph, see [Rud90] and also [Par96] for a concise presentation. The conjecture has been verified by Manfred Einsiedler and Alexander Fish in 2010 for the case where the acting semigroup is sufficiently large in the sense that it has positive lower logarithmic density, see [EF10]. This form of measure rigidity has also been studied for certain reversible dynamical systems, see [EK05] and the references therein. In a different direction, Daniel J. Berend and Roman Muchnik generalised the rigidity result from [Fur67] stated above to compact abelian groups, see [Ber83, Ber84, Muc05].

Coming back to $\times 2, \times 3 : \mathbb{T} \longrightarrow \mathbb{T}$, it is natural to ask: What are the essential features of similar dynamical systems? By Pontryagin duality, $\times 2, \times 3 : \mathbb{T} \longrightarrow \mathbb{T}$ corresponds to $\times 2, \times 3 : \mathbb{Z} \longrightarrow \mathbb{Z}$. In other words, the monoid $\mathbb{N}^2$ acts on the group $\mathbb{Z}$ by multiplication using two relatively prime integers. In an abstract way, we can think of this as a triple (G, P, θ), where

- G is a countably infinite discrete group,
- P is a countably generated, free abelian monoid, and
- $P \overset{\theta}{\curvearrowright} G$ is an action by injective group endomorphisms subject to the *independence condition*: $\theta_p(G) \cap \theta_q(G) = \theta_{pq}(G)$ if and only if p and q are relatively prime in P, i.e. $pP \cap qP = pqP$.

We will refer to triples (G, P, θ) satisfying the three requirements stated above as *irreversible algebraic dynamical systems*. The term *irreversible* is used because $\theta_p \in \mathrm{Aut}(G)$ implies $p = 1_P$, and *algebraic* is used to emphasize the contrast to topological dynamical systems, since the imposed conditions are purely algebraic.

We note that it suffices to check the independence condition for irreversible algebraic dynamical systems for the generators of P. An equivalent characterisation of independence can be given in terms of the isometries $S_{\theta_p}, S_{\theta_q} \in \ell^2(G)$:

$$\theta_p(G) \cap \theta_q(G) = \theta_{pq}(G) \text{ if and only if } S_{\theta_p}^* S_{\theta_q} = S_{\theta_q} S_{\theta_p}^*.$$

The inspiration for the independence condition stems from [CV13, Section 5], where two commuting injective group endomorphisms φ, ψ of a discrete abelian group G with finite cokernel are considered. The maps φ and ψ are said to be independent if $\varphi(G) \cap \psi(G) = \varphi\psi(G)$. It is then shown that independence is equivalent to $\varphi(G) + \psi(G) = G$. This equation is in turn equivalent to the statement that the inclusion $\varphi(G) \hookrightarrow G$ induces an isomorphism $\varphi(G)/(\varphi(G) \cap \psi(G)) \cong G/\psi(G)$.

For the general case, where G need not be abelian, and $G/\varphi(G)$, $G/\psi(G)$ need not be finite, we show that this last equivalence still holds if we only ask for a bijection $\varphi(G)/(\varphi(G) \cap \psi(G)) \longrightarrow G/\psi(G)$, see Proposition 1.1.1. This slight modification is natural since there need not be a group structure on the quotients. But $\varphi(G) \cap \psi(G) = \varphi\psi(G)$ turns out to be weaker than $\varphi(G)\psi(G) = G$, where $\varphi(G)\psi(G) = \{\varphi(g)\psi(g') \mid g, g' \in G\}$, see Example 1.1.12. We will therefore differentiate between independence and what we call strong independence, see Definition 1.1.3.

For abelian G, we examine the dual triple $(\hat{G}, P, \hat{\theta})$, see Section 1.2. In this case, $\hat{G}$ is a compact abelian group and $\hat{\theta}_p$ is a surjective group endomorphism of $\hat{G}$ for every $p \in P$. The (strong) independence condition for θ is stated in terms of $\hat{\theta}$. This is related to the notion of $*$-commutativity introduced in [AR97] and studied for instance in [ER07, Wil10]. Recall that given a set X, two commuting maps $\varphi, \psi : X \longrightarrow X$ are said to $*$-commute if $\varphi : \psi^{-1}(x) \longrightarrow \psi^{-1}(\varphi(x))$ is a bijection for all $x \in X$. For the situation of a pair of commuting surjective group endomorphisms, we find that strong independence is weaker than $*$-commutativity, see Proposition 1.3.2. But we prove in many cases, for example when $G/\theta_p(G)$ is finite for all $p \in P$, that $*$-commutativity is equivalent to independence.

Let us recall that, for abelian G, the quotient $G/\theta_p(G)$ is finite if and only if $\ker \hat{\theta}_p$ is finite. If this is the case for all $p \in P$, then P acts by surjective, local homeomorphisms $\hat{\theta}_p$ on the compact Hausdorff space $\hat{G}$. Due to the equivalence of independence and $*$-commutativity, this action satisfies a $*$-commutativity relation, analogous to the independence condition for irreversible algebraic dynamical systems. This motivates the study of *irreversible $*$-commutative dynamical systems of finite type* (X, P, θ), where P is again a countably generated, free abelian monoid, but X is a compact Hausdorff space on which P acts by regular surjective local homeomorphisms θ_p satisfying:

$$\theta_p \text{ and } \theta_q \text{ } *\text{-commute if and only if } p \text{ and } q \text{ are relatively prime in } P.$$

Analogous to the case of irreversible algebraic dynamical systems, *irreversible* refers to the fact that θ_p is not a homeomorphism of X unless $p = 1_P$. A surjective local homeomorphism is called *regular* if the number of preimages of a singleton is constant on X. Since we only consider compact spaces X, this value is always finite. Such maps are also referred to as covering maps.

Conceptionally, the notion of an irreversible $*$-commutative dynamical system of finite type represents the model type extracted from $\times 2, \times 3 : \mathbb{T} \longrightarrow \mathbb{T}$ from the perspective of topological dynamical systems. It is therefore not surprising that, given an irreversible algebraic dynamical system of finite type (G, P, θ) with abelian G, the dual triple $(\hat{G}, P, \hat{\theta})$ is an irreversible $*$-commutative dynamical system of finite type, see Corollary 1.3.17.

For irreversible algebraic dynamical systems (G, P, θ) as well as for irreversible $*$-commutative dynamical systems of finite type (X, P, θ), we construct and study universal C*-algebras $\mathcal{O}[G, P, \theta]$ and $\mathcal{O}[X, P, \theta]$, by means of generators and relations, in the course of Chapter 2. The main focus is set on $\mathcal{O}[G, P, \theta]$, which is a direct generalization of the C*-algebra $\mathcal{O}[\varphi]$ that appeared in [CV13, Hir02, Vie13]. The C*-algebra $\mathcal{O}[X, P, \theta]$ is a generalization of a certain Exel crossed product $C(X) \rtimes_{\alpha,L} \mathbb{N}$, see [EV06, Section 9]. We show that the structures of $\mathcal{O}[G, P, \theta]$ and $\mathcal{O}[X, P, \theta]$ are consistent with the ones that have been found for $\mathcal{O}[\varphi]$ and $C(X) \rtimes_{\alpha,L} \mathbb{N}$, respectively. Since we focus on $\mathcal{O}[G, P, \theta]$ and the results on $\mathcal{O}[X, P, \theta]$ are mostly used as tools for Chapter 4, let us be more precise concerning the structural properties of $\mathcal{O}[G, P, \theta]$.

Extending [CV13, Lemma 2.4], it is proven that the spectrum G_θ of the (commutative) diagonal subalgebra $\mathcal{D}$ of $\mathcal{O}[G, P, \theta]$ can be interpreted as a completion of G with respect to θ if (G, P, θ) is *minimal* in the sense that $\bigcap_{p \in P} \theta_p(G) = \{1_G\}$, see Lemma 2.2.9. The C*-algebra $\mathcal{O}[G, P, \theta]$ is identified with the semigroup crossed product $\mathcal{D} \rtimes (G \rtimes_\theta P)$, where $(g, p).d = u_g s_p d (u_g s_p)^*$, see Proposition 2.2.18. Using a decomposition theorem for crossed products by semidirect products of monoids, which is established in Section 2.1, the isomorphism between $\mathcal{O}[G, P, \theta]$ and $\mathcal{D} \rtimes (G \rtimes_\theta P)$ gives rise to an isomorphism of $\mathcal{F}$ and $C(G_\theta) \rtimes_\tau G$, where $g.d = u_g d u_g^*$, see Corollary 2.2.19. If G is amenable and (G, P, θ) is minimal, then $\mathcal{F}$ is a generalized Bunce-Deddens algebra in the sense of [Orf10], see Proposition 2.3.2 and [Orf10]. In this case, $\mathcal{F}$ is classified by its Elliott invariant due to a combination of results from [Lin01, MS, Win05], see Corollary 2.3.3. Finally, Corollary 2.2.28 asserts that minimality of (G,P,θ) and amenability of the action $G \overset{\hat{\tau}}{\curvearrowright} G_\theta$ imply that $\mathcal{O}[G,P,\theta]$ is a UCT Kirchberg algebra, hence classifiable by K-theory due to [Kir, Phi00]. Unfortunately, the computation of the K-theory of $\mathcal{O}[G,P,\theta]$ beyond the case of a single group endomorphism, for which this has been accomplished in [CV13], is a hard problem, at least with the techniques currently available.

Chapter 3 provides an alternative approach to the C*-algebras $\mathcal{O}[G,P,\theta]$ and $\mathcal{O}[X,P,\theta]$ as the Cuntz-Nica-Pimsner algebras of discrete product systems of Hilbert bimodules over the semigroup P with coefficients in $C^*(G)$ and $C(X)$, respectively, see Theorem 3.3.4 for (G,P,θ) and Theorem 3.3.7 for (X,P,θ). Discrete product systems form a generalization of the original construction introduced by Mihai Pimsner in [Pim97] for a single Hilbert bimodule. Within the first two sections of Chapter 3, we provide a short introduction to product systems, their representation theory, and the C*-algebras associated to them. We refer to [Fow99, Fow02, Sol06, Yee07, SY10, CLSV11, HLS12] for more information on the subject.

In the last section of Chapter 3, we show how to construct product systems for both irreversible algebraic dynamical systems (G,P,θ) and irreversible $*$-commutative dynamical systems of finite type (X,P,θ). For (G,P,θ), the product system $\mathcal{X}$ associated to it comes with a canonical system of orthonormal bases on its fibres $\mathcal{X}_p, p \in P$, given by $(u_g)_{g\theta_p(G)\in G/\theta_p(G)}$. Similarly, the product system $\mathcal{X}$ associated to (X,P,θ) admits a finite Parseval frame on each fibre $\mathcal{X}_p$, which is constructed by choosing a partition of unity $(f_i) \subset C(X)$ such that θ_p is injective on $\operatorname{supp} f_i$ for all i.

One advantage of realizing $\mathcal{O}[G,P,\theta]$ as the Cuntz-Nica-Pimsner algebras of a product system is that it has a natural Toeplitz extension, called the Nica-Toeplitz algebra. Although this will not be part of this thesis, we want to mention that, jointly with Nathan Brownlowe and Nadia S. Larsen, we proved that the Nica-Toeplitz algebra associated to (G,P,θ) is isomorphic to the (full) semigroup C*-algebra $C^*(G \rtimes_\theta P)$ in the sense of Xin Li, see [Li12, Li13]. Moreover, it coincides with $\mathcal{O}[G,P,\theta]$ for irreversible algebraic dynamical systems (G,P,θ), given that $G/\theta_p(G)$ is infinite for all $p \neq 1_P$. This sheds new light on the result from [Vie13] mentioned in the beginning.

With regards to (X,P,θ), Theorem 3.3.7 grants us access to a gauge-invariant uniqueness theorem from [CLSV11], see Remark 4.1.2. This is an essential tool to characterise topological freeness of (X,P,θ) in terms of $\mathcal{O}[X,P,\theta]$, see Theorem 4.1.9. For an irreversible $*$-commutative dynamical system of finite type (X,P,θ), the following statements are equivalent:

(1) (X,P,θ) is topologically free.

(2) Every non-zero ideal $I \lhd \mathcal{O}[X,P,\theta]$ satisfies $I \cap C(X) \neq 0$.

(3) The natural representation of $\mathcal{O}[X,P,\theta]$ on $\ell^2(X)$ is faithful.

(4) $C(X)$ is a maximal abelian subalgebra (masa) in $\mathcal{O}[X,P,\theta]$.

Thus we achieve a direct generalization of [CS09, Theorem 6], where this result was proven only for singly generated P. The proof we present in this thesis combines the strategy of [CS09] with the gauge-invariant uniqueness theorem from [CLSV11] and techniques from [Exe03b, EV06].

As a useful consequence of Theorem 4.1.9, simplicity of the C*-algebra $\mathcal{O}[X, P, \theta]$ is characterized by minimality of the topological dynamical system (X, P, θ), see Theorem 4.2.11. We also show that minimal irreversible $*$-commutative dynamical systems of finite type are necessarily topologically free, see Proposition 4.2.10. This was observed already in [EV06, Proposition 11.1] in the case where P is singly generated.

Finally, we apply Theorem 4.2.11 to the dual triple $(\hat{G}, P, \hat{\theta})$ of an irreversible algebraic dynamical system of finite type (G, P, θ) with abelian G in order to deduce that simplicity of $\mathcal{O}[G, P, \theta]$ is equivalent to minimality of (G, P, θ), see Corollary 4.2.12. This shows that the conditions required in Theorem 2.2.26 are necessary in the case where G is abelian, and $G/\theta_p(G)$ is finite for all $p \in P$.

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Chapter 1

Irreversible semigroup dynamical systems

In this opening chapter, we will specify the dynamical systems that we are going to analyse and highlight their characteristic features, as well as connections between the different versions. We start off in the first section with a closer examination of the independence condition from [CV13, Section 5] for pairs of commuting injective endomorphisms of an arbitrary group G, see Proposition 1.1.1. This setting is more general than the one considered in [CV13] and leads to a subdivision of the notion of independence into independence and strong independence, see Definition 1.1.3. The two properties are shown to be equivalent in the case where one of the involved endomorphisms has finite index in G, see Proposition 1.1.1. But Example 1.1.12 shows that there are many situations where independence holds, but strong independence does not. We then formulate the concept of irreversible algebraic dynamical systems (G, P, θ) based upon independence, see Definition 1.1.5, and provide a diverse pool of examples.

Section 1.2 is dedicated to the discussion of the dual model $(\hat{G}, P, \hat{\theta})$ for irreversible algebraic dynamical systems in the case where G is commutative. This leads to an complementary notion of (strong) independence for pairs of commuting surjective endomorphisms of an arbitrary group K, see Definition 1.2.7. The fact that there is a close connection between (strong) independence of injective endomorphisms of a discrete abelian group G and (strong) independence of their dualized counterparts, which are surjective endomorphisms of the compact abelian group $\hat{G}$, has already been observed in [CV13, Section 5]. We extend this observation to the general setup in Proposition 1.2.6. This allows us to characterize when a triple (G, P, θ) is a (minimal) commutative irreversible algebraic dynamical system (of finite type), entirely in terms of its dual model $(\hat{G}, P, \hat{\theta})$, see Proposition 1.2.8. We close this section with a brief presentation of the dual models for the commutative examples from Section 1.1.

The beginning of Section 1.3 provides an introduction to $*$-commutativity, see also [ER07, Wil10, MW], and some elementary aspects of regular surjective local homeomorphisms of a compact Hausdorff space X. Here regular means that the map assigning to $x \in X$ the number of preimages under the surjective local homeomorphism is constant, see Definition 1.3.4. Then we define a topological analogue for irreversible algebraic dynamical systems, for which, roughly speaking, independence is replaced by $*$-commutativity, see Definition 1.3.13 for details. By comparing $*$-commutativity with strong independence in the sense of Definition 1.2.7, see Proposition 1.3.16, we are able to conclude that the dual model for commutative irreversible algebraic dynamical systems of finite type falls into the class of irreversible $*$-commutative dynamical systems of finite type, see Corollary 1.3.17. Towards the end of this first chapter we discuss a method from symbolic dynamics to construct examples of irreversible $*$-commutative dynamical systems of finite type. This part builds on the material presented in [ER07, Section 10–14] and, in fact, leads to new examples for commutative irreversible algebraic dynamical systems of finite type.

1.1 Irreversible algebraic dynamical systems

The purpose of this section is to familiarize with the primary object of interest called irreversible algebraic dynamical system in its most general form. Vaguely speaking, such a dynamical system is given by a countably infinite, discrete group G and at most countably many commuting injective, non-surjective group endomorphisms $(\theta_i)_{i \in I}$ of G that are independent in the sense that the intersection of their images is as small as possible. Additionally, we will introduce a minimality condition stating that the intersection of the images of the group endomorphisms from the semigroup generated by $(\theta_i)_{i \in I}$ is trivial. In other words, the group endomorphisms $(\theta_i)_{i \in I}$ (more precisely finite products of these) separate the points in G. At a later stage, namely in Theorem 2.2.26, this condition is shown to be intimately connected to simplicity of the C*-algebra associated to such a dynamical system in Definition 2.2.1.

The following observation is an extension of the concept introduced in [CV13, Section 5]. In contrast to the situation in [CV13], we will require neither the group G to be abelian nor the cokernels of the injective group endomorphisms of G to be finite.

Proposition 1.1.1. *Suppose G is a group. Consider the following statements for two commuting injective group endomorphisms θ_1 and θ_2 of G:*

(i) $\theta_1(G)\theta_2(G) = G$.

(ii) *The inclusion* $\theta_1(G) \hookrightarrow G$ *induces a bijection* $\theta_1(G)/(\theta_1(G) \cap \theta_2(G)) \longrightarrow G/\theta_2(G)$.

(ii') *The inclusion* $\theta_2(G) \hookrightarrow G$ *induces a bijection* $\theta_2(G)/(\theta_1(G) \cap \theta_2(G)) \longrightarrow G/\theta_1(G)$.

(iii) $\theta_1(G) \cap \theta_2(G) = \theta_1\theta_2(G)$.

Then (i),(ii), and (ii') are equivalent and imply (iii). If one of the subgroups $\theta_1(G)$ and $\theta_2(G)$ is of finite index in G, then all these conditions are equivalent.

Proof. Note that we always have $\theta_1(G)\theta_2(G) \subset G$ and $\theta_1(G) \cap \theta_2(G) \supset \theta_1\theta_2(G)$. Moreover, in condition (ii), the inclusion $\theta_1(G) \hookrightarrow G$ induces an injective map $\theta_1(G)/(\theta_1(G) \cap \theta_2(G)) \longrightarrow G/\theta_2(G)$. The corresponding statement holds for (ii').
If (i) holds true, then $G \ni g = \theta_1(g_1)\theta_2(g_2)$ for suitable $g_i \in G$. Hence, the left-coset of $\theta_1(g_1)$ maps to the left-coset of g and (ii) follows.

Conversely, suppose (ii) is valid and pick $g \in G$. Then there is $g_1 \in G$ such that $\theta_1(g_1)\,(\theta_1(G) \cap \theta_2(G)) \mapsto g\theta_2(G)$ via the map from (ii). But since this map comes from the inclusion $\theta_1(G) \longleftrightarrow G$, we have $g\theta_2(G) = \theta_1(g_1)\theta_2(G)$. Thus, there is $g_2 \in G$ such that $g = \theta_1(g_1)\theta_2(g_2)$ showing (i). The equivalence of (i) and (ii') is obtained from the previous argument by swapping θ_1 and θ_2. Given (ii), that is,

$$\theta_1(G)/(\theta_1(G) \cap \theta_2(G)) \xrightarrow{f_1} G/\theta_2(G)$$

is a bijection (induced by the inclusion $\theta_2(G) \hookrightarrow G$), composing f_1^{-1} with the bijection

$$\theta_1(G)/(\theta_1\theta_2(G)) \xrightarrow{f_2} G/\theta_2(G)$$

obtained from applying the injective group endomorphism θ_1 yields a bijection

$$\theta_1(G)/(\theta_1\theta_2(G)) \xrightarrow{f_1^{-1}f_2} \theta_1(G)/(\theta_1(G) \cap \theta_2(G)).$$

Let us assume $\theta_1\theta_2(G) \subsetneqq \theta_1(G) \cap \theta_2(G)$. This means, that there is $g \in \theta_1(G)$ such that $g\theta_1\theta_2(G) \neq \theta_1\theta_2(G)$ but $g\theta_1(G) \cap \theta_2(G) = \theta_1(G) \cap \theta_2(G)$. Noting that $f_1^{-1}f_2$ maps a left-coset $g'\theta_1\theta_2(G)$ to $g'\theta_1(G) \cap \theta_2(G)$, this contradicts injectivity of $f_1^{-1}f_2$. Hence, we must have $\theta_1(G) \cap \theta_2(G) = \theta_1\theta_2(G)$. Similarly, (iii) follows from (ii').
Finally, suppose (iii) holds. By injectivity of θ_1, we have

$$\theta_1(G)/(\theta_1(G) \cap \theta_2(G)) = \theta_1(G)/\theta_1\theta_2(G) \cong G/\theta_2(G).$$

So if $[G : \theta_2(G)]$ is finite, then the injective map from (ii) is necessarily a bijection. Similarly, if $[G : \theta_1(G)]$ is finite, we can run the same argument using (ii'). □

***Remark* 1.1.2.** If the subgroups $\theta_1(G)$ and $\theta_2(G)$ are both normal in G, then $\theta_1(G)\theta_2(G)$ is a normal subgroup of $\theta_i(G), i = 1, 2$, and the bijections in Proposition 1.1.1 (ii) and (ii') are isomorphisms of groups.

Definition 1.1.3. Let G be a group and θ_1, θ_2 commuting, injective group endomorphisms of G. Then θ_1 and θ_2 are said to be **independent**, if they satisfy condition (iii) from Proposition 1.1.1. θ_1 and θ_2 are said to be **strongly independent**, if they satisfy the equivalent conditions (i),(ii') and (ii') from Proposition 1.1.1.

Note that (strong) independence is automatic if θ_1 or θ_2 is a group automorphism.

Lemma 1.1.4. *Let G be a group and suppose $\theta_1, \theta_2, \theta_3$ are commuting, injective group endomorphisms of G. θ_1 is (strongly) independent of $\theta_2\theta_2$ if and only if θ_1 is (strongly) independent of both θ_2 and θ_3.*

Proof. If θ_1 and $\theta_2\theta_3$ are strongly independent, then

$$\theta_1(G)\theta_2(G) \supset \theta_1(G)\theta_2(\theta_3(G)) = G$$

shows that θ_1 and θ_2 are strongly independent. As θ_2 and θ_3 commute, θ_1 is also strongly independent of θ_3. Conversely, if θ_1 is strongly independent of both θ_2 and θ_3, then

$$\begin{array}{rclcl} G & = & \theta_1(G)\theta_2(G) & = & \theta_1(G)\theta_2(\theta_1(G)\theta_3(G)) \\ & = & \theta_1(G\theta_2(G))\theta_2(\theta_3(G)) & \subset & \theta_1(G)\theta_2\theta_3(G), \end{array}$$

so θ_1 and $\theta_2\theta_3$ are strongly independent since the reverse inclusion is trivial.

If θ_1 and $\theta_2\theta_3$ are independent, then commutativity of θ_1, θ_2 and θ_3 in combination with injectivity of θ_3 yield

$$\begin{array}{rclcl} \theta_1(G) \cap \theta_2(G) & = & \theta_3^{-1}(\theta_1\theta_3(G) \cap \theta_2\theta_3(G)) & \subset & \theta_3^{-1}(\theta_1(G) \cap \theta_2\theta_3(G)) \\ & = & \theta_3^{-1}(\theta_1\theta_2\theta_3(G)) & = & \theta_1\theta_2(G). \end{array}$$

Since the reverse inclusion is always true, we conclude that θ_1 and θ_2 are independent. Exchanging the role of θ_2 and θ_3 shows independence of θ_1 and θ_3. Finally, if θ_1 is independent of both θ_2 and θ_3, we get

$$\begin{array}{rclcl} \theta_1(G) \cap \theta_2\theta_3(G) & = & \theta_1(G) \cap \theta_2(G) \cap \theta_2\theta_3(G) & = & \theta_1\theta_2(G) \cap \theta_2\theta_3(G) \\ & = & \theta_2(\theta_1(G) \cap \theta_3(G)) & = & \theta_1\theta_2\theta_3(G) \end{array}$$

by injectivity of θ_2. Thus θ_1 and $\theta_2\theta_3$ are independent. □

If $(P, \leq)$ is a lattice-ordered monoid with unit 1_P, we shall denote the least common multiple and the greatest common divisor of two elements $p, q \in P$ by $p \vee q$ and $p \wedge q$, respectively. p and q are said to be relatively prime (in P) if $p \wedge q = 1_P$ or, equivalently, $p \vee q = pq$. Simple examples of such monoids are countably generated free abelian monoids since such monoids are either isomorphic to $\mathbb{N}^k$ for some $k \in \mathbb{N}$ or $\bigoplus_{\mathbb{N}} \mathbb{N}$.

Definition 1.1.5. An **irreversible algebraic dynamical system** (G, P, θ) consists of

(A) a countably infinite discrete group G with unit 1_G,

(B) a countably generated free abelian monoid P with unit 1_P, and

(C) an action $P \overset{\theta}{\curvearrowright} G$ by injective group endomorphisms with the property that θ_p and θ_q are independent if and only if p and q are relatively prime in P.

(G, P, θ) is said to be

· **minimal**, if $\bigcap_{p \in P} \theta_p(G) = \{1_G\}$,

· **commutative**, if G is commutative,

· **of finite type**, if $[G : \theta_p(G)]$ is finite for all $p \in P$, and

· **of infinite type**, if $[G : \theta_p(G)]$ is infinite for all $p \neq 1_P$.

***Remark* 1.1.6.** The following observations are immediate:

a) Condition (B) means that P is either isomorphic to $\bigoplus_{\mathbb{N}} \mathbb{N}$ or to $\mathbb{N}^k$ for some $k \in \mathbb{N}$.

b) Let us point out that $\theta_{1_P} = id_G$ is the only automorphism of G occuring for this setting. Indeed, if θ_p is an automorphism of G, then it is independent of itself. But $p = 1_P$ is the only element in P that is relatively prime to itself. So unless $P = \{1_P\}$, there is $p \in P$ such that $[G : \theta_p(G)] > 1$. Therefore, $\theta_p(G)$ is a proper subgroup of G. Since θ_p is injective, G needs to be of infinite cardinality.

c) Lemma 1.1.4 shows that the notions of independence and strong independence for injective group endomorphisms are well-behaved with respect to composition. As a consequence, it suffices to check the independence condition from (C) on the generators of P. Moreover, injectivity of the group endomorphisms puts us in position to rephrase this condition by stating that $\theta_p(G) \cap \theta_q(G) = \theta_{p \vee q}(G)$ holds for all $p, q \in P$.

d) If (G, P, θ) is an irreversible algebraic dynamical system of finite type, the independence condition from (C) is equivalent to the requirement that θ_p and θ_q be strongly independent if and only if p and q are relatively prime, see Proposition 1.1.1.

e) Note that $[G : \theta_{pq}(G)] = [G : \theta_p(G)] \cdot [\theta_p(G) : \theta_{pq}(G)] = [G : \theta_p(G)] \cdot [G : \theta_q(G)]$ holds since θ_p is injective. In particular, $[G : \theta_{pq}(G)]$ is finite if and only if both $[G : \theta_p(G)]$ and $[G : \theta_q(G)]$ are finite.

***Remark* 1.1.7.** The minimality condition has been used under the name *exactness* in the case of a commutative G with a single endomorphism with finite cokernel in [CV13]. As explained in [CV13, Remark 2.1], the notion of exactness for a single endomorphism stems from ergodic theory and is a well-studied property for irreversible, measure-preserving transformations. However, for the specific setup that we use, this property was already considered by Ilan Hirshberg in [Hir02], where he called such endomorphisms *pure*. Despite these two available options, we decided to name this property minimality for two reasons:

1. For commutative irreversible algebraic dynamical systems, the corresponding condition for the dualized model $(\hat{G}, P, \hat{\theta})$ really is minimality of the (irreversible) topological dynamical system, see Proposition 1.2.8.

2. Condition (D) is intimately linked to simplicity of the C*-algebras we will construct from the data, see Corollary 2.2.14, Corollary 2.2.19, and Theorem 2.2.26.

***Examples* 1.1.8.** There are various examples for commutative irreversible algebraic dynamical systems and most of them are of finite type. Let us recall that it suffices to check independence of the endomorphisms on the generators of P according to Lemma 1.1.4.

(a) Choose a family $(p_i)_{i\in I} \subset \mathbb{Z}^\times \backslash \mathbb{Z}^* = \mathbb{Z}\backslash\{0,\pm 1\}$ and let $P = |(p_i)_{i\in I}\rangle \subset \mathbb{Z}^\times$ act on $G = \mathbb{Z}$ by $\theta_{p_i}(g) = p_i g$. Since $\mathbb{Z}$ is an integral domain, each θ_{p_i} is an injective group endomorphism of G with $[G : \theta_{p_i}(G)] = p_i$. For $i \neq j$, θ_{p_i} and θ_{p_j} are independent if and only if p_i and p_j are relatively prime in $\mathbb{Z}$. Thus, we get a commutative irreversible algebraic dynamical system of finite type if and only if $(p_i)_{i\in I}$ consists of relatively prime integers. Since the number of factors in its prime factorization is finite for every integer, such irreversible algebraic dynamical systems are automatically minimal.

(b) Let $I \subset \mathbb{N}$, choose relatively prime integers $\{q\} \cup (p_i)_{i\in I} \subset \mathbb{Z} \setminus \{0,\pm 1\}$ and let $G = \mathbb{Z}\left[\frac{1}{q}\right]$. As $\mathbb{Z}\left[\frac{1}{q}\right] = \varinjlim \mathbb{Z}$ with connecting maps given by multiplication with q, and q is relatively prime to each p_i, the arguments from (a) carry over almost verbatim. Thus we get minimal commutative irreversible algebraic dynamical systems of finite type (G, P, θ) which generalize [CV13, Example 2.1.5].

(c) Let $\mathbb{K}$ be a countable field and let $G = \mathbb{K}[T]$ denote the polynomial ring in a single variable T over $\mathbb{K}$. Choose non-constant polynomials $p_i \in \mathbb{K}[T], i \in I$ for some index set I. Multiplying by p_i defines an endomorphism θ_{p_i} of G with $[G : \theta_{p_i}(G)] = |\mathbb{K}|^{\deg(p_i)}$, where $\deg(p_i)$ denotes the degree of $p_i \in \mathbb{K}[T]$. Thus, if we let $P := |(p_i)_{i\in I}\rangle$, then the index of $\theta_p(G)$ in G is finite for all $p \in P$ if and only if $\mathbb{K}$ is finite. It is clear that θ_{p_i} and θ_{p_j} are independent if and only if $(p_i) \cap (p_j) = (p_i p_j)$ holds for the principal ideals (whenever $i \neq j$). Since every $g \in \mathbb{K}[T]$ has finite degree, (G, P, θ) is automatically minimal. Thus, provided $(p_i)_{i\in I}$ has been chosen accordingly, we obtain a minimal commutative irreversible algebraic dynamical system which is of finite type if and only if $\mathbb{K}$ is finite, compare [CV13, Example 2.1.4].

***Example* 1.1.9.** For $G = \mathbb{Z}^d$ with $d \geq 1$, the monoid of injective group endomorphisms of G is isomorphic to the monoid of invertible integral matrices $M_d(\mathbb{Z}) \cap Gl_d(\mathbb{Q})$. For each such endomorphism, the index of its image in G is given by the absolute value of the determinant of the corresponding matrix. In particular, their images always have finite index in G and an endomorphism of G is not surjective precisely if the absolute value of the determinant of the matrix exceeds 1. So let $(T_i)_{i\in I} \subset M_d(\mathbb{Z}) \cap Gl_d(\mathbb{Q})$ be a family of matrices satisfying $|\det T_i| > 1$ for all $i \in I$ and set $P = |(T_i)_{i\in I}\rangle$ as well as $\theta_i(g) = T_i g$. Commutativity of θ_i and θ_j is equivalent to $T_iT_j = T_jT_i$. For $i \neq j$, it is easier to check strong independence of θ_i and θ_j instead of independence. Indeed, since we are dealing with a finite type case, the conditions are equivalent and strong independence takes the form $T_i(\mathbb{Z}^d) + T_j(\mathbb{Z}^d) = \mathbb{Z}^d$, see Proposition 1.1.1. This condition can readily be checked by solving d linear equations. To reduce efforts, these computations can easily be handled by a standard linear algebra program. Thus, if the aforementioned conditions are fulfilled, (G, P, θ) is a commutative irreversible algebraic dynamical system of finite type. If we interpret the integer matrices $(\theta_p)_{p\in P}$ as endomorphisms of the vector space $\mathbb{C}^d$, they have the same generalized eigenspaces for possibly different generalized eigenvalues because they commute. Minimality of $(\mathbb{Z}^d, P, \theta)$ is then equivalent to the property that, for each generalized eigenspace, there is $p \in P$ such that the corresponding generalized eigenvalue for θ_p is strictly larger than one in absolute value.

Example 1.1.8 (a) can be generalized to the case of rings of integers in the following way:

***Example* 1.1.10.** Let $\mathcal{R}$ be the ring of integers in a number field and denote by $\mathcal{R}^\times = \mathcal{R} \setminus \{0_\mathcal{R}\}$ the multiplicative subsemigroup as well as by $\mathcal{R}^* \subset \mathcal{R}^\times$ the group of units in $\mathcal{R}$. Take $G = \mathcal{R}$ and choose a (countable) family $(p_i)_{i\in I} \subset \mathcal{R}^\times \setminus \mathcal{R}^*$. If we set $P = |(p_i)_{i\in I}\rangle$, then this monoid acts on G in a natural way by multiplication, i.e. $\theta_p(g) = pg$ for $g \in G, p \in P$. For $i \neq j$, θ_{p_i} and θ_{p_j} are independent if and only if the principal ideals (p_i) and (p_j) in $\mathcal{R}$ share no common prime ideal. If this is the case, (G, P, θ) constitutes a commutative irreversible algebraic dynamical system of finite type. Since the number of factors in the (unique) prime ideal factorization of (g) in $\mathcal{R}$ is finite for every $g \in G$, minimality is once again automatically satisfied. The argument actually shows that such a construction works whenever $\mathcal{R}$ is a Dedekind domain.

Let us also mention the following example even though, having singly generated P, it has nothing to do with independence. The reason is that Joachim Cuntz and Anatoly Vershik observed in [CV13, Example 2.1.1], that the C*-algebra $\mathcal{O}[G, P, \theta]$ associated to this irreversible algebraic dynamical system is isomorphic to $\mathcal{O}_n$.

***Example* 1.1.11.** For $n \geq 2$, consider the unilateral shift θ_1 acting on $G = \bigoplus_\mathbb{N} \mathbb{Z}/n\mathbb{Z}$ by $(g_0, g_1, \dots) \mapsto (0, g_0, g_1, \dots)$. Since θ_1 is an injective group endomorphism with $[G : \theta_1(G)] = n$, (G, P, θ) with $P = |\theta_1\rangle$ is a minimal commutative irreversible algebraic dynamical system of finite type.

***Example* 1.1.12.** Generalizing Example 1.1.11, suppose P is as required in condition (B) of Definition 1.1.5 and let G_0 be a countable group. Let us assume that G_0 has at least two distinct elements. Then P admits a shift action θ on $G := \bigoplus_P G_0$ given by

$$(\theta_p((g_q)_{q\in P}))_r = \chi_{pP}(r)\ g_{p^{-1}r} \text{ for all } p, r \in P.$$

It is apparent that $\theta_p\theta_q = \theta_q\theta_p$ holds for all $p, q \in P$ and that θ_p is an injective group endomorphism for all $p \in P$. The index $[G : \theta_p(G)]$ is finite for $p \in P \setminus \{1_P\}$ if and only if G_0 is finite and P is singly generated. Indeed, if $p \neq 1_P$, then each element of $\bigoplus_{q \in P \setminus pP} G_0$ yields a distinct left-coset in $G/\theta_p(G)$. Clearly, this group is finite if and only if G_0 is finite and P is singly generated. Given relatively prime p and q in $P \setminus \{1_P\}$, $\theta_p(G)\theta_q(G) \neq G$ since $g_{1_P} = 1_{G_0}$ for all $(g_r)_{r \in P} \in \theta_p(G)\theta_q(G)$ as $1_P \notin pP \cup qP$. Thus, unless P is singly generated, θ does not satisfy the strong independence condition. However, the independence condition is satisfied because $g = (g_r)_{r \in P} \in \theta_p(G) \cap \theta_q(G)$ implies that $g_r \neq 1_{G_0}$ only if $r \in pP \cap qP = pqP$ and thus $g \in \theta_{pq}(G)$.

We have seen in Example 1.1.12 that one cannot expect strong independence for irreversible algebraic dynamical systems of infinite type in general. On the other hand, there are some examples where the subgroups in question have infinite index and the endomorphisms are strongly independent:

***Example* 1.1.13.** Given a family $(G^{(i)}, P, \theta^{(i)})_{i \in \mathbb{N}}$ of irreversible algebraic dynamical systems, we can consider $G := \bigoplus_{i \in \mathbb{N}} G^{(i)}$. If P acts on G component-wise, i.e. $\theta_p(g_i)_{i \in \mathbb{N}} := (\theta_p^{(i)}(g_i))_{i \in \mathbb{N}}$, then (G, P, θ) is an irreversible algebraic dynamical system and $[G : \theta_p(G)]$ is infinite unless $p = 1_P$, see Remark 1.1.6 b). G is commutative if and only if each $G^{(i)}$ is, and (G, P, θ) is minimal if and only if each $(G^{(i)}, P, \theta^{(i)})$ is minimal. If each $(G^{(i)}, P, \theta^{(i)})$ satisfies the strong independence condition, then θ inherits this property as well.

As a final example, we provide more general forms of [Vie13, Example 2.3.9]. These examples are neither commutative irreversible algebraic dynamical systems nor of finite type.

***Example* 1.1.14.** For $2 \leq n \leq \infty$, let $\mathbb{F}_n$ be the free group in n generators $(a_k)_{1 \leq k \leq n}$. Fix $1 \leq d \leq n$ and choose for each $1 \leq i \leq d$ an n-tuple $(m_{i,k})_{1 \leq k \leq n} \subset \mathbb{N}^\times$ such that

1) for each $1 \leq i \leq d$, there exists k such that $m_{i,k} > 1$, and
2) for all $1 \leq i, j \leq d, i \neq j$ and $1 \leq k \leq n$, $m_{i,k}$ and $m_{j,k}$ are relatively prime.

Then $\theta_i(a_k) = a_k^{m_{i,k}}$ defines a group endomorphism of $\mathbb{F}_n$ for each $1 \leq i \leq d$. Noting that the length of an element of $\mathbb{F}_n$ in terms of the generators $(a_k)_{1 \leq k \leq n}$ and their inverses is non-decreasing under θ_i, we deduce that θ_i is injective. It is clear that $\theta_i\theta_j = \theta_j\theta_i$ holds for all i and j. For every $1 \leq i \leq d$, the index $[\mathbb{F}_n : \theta_i(\mathbb{F}_n)]$ is infinite. Indeed, take $1 \leq k \leq n$ such that $m_{i,k} > 1$ according to 1) and pick $1 \leq \ell \leq n$ with $\ell \neq k$. Then the family $((a_k a_\ell)^j)_{j \geq 1}$ yields pairwise distinct left-cosets in $\mathbb{F}_n/\theta_i(\mathbb{F}_n)$ since reduced words of the form $a_k a_\ell b \dots$ with $b \neq a_\ell^{-1}$ are not contained in $\theta_i(\mathbb{F}_n)$. A similar argument shows that θ_i and θ_j are not strongly independent for $i \neq j$: By 1), there are $1 \leq k, \ell \leq n$ such that $m_{i,k} > 1$ and $m_{j,\ell} > 1$. This forces $a_k a_\ell \notin \theta_i(\mathbb{F}_n)\theta_j(\mathbb{F}_n)$. Nonetheless, θ_i and θ_j are independent due to 2). Thus, $G = \mathbb{F}_n$ and $P = |(\theta_i)_{1 \leq i \leq d}\rangle$ acting on G in the obvious way constitutes an irreversible algebraic dynamical system which is neither commutative nor of finite type. Minimality of such irreversible algebraic dynamical systems can easily be characterized by:

3) For each $1 \leq k \leq n$, there exists $1 \leq i \leq d$ satisfying $m_{i,k} > 1$.

There are also examples of irreversible algebraic dynamical systems arising from a refinement of a construction related to symbolic dynamics, see [ER07] for the original exposition. But it is more natural to treat these examples in the framework of irreversible $*$-commutative dynamical systems, which is why we postpone their exposition to Section 1.3, see Example 1.3.23 and Example 1.3.21.

We close this section with two preparatory lemmas which are relevant for the C*-algebraic considerations in Section 2.1 and Section 2.2. The first lemma reflects a crucial feature of the independence assumption.

Lemma 1.1.15. *If (G,P,θ) is an irreversible algebraic dynamical system, then*

$$g\theta_p(G) \cap h\theta_q(G) = \begin{cases} g\theta_p(h')\theta_{p\vee q}(G) & , \text{ if } g^{-1}h \in \theta_p(G)\theta_q(G), \\ \emptyset & , \text{ else} \end{cases}$$

holds for all $g,h \in G$ and $p,q \in P$, where h' is uniquely determined by $g\theta_p(h') \in h\theta_q(G)$ up to right multiplication by elements from $\theta_{p^{-1}(p\vee q)}(G)$.

Proof. If there exist $g_1,g_2 \in G$ such that $g\theta_p(g_1) = h\theta_q(g_2)$, then $g^{-1}h = \theta_p(g_1)\theta_q(g_2^{-1}) \in \theta_p(G)\theta_q(G)$ follows because G is group. Now suppose that $g_3,g_4 \in G$ satisfy $g\theta_p(g_3) = h\theta_q(g_4)$ as well. Since this implies $\theta_p(g_1^{-1}g_3) = \theta_q(g_2^{-1}g_4)$, we appeal to Remark 1.1.6 c) to deduce $\theta_p(g_1^{-1}g_3) \in \theta_{p\vee q}(G)$. Using injectivity of θ_p, this is equivalent to $g_1^{-1}g_3 \in \theta_{p^{-1}(p\vee q)}(G)$. Therefore, $h' = g_1$ is unique up to right multiplication by elements from $\theta_{p^{-1}(p\vee q)}(G)$. $\square$

For the proof of Theorem 2.2.26, we will need the following auxiliary result, which relies on irreversibility of the dynamical system:

Lemma 1.1.16. *Suppose (G,P,θ) is an irreversible algebraic dynamical system and we have $n \in \mathbb{N}, g_i \in G, p_i \in P \setminus \{1_P\}$ for $0 \leq i \leq n$. Then, there exist $g \in g_0\theta_{p_0}(G), p \in p_0P$ satisfying*

$$g\theta_p(G) \subset G \setminus \bigcup_{1\leq i\leq n} \left(g_i \bigcap_{m\in\mathbb{N}} \theta_{p_i^m}(G) \right).$$

Proof. We proceed by induction. Let $n = 1$. As $p_1 \neq e$, we can find $m \in \mathbb{N}$ such that $p_0 \notin p_1^m P$. In other words, we have $p_0 \vee p_1^m \gneqq p_0$. By Lemma 1.1.15,

$$(g_0\theta_{p_0}(G)) \cap (g_1\theta_{p_1^m}(G)) = \begin{cases} g_0\theta_{p_0}(\tilde{g}_1)\theta_{p_0\vee p_1^m}(G) & \text{if } g_0^{-1}g_1 \in \theta_{p_0}(G)\theta_{p_1^m}(G), \\ \emptyset & \text{else,} \end{cases}$$

where $\tilde{g}_1$ is uniquely determined up to $\theta_{p_0^{-1}(p_0 \vee p_1^m)}(G)$. While we simply take $g := g_0$ in the second case, we need to pick $g \in (g_0\theta_{p_0}(G)) \setminus g_0\theta_{p_0}(\tilde{g}_1)\theta_{p_0 \vee p_1^m}(G)$ in the first case. Note that such a g exists as $p_0 \vee p_1^m \gneqq p_0$ by the choice of m. Finally, let $p := p_0 \vee p_1^m$.

The induction step from n to $n+1$ is just a verbatim repetition of the first step: Assume that the statement holds for fixed n. This means that there exist $h \in g_0\theta_{p_0}(G)$ and $q \in p_0P$ such that

$$h\theta_q(G) \subset G \setminus \bigcup_{1 \leq i \leq n} \left(g_i \bigcap_{m \in \mathbb{N}} \theta_{p_i^m}(G) \right).$$

As $p_{n+1} \neq e$, we can find $m \in \mathbb{N}$ such that $q \notin p_{n+1}^m P$. In other words, we have $q \vee p_{n+1}^m \gneqq q$. Recall that

$$(h\theta_q(G)) \cap (g_{n+1}\theta_{p_{n+1}^m}(G)) = \begin{cases} h\theta_q(\tilde{g}_{n+1})\theta_{q \vee p_{n+1}^m}(G) & \text{if } h^{-1}g_{n+1} \notin \theta_q(G)\theta_{p_{n+1}^m}(G), \\ \emptyset & \text{else,} \end{cases}$$

where $\tilde{g}_{n+1}$ is uniquely determined up to $\theta_{q^{-1}(q \vee p_{n+1}^m)}(G)$. In the second case, take $g := h$. For the first case, we choose $g \in (h\theta_q(G)) \setminus h\theta_q(\tilde{g}_{n+1})\theta_{q \vee p_{n+1}^m}(G)$. Note that such a g exists as $q \vee p_{n+1}^m \gneqq q$ by the choice of m. Finally, let $p := q \vee p_{n+1}^m$. Then, it is clear from the construction that we indeed have

$$g\theta_p(G) \subset G \setminus \bigcup_{1 \leq i \leq n+1} \left(g_i \bigcap_{m \in \mathbb{N}} \theta_{p_i^m}(G) \right).$$

□

1.2 The dual picture for commutative systems

In this section, we restrict our focus to commutative irreversible algebraic dynamical systems (G, P, θ). Injective group endomorphisms θ_p of a discrete abelian group G correspond to surjective group endomorphisms $\hat{\theta}_p$ of its Pontryagin dual $\hat{G}$, which is a compact abelian group and the index $[G : \theta_p(G)]$ equals the cardinality of $\ker \hat{\theta}_p$. This motivates the definition of (strong) independence for commuting surjective group endomorphisms, see Definition 1.2.7, which is consistent with the observations from [CV13, Lemma 5.4].

One striking feature of this particular class of irreversible algebraic dynamical systems is that Definition 1.1.5 can be recast using $\hat{G}$ and $\hat{\theta}$, see Proposition 1.2.8. In particular, commutative irreversible algebraic dynamical systems correspond to certain continuous surjective transformations $\hat{\theta}$ of the compact Hausdorff space $\hat{G}$. In Lemma 1.2.13, we show that $\hat{\theta}_p$ is a local homeomorphism if and only if $\ker \hat{\theta}_p$ is finite.

We start with a short review of basic facts about characters on groups, see [DE09] for details and further information.

***Remark* 1.2.1.** Recall that a character χ on a locally compact abelian group G is a continuous group homomorphism $G \xrightarrow{\chi} \mathbb{T}$. The set of characters on G forms a locally compact abelian group $\hat{G}$ when equipped with the topology of uniform convergence on compact subsets of G. Pontryagin duality states that $\hat{\hat{G}} \cong G$. For this result, we interpret $g \in G$ as a character on $\hat{G}$ via $g(\chi) := \chi(g)$. If G is discrete, then $\hat{G}$ is compact and vice versa. Standard examples for this duality phenomenon are

- $\hat{\mathbb{R}} \cong \mathbb{R}$,
- $\widehat{\mathbb{Z}^d} \cong \mathbb{T}^d$ for every $d \in \mathbb{N}$ and
- $\widehat{\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$ for every $n \in \mathbb{N}$.

In the following, we will use some well-known facts connected to annihilators.

Definition 1.2.2. Let G be a locally compact abelian group. For a subset $H \subset G$, the **annihilator** of H is given by $H^\perp := \{\ \chi \in \hat{G} \mid \chi|_H = 1\ \}$.

***Remark* 1.2.3.** The annihilator is always a closed subgroup of $\hat{G}$. A useful fact about annihilators of subgroups H is that we have $\hat{H} \cong \hat{G}/H^\perp$. Additionally, one can show that $(H^\perp)^\perp$ is the smallest closed subgroup of G containing H. So if $H \subset G$ is a closed subgroup, then $(H^\perp)^\perp = H$.

Lemma 1.2.4. *Let G be a locally compact abelian group and $G \xrightarrow{\eta} G$ a group endomorphism. Then $\hat{\eta}(\chi)(g) := \chi \circ \eta(g)$ defines a group endomorphism $\hat{G} \xrightarrow{\hat{\eta}} \hat{G}$ which is continuous if and only if η is. This process has the following properties:*

i) $\hat{\hat{\eta}} = \eta$.

ii) $\eta(G)^\perp = \ker \hat{\eta}$.

iii) $\hat{\eta}(\hat{G}) \subset \hat{G}$ *is dense if and only if* η *is injective.*

iv) $\widehat{\ker \hat{\eta}} \cong \operatorname{coker} \eta$ *if* $\eta(G)$ *is closed.*

Proof. $\hat{\eta}$ is a continuous endomorphism of $\hat{G}$ if and only if η is continuous since the elements of $\hat{G}$ are continuous group homomorphisms by definition. Using Pontryagin duality to identify $\hat{\hat{G}}$ with G, the endomorphism $G \xrightarrow{\hat{\hat{\eta}}} G$ is given by

$$\hat{\hat{\eta}}(g)(\chi) = g(\hat{\eta}(\chi)) = \chi \circ \eta(g) = \eta(g)(\chi) \text{ for all } g \in G, \chi \in \hat{G}.$$

Since $\hat{G}$ separates the points in G, $\hat{\hat{\eta}} = \eta$ follows. ii) is obvious and readily implies iii) if we use i) to swap η and $\hat{\eta}$, i.e. $\hat{\eta}(\hat{G})^\perp = \ker \eta$. For iv), Remark 1.2.3 implies

$$\widehat{\ker \hat{\eta}} \cong G/(\ker \hat{\eta})^\perp = G/(\eta(G)^\perp)^\perp.$$

So if $\eta(G)$ is closed, then $\widehat{\ker \hat{\eta}} \cong \operatorname{coker} \eta$. $\square$

In particular, if G is discrete abelian, then ii) says that $\hat{\eta} : \hat{G} \longrightarrow \hat{G}$ is surjective if and only if $\eta : G \longrightarrow G$ is injective. Moreover, $\eta(G)$ is always closed. If, in addition, $\operatorname{coker} \eta$ is finite, then $\ker \hat{\eta} \cong \widehat{\ker \hat{\eta}} \cong \operatorname{coker} \eta$ follows from Remark 1.2.1 and iv).

Lemma 1.2.5. *If G is a locally compact abelian group and $H_1, H_2 \subset G$ are subgroups, then:*

i) $(H_1 \cdot H_2)^\perp = H_1^\perp \cap H_2^\perp$.

ii) $(H_1 \cap H_2)^\perp = H_1^\perp \cdot H_2^\perp$ *holds if H_1 and H_2 are closed.*

Proof. i) is straightforward. If both H_1 and H_2 are closed, then $H_1 \cap H_2$ is a closed subgroup of G. Thus Remark 1.2.3 shows that ii) is equivalent to

$$H_1 \cap H_2 = ((H_1 \cap H_2)^\perp)^\perp = (H_1^\perp \cdot H_2^\perp)^\perp \overset{i)}{=} (H_1^\perp)^\perp \cap (H_2^\perp)^\perp = H_1 \cap H_2.$$

□

Proposition 1.2.6. *Let G be a discrete abelian group and θ_1, θ_2 be commuting, injective endomorphisms of G. Then the following statements hold:*

i) *θ_1 and θ_2 are strongly independent if and only if* $\ker \hat{\theta}_1 \cap \ker \hat{\theta}_2 = \{1_{\hat{G}}\}$.

ii) *θ_1 and θ_2 are independent if and only if* $\ker \hat{\theta}_1 \cdot \ker \hat{\theta}_2 = \ker \widehat{\theta_1\theta_2}$.

Proof. For strong independence, we compute

$$(\theta_1(G)\theta_2(G))^\perp \overset{1.2.5\ i)}{=} \theta_1(G)^\perp \cap \theta_2(G)^\perp \overset{1.2.4\ ii)}{=} \ker \hat{\theta}_1 \cap \ker \hat{\theta}_2.$$

Therefore, $\theta_1(G)\theta_2(G) = G$ is equivalent to $\ker \hat{\theta}_1 \cap \ker \hat{\theta}_2 = \{1_{\hat{G}}\}$. Similarly, we get

$$(\theta_1(G) \cap \theta_2(G))^\perp \overset{1.2.5\ ii)}{=} \theta_1(G)^\perp \cdot \theta_2(G)^\perp \overset{1.2.4\ ii)}{=} \ker \hat{\theta}_1 \cdot \ker \hat{\theta}_2.$$

Now Lemma 1.2.4 ii) gives $\ker \widehat{\theta_1\theta_2} = \theta_1\theta_2(G)^\perp$, so the two conditions are equivalent. □

This motivates the following definition in analogy to Definition 1.1.3:

Definition 1.2.7. Two commuting, surjective group endomorphisms η_1 and η_2 of a group K are said to be **strongly independent**, if they satisfy $\ker \eta_1 \cap \ker \eta_2 = \{1_K\}$. η_1 and η_2 are called **independent**, if $\ker \eta_1 \cdot \ker \eta_2 = \ker \eta_1\eta_2$ holds true.

It is clear that we have an equivalence between the statements:

(i) η_1 and η_2 are strongly independent.

(ii) η_1 is an injective group endomorphism of $\ker \eta_2$.

(ii') η_2 is an injective group endomorphism of $\ker \eta_1$.

If both $\ker \eta_1$ and $\ker \eta_2$ are finite, then strong independence and independence coincide. Therefore, this definition is consistent with [CV13, Definition 5.5], where the case of endomorphisms (of a compact abelian group K) with finite kernels is treated. Note that there is no conflict with (strong) independence for injective group endomorphisms, see Definition 1.1.3, as all these conditions are trivially satisfied by group automorphisms.

With the observations from Remark 1.2.1, Lemma 1.2.4 and Lemma 1.2.5 at hand, we can now translate the setup from Definition 1.1.5 for commutative irreversible algebraic dynamical systems:

Proposition 1.2.8. *For a discrete abelian group G, a triple (G, P, θ) is a commutative irreversible algebraic dynamical system if and only if*

(A) $\hat{G}$ is a compact abelian group,

(B) P is a countably generated, free, abelian monoid (with unit 1_P), and

(C) $P \overset{\hat{\theta}}{\curvearrowright} \hat{G}$ is an action by surjective group endomorphisms with the property that $\hat{\theta}_p$ and $\hat{\theta}_q$ are independent if and only if p and q are relatively prime in P.

(G, P, θ) is minimal if and only if $\bigcup_{p \in P} \ker \hat{\theta}_p \subset \hat{G}$ is dense. It is of finite (infinite) type if and only if $\ker \hat{\theta}_p$ is (infinite) finite for all $p \in P$ $(p \in P \setminus \{1_P\})$.

Proof. Conditions (A) and (B) of this characterization follow readily from Remark 1.2.1 together with Lemma 1.2.4. Moreover, for any $p \in P$, the equation $(\ker \hat{\theta}_p)^{\perp} = \operatorname{im} \theta_p$ implies $\widehat{\ker \hat{\theta}_p} \cong \operatorname{coker} \theta_p$. Combining Lemma 1.2.4 iii) and Proposition 1.2.6 yields (C). Note that we have $\theta_q(G) \subset \theta_p(G)$ and, correspondingly, $\ker \hat{\theta}_p \subset \ker \hat{\theta}_q$ whenever $q \in pP$. Since P is directed, Lemma 1.2.5 i) and Lemma 1.2.4 ii) imply (D). For the last claim, we recall that a locally compact abelian group is finite if and only if its dual group is finite. Thus $\ker \hat{\theta}_p$ is finite if and only if $\operatorname{coker} \theta_p$ is finite. □

Let us now revisit some commutative irreversible algebraic dynamical systems from Section 1.1:

***Examples* 1.2.9.** The following list corresponds to the one in Example 1.1.8.

(a) For $G = \mathbb{Z}$, a family of relatively prime numbers $(p_i)_{i \in I} \subset \mathbb{Z}^{\times} \backslash \mathbb{Z}^{*} = \mathbb{Z} \backslash \{0, \pm 1\}$ generates a monoid $P = |(p_i)_{i \in I}\rangle \subset \mathbb{Z}^{\times}$ which acts on G by $\theta_{p_i}(g) = p_i g$. In this case, $\hat{G} = \mathbb{T}$ and $\hat{\theta}_p(t) = t^p$ for all $t \in \mathbb{T}$ and $p \in P$.

(b) Let $I \subset \mathbb{N}, 0 \in I$, $(p_i)_{i \in I} \subset \mathbb{Z} \setminus \{0, \pm 1\}$ a family of pairwise relatively prime integers and set $P = |(p_i)_{i \geq 1}\rangle$ as well as $G = \mathbb{Z}\left[\frac{1}{p_0}\right] = \varinjlim \mathbb{Z}$ with connecting maps given by multiplication with p_0. Then this constitutes a minimal commutative irreversible algebraic dynamical system of finite type, see Example 1.1.8 (b). The dual group of G is the solenoid $\hat{G} = \mathbb{Z}_{p_0} = \varprojlim \mathbb{Z}/p_0^k\mathbb{Z}$, on which $\hat{\theta}_p$ is given by multiplication with p.

(c) For a finite field $\mathbb{K}$, let $p_i \in \mathbb{K}[T], i \in I$ (for an index set I) be polynomials in $G = \mathbb{K}[T]$ with the property that $(p_i) \cap (p_j) = (p_i p_j)$ holds for all $i \neq j$. Then the action θ of $P := |(p_i)_{i\in I}\rangle$ given by multiplication with the polynomial itself yields a commutative irreversible algebraic dynamical system of finite type, see Example 1.1.8 (c). Then $\hat{G}$ is the ring of formal power series $\mathbb{K}[[T]]$ over $\mathbb{K}$, compare [CV13, Example 2.1.4], and $\hat{\theta}_p$ is given by multiplication with p in $\mathbb{K}[[T]]$.

***Example* 1.2.10.** Recall that, in Example 1.1.9, we considered $G = \mathbb{Z}^d$ for some $d \geq 1$, a family of pairwise commuting matrices $(T_i)_{i\in I} \subset M_d(\mathbb{Z}) \cap Gl_d(\mathbb{Q})$ satisfying $|\det T_i| > 1$ for all $i \in I$ and set $P = |(T_i)_{i\in I}\rangle$ with $\theta_{T_i}(g) = T_i g$. In this case, we have $\hat{G} = \mathbb{T}^d$ and the endomorphism $\hat{\theta}_p$ is given by the matrix corresponding to θ_p interpreted as an endomorphism of $\mathbb{R}^d/\mathbb{Z}^d \cong \mathbb{T}^d$.

***Example* 1.2.11.** The dual model for the unilateral shift on $G = \bigoplus_{\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$ for $n \geq 2$ from Example 1.1.11 is given by the shift $(x_k)_{k\in\mathbb{N}} \mapsto (x_{k+1})_{k\in\mathbb{N}}$ on $\hat{G} = (\mathbb{Z}/n\mathbb{Z})^{\mathbb{N}}$. The discussion for Example 1.1.12 with the restriction that G_0 be abelian is analogous to this case, where one replaces $\mathbb{N}$ by P and $\mathbb{Z}/n\mathbb{Z}$ by G_0.

***Example* 1.2.12.** In the situation of Example 1.1.13, where we will now require that $(G_n, P, \theta^{(i)})_{i\in\mathbb{N}}$ be a family of commutative irreversible algebraic dynamical systems, $G = \bigoplus_{i\in\mathbb{N}} G_i$ turns into $\hat{G} = \prod_{i\in\mathbb{N}} \hat{G}_i$. For each $p \in P$, the group endomorphism $\hat{\theta}_p$ is given by applying $\theta_p^{(i)}$ to the i-th component of $\hat{G}$. $\ker \hat{\theta}_p$ is infinite for all $p \in P \setminus \{1_P\}$. If each $\theta^{(i)}$ satisfies the strong independence condition from Definition 1.1.3, $\hat{\theta}$ satisfies the strong independence condition from Definition 1.2.7 due to Proposition 1.2.6.

In view of Proposition 1.2.8, it seems that the class of commutative irreversible algebraic dynamical systems can be studied from the perspective of topological dynamical systems. But the next lemma displays a severe difficulty for this strategy in the non-finite case:

Lemma 1.2.13. *Suppose G is a discrete abelian group and η is a group endomorphism of G. Then $\hat{\eta}$ is a local homeomorphism of $\hat{G}$ if and only if* $\operatorname{coker} \eta$ *is finite.*

Proof. Recall that $|\operatorname{coker} \eta| = |\ker \hat{\eta}|$ according to Lemma 1.2.4 ii) and Remark 1.2.3. If $\hat{\eta}$ is a local homeomorphism, then, given $k \in \hat{G}$, there is an open neighborhood U of k such that $\hat{\eta}|_U$ is injective. By compactness of $\hat{G}$, finitely many of these cover $\hat{G}$ and this sets a finite bound for $|\ker \hat{\eta}|$. Conversely, suppose $\ker \hat{\eta}$ is finite. Using the Hausdorff property of $\hat{G}$ finitely many times, we get an open neighborhood V of $1_{\hat{G}}$ such that $V \cap \ker \hat{\eta} = \{1_{\hat{G}}\}$. It follows that $\hat{\eta}|_{kV}$ is injective for all $k \in \hat{G}$. Indeed, if there are $k_1, k_2 \in kV$ satisfying $\hat{\eta}(k_1) = \hat{\eta}(k_2)$, then $\hat{\eta}(k_1^{-1}k_2) \in V \cap \ker \hat{\eta} = \{1_{\hat{G}}\}$, so $k_1 = k_2$. $\square$

1.3 Irreversible $*$-commutative dynamical systems

This section is intended to familiarize the reader with the concept of $*$-commutativity so that we can present dynamical systems built from $*$-commuting surjective local homeomorphisms of a compact Hausdorff space that have a similar flavor as irreversible algebraic dynamical systems. A close connection between strong independence and $*$-commutativity for commuting surjective group endomorphisms is established in Proposition 1.3.2. In particular, this shows that the notion of $*$-commutativity coincides with independence for abelian irreversible algebraic dynamical systems of finite type. However, already the canonical shift action of $\mathbb{N}^2$ on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}^2}$ provides an example where the two generators of the action do not $*$-commute but satisfy the independence condition.

The notion of $*$-commutativity was introduced by Victor Arzumanian and Jean Renault in 1996 for a pair of maps $\eta_1, \eta_2 : X \longrightarrow X$ on an arbitrary set X, see [AR97]. For convenience, we will stick to the following equivalent formulation, see [ER07, Section 10]:

Definition 1.3.1. Suppose X is a set and $\eta_1, \eta_2 : X \longrightarrow X$ are commuting maps. η_1 and η_2 are said to **$*$-commute**, if for every $x_1, x_2 \in X$ satisfying $\eta_1(x_1) = \eta_2(x_2)$, there exists a unique $y \in X$ such that $x_1 = \eta_2(y)$ and $x_2 = \eta_1(y)$.

The visualization of this property goes as follows: The maps η_1 and η_2 $*$-commute if and only if every diagram of the form

$$\begin{tikzcd} & \eta_1(x_1) & \\ x_1 \arrow[ur, "\eta_1"] & & x_2 \arrow[ul, "\eta_2"'] \\ & {} \arrow[ul, "\eta_2"] \arrow[ur, "\eta_1"'] & \end{tikzcd} \quad \begin{array}{c}\text{can be completed}\\ \text{uniquely}\\ \text{by some } y \in X \text{ to}\end{array} \quad \begin{tikzcd} & \eta_1(x_1) & \\ x_1 \arrow[ur, "\eta_1"] & & x_2 \arrow[ul, "\eta_2"'] \\ & y \arrow[ul, "\eta_2"] \arrow[ur, "\eta_1"'] & \end{tikzcd}$$

Proposition 1.3.2. *Let X be a set and $\eta_1, \eta_2 : X \longrightarrow X$ commuting maps. Then the following conditions are equivalent:*

(i) The maps η_1 and η_2 $*$-commute.

(ii) For all $x \in X$, $y_1, y_2 \in \eta_1^{-1}(x)$, $\eta_2(y_1) = \eta_2(y_2)$ implies $y_1 = y_2$.

(iii) For all $x \in X$, the map $\eta_1 : \eta_2^{-1}(x) \longrightarrow \eta_2^{-1}(\eta_1(x))$ is a bijection.

(iii') For all $x \in X$, the map $\eta_2 : \eta_1^{-1}(x) \longrightarrow \eta_1^{-1}(\eta_2(x))$ is a bijection.

Proof. Observe that (ii) is basically a reformulation of (i), so their equivalence is straightforward. In order to see that (i) is equivalent to (iii), a diagram of the form

$$\begin{tikzcd}
& \eta_1(x_1) & \\
x_1 \arrow[ur, "\eta_1"] & & x_2 \arrow[ul, "\eta_2"'] \\
& {} \arrow[ul, "\eta_2"] \arrow[ur, "\eta_1"'] &
\end{tikzcd}$$

clearly gives $x_2 \in \eta_2^{-1}(\eta_1(x_1))$. Thus, if we assume (iii), there is a unique $y \in \eta_2^{-1}(x_1)$ such that $\eta_1(y) = x_2$. In other words, the diagram can be completed uniquely. Conversely, if we assume (i), then, for every $x_2 \in \eta_2^{-1}(\eta_1(x_1))$, we get a unique $y_{x_2} \in \eta_2^{-1}(x_1)$ satisfying $\eta_1(y_{x_2}) = x_2$. Note that since η_1 and η_2 commute, we have $\eta_1(\eta_2^{-1}(x_1)) \subset \eta_2^{-1}(\eta_1(x_1))$. Hence, $x_2 \mapsto y_{x_2}$ is a bijection and, in fact, it is just the inverse map of η_1. The equivalence of (i) and (iii') follows by exchanging the role of x_1, x_2 and η_1, η_2. $\square$

The following result is certainly well-known, but hard to track in the available literature, so we include a short proof based on Proposition 1.3.2.

Lemma 1.3.3. *Let X be a set and suppose $\eta_1, \eta_2, \eta_3 : X \longrightarrow X$ commute. η_1 $*$-commutes with $\eta_2\eta_3$ if and only if η_1 $*$-commutes with both η_2 and η_3.*

Proof. Suppose η_1 $*$-commutes with $\eta_2\eta_3$. We will use the equivalent characterization of $*$-commutativity (ii) from Proposition 1.3.2. If we have $x \in X, y_1, y_2 \in \eta_1^{-1}(x)$ such that $\eta_2(y_1) = \eta_2(y_2)$, then $\eta_2\eta_3(y_1) = \eta_2\eta_3(y_2)$ forces $y_1 = y_2$. Thus η_1 and η_2 $*$-commute. For η_1 and η_3, we note that the situation is symmetric in η_2 and η_3. If η_1 $*$-commutes with both η_2 and η_3, then η_1 $*$-commutes with $\eta_2\eta_3$ by the equivalence of $*$-commutativity and condition (iii') in Proposition 1.3.2. $\square$

Given a compact Hausdorff space X, a first step away from reversibility is to consider local homeomorphisms instead of homeomorphisms. Let us recall that if $\eta : X \longrightarrow X$ is a local homeomorphism, then $|\eta^{-1}(x)|$ is finite for all $x \in X$. Indeed, the collection of all open subsets U of X on which η is injective constitutes an open cover of X. By compactness of X, this can be reduced to a finite number which bounds $|\eta^{-1}(x)|$.

We will be interested in surjective local homeomorphisms $\eta : X \longrightarrow X$ for which the cardinality of the preimage of a point is constant on X. Such transformations will be called regular. They also appear in [CS09] under the name *covering map.*

Definition 1.3.4. Let X be a compact Hausdorff space. A surjective local homeomorphism $\eta : X \longrightarrow X$ is said to be **regular**, if $|\eta^{-1}(x)| = |\eta^{-1}(y)|$ holds for all $x, y \in X$.

Via $f \mapsto f \circ \eta$, such a transformation yields an injective $*$-homomorphism α of $C(X)$ which has a left-inverse in the monoid formed by the positive linear maps $X \longrightarrow X$ with composition. This map can be defined abstractly on the C*-algebraic level:

Definition 1.3.5. Given a C*-algebra A and a $*$-endomorphism α of A, a positive linear map $L : A \longrightarrow A$ is called a **transfer operator** for α, if it satisfies $L(\alpha(a)b) = aL(b)$ for all $a, b \in A$. If A is unital, L is said to be **normalized** provided that $L(1) = 1$.

***Example* 1.3.6.** If X is a compact Hausdorff space and $\eta : X \longrightarrow X$ is a regular surjective local homeomorphism with $N_\eta := |\eta^{-1}(x)|$, where $x \in X$ is arbitrary, then

$$L(f)(x) := \frac{1}{N_\eta} \sum_{y \in \eta^{-1}(x)} f(y)$$

defines a transfer operator for the injective $*$-homomorphism α of $C(X)$ given by $f \mapsto f \circ \eta$. Indeed, L is a positive linear map and, for $f, g \in C(X)$ and $x \in X$, we have

$$L(\alpha(f)g)(x) = \frac{1}{N_\eta} \sum_{y \in \eta^{-1}(x)} f(\eta(y))g(y) = (fL(g))(x).$$

***Example* 1.3.7.** Let G be a discrete abelian group and η an injective group endomorphism of G with $[G : \eta(G)] < \infty$. Then $\hat{\eta}$ is a local homeomorphism of $\hat{G}$ by Lemma 1.2.13. It is clear that $\hat{\eta}$ is surjective and every $k \in \hat{G}$ has precisely $|\ker \hat{\eta}| = [G : \eta(G)]$ preimages under $\hat{\eta}$. Thus, $\hat{\eta}$ is regular. If L is the transfer operator for $\hat{\eta}$ as in Example 1.3.6 and $(u_g)_{g \in G}$ denote the standard generators of $C^*(G)$ (which we identify with $C(\hat{G})$), then

$$L(u_g) = \chi_{\eta(G)}(g) u_{\eta^{-1}(g)}$$

holds for all $g \in G$. Indeed, if $g \in \eta(G)$, then

$$L(u_g) = L \circ \alpha(u_{\eta^{-1}(g)}) = u_{\eta^{-1}(g)},$$

where α denotes the endomorphism $u_g \mapsto u_{\eta(g)}$ (which is the same as $f \mapsto f \circ \hat{\eta}$ for $f \in C(\hat{G})$). For the case $g \notin \eta(G)$, let $k \in \hat{G}$ and note that $\hat{\eta}^{-1}(k) = \ell_0 \ker \hat{\eta}$ holds for every $\ell_0 \in \hat{\eta}^{-1}(k)$. Hence, we get

$$L(u_g)(k) = \frac{1}{|\ker \hat{\eta}|} \sum_{\ell \in \hat{\eta}^{-1}(k)} u_g(\ell) = \frac{1}{|\ker \hat{\eta}|} u_g(\ell_0) \sum_{\ell \in \ker \hat{\eta}} u_g(\ell) = 0,$$

since the sum over a finite, nontrivial subgroup of $\mathbb{T}$ vanishes.

The following lemma is a standard fact on how to obtain a conditional expectation from a normalized transfer operator.

Lemma 1.3.8. *Suppose A is a unital C^*-algebra, α is a unital $*$-endomorphism of A and L is a normalized transfer operator for α. Then $E := \alpha \circ L$ is a conditional expectation from A onto $\alpha(A)$.*

Proof. By [BO08, Theorem 5.9], it suffices to show that E is a contractive projection from A onto $\alpha(A)$, i.e. a linear map satisfying $E(a) = a$ for all $a \in \alpha(A)$ and $\|E\| \leq 1$. The first part follows immediately from $L(\alpha(a)) = a$, see Definition 1.3.5. For $\|E\| \leq 1$, note that E is a positive linear map. So if $a \in A_+$, then $\|E(a)\| \leq \|E(\|a\|1)\| = \|a\|$ since $E(1) = 1$. For arbitrary $a \in A$, we have $E(a)E(a^*) \leq E(aa^*)$ (use $E(bb^*) \geq 0$ for $b := a - E(a)$) and thus

$$\|E(a)\|^2 = \|E(a)E(a^*)\| \leq \|E(aa^*)\| \leq \|aa^*\| = \|a\|^2.$$

□

The next lemma is a reformulation of [EV06, Proposition 8.6]:

Lemma 1.3.9. *Let $\eta : X \longrightarrow X$ be a regular surjective local homeomorphism of a compact Hausdorff space X with $N_\eta := |\eta^{-1}(x)|$, where $x \in X$ is arbitrary. Denote by L the natural transfer operator for the induced injective endomorphism α of $C(X)$. Then there exists a finite, open cover $\mathcal{U} = (U_i)_{1 \leq i \leq n}$ of X such that the restriction of η to each U_i is injective. If $(v_i)_{1 \leq i \leq n}$ is a partition of unity for X subordinate to $\mathcal{U}$, then $\nu_i := (N_\eta v_i)^{\frac{1}{2}}$ satisfies*

$$\sum_{1 \leq i \leq n} \nu_i \alpha \circ L(\nu_i f) = f \text{ for all } f \in C(X).$$

Proof. Since η is a local homeomorphism, the open subsets on which η is injective form a cover of X which can be reduced to a finite, open cover $\mathcal{U}$ by compactness of X. It is well-known that, for every such cover, there exists a partition of unity $(v_i)_{1 \leq i \leq n}$ for X subordinate to $\mathcal{U}$. Given $f \in C(X)$, we get

$$\begin{aligned}
\sum_{1 \leq i \leq n} \nu_i \alpha \circ L(\nu_i f)(x) &= \sum_{1 \leq i \leq n} \sum_{y \in \eta^{-1}(\eta(x))} \underbrace{v_i^{\frac{1}{2}}(x) v_i^{\frac{1}{2}}(y)}_{= \delta_{xy}} f(y) \\
&= \sum_{1 \leq i \leq n} v_i(x) f(x) \\
&= f(x)
\end{aligned}$$

for all $x \in X$, where we used injectivity of $\eta|_{\operatorname{supp} v_i}$. □

The equation proved in Lemma 1.3.9 can be interpreted as a reconstruction formula. The conclusion of this result will be relevant for Section 2.4. Before we return to $*$-commuting maps, we add another small observation which is of independent interest.

Lemma 1.3.10. *Let $\eta_1, \eta_2 : X \longrightarrow X$ be commuting continuous maps of a compact Hausdorff space X. Assume that there are two finite open covers $\mathcal{U}_1 = (U_{1,i})_{i \in I_1}$ and $\mathcal{U}_2 = (U_{2,i})_{i \in I_2}$ of X such that $\eta_1|_{U_{1,i}}$ is injective for all $i \in I_1$ and $\eta_2|_{U_{2,i}}$ is injective for all $i \in I_2$. Then*

$$\mathcal{U}_1 \vee_{\eta_1} \mathcal{U}_2 := \left\{ \ U_{1,i_1} \cap \eta_1^{-1}(U_{2,i_2}) \mid i_1 \in I_1, \ i_2 \in I_2 \ \right\}$$

is a finite open cover of X such that the restriction of $\eta_1\eta_2$ to every element of $\mathcal{U}_1 \vee_{\eta_1} \mathcal{U}_2$ is injective. Furthermore, suppose $(v_{1,i})_{i \in I_1}$ and $(v_{2,i})_{i \in I_2}$ are partitions of unity for X subordinate to $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively. If α_1 denotes the endomorphism of $C(X)$ given by $f \mapsto f \circ \eta_1$, then $(v_{i_1,i_2})_{i_1 \in I_1, i_2 \in I_2}$, where $v_{i_1,i_2} := v_{1,i_1}\alpha_1(v_{2,i_2})$ defines a partition of unity subordinate to $\mathcal{U}_1 \vee_{\eta_1} \mathcal{U}_2$.

Proof. First of all, $\mathcal{U}_1 \vee_{\eta_1} \mathcal{U}_2$ consists of open sets by continuity of η_1 and it is clear that these sets cover X. If we let $U' := U_{1,i_1} \cap \eta_1^{-1}(U_{2,i_2})$, we get a commutative diagram:

$$\begin{tikzcd} U' \arrow[rr, "\eta_1\eta_2"] \arrow[dr, "\eta_1"'] & & X \\ & U_{2,i_2} \arrow[ur, "\eta_2"'] & \end{tikzcd}$$

As η_1 is injective on U_{1,i_1} and η_2 is injective on U_{2,i_2}, it follows that $\eta_1\eta_2$ is injective on $U_{1,i_1} \cap \eta_1^{-1}(U_{2,i_2})$ for all i_1, i_2. For the second part, we observe that

$$\sum_{\substack{i_1 \in I_1 \\ i_2 \in I_2}} v_{i_1,i_2}(x) = \underbrace{\sum_{i_1 \in I_1} v_{1,i_1}(x)}_{=1} \underbrace{\sum_{i_2 \in I_2} v_{2,i_2}(\eta_1(x))}_{=1} = 1$$

holds for all $x \in X$ and

$$\operatorname{supp} v_{i_1,i_2} = \operatorname{supp} v_{1,i_1} \cap \eta_1^{-1}(\operatorname{supp} v_{2,i_2}) \subset U_{1,i_1} \cap \eta_1^{-1}(U_{2,i_2}).$$

□

***Remark* 1.3.11.** In particular, Lemma 1.3.10 applies to commuting regular surjective local homeomorphisms by Lemma 1.3.9. The idea is to think of $\mathcal{U}_1 \vee_{\eta_1} \mathcal{U}_2$ as a common refinement of $\mathcal{U}_1$ and $\mathcal{U}_2$ with respect to η_1. But note that this construction is clearly not symmetric in η_1 and η_2.

The next proposition shows that, for two $*$-commuting regular transformations η_1 and η_2, the transfer operator L_1 for the endomorphism α_1 of $C(X)$ induced by η_1 commutes with α_2 (which is induced by η_2). This will be useful in Theorem 3.3.7.

Proposition 1.3.12. *Suppose X is a compact Hausdorff space and $\eta_1, \eta_2 : X \longrightarrow X$ are regular surjective local homeomorphisms. Let α_i denote the endomorphism of $C(X)$ induced by η_i and be L_1 the natural transfer operator for α_1 as constructed in Example 1.3.6. Then η_1 and η_2 $*$-commute if and only if L_1 and α_2 commute.*

Proof. Assume that η_1 and η_2 $*$-commute. Using (iii') from Proposition 1.3.2, this is a straightforward computation. For $f \in C(X)$ and $x \in X$, we get

$$L_1(\alpha_2(f))(x) = \frac{1}{N_1} \sum_{z \in \eta_2(\eta_1^{-1}(x))} f(z) = \frac{1}{N_1} \sum_{z \in \eta_1^{-1}(\eta_2(x))} f(z) = \alpha_2(L_1(f))(x).$$

If η_1 and η_2 do not $*$-commute, there is $x \in X$ such that $\eta_2(\eta_1^{-1}(x))$ is a proper subset of $\eta_1^{-1}(\eta_2(x))$ because η_1 is regular. This forces $\alpha_2(L_1(f))(x) \neq L_1(\alpha_2(f))(x)$ for every constant function $f \in C(X)$, e.g. $f = 1$. $\square$

Next, we will define the analogue of an irreversible algebraic dynamical system of finite type based on $*$-commuting regular transformations of a compact Hausdorff space X, compare Definition 1.1.5. As X is compact, we cannot get anything beyond the finite type case here. We note that more general dynamical systems of this nature have been considered in [FPW13], where X is allowed to be locally compact. In their approach, regularity is relaxed to the requirement that there is a uniform finite bound on the number of preimages of a single point, see [FPW13, Definition 3.2].

Definition 1.3.13. An **irreversible $*$-commutative dynamical system of finite type** is a triple (X, P, η) consisting of

(A) a compact Hausdorff space X,

(B) a countably generated free abelian monoid P with unit 1_P and

(C) an action $P \overset{\eta}{\curvearrowright} X$ by regular surjective local homeomorphisms with the following property: η_p and η_q $*$-commute if and only if p and q are relatively prime in P.

Before considering examples, let us relate the notion of $*$-commutativity to the notion of strong independence introduced in Definition 1.2.7. This will provide examples for irreversible $*$-commutative dynamical systems of finite type coming from commutative irreversible algebraic dynamical systems of finite type.

In addition, we will see that independence is directly connected to $*$-commutativity in the case of surjective group endomorphisms. In fact, independence turns out to be weaker in principle, but the two conditions are equivalent if the kernel of one of the surjective group endomorphisms is a co-Hopfian group. Co-Hopfian groups have first been studied under the name "S-groups" in [Bae44] and we refer to [GG12, ER05] as well as [dlH00, Section 22 of Chapter III] for more information on the subject.

Definition 1.3.14. A group K is said to be co-Hopfian if every injective group endomorphism $\eta : K \hookrightarrow K$ is already an automorphism of K.

***Remark* 1.3.15.** One can rephrase the condition by saying that a group is co-Hopfian if it does not admit nontrivial group embeddings into itself. Likewise, there is a notion of Hopfian groups, called Q-groups in [Bae44], which describes the class of groups with the property that every surjective group endomorphism is automatically an automorphism of the group. In other words, a group is Hopfian if it is not isomorphic to any of its proper subquotients. Both concepts stem from a rigidity property of finite groups. It is important to point out that both classes are much larger than the class of finite groups. For instance, $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}, SL(n,\mathbb{Z})$ for $n \geq 3$ and the fundamental group of any closed surface of genus at least 2 are co-Hopfian, see [Bae44] and [dlH00, Chapter III, Section 22] for more information.

Proposition 1.3.16. *Suppose K is a group and η_1, η_2 are commuting surjective endomorphisms of K. If η_1 and η_2 $*$-commute, then η_1 and η_2 are strongly independent. If $\eta_2 : \ker \eta_1 \longrightarrow \ker \eta_1$ or $\eta_1 : \ker \eta_2 \longrightarrow \ker \eta_2$ is surjective, then the converse holds as well. In particular, this is the case if $\ker \eta_1$ or $\ker \eta_2$ is co-Hopfian.*

Proof. Note that we have $\eta_i^{-1}(k) = k' \ker \eta_i$ for all $k \in K$ where $k' \in \eta_i^{-1}(k)$ is chosen arbitrarily. According to Proposition 1.3.2, η_1 and η_2 $*$-commute precisely if $\eta_1 : \eta_2^{-1}(k) \longrightarrow \eta_1(\eta_2^{-1}(k))$ is bijective for all $k \in K$. Since η_1 and η_2 are group endomorphisms, this is equivalent to the requirement that η_1 is an automorphism of the subgroup $\ker \eta_2$. Indeed, this is clearly necessary and if it is true, then $\eta_1 : \eta_2^{-1}(k) \longrightarrow \eta_1(\eta_2^{-1}(k))$ is a bijection because $\eta_2^{-1}(\eta_1(k)) = \eta_1(k') \ker \eta_2$ and $\eta_1(\eta_2^{-1}(k)) = \eta_1(k')\eta_1(\ker \eta_2)$. In particular, we have $\ker \eta_1 \cap \ker \eta_2 = \{1_K\}$, so η_1 and η_2 are strongly independent in the sense of Definition 1.2.7. Moreover, we see that strong independence corresponds to injectivity of η_1 and η_2 on $\ker \eta_2$ and $\ker \eta_1$, respectively. Hence, if one of these maps is surjective, we get $*$-commutativity of η_1 and η_2. By definition, this is for granted if one knows that one of the kernels is a co-Hopfian group. $\square$

There are interesting examples of dynamical systems built from $*$-commuting transformations, see for instance [ER07, Sections 10–14] and [Wil10, MW]. On the other hand, $*$-commutativity is also considered to be a severe restriction. While $*$-commutativity implies strong independence in the case of surjective group endomorphisms, there are examples for commutative irreversible algebraic dynamical systems that do not satisfy the strong independence condition, see for instance Example 1.1.12. Thus we conclude that, in principle, the notion of independence is less restrictive than $*$-commutativity.

Comparing Definition 1.3.13 with Definition 1.1.5, we make the following observation based on Proposition 1.2.8, Lemma 1.2.13 and Proposition 1.3.16:

Corollary 1.3.17. *Let G be a discrete abelian group, P a monoid and $P \overset{\theta}{\curvearrowright} G$ an action by group endomorphisms. (G, P, θ) is a commutative irreversible algebraic dynamical system of finite type if and only if $(\hat{G}, P, \hat{\theta})$ is an irreversible $*$-commutative dynamical system of finite type.*

For the remainder of this section, we would like to direct the reader's attention to another intriguing class of examples for irreversible algebraic dynamical systems, namely to dynamical systems arising from cellular automata. This part builds on [ER07, Section 14] and can be considered as a natural extension of the observations presented there. In the following, let $X = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ and σ denote the unilateral shift, i.e. $\sigma(x)_k = x_{k+1}$ for all $k \in \mathbb{N}$ and $x \in X$. Moreover, let $X_n = (\mathbb{Z}/2\mathbb{Z})^n$ for $n \in \mathbb{N}$ and suppose we are given $D \subset X_n$. Then we can define a transformation η_D of X by the sliding window method

$$(\eta_D(x))_k = \chi_D(x_k, x_{k+1}, \ldots, x_{k+n-1}).$$

In other words, the entry at place k becomes 1 if the word of length n starting at place k belongs to the so-called dictionary D. It is interesting to analyze the extent to which properties of η_D can be expressed in terms of its dictionary. One outcome of such considerations are the following two definitions:

Definition 1.3.18. For $n \in \mathbb{N}$, a subset $D \subset X_n$ is called a **dictionary**. D is called **progressive**, if for any $x \in X_{n-1}$, there is a unique $x_n \in X_1$ such that $(x_1, \ldots, x_n) \in D$. D is called **admissible**, if it is progressive and has the property that, for $x, y, z \in X_n$, $x + y = z \in D$ implies that either $x \in D$ or $y \in D$ holds.

Let us observe that $X_n \backslash D$ is a group of order 2^{n-1} for every admissible dictionary D. It is clear that η_D is continuous on X and commutes with σ for every dictionary D. Morton L. Curtis, Gustav A. Hedlund and Roger Lyndon have shown in [Hed69] that any continuous self-map of X which commutes with the shift σ corresponds to a cellular automaton (Even though the article is authored by Hedlund only, he credits Curtis and Lyndon as co-discoverers in the introduction.). Thus (X, η_D) can be identified as a cellular automaton. It is shown in [ER07, Theorem 14.3] that for progressive D, the transformation η_D is a surjective local homeomorphism of X. This allows us to deduce:

Proposition 1.3.19. *If $D \subset X_n$ is an admissible dictionary, then η_D is a continuous surjective group endomorphism of X that commutes with σ. $\ker \eta_D$ is isomorphic to the group $X_n \backslash D$ and thus consists of 2^{n-1} elements.*

Proof. The only thing that remains to be proven is that η_D is a group endomorphism of X with finite kernel. The first part follows readily from the additional requirement that whenever we have $x + y = z$ for some $x, y, z \in X_n$, $z \in D$ implies that either $x \in D$ or $y \in D$ holds. For the assertion concerning the kernel of η_D, note that $x \in \ker \eta_D$ means that we have $(x_k, x_{k+1}, \ldots, x_{k+n-1}) \notin D$ for all $k \in \mathbb{N}$. But if the first n entries form a word that is not contained in D, sliding the window forward once we see that there is precisely one option for the entry at place $n+1$ to arrange for a word that is contained in D as D is progressive. Conversely, since our alphabet is just $\{0, 1\}$, the previous entries determine the last entry uniquely if we assume that the word they form is not contained in D. Therefore, an element in $\ker \eta_D$ is given by its first n components which yield an element of $X_n \backslash D$. $\square$

***Remark* 1.3.20.** In view of Proposition 1.3.2, we are now in position to provide new examples for commutative irreversible algebraic dynamical systems of finite type in terms of their dual pictures. We note that it is easier to check strong independence of σ and η_D than examining $*$-commutativity of these for an admissible D. Indeed, $\ker \sigma$ is easily determined and Proposition 1.3.19 provides us with an explicit description of $\ker \eta_D$.

Concerning the quest for further examples where the transformations are independent but do not $*$-commute, Proposition 1.3.19 shows that this is impossible within the current setting since the kernels are finite. The first example we consider is the so-called Ledrappier shift, see [ER07, Section 11]:

***Example* 1.3.21.** Let Y be the subshift of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}^2}$ given by all sequences $y = (y_n)_{n \in \mathbb{N}^2}$ s.t. $y_n + y_{n+e_1} + y_{n+e_2} = 0 \in \mathbb{Z}/2\mathbb{Z}$ for all $n \in \mathbb{N}^2$. $\mathbb{N}^2 \overset{\eta}{\curvearrowright} Y$ is given by the coordinate shifts $\eta_{e_i}(y_n)_n = (y_{n+e_i})_n,\ i = 1, 2$. The four basic blocks in Y are:

$$\begin{array}{ll} \boxed{0} & \\ \boxed{0} & \boxed{0} \end{array} \qquad \begin{array}{ll} \boxed{1} & \\ \boxed{0} & \boxed{1} \end{array} \qquad \begin{array}{ll} \boxed{0} & \\ \boxed{1} & \boxed{1} \end{array} \qquad \begin{array}{ll} \boxed{1} & \\ \boxed{1} & \boxed{0} \end{array}$$

Observe that, for any given $y \in Y$ and every path $(n_m)_{m \in \mathbb{N}}$ with $n_{m+1} \in \{n_m + e_1, n_m + e_2\}$, the sequence $(y_{n_m})_{m \in \mathbb{N}}$ determines y completely. Conversely, one can show inductively, that for every path $(n_m)_{m \in \mathbb{N}}$ and sequence $(y_m)_{m \in \mathbb{N}}$ with $y_m \in \mathbb{Z}/2\mathbb{Z}$, there is an $y \in Y$ with $y_{n_m} = y_m$ for all m. One consequence of this is that there is a homeomorphism $Y \longrightarrow X = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ given by restricting to the base row, i.e. $(y_{m,n})_{m,n \in \mathbb{N}} \mapsto (y_{n,0})_{n \in \mathbb{N}}$. Under this homeomorphism to the Bernoulli space, η_{e_1} corresponds to the shift σ on X and η_{e_2} corresponds to $x \mapsto x + \sigma(x) = (x_n + x_{n+1})_{n \in \mathbb{N}}$ for $x \in X$. In view of the example from cellular automata, it is quite intriguing to notice that the Ledrappier shift fits into the picture quite nicely: The conjugate map to the vertical shift is nothing but η_D for the admissible dictionary $D = \{(0,1),(1,0)\}$. In fact, (X, η_D) is the most basic non-trivial example of a cellular automaton coming from an admissible dictionary. By Proposition 1.3.19, θ_D $*$-commutes with the shift, so η_{e_1} and η_{e_2} $*$-commute. Hence the Ledrappier shift gives rise to a commutative irreversible algebraic dynamical system of finite type.

***Remark* 1.3.22.** We have seen that the Ledrappier shift can be obtained from an admissible dictionary. In fact, there is only one admissible dictionary D for words of length 2 such that the induced transformation η_D $*$-commutes with shift σ. So the Ledrappier shift constitutes a minimal non-trivial example of a commutative irreversible algebraic dynamical system of finite type arising from a cellular automaton.

Reversing the perspective, the Ledrappier shift is formed out of the cellular automaton (X, η_D) by stacking the orbit. This is to say that for $x \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ the k-th row of the corresponding element in $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}^2}$ is given by $\eta_D^k(x)$. Building on this observation, we may always construct a subshift of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}^2}$ out of a progressive dictionary. This may turn out to be a source of potentially interesting subshifts of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}^2}$. Let us now look at what happens for dictionaries using longer words:

***Example* 1.3.23.** Let $D_1, D_2 \subset X_3$ be the dictionaries

$$\begin{array}{rcl} D_1 &=& \{(0,0,1),(1,0,0),(0,1,1),(1,1,0)\} \\ & \text{and} & \\ D_2 &=& \{(0,0,1),(1,0,0),(0,1,0),(1,1,1)\}. \end{array}$$

Then D_1 and D_2 are admissible dictionaries. Hence, η_{D_1} and η_{D_2} are surjective group endomorphisms of $X = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ that commute with the shift σ and

$$\begin{array}{rcl} \ker \eta_{D_1} &=& \{0, 1, (\overline{0,1}, \dots), (\overline{1,0}, \dots)\}, \\ \ker \eta_{D_2} &=& \{0, (\overline{1,0,1}, \dots), (\overline{0,1,1}, \dots), (\overline{1,1,0}, \dots)\}, \end{array}$$

where we write $(\overline{a,b,c}, \dots)$ for the periodic word $(a,b,c,a,b,c,\dots)$. Apparently, we have $\ker \sigma = \{0, (1, \overline{0}, \dots)\}$, so σ and η_{D_i} are strongly independent for $i = 1,2$. By Proposition 1.3.2, they also $*$-commute. Hence, each D_i gives rise to a commutative irreversible algebraic dynamical system of finite type (G, P, θ) with $G = \hat{X}$ and $P = |\sigma, \eta_{D_i}\rangle \cong \mathbb{N}^2$ acting by their dual endomorphisms. Noting that $\ker \eta_{D_1} \cap \ker \eta_{D_2}$ is trivial, we also get a commutative irreversible algebraic dynamical system of finite type for $P = |\sigma, \eta_{D_1}, \eta_{D_2}\rangle \cong \mathbb{N}^3$.

***Remark* 1.3.24.** In fact, D_1 and D_2 are the only admissible dictionaries for words of length 3 for which the induced transformation $*$-commutes with σ. Indeed, every such admissible dictionary D needs to contain $(0,0,1)$ and $(1,0,0)$. If $(0,0,0) \in D$, then D cannot induce a group homomorphism. Likewise, if we had $(1,0,0) \notin D$, then $\ker \eta_D$ would contain $\ker \sigma$. In particular, their intersection would be non-trivial. Now, if $(0,1,1) \in D$, then this forces $(1,1,0) \in D$ since $(0,1,1)+(1,1,1) = (1,0,0) \in D$. Similarly, $(0,1,0) \in D$, then this forces $(1,1,1) \in D$ since $(0,1,0) + (1,1,0) = (1,0,0) \in D$. One can check that there are precisely two additional admissible dictionaries $D_3, D_4 \subset X_3$ given by

$$\begin{array}{rcl} D_3 &=& \{(0,0,1),(1,0,1),(0,1,0),(1,1,0)\} \\ & \text{and} & \\ D_4 &=& \{(0,0,1),(1,0,1),(0,1,1),(1,1,1)\}. \end{array}$$

Thus, there are four admissible dictionaries for word length 3, two of which induce surjective group endomorphisms of X that $*$-commute with the shift σ. The corresponding group endomorphisms of X are

$$\begin{array}{rcl} \eta_1(x) &=& x + \sigma^2(x) \\ \eta_2(x) &=& x + \sigma(x) + \sigma^2(x) \end{array} \quad \text{and} \quad \begin{array}{rcl} \eta_3(x) &=& \sigma(x) + \sigma^2(x) \\ \eta_4(x) &=& \sigma^2(x). \end{array}$$

This simple description raises the question whether it might be possible to characterize admissibility of a dictionary $D \subset X_n$ for general $n \geq 2$ and $*$-commutativity of η_D with σ in a more accessible way.

***Remark* 1.3.25.** In [ER07, Example 14.4], Ruy Exel and Jean Renault provided an example of a progressive dictionary which does not induce a transformation that $*$-commutes with the shift, namely

$$D = \{(0,0,0),(1,0,0),(0,1,0),(1,1,1)\}.$$

This is stated implicitly in [ER07, Corollary 14.5] and follows from [ER07, Theorem 10.4 and Proposition 14.1]. However, this dictionary does not give a group homomorphism of X because it contains the neutral element of X and hence $\theta_D(0) \neq 0$. The dictionary D_2 from Example 1.3.23 is a slight variation of [ER07, Example 14.4] designed to produce a group homomorphism.

Chapter 2

C*-algebras for irreversible semigroup dynamical systems

Within this chapter, we will associate a C*-algebra $\mathcal{O}[G,P,\theta]$ to each irreversible algebraic dynamical system (G,P,θ) and study its internal structure, see Section 2.2. Once again, the finite type case deserves special attention, see Section 2.3. Analogously, we construct a C*-algebra $\mathcal{O}[X,P,\theta]$ for irreversible $*$-commutative dynamical systems of finite type, see Section 2.4.

We start in Section 2.1 with a general observation concerning the decomposability of crossed products by semidirect products of semigroups: Let S and T be discrete monoids and $T \overset{\theta}{\curvearrowright} S$ be an action by monoidal endomorphisms. Then we can form the semidirect product $S \rtimes_\theta T$ given by $S \times T$ with composition $(s,t)(s',t') = (s\theta_t(s'),tt')$. Now if A is unital C*-algebra and α is an action of $S \rtimes_\theta T$ on A by endomorphisms, then we can consider the semigroup crossed products $A \rtimes_{\alpha|_S} S$ and $A \rtimes_\alpha (S \rtimes_\theta T)$. We prove that, in case $\{1_A - \alpha_{(s,1_T)}(1_A) \mid s \in S\} \subset \bigcap_{t\in T} \ker \alpha_{(1_S,t)}$ holds true, there is a T-action $\tilde{\alpha}$ on $A \rtimes_{\alpha|_S} S$ (naturally induced by α and θ) so that $A \rtimes_\alpha (S \rtimes_\theta T) \cong \big(A \rtimes_{\alpha|_S} S\big) \rtimes_{\tilde{\alpha}} T$, see Theorem 2.1.5. This will be useful for the analysis of $\mathcal{O}[G,P,\theta]$, see Corollary 2.2.19.

The C*-algebraic model $\mathcal{O}[G,P,\theta]$ introduced in Section 2.2 is based on an examination of the natural representation of (G,P,θ) by unitary and isometric linear operators on $\ell^2(G)$, and is inspired by [CV13, Vie13, Li12]. The properties of $\mathcal{O}[G,P,\theta]$ are closely linked to the structures of its core subalgebra $\mathcal{F}$, which can be described as the fixed-point algebra under the natural gauge action of the dual group L of $H = P^{-1}P$, and its (commutative) diagonal subalgebra $\mathcal{D}$. In Lemma 2.2.9, we show that the spectrum G_θ of $\mathcal{D}$ can be interpreted as a completion of G with respect to θ if (G,P,θ) is minimal. This extends [CV13, Lemma 2.4] to minimal irreversible algebraic dynamical systems. As the next step, we prove that $\mathcal{O}[G,P,\theta]$ is canonically isomorphic to $\mathcal{D} \rtimes (G \rtimes_\theta P)$, see Proposition 2.2.18. Combining this with Theorem 2.1.5 yields an isomorphism between

$\mathcal{F}$ and $C(G_\theta) \rtimes_\tau G$, see Corollary 2.2.19, where the action τ is given by left translation. It is apparent from the construction that $\mathcal{O}[G,P,\theta]$ shares some flavour with $\mathcal{O}_n$, where $1 \leq n \leq \infty$. Therefore, it is natural to ask whether $\mathcal{O}[G,P,\theta]$ is purely infinite and simple under certain conditions. In Theorem 2.2.26, a positive answer is provided that invokes minimality of (G,P,θ) and amenability of the action $G \overset{\hat{\tau}}{\curvearrowright} G_\theta$ as sufficient conditions. Next, we show that these two conditions force $\mathcal{O}[G,P,\theta]$ to be a UCT Kirchberg algebra, see Corollary 2.2.28.

Section 2.2 constitutes a short interlude, where we specialize to the structure of the core $\mathcal{F}$ for irreversible algebraic dynamical systems of finite type. If G is amenable and (G,P,θ) is minimal, then $\mathcal{F}$ is a generalized Bunce-Deddens algebra in the sense of [Orf10], see Proposition 2.3.2. In this situation, we briefly discuss how classification of $\mathcal{F}$ by its Elliott invariant is achieved based on [Lin01, MS, Win05], see Corollary 2.3.3. Partly, this has been achieved earlier through [Car11]. In addition to that, we extend the observation mentioned after [CV13, Lemma 2.5] to minimal commutative irreversible algebraic dynamical systems of finite type, see Corollary 2.3.4.

Finally, we turn to the construction of the C*-algebra $\mathcal{O}[X,P,\theta]$ for irreversible $*$-commutative dynamical systems of finite type (X,P,θ). The choice of relations for $\mathcal{O}[X,P,\theta]$ essentially builds on the insights gained from $\mathcal{O}[G,P,\theta]$ and the approaches in [Exe03a, CS09]. Therefore, it is not surprising that Proposition 2.4.3 establishes a consistency result for the two C*-algebraic construction in the case of commutative irreversible algebraic dynamical systems of finite type. Next, we provide an explicit representation of $\mathcal{O}[X,P,\theta]$ on $\ell^2(X)$, see Proposition 2.4.4. From there on, the core subalgebra $\mathcal{F}$ for $\mathcal{O}[X,P,\theta]$ is analysed in search for results that partly mimic crucial parts of the findings from Section 2.2, see Proposition 2.4.9, Corollary 2.4.13, and Lemma 2.4.14. Many of these observations will play a role in the course of Chapter 4.

2.1 Crossed products by semidirect products of semigroups

Within this section, we will establish a result about viewing a crossed product of a C*-algebra by a semidirect product of discrete monoids as an iterated crossed product, see Theorem 2.1.5. This extends the well-known result for semidirect products of locally compact groups in the discrete case, see [Wil07, Proposition 3.11], and is essential for the proof of Corollary 2.2.19.

For convenience, we will restrict our attention to the case of unital coefficient algebras and include the basic definitions for semigroup crossed products based on covariant pairs of representations. We refer to [Lar10] for a more extensive treatment of the subject.

All semigroups will be left cancellative and discrete. In the following, let $\mathrm{Isom}(B)$ denote the semigroup of isometries in a unital C*-algebra B.

Definition 2.1.1. Let S be a semigroup and A a unital C*-algebra with an action $S \overset{\alpha}{\curvearrowright} A$ by endomorphisms. A **covariant pair** (π_A, π_S) for (A, S, α) is given by a unital C*-algebra B together with a unital $*$-homomorphism $\pi_A : A \longrightarrow B$ and a semigroup homomorphism $\pi_S : S \longrightarrow \mathrm{Isom}(B)$ subject to the covariance condition:

$$\pi_S(s)\pi_A(a)\pi_S(s)^* = \pi_A(\alpha_s(a)) \text{ for all } a \in A, s \in S.$$

Definition 2.1.2. Let S be a semigroup and A a unital C*-algebra with an action $S \overset{\alpha}{\curvearrowright} A$ by endomorphisms. The **crossed product** for (A, S, α), denoted by $A \rtimes_\alpha S$, is the C*-algebra generated by a covariant pair (ι_A, ι_S) which is universal in the sense that whenever (π_A, π_S) is a covariant pair for (A, S, α), it factors through (ι_A, ι_S). That is to say, there is a surjective $*$-homomorphism $\overline{\pi} : A \rtimes_\alpha S \longrightarrow C^*(\pi_A(A), \pi_S(S))$ satisfying $\pi_A = \overline{\pi} \circ \iota_A$ and $\pi_S = \overline{\pi} \circ \iota_S$. $A \rtimes_\alpha S$ is uniquely determined up to canonical isomorphism by this universal property.

This crossed product may be 0, as for instance for $A = C_0(\mathbb{N}) = \{(a_n)_{n\in\mathbb{N}} \mid a_n \in \mathbb{C}, \lim\limits_{n\to\infty} |a_n| = 0\}$ and $S = \mathbb{N}$ acting by the unilateral shift $\alpha_1((a_n)_{n\in\mathbb{N}}) = (a_{n+1})_{n\in\mathbb{N}}$, see [Sta93, Example 2.1(a)]. But it is known that the coefficient algebra A embeds into $A \rtimes_\alpha S$ provided that S acts by injective endomorphisms and is right-reversible, i.e. $Ss \cap St \neq \emptyset$ for all $s, t \in S$, see [DFK, Lemma 5.2.1].

Suppose that S and T are semigroups with an action $T \overset{\theta}{\curvearrowright} S$ by semigroup homomorphisms of S. Then we can form the semidirect product $S \rtimes_\theta T$, which is the semigroup given by $S \times T$ with $ax + b$-composition rule:

$$(s, t)(s', t') = (s\theta_t(s'), tt')$$

Now suppose further that S and T are monoids and that α is an action of $S \rtimes_\theta T$ on a unital C*-algebra A. Then the semigroup crossed product $A \rtimes_\alpha (S \rtimes_\theta T)$ is given by a unital $*$-homomorphism

$$\iota_{A, S\rtimes_\theta T} : A \longrightarrow A \rtimes_\alpha (S \rtimes_\theta T)$$

and a semigroup homomorphism

$$\iota_{S\rtimes_\theta T} : S \rtimes_\theta T \longrightarrow \mathrm{Isom}(A \rtimes_\alpha (S \rtimes_\theta T)).$$

On the other hand, we can consider $A \rtimes_{\alpha|_S} S$ given by a unital $*$-homomorphism

$$\iota_{A,S} : A \longrightarrow A \rtimes_{\alpha|_S} S$$

and a semigroup homomorphism

$$\iota_S : S \longrightarrow \mathrm{Isom}(A \rtimes_{\alpha|_S} S).$$

A natural question in this situation is whether α and θ give rise to an action $T \overset{\tilde{\alpha}}{\curvearrowright} A \rtimes_{\alpha|_S} S$. The next lemma provides a positive answer for the case where α satisfies

$$\{1_A - \alpha_{(s,1_T)}(1_A) \mid s \in S\} \subset \bigcap_{t \in T} \ker \alpha_{(1_S,t)}.$$

For the sake of readability, let $p_{(s,t)} := \iota_{A,S}(\alpha_{(s,t)}(1_A))$ for $s \in S, t \in T$ and we will simply write p_t for $p_{(1_S,t)}$. We observe that the aforementioned condition is equivalent to

$$p_{(\theta_t(s),t)} = p_t \text{ for all } s \in S, t \in T.$$

Lemma 2.1.3. *Suppose that S and T are monoids with an action $T \overset{\theta}{\curvearrowright} S$ by semigroup homomorphisms of S. Let α be an action of $S \rtimes_\theta T$ on a unital C*-algebra A by endomorphisms. For $t \in T$, let*

$$\tilde{\alpha}_t(\iota_{A,S}(a)\iota_S(s)) := \iota_{A,S}(\alpha_{(1_S,t)}(a))\iota_S(\theta_t(s)) \textit{ for } a \in A, s \in S.$$

$\tilde{\alpha}_t$ is an endomorphism from $A \rtimes_{\alpha|_S} S \longrightarrow p_t(A \rtimes_{\alpha|_S} S)p_t$ provided that

$$1_A - \alpha_{(s,1_T)}(1_A) \in \ker \alpha_{(1_S,t)} \textit{ for all } s \in S.$$

In particular, if this holds for all $t \in T$, i.e.

$$1_A - \alpha_{(s,1_T)}(1_A) \in \bigcap_{t \in T} \ker \alpha_{(1_S,t)} \textit{ for all } s \in S,$$

then $\tilde{\alpha}$ defines an action of T on $A \rtimes_{\alpha|_S} S$.

Proof. Note that

$$\tilde{\alpha}_t(\iota_S(s)) = \tilde{\alpha}_t(\iota_{A,S}(1_A)\iota_S(s)) = p_t\iota_S(\theta_t(s))$$

is valid for all $s \in S, t \in T$ since $\iota_{A,S}$ is unital. Suppose $t \in T$ satisfies

$$1_A - \alpha_{(s,1_T)}(1_A) \in \ker \alpha_{(1_S,t)} \text{ for all } s \in S.$$

This is equivalent to $p_{(\theta_t(s),t)} = p_t$ for all $s \in S$. Hence, p_t commutes with $\iota_S(\theta_t(s))$ since

$$\iota_S(\theta_t(s))p_t = \iota_S(\theta_t(s))p_t\iota_S(\theta_t(s))^*\iota_S(\theta_t(s)) = p_{(\theta_t(s),t)}\iota_S(\theta_t(s)) = p_t\iota_S(\theta_t(s)).$$

To prove that $\tilde{\alpha}_t$ is an endomorphism of $A \rtimes_{\alpha|_S} S$, we show that

$$\big(\iota_{A,S} \circ \alpha_{(1_S,t)}, p_t(\iota_S \circ \theta_t(\cdot))\big)$$

is a covariant pair for $(A, S, \alpha|_S)$. It is then easy to see that the induced map coming from the universal property of the crossed product is precisely $\tilde{\alpha}_t$ and maps $A \rtimes_{\alpha|_S} S$ onto the corner $p_t \left(A \rtimes_{\alpha|_S} S\right) p_t$.

Firstly, $\iota_{A,S} \circ \alpha_{(1_S,t)}$ is a unital $*$-homomorphism from A to $p_t \left(A \rtimes_{\alpha|_S} S\right) p_t$. In addition, $p_t(\iota_S \circ \theta_t(\cdot))$ maps S to the isometries in $p_t \left(A \rtimes_{\alpha|_S} S\right) p_t$ because

$$(p_t \iota_S(\theta_t(s)))^* p_t \iota_S(\theta_t(s)) = \iota_S(\theta_t(s))^* p_t \iota_S(\theta_t(s)) = \iota_S(\theta_t(s))^* \iota_S(\theta_t(s)) p_t = p_t.$$

This map turns out to be a semigroup homomorphism as

$$p_t \iota_S(\theta_t(s_1)) p_t \iota_S(\theta_t(s_2)) = p_t^2 \iota_S(\theta_t(s_1)) \iota_S(\theta_t(s_2)) = p_t \iota_S(\theta_t(s_1 s_2)).$$

Finally, for $a \in A$ and $s \in S$, we compute

$$\begin{aligned} p_t \iota_S(\theta_t(s)) \iota_{A,S}(\alpha_{(1_S,t)}(a)) (p_t \iota_S(\theta_t(s)))^* &= p_t \iota_{A,S}(\alpha_{(\theta_t(s),t)}(a)) p_t \\ &= \iota_{A,S}(\alpha_{(1_S,t)}(\alpha_{(s,1_T)}(a))). \end{aligned}$$

Thus, $\left(\iota_{A,S} \circ \alpha_{(1_S,t)}, p_t(\iota_S \circ \theta_t(\cdot))\right)$ forms a covariant pair for $(A, S, \alpha|_S)$. In particular, the induced map $\tilde{\alpha}_t$ is an endomorphism of $A \rtimes_{\alpha|_S} S$.

Conversely, assume that $\tilde{\alpha}_t$ defines an endomorphism of $A \rtimes_{\alpha|_S} S$. Then $(\tilde{\alpha}_t \circ \iota_{A,S}, \tilde{\alpha}_t \circ \iota_S)$ forms a covariant pair for $(A, \alpha|_S, S)$ mapping A and S to the C*-algebra $B := \tilde{\alpha}_t(A \rtimes_{\alpha|_S} S)$. Note that the unit inside this C*-algebra is p_t. In particular, we have a semigroup homomorphism $\tilde{\alpha}_t \circ \iota_S : S \longrightarrow \mathrm{Isom}(B)$. This forces

$$p_t = \tilde{\alpha}_t(\iota_S(s))^* \tilde{\alpha}_t(\iota_S(s)) = \iota_S(\theta_t(s))^* p_t \iota_S(\theta_t(s)) = p_{(\theta_t(s),t)}$$

for all $s \in S$, which is equivalent to

$$\{1_A - \alpha_{(s,1_T)}(1_A) \mid s \in S\} \subset \ker \alpha_{(1_S,t)}.$$

Since $\alpha|_T$ and θ are semigroup homomorphisms, it is clear that $\tilde{\alpha}$ defines an action of T on $A \rtimes_{\alpha|_S} S$ provided that the imposed condition holds for every $t \in T$. $\square$

***Remark* 2.1.4.** It would be interesting to know whether the condition from Lemma 2.1.3 is actually necessary. This would be the case if $p_t \leq p_{(\theta_t(s),1_T)}$ was true for $s, t \in S$. Note that we do have $p_{(\theta_t(s),t)} \leq p_t$ and $p_{(\theta_t(s),t)} \leq p_{(\theta_t(s),1_T)}$.

Now, given the hypotheses of Lemma 2.1.3 are satisfied, $T \overset{\tilde{\alpha}}{\curvearrowright} A \rtimes_{\alpha|_S} S$ gives rise to an iterated semigroup crossed product $\left(A \rtimes_{\alpha|_S} S\right) \rtimes_{\tilde{\alpha}} T$ and it is a natural task to relate this crossed product to $A \rtimes_\alpha (S \rtimes_\theta T)$. The next result shows that indeed, this decomposition procedure recovers the original crossed product.

Theorem 2.1.5. *Suppose S and T are monoids together with an action $T \overset{\theta}{\curvearrowright} S$ by semigroup homomorphisms of S as well as an action α of $S \rtimes_\theta T$ on a unital C*-algebra A by endomorphisms. If*

$$\{1_A - \alpha_{(s,1_T)}(1_A) \mid s \in S\} \subset \bigcap_{t \in T} \ker \alpha_{(1_S,t)}$$

holds true, then there is a canonical isomorphism

$$\begin{array}{rcl} A \rtimes_\alpha (S \rtimes_\theta T) & \xrightarrow{\pi} & \big(A \rtimes_{\alpha|_S} S\big) \rtimes_{\tilde{\alpha}} T, \\ \iota_{A,S\rtimes_\theta T}(a) & \mapsto & \iota_{A\rtimes S} \circ \iota_{A,S}(a) \\ \iota_{S\rtimes_\theta T}(s,t) & \mapsto & (\iota_{A\rtimes S} \circ \iota_S)(s)\iota_T(t) \end{array}$$

where $\tilde{\alpha}$ is given by $\tilde{\alpha}_t(\iota_{A,S}(a)\iota_S(s)) = \iota_A(\alpha_{(1_S,t)}(a))\iota_S(\theta_t(s))$.

Proof. Recall that $(\iota_{A,S\rtimes_\theta T}, \iota_{S\rtimes_\theta T}), (\iota_{A,S}, \iota_S)$ and $(\iota_{A\rtimes S}, \iota_T)$ denote the universal pairs for covariant pairs for $(A, S\rtimes_\theta T, \alpha)$, $(A, S, \alpha|_S)$ and $(A \rtimes_{\alpha|_S} S, T, \tilde{\alpha})$, respectively. In other words, their images generate the corresponding crossed products. The strategy is governed by the following claims:

1) $(\iota_{A\rtimes S} \circ \iota_{A,S}, (\iota_{A\rtimes S} \circ \iota_S) \times \iota_T)$ forms a covariant pair for $(A, S\rtimes_\theta T, \alpha)$.

2) $(\iota_{A,S\rtimes_\theta T} \times \iota_{S\rtimes_\theta T}|_S, \iota_{S\rtimes_\theta T}|_T)$ forms a covariant pair for $(A \rtimes_{\alpha|_S} S, T, \tilde{\alpha})$.

If we assume 1) and 2), then 1) and the universal property of $A \rtimes_\alpha (S\rtimes_\theta T)$ give a $*$-homomorphism

$$\begin{array}{rcl} A \rtimes_\alpha (S \rtimes_\theta T) & \xrightarrow{\pi} & (A \rtimes_{\alpha|_S} S) \rtimes_{\tilde{\alpha}} T \\ \iota_{A,S\rtimes_\theta T}(a) & \mapsto & \iota_{A\rtimes S} \circ \iota_{A,S}(a) \\ \iota_{S\rtimes_\theta T}(s,t) & \mapsto & (\iota_{A\rtimes S} \circ \iota_S)(s)\iota_T(t) \end{array}$$

Since S and T both have an identity, the induced map equals π. Note that the pair from 2) is the natural candidate to provide an inverse for π. Indeed, if 2) is valid, then the two induced $*$-homomorphisms are mutually inverse on the standard generators of the C*-algebras on both sides. Thus it remains to establish 1) and 2).

For step 1), note that $\iota_{A\rtimes S} \circ \iota_{A,S}$ is a unital $*$-homomorphism and $\iota_{A\rtimes S} \circ \iota_S$ defines a semigroup homomorphism from S to the isometries in $(A \rtimes_{\alpha|_S} S) \rtimes_{\tilde{\alpha}} T$. The covariance condition for $(T, \tilde{\alpha})$ yields

$$\iota_T(t)\iota_{A\rtimes S} \circ \iota_S(s) = \tilde{\alpha}(\iota_{A\rtimes S} \circ \iota_S(s))\iota_T(t) = \iota_{A\rtimes S} \circ \iota_S(\theta_t(s))\iota_T(t).$$

Therefore, $(\iota_{A\rtimes S} \circ \iota_S) \times \iota_T$ is well-behaved with respect to the semidirect product structure on $S \times T$ coming from θ, so we get a semigroup homomorphism

$$(\iota_{A\rtimes S} \circ \iota_S) \times \iota_T : S\rtimes_\theta T \longrightarrow \mathrm{Isom}((A \rtimes_{\alpha|_S} S) \rtimes_{\tilde{\alpha}} T).$$

Now let $a \in A, s \in S$ and $t \in T$. Then we compute

$$\begin{aligned} &((\iota_{A\rtimes S} \circ \iota_S) \times \iota_T)(s,t)\iota_{A\rtimes S} \circ \iota_{A,S}(a)((\iota_{A\rtimes S} \circ \iota_S) \times \iota_T)(s,t)^* \\ &= \iota_{A\rtimes S} \circ \iota_S(s)\iota_T(t)\iota_{A\rtimes S} \circ \iota_{A,S}(a)\iota_T(t)^*\iota_{A\rtimes S} \circ \iota_S(s)^* \\ &= \iota_{A\rtimes S} \circ \iota_S(s)\iota_{A\rtimes S} \circ \iota_{A,S}(\alpha_{(1_S,t)}(a))\iota_{A\rtimes S} \circ \iota_S(s)^* \\ &= \iota_{A\rtimes S} \circ \iota_{A,S}(\alpha_{(s,1_T)(1_S,t)}(a)) \\ &= \iota_{A\rtimes S} \circ \iota_{A,S}(\alpha_{(s,t)}(a)), \end{aligned}$$

which completes 1). For part 2), we remark that $(\iota_{A,S\rtimes_\theta T}, \iota_{S\rtimes_\theta T}|_S)$ is a covariant pair for $(A, S, \alpha|_S)$. Since $\iota_{A,S\rtimes_\theta T}$ and $\iota_{A,S}$ are unital, the induced map is unital as well. Moreover, $\iota_{S\rtimes_\theta T}|_T$ is a semigroup homomorphism mapping T to the isometries in $A\rtimes_\alpha(S\rtimes_\theta T)$. Thus, we are left with the covariance condition. Note that it suffices to check the covariance condition on the standard generators of $A \rtimes_{\alpha|_S} S$. For $a \in A, s \in S$ and $t \in T$, we get

$$\begin{aligned}
&\iota_{S\rtimes_\theta T}(1_S,t)\iota_{A,S\rtimes_\theta T}(a)\iota_{S\rtimes_\theta T}(s,1_T)\iota_{S\rtimes_\theta T}(1_S,t)^* \\
&\quad = \iota_{S\rtimes_\theta T}(1_S,t)\iota_{A,S\rtimes_\theta T}(a)\iota_{S\rtimes_\theta T}(1_S,t)^*\iota_{S\rtimes_\theta T}(1_S,t)\iota_{S\rtimes_\theta T}(s,1_T)\iota_{S\rtimes_\theta T}(1_S,t)^* \\
&\quad = \iota_{A,S\rtimes_\theta T}(\alpha_{(1_S,t)}(a))\iota_{S\rtimes_\theta T}(\theta_t(s),1_T)p_t \\
&\quad = \iota_{A,S\rtimes_\theta T}(\alpha_{(1_S,t)}(a))\iota_{S\rtimes_\theta T}(\theta_t(s),1_T) \\
&\tilde{\alpha}_t(\iota_{A,S\rtimes_\theta T}(a)\iota_{S\rtimes_\theta T}(s,1_T)).
\end{aligned}$$

Hence 1) and 2) are both valid, so the proof is complete. □

We close this preparatory section with a remark on the condition $p_{(\theta_t(s),t)} = p_t$.

***Remark* 2.1.6.**

a) The previous observations should carry over to the setting where A is non-unital, representations are non-degenerate and α is required to be extendible, see [Lar10] for more information on the conditions.

b) The condition $p_{(\theta_t(s),t)} = p_t$ for all $s \in S$ and $t \in T$ is satisfied if $\alpha|_S$ is unital. This follows from
$$\alpha_{(\theta_t(s),t)}(1_A) = \alpha_{(1_S,t)}(\alpha_{(s,1_T)}(1_A)) = \alpha_{(1_S,t)}(1_A).$$
In particular, this is the case whenever S is a group. If $\alpha|_T$ consists of injective endomorphisms, then $p_{(\theta_t(s),t)} = p_t$ holds if and only if $\alpha|_S$ is unital.

2.2 Fundamental results for irreversible algebraic dynamical systems

In this section, we associate a universal C*-algebra $\mathcal{O}[G,P,\theta]$ to every irreversible algebraic dynamical system (G,P,θ). The general approach is inspired by the methods of [CV13] for the case of a single group endomorphism with finite cokernel of a discrete abelian group. Partly, these ideas can even be traced back to [Cun77]. Note however, that we are going to use a different spanning family than the one used in [CV13].

We will examine structural properties of $\mathcal{O}[G,P,\theta]$ as well as of two nested subalgebras: the core $\mathcal{F}$ and the diagonal $\mathcal{D}$. In Lemma 2.2.9, a description of the spectrum G_θ of the diagonal $\mathcal{D}$ is provided, which allows us to regard G_θ as a completion of G with respect to θ in the case where (G,P,θ) is minimal, compare [CV13, Lemma 2.4].

Based on the description of G_θ, the action $\hat{\tau}$ of G on G_θ coming from $\tau_g(e_{h,p}) = e_{gh,p}$ is shown to be always minimal. Moreover, we prove that topological freeness of $\hat{\tau}$ corresponds to minimality of (G,P,θ), see Proposition 2.2.13. As an immediate consequence we deduce that $\mathcal{D} \rtimes_\tau G$ is simple if and only if (G,P,θ) is minimal and $\hat{\tau}$ is amenable, see Corollary 2.2.14. This crossed product is actually isomorphic to $\mathcal{F}$, see Corollary 2.2.19.

We remark that our strategy of proof differs from the one of [CV13] because we start by establishing an isomorphism between $\mathcal{O}[G,P,\theta]$ and $\mathcal{D} \rtimes (G \rtimes_\theta P)$, compare Proposition 2.2.18 and [CV13, Lemma 2.5 and Theorem 2.6]. By Theorem 2.1.5, we deduce that $\mathcal{O}[G,P,\theta]$ is isomorphic to the semigroup crossed product $\mathcal{F} \rtimes P$. So we get

$$\mathcal{O}[G,P,\theta] \cong \mathcal{D} \rtimes (G \rtimes_\theta P) \cong \mathcal{F} \rtimes P.$$

One advantage of this strategy is that we are able to establish these isomorphisms in greater generality, i.e. without minimality of (G,P,θ) and amenability of $\hat{\tau}$ which would give simplicity of both $\mathcal{F}$ and $\mathcal{O}[G,P,\theta]$.

Similar to [CV13], we then conclude that, whenever (G,P,θ) is minimal and the action $G \overset{\hat{\tau}}{\curvearrowright} G_\theta$ is amenable, the C*-algebra $\mathcal{O}[G,P,\theta]$ is a unital UCT Kirchberg algebra, see Theorem 2.2.26 and Corollary 2.2.28. Thus $\mathcal{O}[G,P,\theta]$ is classified by its K-theory in this case due to the important classification results of Christopher Phillips and Eberhard Kirchberg, see [Kir].

There is more to be said about the structure of $\mathcal{F}$ and $\mathcal{D}$ in the case of (commutative) irreversible algebraic dynamical systems of finite type, which forms the major part of Section 2.3.

Throughout this section, (G,P,θ) will represent an irreversible algebraic dynamical system unless specified otherwise. Let $(\xi_g)_{g\in G}$ denote the canonical orthonormal basis of $\ell^2(G)$. For $g \in G$ and $p \in P$, define operators U_g and S_p on $\ell^2(G)$ by $U_g(\xi_{g'}) := \xi_{gg'}$ and $S_p(\xi_{g'}) := \xi_{\theta_p(g')}$ for $g' \in G$. Then $(U_g)_{g\in G}$ is a unitary representation of the group G and $S_p^*(\xi_{g'}) = \chi_{\theta_p(G)}(g')\xi_{\theta_p^{-1}(g')}$ for $g' \in G$, so $(S_p)_{p\in P}$ is a representation of the semigroup P by isometries. Furthermore, these operators satisfy

$$\begin{array}{llcll} \text{(CNP 1)} & S_pU_g(\xi_{g'}) & = & \xi_{\theta_p(gg')} = U_{\theta_p(g)}S_p(\xi_{g'}), & \\ & & \text{and} & & \\ \text{(CNP 3)} & \sum\limits_{[g]\in G/\theta_p(G)} E_{g,p}(\xi_{g'}) & = & \xi_{g'} & \text{if } [G:\theta_p(G)] < \infty, \end{array}$$

where $E_{g,p} = U_gS_pS_p^*U_g^*$. In fact, (CNP 3) holds also in the case of an infinite index $[G:\theta_p(G)]$, as $(\sum_{[g]\in F} E_{g,p})_{F\subset G/\theta_p(G)}$ converges to the identity on $\ell^2(G)$ as $F \nearrow G/\theta_p(G)$ with respect to the strong operator topology. But this convergence does not hold in norm because each $E_{g,p}$ is a non-zero projection. In view of our motivation to construct a universal C*-algebra based on this model, it is therefore reasonable to restrict this relation to the case where $[G:\theta_p(G)]$ is finite.

As the numbering indicates, we are interested in an additional relation (CNP 2) which will increase the accessibility of the universal model: If G was trivial, this would simply be the condition that S_p and S_q doubly commute for all relatively prime p and q in P, i.e. $S_p^*S_q = S_qS_p^*$. This condition has been employed successfully for quasi-lattice ordered groups, see [Nic92, Section 3] and also [LR96] for more information. But as G is an infinite group, this will not be sufficient.

Moreover, we want to ensure that, within the universal model to be built, an expression corresponding to $S_p^*U_gS_p$ belongs to $C^*(G)$. This property has been used extensively in the context of semigroup crossed products involving transfer operators, see [Exe03a, Lar10].

An entirely different way to put it is that we aim for a better understanding of the structure of the commutative subalgebra $C^*(\{ E_{g,p} \mid g \in G,\ p \in P \})$ inside $\mathcal{L}(\ell^2(G))$. In a much more general framework, this has been considered by Xin Li, see [Li12] and resulted in a new definition of semigroup C*-algebras for discrete left cancellative semigroups with identity. One particular strength of his notion is the close connection between amenability of semigroups and nuclearity of their C*-algebras, see [Li13].

All of these three instances suggest that a closer examination of the terms $S_p^*U_gS_q$ is in order. For $g = \theta_p(g_1)\theta_q(g_2)$ with $g_1, g_2 \in G$, we get $S_p^*U_gS_q = U_{g_1}S_{(p\wedge q)^{-1}q}S^*_{(p\wedge q)^{-1}p}U_{g_2}$. On the other hand, $g \notin \theta_p(G)\theta_q(G)$ is equivalent to $g\theta_q(G) \cap \theta_p(G) = \emptyset$, which forces $S_p^*U_gS_q = 0$. Thus we get

$$\text{(CNP 2)}\quad S_p^*U_gS_q = \begin{cases} U_{g_1}S_{(p\wedge q)^{-1}q}S^*_{(p\wedge q)^{-1}p}U_{g_2} & \text{if } g = \theta_p(g_1)\theta_q(g_2), \\ 0 & \text{else.} \end{cases}$$

for all $g \in G, p, q \in P$. These observations motivate the following definition:

Definition 2.2.1. $\mathcal{O}[G, P, \theta]$ is the universal C*-algebra generated by a unitary representation $(u_g)_{g\in G}$ of the group G and a representation $(s_p)_{p\in P}$ of the semigroup P by isometries subject to the relations:

$$\text{(CNP 1)}\qquad s_pu_g = u_{\theta_p(g)}s_p$$

$$\text{(CNP 2)}\quad s_p^*u_gs_q = \begin{cases} u_{g_1}s_{(p\wedge q)^{-1}q}s^*_{(p\wedge q)^{-1}p}u_{g_2} & \text{if } g = \theta_p(g_1)\theta_q(g_2), \\ 0, & \text{else.} \end{cases}$$

$$\text{(CNP 3)}\qquad 1 = \sum_{[g]\in G/\theta_p(G)} e_{g,p} \quad \text{if } [G : \theta_p(G)] < \infty,$$

where $e_{g,p} = u_gs_ps_p^*u_g^*$.

Proposition 2.2.2. *Then $\mathcal{O}[G, P, \theta]$ has a canonical non-trivial representation on $\ell^2(G)$ given by $u_g \mapsto U_g,\ s_p \mapsto S_p$. In particular, $\mathcal{O}[G, P, \theta]$ is non-zero.*

***Remark* 2.2.3.**

a) The presence of (CNP 1) guarantees that the expression in (CNP 2) is independent of the choice of g_1 and g_2 satisfying $g = \theta_p(g_1)\theta_q(g_2)$. To see this, suppose g_3 and g_4 satisfy $g = \theta_p(g_3)\theta_q(g_4)$ as well. Since G is a group, $\theta_p(g_1^{-1}g_3) = \theta_q(g_2g_4^{-1})$ follows. This is equivalent to $\theta_{(p\wedge q)^{-1}p}(g_1^{-1}g_3) = \theta_{(p\wedge q)^{-1}q}(g_2g_4^{-1})$ by injectivity of $\theta_{p\wedge q}$.

   As $(p\wedge q)^{-1}p$ and $(p\wedge q)^{-1}q$ are relatively prime, condition (C) from Definition 1.1.5 implies $g_1^{-1}g_3 \in \theta_{(p\wedge q)^{-1}q}(G)$ and $g_2g_4^{-1} \in \theta_{(p\wedge q)^{-1}p}(G)$. Applying injectivity of $\theta_{p\vee q}$ to $\theta_{(p\wedge q)^{-1}p}(g_1^{-1}g_3)\theta_{(p\wedge q)^{-1}q}(g_4g_2^{-1}) = 1_G$ yields

$$\theta^{-1}_{(p\wedge q)^{-1}q}(g_1^{-1}g_3)\theta^{-1}_{(p\wedge q)^{-1}p}(g_2g_4^{-1}) = 1_G.$$

   Therefore we conclude

$$\begin{aligned} u_{g_3}s_{(p\wedge q)^{-1}q}s^*_{(p\wedge q)^{-1}p}u_{g_4} &= u_{g_1}u_{g_1^{-1}g_3}s_{(p\wedge q)^{-1}q}s^*_{(p\wedge q)^{-1}p}u_{g_4g_2^{-1}}u_{g_2} \\ &= u_{g_1}s_{(p\wedge q)^{-1}q}u_{\theta_q^{-1}(g_1^{-1}g_3)\theta_p^{-1}(g_4g_2^{-1})}s^*_{(p\wedge q)^{-1}p}u_{g_2} \\ &= u_{g_1}s_{(p\wedge q)^{-1}q}s^*_{(p\wedge q)^{-1}p}u_{g_2}. \end{aligned}$$

b) For $p \in P$ and $g_1, g_2 \in G$ such that $g_1\theta_p(G) = g_2\theta_p(G)$, (CNP 1) implies

$$e_{g_2,p} = u_{g_1}u_{g_1^{-1}g_2}s_ps_p^*u_{g_2}^* = u_{g_1}s_ps_p^*u_{g_1^{-1}g_2}u_{g_2}^* = e_{g_1,p}$$

   Thus the summation in (CNP 3) makes sense.

c) Condition (CNP 2) includes the following two relations as special cases:

$$\begin{array}{lcll} s_p^*s_q & = & s_qs_p^* & \text{for all relatively prime } p,q \in P. \\ s_p^*u_gs_p & = & \chi_{\theta_p(G)}(g)u_{\theta_p^{-1}(g)} & \text{for all } g \in G, p \in P. \end{array}$$

Lemma 2.2.4. *The linear span of $(u_gs_ps_q^*u_h)_{g,h\in G,p,q\in P}$ is dense in $\mathcal{O}[G,P,\theta]$.*

Proof. The family $(u_gs_ps_q^*u_h)_{g,h\in G,p,q\in P}$ includes the generators $(u_g)_{g\in G}, (s_p)_{p\in P}$ and is closed under involution. We claim that the family is also multiplicatively closed (unless the product is zero). Due to (CNP 2), an expression $s_q^*u_hs_p$ either vanishes or takes the form $u_{h_1}s_{p_1}s_{p_2}^*u_{h_2}$ for some $h_i \in G, p_i \in P$. In view of (CNP 1), this yields the claim. ☐

Lemma 2.2.5. *The projections $(e_{g,p})_{g\in G,p\in P}$ commute. More precisely, for $g,h \in G$ and $p,q \in P$, we have*

$$e_{g,p}e_{h,q} = \begin{cases} e_{g\theta_p(h'),p\vee q} & \text{if } g^{-1}h \in \theta_p(G)\theta_q(G), \\ 0 & \text{else,} \end{cases}$$

where $h' \in G$ is determined uniquely up to multiplication from the right by elements of $\theta_{p^{-1}(p\vee q)}(G)$ by the condition that $g\theta_p(h') \in h\theta_q(G)$.

Proof. For $g,h \in G$ and $p,q \in P$, the product $e_{g,p}e_{h,q}$ is non-zero only if $g^{-1}h \in \theta_p(G)\theta_q(G)$ by (CNP 2). So let us assume that $g^{-1}h \in \theta_p(G)\theta_q(G)$ holds. Then there are $g',h' \in G$ such that $g^{-1}h = \theta_p(h')\theta_q(g')$. As G is a group, this is equivalent to $h\theta_q(g')^{-1} = g\theta_p(h')$. Thus we get

$$e_{g,p}e_{h,q} = u_{g\theta_p(h')}s_p s_{(p\wedge q)^{-1}q}s^*_{(p\wedge q)^{-1}p}s^*_q u^*_{h\theta_q(g')^{-1}} = e_{g\theta_p(h'),p\vee q}.$$

Clearly, this also proves that the two projections commute. The uniqueness assertion follows from (CNP 2). □

Definition 2.2.6. The C*-subalgebra $\mathcal{D}$ of $\mathcal{O}[G,P,\theta]$ generated by the commuting projections $(e_{g,p})_{g\in G,p\in P}$ is called the **diagonal**. In addition, let $\mathcal{D}_p := C^*(\{e_{g,q} \mid [g] \in G/\theta_p(G), p \in qP\})$ denote the C*-subalgebra of $\mathcal{D}$ corresponding to $p \in P$.

Lemma 2.2.7. *For all $p,q \in P$, $p \in qP$ implies $\mathcal{D}_q \subset \mathcal{D}_p$. $\mathcal{D}$ is the closure of $\bigcup_{p\in P}\mathcal{D}_p$. If $[G:\theta_p(G)]$ is finite, then*

$$\mathcal{D}_p = \operatorname{span}\{e_{g,p} \mid [g] \in G/\theta_p(G)\} \cong \mathbb{C}^{[G:\theta_p(G)]}.$$

Proof. The first assertion follows from the definition of $\mathcal{D}_p$. Lemma 2.2.5 implies that $\mathcal{D}$ is the closure of the span of $(e_{g,q})_{g\in G,q\in P}$. Likewise, $\mathcal{D}_p$ is the closure of the span of the projections $(e_{g,q})_{g\in G,p\in qP}$. This establishes the second claim. Finally, suppose $[G:\theta_p(G)]$ is finite and let $g \in G, q \in P$ such that there is $r \in P$ satisfying $p = qr$. Note that $[G:\theta_r(G)]$ is finite since $[G:\theta_p(G)]$ is finite. Then (CNP 3) gives

$$e_{g,q} = u_g s_q \left(\sum_{[g']\in G/\theta_r(G)} e_{g',r}\right) s_q u_g^* = \sum_{[g']\in G/\theta_r(G)} e_{g\theta_q(g'),p}.$$

□

Let us make the following non-trivial observation:

Lemma 2.2.8. *Suppose $(g,p) \in G\times P$ and a finite subset F of $G\times P$ are chosen in such a way that $e_{g,p}\prod_{(h,q)\in F}(1-e_{h,q})$ is non-zero. Then there exists $(g',p') \in G\times P$ satisfying $e_{g',p'} \leq e_{g,p}\prod_{(h,q)\in F}(1-e_{h,q})$.*

Proof. If F is empty, then $\prod_{(h,q)\in F}(1-e_{h,q}) = 1$ by convention, so there is nothing to show. Now let F be non-empty. For $(h,q) \in F$, let us decompose q uniquely as $q = q^{(fin)}q^{(inf)}$, where $[G:\theta_{q^{(fin)}}(G)]$ is finite and we require that, for each $r \in P$ with $q \in rP$, finiteness of $[G:\theta_r(G)]$ implies $q^{(fin)} \in rP$. In other words, $[G:\theta_r(G)]$ is infinite for every $r \neq 1_P$ with $q^{(inf)} \in rP$. Using (CNP 3) for $q^{(fin)}$ and Lemma 2.2.5, we compute

$$\begin{aligned} 1-e_{h,q} &= (1-e_{h,q^{(fin)}}e_{h,q^{(inf)}})\sum_{[k]\in G/\theta_{q^{(fin)}}(G)} e_{k,p^{(fin)}} \\ &= e_{h,q^{(fin)}}(1-e_{h,q^{(inf)}}) + \sum_{\substack{[k]\in G/\theta_{q^{(fin)}}(G)\\ [k]\neq[h]}} e_{k,q^{(fin)}}. \end{aligned}$$

Therefore, we can rewrite the initial product as

$$e_{g,p} \prod_{(h,q)\in F} (1 - e_{h_i,q_i}) = \sum_{(\tilde{g},\tilde{p})\in\tilde{F}} e_{\tilde{g},\tilde{p}} \prod_{(h,q)\in F_{(\tilde{g},\tilde{p})}} (1 - e_{h,q}),$$

where

- $\tilde{F}$ is a finite subset of $G{\times}P$,
- $e_{\tilde{g},\tilde{p}} \leq e_{g,p}$ for all $(\tilde{g},\tilde{p}) \in \tilde{F}$,
- the projections $(e_{\tilde{g},\tilde{p}})_{(\tilde{g},\tilde{p})\in\tilde{F}}$ are mutually orthogonal,
- for each $(\tilde{g},\tilde{p}) \in \tilde{F}$, $F_{(\tilde{g},\tilde{p})}$ is a finite subset of $G{\times}P$, and
- each $(h,q) \in F_{(\tilde{g},\tilde{p})}$ satisfies $q = q^{(inf)}$ and $\tilde{p} \notin qP$.

Since the product $e_{g,p}\prod_{(h,q)\in F}(1-e_{h_i,q_i})$ on the left hand side is non-zero, there is $(g_0,p_0) \in \tilde{F}$ such that $e_{g_0,p_0}\prod_{(h,q)\in F_{(g_0,p_0)}}(1-e_{h,q})$ is non-zero. Without loss of generality, we may assume that $e_{g_0,p_0}e_{h,q}$ is non-zero for all $(h,q) \in F_{(g_0,p_0)}$. Consider $F_P := \{p_0 \vee q \mid (h,q) \in F_{(g_0,p_0)}$ for some $h \in G\}$. Pick $p_1 \in F_P$ which is minimal in the sense that for any other $r \in F_P$, $p_1 \in rP$ implies $r = p_1$. Let $(h_1,q_1),\dots,(h_n,q_n) \in F_{(g_0,p_0)}$ denote the elements satisfying $p_0 \vee q_i = p_1$. According to Lemma 2.2.5, we have

$$e_{g_0,p_0}e_{h_i,q_i} = e_{g_0\theta_{p_0}(g_i'),p_1} \text{ for a suitable } g_i' \in G \text{ (for } i = 1,\dots,n).$$

Note that $p_0^{-1}p_1 \neq 1_P$ and $q_1 = q_1^{(inf)} \in p_0^{-1}p_1P$, so $[G : \theta_{p_0^{-1}p_1}(G)]$ is infinite. Hence there exists $g_1 \in g_0\theta_{p_0}(G)$ with

$$e_{g_1,p_1} \leq e_{g_0,p_0} \text{ and } e_{g_1,p_1}e_{h_i,q_i} = 0 \text{ for } i = 1,\dots,n.$$

Setting

$$F_{(g_1,p_1)} := \{(h,q) \in F_{(g_0,p_0)} \mid e_{h,q}e_{g_1,p_1} \neq 0\} \subsetneqq F_{(g_0,p_0)},$$

we observe that

$$e_{g_1,p_1} \prod_{(h,q)\in F_{(g_1,p_1)}} (1 - e_{h,q}) \neq 0$$

follows from the initial statement for (g_0,p_0) and $F_{(g_0,p_0)}$ since we have chosen p_1 in a minimal way. Indeed, if the product was trivial, then there would be $(h,q) \in F_{(g_1,p_1)}$ with $e_{h,q} \geq e_{g_1,p_1}$. By Lemma 2.2.5, this would force $p_1 \in qP$ and therefore $p_1 \in (p_1 \vee q)P \subset (p_0 \vee q)P$, which cannot be true since p_1 was chosen in a minimal way.

Thus, we can iterate the process used to obtain (g_1,p_1) and $F_{(g_1,p_1)}$ for (g_0,p_0) and $F_{(g_0,p_0)}$. After finitely many steps, we arrive at an element $(g_n,p_n) =: (g',p')$ with the property that $e_{g',p'} \leq e_{g_0,p_0}$ is orthogonal to $e_{h,q}$ for all $(h,q) \in F_{(g_0,p_0)}$. This establishes the claim. $\square$

The opportunity to pass to smaller subprojections provided through Lemma 2.2.8 will be crucial for the proof of pure infiniteness and simplicity of $\mathcal{O}[G,P,\theta]$, see Theorem 2.2.26 and in particular Lemma 2.2.25. A first application of this observation lies in the determination of the spectrum of $\mathcal{D}$:

Lemma 2.2.9. *The spectrum of $\mathcal{D}$, denoted by G_θ, is a totally disconnected, compact Hausdorff space. A basis for the topology on G_θ is given by the cylinder sets*

$$Z_{(g,p),(h_1,q_1),\ldots,(h_n,q_n)} = \{\chi \in G_\theta \mid \chi(e_{g,p}) = 1,\ \chi(e_{h_i,q_i}) = 0 \text{ for all } i\},$$

where $n \in \mathbb{N}, g, h_1, \ldots, h_n \in G$ and $p, q_1, \ldots, q_n \in P$. Moreover,

$$\iota(g) \in Z_{(g',p),(h_1,q_1),\ldots,(h_n,q_n)} \Longleftrightarrow g \in g'\theta_p(G) \text{ and } g \notin h_i\theta_{q_i}(G) \text{ for all } i$$

defines a map $\iota : G \longrightarrow G_\theta$ with dense image. ι is injective if and only if (G,P,θ) is minimal.

Proof. G_θ is a totally disconnected, compact Hausdorff space since $\mathcal{D}$ is a unital C*-algebra generated by commuting projections. The statement concerning the basis for the topology on G_θ follows from Lemma 2.2.7. To see that ι has dense image, let $\chi \in G_\theta$. As the cylinder sets form a basis for the topology of G_θ, every open neighbourhood of χ contains a cylinder set $Z_{(g,p),(h_1,q_1),\ldots,(h_n,q_n)}$ with $\chi \in Z_{(g,p),(h_1,q_1),\ldots,(h_n,q_n)}$. This means that $e_{g,p}\prod_{i=1}^n(1-e_{h_i,q_i})$ is non-zero. Hence we can apply Lemma 2.2.8 to obtain $(g',p') \in G\times P$ satisfying $e_{g',p'} \leq e_{g,p}\prod_{i=1}^n(1-e_{h_i,q_i})$. In other words, $\iota(g') \in Z_{(g,p),(h_1,q_1),\ldots,(h_n,q_n)}$, so $\iota(G)$ is a dense subset of G_θ.

Now given $g,h \in G$, we observe that $\iota(g) = \iota(h)$ is equivalent to $g^{-1}h \in \bigcap_{p\in P}\theta_p(G)$ because the cylinder sets form a basis of the topology on the Hausdorff space G_θ. Therefore ι is injective precisely if (G,P,θ) is minimal. $\square$

***Remark* 2.2.10.** By the preceding lemma, G_θ is a completion of G with respect to θ whenever (G,P,θ) is minimal.

There is a canonical action τ of G on $\mathcal{D}$ given by $\tau_g(e_{h,p}) = e_{gh,p}$ for $g,h \in G$ and $p \in P$. Known results, as for instance [CV13, Lemma 2.5], indicate that $\mathcal{D} \rtimes_\tau G$ ought to be simple provided that the irreversible algebraic dynamical system (G,P,θ) is minimal. Of course, this can only be true if $G \overset{\tau}{\curvearrowright} \mathcal{D}$ is regular, that is, $\mathcal{D} \rtimes_\tau G \cong \mathcal{D} \rtimes_{\tau,r} G$ via the canonical map. Building on the results of [AD87], this can be rephrased as amenability of the action $G \overset{\hat{\tau}}{\curvearrowright} G_\theta$, see also [BO08, Theorem 4.4.3] for a concise exposition. Moreover, the map ι from Lemma 2.2.9 satisfies $\hat{\tau}_g(\iota(h)) = \iota(gh)$ for all $g,h \in G$.

If $\hat{\tau}$ is amenable, the celebrated result of [AS94] states that the crossed product $C(G_\theta) \rtimes_\tau G$ is simple if and only if the action $G \overset{\hat{\tau}}{\curvearrowright} G_\theta$ is minimal and topologically free. As it turns out, minimality of (G,P,θ) corresponds precisely to these two properties. For convenience, let us recall the standard definitions of topological freeness and minimality for group actions.

Definition 2.2.11. Let X be a topological space and G a group. A group action $G \curvearrowright X$ is said to be **topologically free**, if the set $X^g = \{x \in X \mid g.x = x\}$ has empty interior for $g \in G \setminus \{1_G\}$.

Definition 2.2.12. Let X be a topological space and G a group. A group action $G \curvearrowright X$ is said to be **minimal**, if the orbit $\mathcal{O}(x) = \{g.x \mid g \in G\}$ is dense in X for every $x \in X$.

Equivalently, an action is minimal if the only invariant open (closed) subsets of X are $\emptyset$ and X.

Proposition 2.2.13. *The action $G \overset{\hat{\tau}}{\curvearrowright} G_\theta$ is minimal. It is topologically free if and only if (G,P,θ) is minimal.*

Proof. On $\iota(G)$, which is dense in G_θ by Lemma 2.2.9, $\hat{\tau}$ is simply given by translation from the left. Hence $\hat{\tau}$ is minimal. For the second part, we note that $\tau_g = \mathrm{id}_{\mathcal{D}}$ holds for every $g \in \bigcap_{p \in P} \theta_p(G)$. Thus, if (G,P,θ) is not minimal, there is $g \neq 1_G$ such that $G_\theta^g = G_\theta$, so $\hat{\tau}$ is not topologically free. If (G,P,θ) is minimal, then $\hat{\tau}$ acts freely on $\iota(G)$ because ι is injective and G is left-cancellative. Since $\iota(G)$ is dense in G_θ, we conclude that $\hat{\tau}$ is topologically free. □

Corollary 2.2.14. *The crossed product $\mathcal{D} \rtimes_\tau G$ is simple if and only if (G,P,θ) is minimal and $G \overset{\hat{\tau}}{\curvearrowright} G_\theta$ is amenable.*

Proof. Due to a central result from [AD87], amenability of the action is equivalent to regularity of the crossed product. Hence Proposition 2.2.13 and [AS94, Corollary following Theorem 2] establish the claim. □

Definition 2.2.15. The **core** $\mathcal{F}$ is the C*-subalgebra of $\mathcal{O}[G,P,\theta]$ generated by $\mathcal{D}$ and $(u_g)_{g \in G}$.

Lemma 2.2.16. *The linear span of $(u_g s_p s_p^* u_h^*)_{g,h \in G, p \in P}$ is dense in $\mathcal{F}$.*

Proof. This follows immediately from the calculations for Lemma 2.2.4. □

***Remark* 2.2.17.** For every irreversible algebraic dynamical system (G,P,θ), P is a discrete abelian Ore semigroup. Therefore its enveloping group $P^{-1}P$ is discrete abelian. Let us denote the dual group of $P^{-1}P$ by L, which is a compact abelian group by Pontryagin duality. Furthermore, L acts on $\mathcal{O}[G,P,\theta]$ via the so-called gauge-action γ given by

$$\gamma_\ell(u_g) = u_g \text{ and } \gamma_\ell(s_p) = \ell(p)s_p, \text{ for } g \in G, p \in P \text{ and } \ell \in L.$$

a) The fixed-point algebra of γ coincides with $\mathcal{F}$.

b) If μ denotes the normalized Haar measure on L, then

$$E_1(u_g s_p s_q^* u_h^*) := \int\limits_{\ell \in L} \gamma_\ell(u_g s_p s_q^* u_h^*) d\mu(\ell) = \delta_{pq} u_g s_p s_p^* u_h^*$$

defines a faithful conditional expectation $\mathcal{O}[G,P,\theta] \xrightarrow{E_1} \mathcal{F}$ as γ is strongly continuous.

The similarity between $\mathcal{F}$ and $\mathcal{D} \rtimes_\tau G$ is apparent. If one assumes that $\mathcal{D} \rtimes_\tau G$ is simple, which by Corollary 2.2.14 means that the irreversible algebraic dynamical system (G,P,θ) is minimal, it is easy to show that these two algebras are isomorphic. This strategy has been pursued in [CV13, Lemma 2.5].

However, we will show in Corollary 2.2.19 that this identification holds in full generality. To do so, we will first derive a semigroup crossed product description $\mathcal{O}[G,P,\theta] \cong \mathcal{D} \rtimes (G \rtimes_\theta P)$, which is of independent interest, compare [CV13, Theorem 2.6]. Also, if (G,P,θ) is of infinite type, that is, $[G : \theta_p(G)]$ is infinite for all $p \neq 1_P$, then this result reproduces the standard picture $C^*(S) \cong \mathcal{D}_S \rtimes S$ for C*-algebras of left cancellative semigroups S in the case where $S = G \rtimes_\theta P$, compare [Li12, Lemma 2.14].

In order to get down to $\mathcal{F}$ and $\mathcal{D} \rtimes_\tau G$, respectively, we observe that a crossed product coming from a semidirect product of discrete semigroups can be displayed as an iterated semigroup crossed product under a certain condition, see Theorem 2.1.5. This condition will be satisfied as G is a group, see Remark 2.1.6 b).

Proposition 2.2.18. *Let the semidirect product $G \rtimes_\theta P$ act on $\mathcal{D}$ by $(g,p).e_{h,q} = e_{g\theta_p(h),pq}$ and suppose $(v_{(g,p)})_{(g,p) \in G \rtimes_\theta P}$ is the family of isometries in $\mathcal{D} \rtimes (G \rtimes_\theta P)$ implementing the action of the semigroup, that is, $v_{(g,p)} e_{h,q} v_{(g,p)}^* = e_{g\theta_p(h),pq}$. Then the map*

$$\begin{array}{ccc} \mathcal{O}[G,P,\theta] & \xrightarrow{\varphi} & \mathcal{D} \rtimes (G \rtimes_\theta P) \\ u_g s_p & \mapsto & v_{(g,p)} \end{array}$$

is an isomorphism.

Proof. Recall from Definition 2.2.1 that $\mathcal{O}[G,P,\theta]$ is the universal C*-algebra generated by a unitary representation $(u_g)_{g \in G}$ of the group G and a semigroup of isometries $(s_p)_{p \in P}$ subject to the relations (CNP 1)–(CNP 3). Hence, in order to show that φ defines a surjective $*$-homomorphism, it suffices to show that for every $g \in G$, the isometry $v_{(g,1_P)}$

is a unitary, and that the families $(v_{(g,1_P)})_{g\in G}, (v_{(1_G,p)})_{p\in P}$ satisfy (CNP 1)–(CNP 3):

$$v_{(g,1_P)}v_{(g^{-1},1_P)} = v_{(g,1_P)(g^{-1},1_P)} = v_{(1_G,1_P)} = 1$$

$$\text{(CNP 1)} \qquad v_{(1_G,p)}v_{(g,1_P)} = v_{(1_G,p)(g,1_P)} = v_{(\theta_p(g),p)} = v_{(\theta_p(g),1_P)}v_{(1_G,p)}$$

$$\text{(CNP 2)} \qquad v^*_{(1_G,p)}v_{(g,1_P)}v_{(1_G,q)} \overset{!}{=} \chi_{\theta_p(G)\theta_q(G)}(g)\ v_{(g_1,(p\wedge q)^{-1}q)}v^*_{(g_2^{-1},(p\wedge q)^{-1}p)}$$
$$\text{where } g = \theta_p(g_1)\theta_q(g_2).$$

$$\iff \qquad v_{(1_G,p)}v^*_{(1_G,p)}v_{(g,q)}v^*_{(g,q)} \overset{!}{=} \chi_{\theta_p(G)\theta_q(G)}(g)\ v_{(\theta_p(g_1),p\vee q)}v^*_{(g\theta_q(g_2^{-1}),p\vee q)}$$

$$\iff \qquad e_{1_G,p}e_{g,q} \overset{!}{=} \chi_{\theta_p(G)\theta_q(G)}(g)\ e_{(g\theta_q(g_2^{-1}),p\vee q)}$$

as $g = \theta_p(g_1)\theta_q(g_2)$ gives $\theta_p(g_1) = g\theta_q(g_2^{-1})$. This last equation holds by Lemma 2.2.5, so (CNP 2) is satisfied as well. (CNP 3) is a relation that is encoded inside $\mathcal{D}$, so it is satisfied as the range projection of the isometry $v_{(g,p)}$ coincides with $e_{g,p}$. To show that φ is injective, we note that the isometries $u_g s_p$ satisfy the covariance relation for the dynamical system $G \rtimes_\theta P \curvearrowright \mathcal{D}$:

$$u_g s_p e_{h,q}(u_g s_p)^* = e_{g\theta_p(h),pq} = (g,p).e_{h,q}.$$

Hence, there is a surjective $*$-homomorphism from $\mathcal{D}\rtimes(G\rtimes_\theta P)$ to $\mathcal{O}[G,P,\theta]$ sending $v_{(g,p)}$ to $u_g s_p$. Apparently, the two $*$-homomorphisms are mutually inverse, so φ is an isomorphism. □

This description allows us to deduce several relevant properties of $\mathcal{O}[G,P,\theta]$ and its core subalgebra $\mathcal{F}$.

Corollary 2.2.19. *The isomorphism φ from Proposition 2.2.18 restricts to an isomorphism $\mathcal{F} \xrightarrow{\varphi} \mathcal{D}\rtimes G$. In particular, we have a canonical isomorphism $\mathcal{O}[G,P,\theta] \cong \mathcal{F}\rtimes P$.*

Proof. The first claim follows immediately from Proposition 2.2.18 together with Theorem 2.1.5 and Remark 2.1.6. The second assertion is implied by Lemma 2.2.16. □

Proposition 2.2.20. *If $G \overset{\hat{\tau}}{\curvearrowright} G_\theta$ is amenable, then both $\mathcal{F}$ and $\mathcal{O}[G,P,\theta]$ are nuclear and satisfy the universal coefficient theorem (UCT).*

Proof. As $\mathcal{F} \cong \mathcal{D} \rtimes G$ by Corollary 2.2.19 and $G \overset{\hat{\tau}}{\curvearrowright} G_\theta$ is amenable, $\mathcal{F}$ is nuclear by results of Claire Anatharaman-Delaroche, see [AD87] or [BO08, Theorem 4.3.4]. Similarly, amenability of $G \overset{\hat{\tau}}{\curvearrowright} G_\theta$ passes to the corresponding transformation groupoid $\mathcal{G}$. Thus, we can rely on results of Jean-Louis Tu, see [Tu99], to deduce that $\mathcal{F} \cong \mathcal{D} \rtimes_\tau G \cong C^*(\mathcal{G})$ satisfies the UCT. The class of separable nuclear C*-algebras that satisfy the UCT is closed under crossed products by $\mathbb{N}$ and inductive limits. Recall that either $P \cong \mathbb{N}^k$ for some $k \in \mathbb{N}$ or $P \cong \bigoplus_{\mathbb{N}} \mathbb{N}$ according to condition (B) of Definition 1.1.5. Hence the claims concerning $\mathcal{O}[G,P,\theta]$ follow from $\mathcal{O}[G,P,\theta] \cong \mathcal{F} \rtimes P$, see Corollary 2.2.19. □

Corollary 2.2.21. *The map* $E_2(u_g s_p s_p^* u_h^*) := \delta_{gh}\ e_{g,p}$ *defines a conditional expectation* $\mathcal{F} \xrightarrow{E_2} \mathcal{D}$ *which is faithful if and only if* $G \overset{\hat{\tau}}{\curvearrowright} G_\theta$ *is amenable.*

Proof. Due to Corollary 2.2.19, $\mathcal{F}$ is canonically isomorphic to $\mathcal{D} \rtimes_\tau G$. Since G is discrete, the reduced crossed product $\mathcal{D} \rtimes_{\tau,r} G$ has a faithful conditional expectation given by evaluation at 1_G. The map E_2 is nothing but the composition of

$$\mathcal{F} \cong \mathcal{D} \rtimes_\tau G \twoheadrightarrow \mathcal{D} \rtimes_{\tau,r} G \xrightarrow{ev_{1_G}} \mathcal{D}.$$

By [AD87], the canonical surjection $\mathcal{D} \rtimes_\tau G \twoheadrightarrow \mathcal{D} \rtimes_{r,\tau} G$ is an isomorphism if and only if $G \overset{\hat{\tau}}{\curvearrowright} G_\theta$ is amenable. Thus the conditional expectation E_2 is faithful if and only if the action $\hat{\tau}$ is amenable. □

Corollary 2.2.22. *The map* $E(u_g s_p s_q^* u_h^*) := \delta_{pq}\delta_{gh} e_{g,p}$ *defines a conditional expectation* $\mathcal{O}[G,P,\theta] \xrightarrow{E} \mathcal{D}$ *which is faithful if and only if* $G \overset{\hat{\tau}}{\curvearrowright} G_\theta$ *is amenable.*

Proof. Clearly, $E = E_2 \circ E_1$, so the result follows from Remark 2.2.17 and Corollary 2.2.21. □

Note that if G happens to be amenable, the faithful conditional expectation E can be obtained directly by showing that the left Ore semigroup $G \rtimes_\theta P$ has an amenable enveloping group. Before we can turn to simplicity of $\mathcal{O}[G,P,\theta]$, we need the following general observations:

Definition 2.2.23. Given a family of commuting projections $(E_i)_{i \in I}$ in a unital C*-algebra B and finite subsets $A \subset F$ of I, let

$$Q_{F,A}^E := \prod_{i \in A} E_i \prod_{j \in F \setminus A} (1 - E_j).$$

Products indexed by $\emptyset$ are treated as 1 by convention.

Lemma 2.2.24. *Suppose* $(E_i)_{i \in I}$ *is a family of commuting projections in a unital C*-algebra* B*,* $A \subset F$ *are finite subsets of* I*. Then each* $Q_{F,A}^E$ *is a projection,* $\sum\limits_{A \subset F} Q_{F,A}^E = 1$ *and, for given coefficients* $\lambda_i \in \mathbb{C}, i \in F$*, we have*

$$\sum_{i \in F} \lambda_i E_i = \sum_{A \subset F} \Big(\sum_{i \in A} \lambda_i \Big) Q_{F,A}^E,$$

as well as

$$\Big\| \sum_{i \in F} \lambda_i E_i \Big\| = \max_{\substack{A \subset F \\ Q_{F,A}^E \neq 0}} \Big| \sum_{i \in A} \lambda_i \Big|.$$

Proof. Since the projections E_i commute, $Q^E_{F,A}$ is a projection. The second assertion is obtained via

$$1 = \prod_{i \in F} (E_i + 1 - E_i) = \sum_{A \subset F} Q^E_{F,A}.$$

The two equations from the claim follow immediately from this. □

Lemma 2.2.25. *For $d = \sum\limits_{1 \leq i \leq n} \lambda_i e_{g_i,p_i} \in \mathcal{D}_+$ with $\lambda_i \in \mathbb{C}$ and $(g_i, p_i) \in G \times P$, there exist $(g, p) \in G \times P$ satisfying $de_{g,p} = \|d\| e_{g,p}$.*

Proof. The element d belongs to $C^*\left(\left\{Q^e_{F,A} \mid A \subset F = \{(g_i, p_i) \mid 1 \leq i \leq n\}\right\}\right)$, which is commutative by Lemma 2.2.5. Then Lemma 2.2.24 says that there exists $A \subset F$ such that $Q^e_{F,A}$ is non-zero and $dQ^e_{F,A} = \|d\| Q^e_{F,A}$. In particular, $\prod_{(g,p) \in A} e_{g,p}$ is non-zero, so Lemma 2.2.5 implies that there is $(g_A, p_A) \in G \times P$ such that $\prod_{(g,p) \in A} e_{g,p} = e_{g_A,p_A}$. Thus, we can apply Lemma 2.2.8 to $e_{g_A,p_A} \prod_{(h,q) \in F \setminus A} (1 - e_{h,q}) = Q^e_{F,A} \neq 0$ and the proof is complete. □

We point out that the hard part of the proof for Lemma 2.2.25 is hidden in Lemma 2.2.8.

Theorem 2.2.26. *If (G, P, θ) is minimal and the action $G \overset{\hat{\tau}}{\curvearrowright} \mathcal{D}$ is amenable, then $\mathcal{O}[G, P, \theta]$ is purely infinite and simple.*

Proof. Recall that the linear span of $(u_g s_p s_q^* u_h^*)_{g,h \in G, p,q \in P}$ is dense in $\mathcal{O}[G, P, \theta]$ according to Lemma 2.2.4. Every element z from this linear span can be displayed as

$$z = \sum_{i=1}^{m_1} c_i e_{g_i,p_i} + \sum_{i=m_1+1}^{m_2} c_i u_{g_i} s_{p_i} s_{p_i}^* u_{h_i}^* + \sum_{i=m_2+1}^{m_3} c_i u_{g_i} s_{p_i} s_{q_i}^* u_{h_i}^*,$$

where $c_i \in \mathbb{C}$,

a) $g_i \neq h_i$ for $m_1 + 1 \leq i \leq m_2$, and

b) $p_i \neq q_i$ for $m_2 + 1 \leq i \leq m_3$.

By Corollary 2.2.22, we have $E(z) = \sum\limits_{i=1}^{m_1} c_i e_{g_i,p_i} \in \mathcal{D}$. If we assume z to be non-zero and positive, which we will do from now on, then $E(z) > 0$ as E is a faithful conditional expectation. Applying Lemma 2.2.25 to $E(z)$ yields $(g, p) \in G \times P$ such that

c) $E(z) e_{g,p} = \|E(z)\| e_{g,p}$.

In order to prove simplicity and pure infiniteness of $\mathcal{O}[G, P, \theta]$, it suffices to establish the following claim: There exist $(\tilde{g}, \tilde{p}) \in G \times P$ satisfying

(a) $e_{\tilde{g},\tilde{p}} \leq e_{g,p}$,

(b) $e_{\tilde{g},\tilde{p}} u_{g_i} s_{p_i} s_{p_i}^* u_{h_i}^* e_{\tilde{g},\tilde{p}} = 0$ for $m_1 + 1 \leq i \leq m_2$ and

(c) $e_{\tilde{g},\tilde{p}} u_{g_i} s_{p_i} s_{q_i}^* u_{h_i}^* e_{\tilde{g},\tilde{p}} = 0$ for $m_2 + 1 \leq i \leq m_3$.

Indeed, if this can be done, then we get

$$e_{\tilde{g},\tilde{p}} z e_{\tilde{g},\tilde{p}} \overset{(b),(c)}{=} e_{\tilde{g},\tilde{p}} E(z) e_{\tilde{g},\tilde{p}} \overset{c),(a)}{=} \|E(z)\| e_{\tilde{g},\tilde{p}}.$$

Now for $x \in \mathcal{O}[G,P,\theta]$ positive and non-zero, let $\varepsilon > 0$ and choose a positive, non-zero element z, which is a finite linear combination of elements $u_{g'} s_{p'} s_{q'}^* u_{h'}^*$, to approximate x up to ε. Then $\|E(z)\|$ is a non-zero positive element of $\mathcal{D}$. Thus, choosing $e_{\tilde{g},\tilde{p}}$ as above, we see that $e_{\tilde{g},\tilde{p}} z e_{\tilde{g},\tilde{p}} = \|E(z)\| e_{\tilde{g},\tilde{p}}$ is invertible in $e_{\tilde{g},\tilde{p}} \mathcal{O}[G,P,\theta] e_{\tilde{g},\tilde{p}}$. If $\|x - z\|$ is sufficiently small, this implies that $e_{\tilde{g},\tilde{p}} x e_{\tilde{g},\tilde{p}}$ is positive and invertible in $e_{\tilde{g},\tilde{p}} \mathcal{O}[G,P,\theta] e_{\tilde{g},\tilde{p}}$ as well because $\|E(z)\| \overset{\varepsilon \to 0}{\longrightarrow} \|E(x)\| > 0$. Hence, if we denote its inverse by y, then

$$\left(y^{\frac{1}{2}} u_{\tilde{g}} s_{\tilde{p}}\right)^* e_{\tilde{g},\tilde{p}} x e_{\tilde{g},\tilde{p}} \left(y^{\frac{1}{2}} u_{\tilde{g}} s_{\tilde{p}}\right) = 1.$$

We claim that there is a pair $(\tilde{g},\tilde{p}) \in G \times P$ satisfying (a)–(c). Let $(g',p') \in g\theta_p(G) \times pP$ and $m_1 + 1 \leq i \leq m_2$. Noting that $u_{g_i} s_{p_i} s_{p_i}^* u_{h_i}^* = u_{g_i h_i^{-1}} e_{h_i,p_i}$, Lemma 2.2.5 implies

$$\begin{aligned} e_{g',p'} u_{g_i h_i^{-1}} e_{h_i,p_i} e_{g',p'} &= e_{g',p'} u_{g_i h_i^{-1}} e_{g',p'} e_{h_i,p_i} \\ &= \chi_{\theta_{p'}(G)}((g')^{-1} g_i h_i^{-1} g')\ u_{g_i h_i^{-1}} e_{g',p'} e_{h_i,p_i}. \end{aligned}$$

According to a), we have $(g')^{-1} g_i h_i^{-1} g' \neq 1_G$. Thus, minimality of (G,P,θ) provides $p_i' \in pP$ with the property that $(g')^{-1} g_i h_i^{-1} g' \notin \theta_{p_i'}(G)$. So if we take $p^{(b)} := \bigvee_{i=m_1+1}^{m_2} p_i'$, then (a) and (b) of the claim hold for all $(g',p') \in g\theta_p(G) \times p^{(b)}P$. Let us assume that $p' \geq p^{(b)} \vee \bigvee_{i=m_2+1}^{m_3} p_i \vee q_i$ and $g' \in g \circ \theta_{p'}(G)$. Then condition (c) holds for (g',p') if and only if

$$\begin{aligned} 0 &= s_{p'}^* u_{(g')^{-1} g_i} s_{p_i} s_{q_i}^* u_{h_i^{-1} g'} s_{p'} \\ &= \chi_{\theta_{p_i}(G)}((g')^{-1} g_i) \chi_{\theta_{q_i}(G)}(h_i^{-1} g') s_{p_i^{-1} p'}^* u_{\theta_{p_i}^{-1}((g')^{-1} g_i) \theta_{q_i}^{-1}(h_i^{-1} g')} s_{q_i^{-1} p'} \end{aligned}$$

is valid for all $m_2 + 1 \leq i \leq m_3$. This is precisely the case if at least one of the conditions

- $(g')^{-1} g_i \in \theta_{p_i}(G)$,
- $(g')^{-1} h_i \in \theta_{q_i}(G)$, or
- $\theta_{p_i}^{-1}((g')^{-1} g_i) \theta_{q_i}^{-1}(h_i^{-1} g') \in \theta_{(p_i \vee q_i)^{-1} p'}(G)$

fails for each i. Suppose, we have an index i for which the first two conditions are satisfied. Using injectivity of $\theta_{p_i \vee q_i}$, the third condition can be transformed into

$$\theta_{r_q}((g')^{-1}g_i)\theta_{r_p}(h_i^{-1}g') \in \theta_{p'}(G),$$

where $r_p := (p_i \wedge q_i)^{-1}p_i$ and $r_q := (p_i \wedge q_i)^{-1}q_i$. Condition b) implies $r_p \wedge r_q = 1_P \neq r_p r_q$. Furthermore, we have

$$\theta_{r_q}((g')^{-1}g_i)\theta_{r_p}(h_i^{-1}g') = 1_G \Longleftrightarrow \theta_{r_q}(g')\theta_{r_p}(g')^{-1} = \theta_{r_q}(g_i)\theta_{r_p}(h_i^{-1}).$$

Let us examine the range of the map $G \xrightarrow{f_i} G$ given by $g \mapsto \theta_{r_q}(g)\theta_{r_p}(g)^{-1}$. Note that f_i need not be a group homomorphism unless G is abelian, in which case the following part can be shortened. If $k_1, k_2 \in G$ have the same image under f_i, then $\theta_{r_p}(k_2^{-1}k_1) = \theta_{r_q}(k_2^{-1}k_1)$. By (C1) from Definition 1.1.5, this gives

$$k_2^{-1}k_1 \in \theta_{r_p}(G) \cap \theta_{r_q}(G) = \theta_{r_p r_q}(G).$$

But if $k_2^{-1}k_1 = \theta_{r_p r_q}(k_3)$ holds true for some $k_3 \in G$, then $\theta_{r_p}(k_2^{-1}k_1) = \theta_{r_q}(k_2^{-1}k_1)$ implies that $\theta_{r_p}(k_3) = \theta_{r_q}(k_3)$ holds as well because P is commutative and $\theta_{q_{i,1}q_{i,2}}$ is injective. By means of induction, we deduce $k_2^{-1}k_1 \in \bigcap_{n\in\mathbb{N}} \theta_{(r_p r_q)^n}(G)$.

Hence $f_i^{-1}(\theta_{r_p}(h_i)\theta_{r_q}(g_i^{-1}))$ is either empty, in which case there is nothing to do, or it is of the form $\tilde{g}_i \bigcap_{n\in\mathbb{N}} \theta_{(r_p r_q)^n}(G)$ for a suitable $\tilde{g}_i \in G$. But for the collection of those i for which the preimage in question is non-empty, we can apply Lemma 1.1.16 to obtain $\tilde{g} \in g\theta_{p'}(G)$ such that $f_i(\tilde{g}) \neq \theta_{r_p}(h_i)\theta_{r_q}(g_i^{-1})$ for all relevant i.

By condition (C2) from Definition 1.1.5, we can choose $\tilde{p} \geq p'$ large enough so that these elements are still different modulo $\theta_{(p_i \vee q_i)^{-1}\tilde{p}}(G)$ for all i. In this case, we get

$$\theta_{p_i}^{-1}(\tilde{g}^{-1}g_i)\theta_{q_i}(h_i^{-1}\tilde{g}) \notin \theta_{(p_i\vee q_i)^{-1}\tilde{p}}(G) \text{ for all } m_2 + 1 \leq i \leq m_3,$$

so $(\tilde{g}, \tilde{p})$ satisfies (c). In other words, we have proven that the pair $(\tilde{g}, \tilde{p})$ satisfies (a)–(c). Thus, $\mathcal{O}[G,P,\theta]$ is purely infinite and simple. □

From this result, we easily get the following corollaries:

Corollary 2.2.27. *If (G,P,θ) is minimal and the action $G \overset{\hat{\tau}}{\curvearrowright} G_\theta$ is amenable, then the canonical representation $\lambda : \mathcal{O}[G,P,\theta] \longrightarrow \mathcal{L}(\ell^2(G))$ from Proposition 2.2.2 is faithful.*

Proof. This follows readily from Proposition 2.2.2 and simplicity of $\mathcal{O}[G,P,\theta]$. □

Combining Lemma 2.2.4, Theorem 2.2.26 and Proposition 2.2.20, we get:

Corollary 2.2.28. *If (G,P,θ) is minimal and the action $G \overset{\hat{\tau}}{\curvearrowright} G_\theta$ is amenable, then $\mathcal{O}[G,P,\theta]$ is a unital UCT Kirchberg algebra.*

Thus, minimal irreversible algebraic dynamical systems (G, P, θ) with amenable action $\hat{\tau}$ yield C*-algebras $\mathcal{O}[G, P, \theta]$ that are classified by their K-theory, see [Kir, Phi00]. Let us come back to some of the examples from Section 1.1 and briefly describe the structure of the C*-algebras obtained in the various cases:

***Examples* 2.2.29.**

(a) Let $G = \mathbb{Z}$, $(p_i)_{i \in I} \subset \mathbb{Z} \setminus \{0, \pm 1\}$ be a family of relatively prime integers, and set $P = |(p_i)_{i \in I}\rangle \subset \mathbb{Z}^\times$, which acts on G by $\theta_i(g) = p_i g$. We know from the considerations in Example 1.1.8 (a) that (G, P, θ) is minimal, so $\mathcal{O}[G, P, \theta]$ is a unital UCT Kirchberg algebra. If we denote $p := \prod_{i \in I} |p_i| \in \mathbb{N} \cup \{\infty\}$, then G_θ can be identified with the p-adic completion $\mathbb{Z}_p = \varprojlim (\mathbb{Z}/q\mathbb{Z}, \theta_q)_{q \in P}$ of $\mathbb{Z}$. Moreover, $\mathcal{F}$ is the Bunce-Deddens algebra of type p^∞, compare [Orf10] and see [BD75] for the classification of Bunce-Deddens algebras by supernatural numbers.

(b) Let $I \subset \mathbb{N}$, choose $\{q\} \cup (p_i)_{i \in I} \subset \mathbb{Z} \setminus \{0, \pm 1\}$ relatively prime, $P = |(p_i)_{i \in I}\rangle$, set $G = \mathbb{Z}\left[\frac{1}{q}\right]$, and let $\theta_p(g) = pg$ for all $g \in G, p \in P$. As for (a), $\mathcal{O}[G, P, \theta]$ is a unital UCT Kirchberg algebra by Example 1.1.8 (b) and Corollary 2.2.28. If we let $p := \prod_{i \in I} |p_i| \in \mathbb{N} \cup \{\infty\}$, then G_θ can be thought of as a p-adic completion of $\mathbb{Z}\left[\frac{1}{q}\right]$ and $\mathcal{F} \cong \mathcal{D} \rtimes_\tau \mathbb{Z}\left[\frac{1}{q}\right]$.

***Example* 2.2.30.** We have seen in Example 1.1.11 that for $n \geq 2$, the dynamical system given by the unilateral shift on $G = \bigoplus_{\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$ is a minimal commutative irreversible algebraic dynamical system of finite type. It has been observed in [CV13] that $\mathcal{O}[G, P, \theta]$ is isomorphic to $\mathcal{O}_n$ in a canonical way: If $e_1 = (1, 0, 0, \ldots,) \in G$, $s \in \mathcal{O}[G, P, \theta]$ denotes the generating isometry for P and $s_1, \ldots, s_n$ are the generating isometries of $\mathcal{O}_n$, then this isomorphism is given by $u_{ke_1} s \mapsto s_k$ for $k = 1, \ldots, n$. In particular, $\mathcal{F}$ is the UHF algebra of type n^∞ and G_θ is homeomorphic to the space of infinite words using the alphabet $\{1, \ldots, n\}$.

***Example* 2.2.31.** Given a family $(G^{(i)}, P, \theta^{(i)})_{i \in \mathbb{N}}$, where each $(G^{(i)}, P, \theta^{(i)})$ is an irreversible algebraic dynamical system, we can consider $G := \bigoplus_{i \in \mathbb{N}} G^{(i)}$, on which P acts component-wise. Assume that each $(G^{(i)}, P, \theta^{(i)})$ and hence (G, P, θ) is minimal, compare Example 1.1.13. We have $G_\theta \cong \prod_{i \in I} G^{(i)}_{\theta^{(i)}}$. Thus the action $G \overset{\hat{\tau}}{\curvearrowright} G_\theta$ is amenable if and only if $G_i \overset{\hat{\tau}_i}{\curvearrowright} G^{(i)}_{\theta^{(i)}}$ is amenable for each $i \in I$. As G is commutative (amenable) if and only if each $G^{(i)}$ is, there are various cases where amenability of $\hat{\tau}$ is for granted. In such situations, $\mathcal{O}[G, P, \theta]$ is a unital UCT Kirchberg algebra.

***Example* 2.2.32.** For the examples arising from free group $\mathbb{F}_n$ with $2 \leq n \leq \infty$, see Example 1.1.14, we are able to provide criteria (1)–(3) to ensure that we obtain minimal irreversible algebraic dynamical systems. Hence G_θ can be interpreted as a certain completion of $\mathbb{F}_n$ with respect to θ. Now $\mathbb{F}_n$ is far from being amenable, but the action

$\mathbb{F}_n \overset{\hat{\tau}}{\curvearrowright} G_\theta$ could still be amenable: The free groups are known to be exact. By a famous result of Narutaka Ozawa, exactness of a discrete group is equivalent to amenability of the left translation action on its Stone-Čech compactification, see [Oza00]. Recently, Mehrdad Kalantar and Matthew Kennedy have shown that exactness of a discrete group is also determined completely by amenability of the natural action on its Furstenberg boundary, see [KK] for details. The latter space is usually substantially smaller than the Stone-Čech compactification and their methods may give some insights into the question of amenability in the context of the examples presented here.

2.3 A closer look at dynamical systems of finite type

This section provides a more detailed presentation of the case where (G, P, θ) is of finite type. In particular, we exhibit additional structural properties of the spectrum G_θ of the diagonal $\mathcal{D}$ in $\mathcal{O}[G, P, \theta]$. For instance, the assumption that $\theta_p(G) \subset G$ is normal for every $p \in P$ causes G_θ to inherit the group structure from G. This turns G_θ into a profinite group. If, in addition, (G, P, θ) is minimal and G is amenable, then $\mathcal{F}$ falls into the class of generalized Bunce-Deddens algebras, see [Orf10, Car11] for details. Due to [Lin01, MS, Win05], they belong to a large class of C*-algebras that can be classified by K-theory.

We are particularly interested in the case where G is abelian, or at least amenable. For such dynamical systems, the situation is significantly easier as $\theta_p(G) \subset G$ is normal for all $p \in P$ and the action $\hat{\tau}$ is always amenable. In fact, the structure of $\mathcal{D}$ and $\mathcal{F}$ is quite similar to the one discovered in the singly generated case, compare [CV13, Section 2]. More explicitly, G_θ is a compact abelian group and we have a chain of isomorphisms:

$$\mathcal{F} \cong C(G_\theta) \rtimes_\tau G \cong C(\hat{G}) \rtimes_{\hat{\tau}} \hat{G}_\theta$$

Throughout this section, we will assume that (G, P, θ) is an irreversible algebraic dynamical system of finite type.

***Remark* 2.3.1.** Recall from Remark 2.2.10 that G_θ can be thought of as a completion of G with respect to θ provided that (G, P, θ) is minimal. The map ι from Lemma 2.2.9 transports more structure under additional hypotheses:

a) If $\theta_p(G)$ is normal in G for all $p \in P$, then $G_\theta = \varprojlim_{p \in P} \operatorname{coker} \theta_p$ is a profinite group.

b) If (G, P, θ) is minimal and $\theta_p(G)$ is normal in G for all $p \in P$, then ι is a dense embedding of groups. In particular, $G \overset{\hat{\tau}}{\curvearrowright} G_\theta$ is the left translation action of a dense subgroup in G_θ.

c) G_θ is an abelian group if and only if G is an abelian group. So if (G, P, θ) is a minimal commutative irreversible algebraic dynamical system, then G_θ is a compact abelian group and $\mathcal{F}$ has a unique tracial state by b). This follows from a straightforward adaptation of the corresponding part of the proof for [CV13, Lemma 2.5]

Proposition 2.3.2. *Suppose (G, P, θ) is minimal and G is amenable. Then $\mathcal{F}$ is a generalized Bunce-Deddens algebra.*

Proof. This follows directly from the construction of the generalized Bunce-Deddens algebras presented in [Orf10, Section 2]: Choose an arbitrary, increasing, cofinal sequence $(p_n)_{n\in\mathbb{N}} \subset P$, where cofinal means that, for every $q \in P$, there exists an $n \in \mathbb{N}$ such that $p_n \in qP$. Then $(\theta_{p_n}(G))_{n\in\mathbb{N}}$ is a family of nested, normal subgroups of finite index in G. This family is separating for G by minimality of (G, P, θ). □

In particular, these assumptions force $\mathcal{F}$ to be unital, nuclear, separable, simple, quasidiagonal, and to have real rank zero, stable rank one, strict comparison for projections as well as a unique tracical state, see [Orf10]. As the combination of real rank zero and strict comparison for projections yields strict comparison (for positive elements), the prerequisites for [MS, Theorem 1.1] are met, so $\mathcal{F}$ also has finite decomposition rank. This establishes the remaining step to achieve classification of the core $\mathcal{F}$ by means of its Elliott invariant $(K_0(\mathcal{F}), K_0(\mathcal{F})_+, [1_\mathcal{F}], K_1(\mathcal{F}))$ thanks to results of Huaxin Lin and Wilhelm Winter, see [Win05, Corollary 6.5(i)] and [Lin01]:

Corollary 2.3.3. *Let (G_i, P_i, θ_i) be minimal and G_i be amenable for $i = 1, 2$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ denote the respective cores, then $\mathcal{F}_1 \cong \mathcal{F}_2$ holds if and only if*

$$(K_0(\mathcal{F}_1), K_0(\mathcal{F}_1)_+, [1_{\mathcal{F}_1}], K_1(\mathcal{F}_1)) \cong (K_0(\mathcal{F}_2), K_0(\mathcal{F}_2)_+, [1_{\mathcal{F}_2}], K_1(\mathcal{F}_2)) .$$

We close this section by presenting an intriguing isomorphism of group crossed products on the level of $\mathcal{F}$, compare [CV13, Lemma 2.5]:

Corollary 2.3.4. *Suppose (G, P, θ) is minimal and G is abelian. Then G_θ is a compact abelian group and there is an action $\bar{\tau}$ of its dual group $\hat{G}_\theta$ on $C(\hat{G})$ such that we have canonical isomorphisms*

$$\mathcal{F} \cong C(G_\theta) \rtimes_\tau G \cong C(\hat{G}) \rtimes_{\bar{\tau}} \hat{G}_\theta.$$

Proof. The first isomorphism has been achieved in Corollary 2.2.19. For the second part, let $\bar{\tau}_{\chi_\theta}(\chi)(g) := \chi_\theta(\iota(g))\chi(g)$ for $\chi_\theta \in \hat{G}_\theta, \chi \in \hat{G}$ and $g \in G$. Since $G \xrightarrow{\iota} G_\theta$ is a group homomorphism, $\bar{\tau}_{\chi_\theta}(\chi)$ defines a character of G. Clearly, $\hat{\tau}$ is compatible with the group structure on $\hat{G}_\theta$. According to Remark 2.3.1 b) the group homomorphism ι identifies G with a dense subgroup of G_θ. In this case the characters on G_θ are in one-to-one correspondence with the characters on G. Note that this correspondence is precisely given by regarding characters on G_θ as characters on G using ι. Therefore, $\bar{\tau}$ defines an action of $\hat{G}_\theta$ by homeomorphisms of the compact space $\hat{G}$. Once we know that $\bar{\tau}$ defines an action, we readily see that there is a canonical surjective $*$-homomorphism $C(G_\theta) \rtimes_\tau G \twoheadrightarrow C(\hat{G}) \rtimes_{\bar{\tau}} \hat{G}_\theta$. As $C(G_\theta) \rtimes_\tau G$ is simple, this map is an isomorphism. $\square$

2.4 Fundamental results for irreversible $*$-commutative dynamical systems

This section is devoted to the construction of universal C*-algebras for irreversible $*$-commutative dynamical systems of finite type (X, P, θ). We show that this construction is consistent with the natural realization of (X, P, θ) as operators on $\ell^2(X)$, see Proposition 2.4.4. Moreover, we show that, for commutative irreversible algebraic dynamical systems of finite type (G, P, θ), there is a natural isomorphism between $\mathcal{O}[G, P, \theta]$ and $\mathcal{O}[\hat{G}, P, \hat{\theta}]$, see Proposition 2.4.3. In analogy to the case of irreversible algebraic dynamical systems, we establish a few elementary properties of this C*-algebra and its core subalgebra $\mathcal{F}$. A fair amount of the results from this section is relevant for Chapter 4.

Throughout this section, let (X, P, θ) denote an irreversible $*$-commutative dynamical system, unless specified otherwise. Recall that, for $p \in P$, the endomorphism α_p of $C(X)$ and its transfer operator L_p are given by

$$\alpha_p(f)(x) = f(\theta_p(x)) \text{ and } L_p(f)(x) = \frac{1}{N_p} \sum_{y \in \theta_p^{-1}(x)} f(y) \text{ for all } x \in X, f \in C(X),$$

where $N_p = |\theta_p^{-1}(x)|$. Moreover, we let $E_p := \alpha_p \circ L_p : C(X) \longrightarrow \alpha_p(C(X))$ denote the associated conditional expectation.

Definition 2.4.1. $\mathcal{O}[X, P, \theta]$ is the universal C*-algebra generated by $C(X)$ and a representation of the monoid P by isometries $(s_p)_{p \in P}$ subject to the relations:

(I) $\quad s_p f = \alpha_p(f) s_p \quad$ for all $f \in C(X), p \in P$.

(II) $\quad s_p^* f s_p = L_p(f) \quad$ for all $f \in C(X), p \in P$.

(III) $\quad s_p^* s_q = s_q s_p^* \quad$ if p and q are relatively prime in P.

(IV) If $p \in P$ and $f_{i,1}, f_{i,2} \in C(X), 1 \le i \le n$, satisfy the reconstruction formula

$$\sum_{1 \le i \le n} f_{i,1} E_p(\bar{f}_{i,2} f) = f \qquad \text{for all } f \in C(X),$$

then $$\sum_{1 \le i \le n} f_{i,1} s_p s_p^* \bar{f}_{i,2} = 1.$$

The next lemma explains the motivation behind relation (IV).

Lemma 2.4.2. *For every $p \in P$, the validity of relation (IV) from Definition 2.4.1 is independent of the choice of the family $(f_{i,j})_{1\leq i\leq m, j=1,2}$ satisfying the reconstruction formula. In particular, if $\mathcal{U} = (U_i)_{1\leq i\leq n}$ is a finite open cover of X such that the restriction of θ_p to each U_i is injective and $(v_i)_{1\leq i\leq n}$ is a partition of unity for X subordinate to $\mathcal{U}$, then*

$$\sum_{1\leq i\leq n} \nu_i s_p s_p^* \nu_i = 1$$

holds for $\nu_i = (N_p v_i)^{\frac{1}{2}}$.

Proof. For the first part, let $(f_{i,j})_{1\leq i\leq m, j=1,2}$ and $(g_{k,\ell})_{1\leq k\leq n, \ell=1,2}$ be two families in $C(X)$ that both satisfy the reconstruction formula for all $f \in C(X)$. Now if relation (IV) from Definition 2.4.1 holds for $(f_{i,j})_{1\leq i\leq m, j=1,2}$, then

$$\begin{aligned} \sum_{1\leq k\leq n} g_{k,1} s_p s_p^* \bar{g}_{k,2} &= \sum_{1\leq k\leq n} g_{k,1} s_p s_p^* \bar{g}_{k,2} \sum_{1\leq i\leq m} f_{i,1} s_p s_p^* \bar{f}_{i,2} \\ &= \sum_{1\leq i\leq m} \sum_{1\leq k\leq n} g_{k,1} E_p(\bar{g}_{k,2} f_{i,1}) s_p s_p^* \bar{f}_{i,2} \\ &= \sum_{1\leq i\leq m} f_{i,1} s_p s_p^* \bar{f}_{i,2} = 1. \end{aligned}$$

The second claim follows from Lemma 1.3.9. □

Since finite open covers of the form appearing in Lemma 2.4.2 always exist for surjective local homeomorphisms of compact Hausdorff spaces, see Lemma 1.3.9, there are in fact functions $f_{i,j}$ satisfying the reconstruction formula for each $p \in P$. Thus, relation (IV) is non-void.

There is a close connection to the defining relations (CNP 1)–(CNP 3) for the C*-algebras associated to irreversible algebraic dynamical systems (G, P, θ), compare Definition 2.2.1. We will now show that the two constructions yield the same C*-algebra if both methods are applicable, that is, if (G, P, θ) of finite type and G is commutative, see Corollary 1.3.17. Recall that the dual model $(\hat{G}, P, \hat{\theta})$ is an irreversible $*$-commutative dynamical system of finite type in this case.

Proposition 2.4.3. *Let (G, P, θ) be a commutative irreversible algebraic dynamical system of finite type. If $(u_g)_{g\in G}$ and $(s_p)_{p\in P}$ denote the canonical generators of $\mathcal{O}[G, P, \theta]$ and $(w_g)_{g\in G}$ and $(v_p)_{p\in P}$ denote the canonical generators of $\mathcal{O}[\hat{G}, P, \hat{\theta}]$, then*

$$\begin{array}{ccc} \mathcal{O}[G, P, \theta] & \xrightarrow{\varphi} & \mathcal{O}[\hat{G}, P, \hat{\theta}] \\ u_g s_p & \mapsto & w_g v_p \end{array}$$

is an isomorphism.

Proof. It is clear that $(w_g)_{g\in G}$ and $(v_p)_{p\in P}$ satisfy (CNP 1). (CNP 3) follows from (IV) since we can easily check the reconstruction formula required in (IV) on each w_g and note that $C(\hat{G})$ can be identified with the closed linear span of $(w_g)_{g\in G}$. It remains to prove (CNP 2), that is,

$$v_p^* w_g v_q = \chi_{\theta_p(G)\theta_q(G)}(g)\ w_{g_1} v_{(p\wedge q)^{-1}q} v_{(p\wedge q)^{-1}p}^* w_{g_2} \text{ for all } g \in G \text{ and } p,q \in P,$$

for $g = \theta_p(g_1)\theta_q(g_2)$, and $v_p^* w_g v_q = 0$ otherwise. The case $g \in \theta_p(G)\theta_q(G)$ follows in a straightforward manner from (I) and (III), so suppose $g \notin \theta_p(G)\theta_q(G)$. Since (G,P,θ) is of finite type, Proposition 1.1.1 yields $\theta_{(p\wedge q)^{-1}p}(G)\theta_{(p\wedge q)^{-1}q}(G) = G$. Hence we get

$$g \notin \theta_{p\wedge q}(\theta_{(p\wedge q)^{-1}p}(G)\theta_{(p\wedge q)^{-1}q}(G)) = \theta_{p\wedge q}(G)$$

and with the help of Example 1.3.7 we conclude that

$$v_p^* w_g v_q = v_{(p\wedge q)^{-1}p}^* v_{p\wedge q}^* w_g v_{p\wedge q} v_{(p\wedge q)^{-1}q} \overset{(II)}{=} v_{(p\wedge q)^{-1}p}^* L_{p\wedge q}(w_g) v_{(p\wedge q)^{-1}q} = 0.$$

Thus we have shown that φ is a surjective $*$-homomorphism. In order to see that φ is an isomorphism, it suffices to check that $C^*((u_g)_{g\in G}) \cong C(\hat{G})$ and $(s_p)_{p\in P}$ satisfy (I)–(IV). Condition (I) is nothing but (CNP 1). Conditions (II) and (III) follow from (CNP 2) using Remark 2.2.3 c) and Example 1.3.7. Finally, (IV) can be deduced from (CNP 3) with the help of Lemma 2.4.2. □

We have seen in Lemma 2.4.2 that we can always choose elements $f_{i,j}$ satisfying the reconstruction formula for (IV) in such a way that we get a C*-algebraic partition of unity in $\mathcal{O}[X,P,\theta]$, that is, the corresponding elements are positive and sum up to one. Unless X is totally disconnected, this may produce a number of genuine positive elements exceeding the actual number of preimages a single point has. For example, the minimal number of elements appearing in a partition of unity as in Lemma 2.4.2 for the map $\times 2 : \mathbb{T} \longrightarrow \mathbb{T}$ is three.

One particular feature of commutative irreversible algebraic dynamical systems of finite type compared to arbitrary irreversible $*$-commutative dynamical systems of finite type is that we can choose the elements satisfying the reconstruction formula for (IV) in a different manner using the algebraic structure. This allows us to reduce the number of positive elements needed to the optimal value, that is, the size of the kernel of the group endomorphism on $\hat{G}$. Moreover, the elements forming the C*-algebraic partition of unity are projections in this case.

Now that we have already established some connections to Section 2.2, let us start with an analysis of basic properties of the C*-algebra $\mathcal{O}[X,P,\theta]$. First of all, there is a natural representation of $\mathcal{O}[X,P,\theta]$ on $\ell^2(X)$, whose standard orthonormal basis will be denoted by $(\xi_x)_{x\in X}$:

Proposition 2.4.4. *Let* $M_f\xi_x := f(x)\xi_x$ *and* $S_p\xi_x = N_p^{-\frac{1}{2}} \sum_{y\in\theta_p^{-1}(x)} \xi_y$ *for* $x \in X, f \in C(X)$ *and* $p \in P$. *Then the map*

$$\begin{array}{ccc} \mathcal{O}[X,P,\theta] & \xrightarrow{\varphi} & \mathcal{L}\left(\ell^2(X)\right) \\ fs_p & \mapsto & M_fS_p \end{array}$$

is a representation of $\mathcal{O}[X,P,\theta]$, *which is faithful on* $C(X)$.

Proof. Firstly, $S_p^*(\xi_x) = N_p^{-\frac{1}{2}}\xi_{\theta_p(x)}$ for all $p \in P$ and $x \in X$ since

$$\langle S_p^*(\xi_x), \xi_y\rangle = \langle \xi_x, S_p(\xi_y)\rangle = \chi_{\theta_p^{-1}(y)}(x)\ N_p^{-\frac{1}{2}}.$$

Thus, S_p is an isometry. $(S_p)_{p\in P}$ is a representation of P because

$$\begin{aligned} S_pS_q(\xi_x) &= N_q^{-\frac{1}{2}} \sum_{y\in\theta_q^{-1}(x)} S_p(\xi_y) \\ &= (N_pN_q)^{-\frac{1}{2}} \sum_{\substack{y\in\theta_q^{-1}(x)\\ z\in\theta_p^{-1}(y)}} \xi_z \\ &= (N_{pq})^{-\frac{1}{2}} \sum_{z\in\theta_{pq}^{-1}(x)} \xi_z \\ &= S_{pq}(\xi_x). \end{aligned}$$

(I) If p and q are relatively prime in P, then θ_p and θ_q $*$-commute according to Definition 1.3.13 (C). Using the equivalent condition (iii) from Proposition 1.3.2, we obtain

$$S_p^*S_q(\xi_x) = N_{pq}^{-\frac{1}{2}} \sum_{y\in\theta_p(\theta_q^{-1}(x))} \xi_y = N_{pq}^{-\frac{1}{2}} \sum_{y\in\theta_q^{-1}(\theta_p(x))} \xi_y = S_qS_p^*(\xi_x),$$

so S_p and S_q doubly commute.

(II) $S_pM_f = M_{\alpha_p(f)}S_p$ is readily verified for all $f \in C(X)$ and $p \in P$.

(III) $S_p^*M_fS_p = M_{L_p(f)}$ is also straightforward.

(IV) For $\nu_i = (N_pv_i)^{\frac{1}{2}}$, where $(v_i)_{1\leq i\leq n}$ is a partition of unity such that $\theta_p|_{\operatorname{supp} v_i}$ is injective for all i (as in Lemma 2.4.2), we compute

$$\begin{aligned} \sum_{1\leq i\leq n} M_{\nu_i}S_pS_p^*M_{\nu_i}(\xi_x) &= \sum_{1\leq i\leq n}\ \sum_{y\in\theta_p^{-1}(\theta_p(x))} \underbrace{(v_i(y)v_i(x))^{\frac{1}{2}}}_{\delta_{x\ y}}\ \xi_y \\ &= \sum_{1\leq i\leq n} v_i(x)\xi_x \\ &= \xi_x. \end{aligned}$$

We infer from Lemma 2.4.2 that this yields (IV) since the proof provided there only uses the additional property (II), which we have already established for S_p and M_f.

Thus, φ is a $*$-homomorphism by the universal property of $\mathcal{O}[X,P,\theta]$ and it is clear that φ is faithful on $C(X)$. $\square$

Lemma 2.4.5. *The linear span of* $(fs_ps_q^*g)_{f,g\in C(X),p,q\in P}$ *is dense in* $\mathcal{O}[X,P,\theta]$.

Proof. The set is closed under taking adjoints and contains the generators, so we only have to show that it is multiplicatively closed. Let $p_i,q_i\in P,\ f_i,g_i\in C(X)$ and $a_i:=f_is_{p_i}s_{q_i}^*g_i$ for $i=1,2$. Additionally, choose a partition of unity $(v_j)_{1\leq j\leq n}$ subordinate to a finite open cover $(U_j)_{1\leq j\leq n}$ of X such that $\theta_{q_1\vee p_2}|_{U_j}$ is injective and $\nu_j:=(N_{q_1\vee p_2}\ v_j)^{\frac{1}{2}}$ for all j. Then, we get

$$\begin{aligned}
a_1a_2 &\overset{\text{(IV)}}{=} a_1\sum_{1\leq j\leq n}\nu_js_{q_1\vee p_2}s_{q_1\vee p_2}^*\nu_ja_2\\
&\overset{\text{(II)}}{=} \sum_{1\leq j\leq n} f_1s_{p_1}L_{q_1}(g_1\nu_j)s_{q_1^{-1}(q_1\vee p_2)}s_{p_2^{-1}(q_1\vee p_2)}^*L_{p_2}(\nu_jf_2)s_{q_2}^*g_2\\
&\overset{\text{(I)}}{=} \sum_{1\leq j\leq n} f_1\ \alpha_{p_1}\circ L_{q_1}(g_1\nu_j)s_{p_1q_1^{-1}(q_1\vee p_2)}s_{q_2p_2^{-1}(q_1\vee p_2)}^*\alpha_{q_2}\circ L_{p_2}(\nu_jf_2)g_2.
\end{aligned}$$

$\square$

The remainder of this section will deal with faithfulness of conditional expectations related to a core subalgebra of $\mathcal{O}[X,P,\theta]$ similar to the one introduced in Definition 2.2.15 for $\mathcal{O}[G,P,\theta]$. Recall that the enveloping group $H=P^{-1}P$ of P is discrete abelian. If we denote its Pontryagin dual by L, which is then a compact abelian group, we get a so-called gauge action γ of L on $\mathcal{O}[X,P,\theta]$ by

$$\gamma_\ell(f)=f \text{ and } \gamma_\ell(s_p)=\ell(p)s_p \text{ for } f\in C(X), p\in P \text{ and } \ell\in L.$$

It is well-known that actions of this form are strongly continuous.

Definition 2.4.6. The fixed point algebra $\mathcal{O}[X,P,\theta]^\gamma$ for the gauge action γ, denoted by $\mathcal{F}$, is called the **core** of $\mathcal{O}[X,P,\theta]$. In addition, let

$$\mathcal{F}_p:=C^*\left(\{fs_ps_p^*g\mid f,g\in C(X)\}\right)$$

denote the subalgebra of $\mathcal{F}$ corresponding to $p\in P$.

Lemma 2.4.7. *Let μ denote the normalized Haar measure of the compact abelian group L. Then $E_1(a):=\int\limits_{\ell\in L}\gamma_\ell(a)\ d\mu(\ell)$ defines a faithful conditional expectation $\mathcal{O}[X,P,\theta]\xrightarrow{E_1}\mathcal{F}$.*

Proof. If $a\in\mathcal{O}[X,P,\theta]$ is positive and non-zero, then there is a state ψ on $\mathcal{O}[X,P,\theta]$ such that $\psi(a)=\|a\|$. Since $\gamma_\ell(a)\geq 0$ in $\mathcal{O}[X,P,\theta]$, we have $\psi(\gamma_\ell(a))\geq 0$ for all $\ell\in L$. Thus, $\psi(a)=\|a\|>0$ together with strong continuity of γ implies

$$\psi(E_1(a))=\int\limits_{\ell\in L}\psi(\gamma_\ell(x))\ d\mu(\ell)>0.$$

$\square$

Proposition 2.4.8. *$\mathcal{F}$ is the closed linear span of $(fs_ps_p^*g)_{f,g\in C(X),p\in P}$. $\mathcal{F}_p \subset \mathcal{F}_q$ holds whenever $q \in pP$ and hence $\mathcal{F} = \overline{\bigcup_{p\in P}\mathcal{F}_p}$.*

Proof. Clearly, every element $fs_ps_p^*g$ is fixed by γ. Conversely, if $a \in \mathcal{F}$, we can approximate a by finite linear combinations of elements $f_is_{p_i}s_{q_i}^*g_i$ according to Lemma 2.4.5. Relying on the conditional expectation E_1 from Lemma 2.4.7, we know that it suffices to take those $f_is_{p_i}s_{q_i}^*g_i$ satisfying $p_i = q_i$.

If $q \in pP$ holds true, then we can employ (IV) for $p^{-1}q$ to deduce $\mathcal{F}_p \subset \mathcal{F}_q$. The last claim is an immediate consequence of this. □

The next observation and its proof are based on [EV06, Proposition 7.9].

Proposition 2.4.9. *For $p \in P$, the subalgebra $\mathcal{F}_p$ of $\mathcal{F}$ satisfies*

$$\begin{array}{rcl} \mathcal{F}_p & = & \mathrm{span}\{fs_ps_p^*g \mid f,g \in C(X)\} \\ & \textit{and} & \\ (\mathcal{F}_p)_+ & = & \mathrm{span}\{fs_ps_p^*\bar{f} \mid f \in C(X)\}. \end{array}$$

Proof. The right hand side of the first equation is multiplicatively closed as

$$f_1s_ps_p^*g_1\ f_2s_ps_p^*g_2 \overset{(II),(I)}{=} f_1E_p(g_1f_2)s_ps_p^*g_2.$$

Let $a \in \mathcal{F}_p, \varepsilon > 0$ and choose $m \in \mathbb{N}$ and $f_k, g_k \in C(X), 1 \leq k \leq m$ such that

$$\|\sum_{k=1}^m f_ks_ps_p^*g_k - a\| < \varepsilon.$$

Pick $(\nu_i)_{1\leq i\leq n}$ coming from a suitable partition of unity of X for θ_p as in Lemma 1.3.9. In other words, the family $(\nu_i)_{1\leq i\leq n}$ satisfies (IV) from Definition 2.4.1. Then we obtain

$$\begin{aligned} \sum_{k=1}^m f_ks_ps_p^*g_k &= \sum_{i=1}^n \nu_is_ps_p^*\nu_i \sum_{k=1}^m f_ks_ps_p^*g_k \sum_{j=1}^n \nu_js_ps_p^*\nu_j \\ &= \sum_{i,j=1}^n h_{i,j}\ \nu_is_ps_p^*\nu_j, \end{aligned}$$

where $h_{i,j} = \sum_{1\leq k\leq m} E_p(\nu_if_k)E_p(g_k\nu_j)$, so n^2 summands suffice to approximate a up to ε.

For the second part, let $a \in (\mathcal{F}_p)_+$. Then $a = b^*b$ holds for some $b \in \mathcal{F}_p$. From the first part, we know that $b = \sum_{i=1}^m f_is_ps_p^*g_i$ for some $m \in \mathbb{N}$ and suitable $f_i, g_i \in C(X)$. Therefore,

$$a = \sum_{i,j=1}^m \bar{g}_is_ps_p^*\bar{f}_if_js_ps_p^*g_j = \sum_{i,j=1}^m \bar{g}_iE_p(\bar{f}_if_j)s_ps_p^*g_j.$$

Recall that $E_p : C(X) \longrightarrow \alpha_p(C(X))$ is a conditional expectation and hence completely positive, see [BO08, Theorem 5.9]. Thus $(\bar{f}_if_j)_{i,j} \in M_m(C(X))_+$ implies that there is

$c = (c_{ij})_{1\leq i,j\leq m} \in M_m(E_p(C(X)))$ satisfying $\left(E_p(\bar{f}_i f_j)\right)_{1\leq i,j\leq m} = c^*c$. Setting $h_k = \sum_{i=1}^m \overline{\alpha_p(c_{ki})g_i}$ for $1 \leq k \leq m$, we obtain

$$\begin{aligned} a &= \sum_{i,j=1}^m \bar{g}_i \left(\sum_{k=1}^m \bar{c}_{ki}c_{kj}\right) s_p s_p^* g_j \\ &= \sum_{k=1}^m \left(\sum_{i=1}^m \overline{g_i c_{ki}}\right) s_p s_p^* \left(\sum_{j=1}^m c_{kj}g_j\right) \\ &= \sum_{k=1}^m h_k s_p s_p^* \bar{h}_k. \end{aligned}$$

□

We need some results related to finite index endomorphisms. Since we do not assume that the reader is familiar with this notion, we shall recall it briefly and state the required results without proofs from [Exe03b]:

Definition 2.4.10 ([Wat90, 1.2.2,2.1.6], [Exe03b, 8.1])**.** Let A be a C*-algebra. A pair (α, E) consisting of a $*$-endomorphism α of A and a conditional expectation $A \xrightarrow{E} \alpha(A)$ is said to be a **finite-index endomorphism**, if there are $\nu_1, \ldots, \nu_n \in A$ such that

$$\sum_{1\leq i\leq n} \nu_i E(\nu_i^* a) = a \text{ for all } a \in A.$$

***Remark* 2.4.11.** Concrete examples of this situation are provided by regular surjective local homeomorphisms η of compact Hausdorff spaces X, see Section 1.4. In this case, we have $A = C(X)$, $\alpha(f)(x) = f(\eta(x))$ and $E = \alpha \circ L$, where L is the natural transfer operator constructed in Example 1.3.6. To see this, observe that the requirement in Definition 2.4.10 is nothing but the reconstruction formula established in Lemma 1.3.9. From this perspective, finite-index endomorphisms can be thought of as irreversible C*-dynamical systems (A, α, E) that admit a finite Parseval frame.

The following proposition is a reformulation of some results from [Exe03b] in terms of the terminology used within this exposition.

Proposition 2.4.12 ([Exe03b, 8.6,8.8])**.** *The map $E_2 : \mathcal{F} \longrightarrow C(X)$ given by $f s_p s_p^* g \mapsto N_p^{-1} fg$ is a conditional expectation. Moreover, it is the only conditional expectation from $\mathcal{F}$ to $C(X)$ as the latter is commutative.*

As a straightforward consequence of Proposition 2.4.12 and Proposition 2.4.9 we get:

Corollary 2.4.13. *The map $G := E_2 \circ E_1$ is a conditional expectation from $\mathcal{O}[X, P, \theta]$ to $C(X)$, whose restriction to $\mathcal{F}_p$ is faithful for all $p \in P$.*

Proof. By Proposition 2.4.9, every element $a \in (\mathcal{F}_p)_+$ is of the form $a = \sum_{i=1}^n f_i s_p s_p^* \bar{f}_i$ for suitable $n \in \mathbb{N}$ and $f_i \in C(X)$. Then

$$0 = G(a) = N_p^{-1} \sum_{i=1}^n |f_i|^2$$

implies $f_i = 0$ for all i, so $a = 0$. Thus G is faithful on $\mathcal{F}_p$. $\square$

Although the conditional expectation G from Corollary 2.4.13 may fail to be faithful, it satisfies the following weaker condition, which turns out to be useful in the proof of the main result Theorem 4.1.9.

Lemma 2.4.14. *If $a \in \mathcal{O}[X,P,\theta]_+$ satisfies $G(bab^*) = 0$ for all $b \in \mathcal{F}$, then $a = 0$.*

Proof. Let us assume $a \in \mathcal{F}$ at first and suppose $G(bab^*) = 0$ holds for all $b \in \mathcal{F}$. This implies $G(bac) = 0$ for all $b, c \in \mathcal{F}$ as

$$|G(bac)| \leq G(bacc^*ab^*)^{\frac{1}{2}} \leq \|a^{\frac{1}{2}}c\| G(bab^*)^{\frac{1}{2}} = 0.$$

For $a \neq 0$, $I := \{d \in \mathcal{F} \mid G(bdc) = 0 \text{ for all } b, c \in \mathcal{F}\}$ is a non-trivial ideal in $\mathcal{F}$. By $\mathcal{F} = \overline{\bigcup_{p\in P} \mathcal{F}_p}$, see Proposition 2.4.8, it follows that $I \cap \mathcal{F}_p \neq 0$ for some $p \in P$, so there is some $d \in (\mathcal{F}_p)_+ \setminus \{0\}$ such that $G(d) = 0$. But Proposition 2.4.9 shows that $d = \sum_{i=1}^n f_i s_p s_p^* \bar{f}_i$ for some $n \in \mathbb{N}$ and suitable $f_i \in C(X)$, so $0 = G(b) = N_p^{-1}\ \sum_{i=1}^n |f_i|^2 \neq 0$ yields a contradiction. Thus, we conclude that, for $a \in \mathcal{F}_+$, $G(bab^*) = 0$ for all $b \in \mathcal{F}$ implies $a = 0$. Now let $a \in \mathcal{O}[X,P,\theta]_+$ be arbitrary. Then

$$0 = G(bab^*) = G \circ E_1(bab^*) = G(bE_1(a)b^*) \text{ for all } b \in \mathcal{F},$$

so $E_1(a) = 0$ by what we have just shown. But this forces $a = 0$ since E_1 is faithful according to Lemma 2.4.7. $\square$

Chapter 3

Product systems of Hilbert bimodules over discrete semigroups

In this chapter we will present an alternative construction of the C*-algebras $\mathcal{O}[G, P, \theta]$ and $\mathcal{O}[X, P, \theta]$ as the Cuntz-Nica-Pimsner algebras of discrete product systems of Hilbert bimodules over P which arise in a natural way from (G, P, θ) and (X, P, θ), respectively. Discrete product systems of Hilbert bimodules have been used extensively to construct and study more general C*-algebras in the spirit of [Pim97], see [Fow99, Fow02, Yee07, SY10, HLS12, FPW13]. But there are also connections to von Neumann algebras, see for instance [Sol06].

The product systems we will construct for irreversible algebraic dynamical systems (G, P, θ) admit a coherent system of orthonormal bases in which the orthonormal basis of the p-th fiber corresponds to $G/\theta_p(G)$. Hence this orthonormal basis is finite if and only if $\theta_p(G)$ has finite index in G. Using this feature, we show in Theorem 3.3.4 that the Cuntz-Nica-Pimsner algebra of the product system associated to (G, P, θ) coincides with the algebra $\mathcal{O}[G, P, \theta]$ from Definition 2.2.1.

Interestingly, the framework of product systems allows us to treat irreversible $*$-commutative dynamical systems of finite type in a similar manner. However, we are forced to work with Parseval frames instead of orthonormal bases (on the fibers of the product systems) because the use of partitions of unity subordinate to suitable open covers of X does not produce orthogonal elements, unless we can choose the covers to consist of clopen (disjoint) sets. Nevertheless, there is sufficient structure to show that, also in this case, $\mathcal{O}[X, P, \theta]$ is canonically isomorphic to the Cuntz-Nica-Pimsner algebra of the product system of Hilbert bimodules associated to (X, P, θ). This is established through Theorem 3.3.7.

For convenience, we start with a short summary of the relevant ideas and facts concerning product systems of Hilbert bimodules over discrete semigroups. Most of these results are true in greater generality, but we will reduce our exposition to a minimal level for the sake of brevity. The interested reader may find a profound introduction to Hilbert bimodules in [Lan95]. For more details about discrete product systems of Hilbert bimodules, we refer to [Fow02, SY10, HLS12].

3.1 Product systems with orthonormal bases

Unless specified otherwise, let A be a unital C*-algebra and P a discrete, left cancellative, commutative monoid with unit 1_P. There is a natural partial order on P defined by $p \leq q$ if $q \in pP$ and we will assume P to be lattice-ordered with respect to this partial order. That is to say, for $p, q \in P$ there exists a unique least common upper bound $p \vee q \in P$. Hence, there is also a unique greatest common lower bound $p \wedge q = (p \vee q)^{-1}pq$ for p and q. In particular, this condition forces $P^* = \{1_P\}$. We point out that all these requirements are satisfied for countably generated, free abelian monoids.

Definition 3.1.1. A **right pre-Hilbert A-module** is a $\mathbb{C}$-vector space $\mathcal{H}$ equipped with a right A-module structure and a bilinear map $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow A$, which is linear in the second component, such that the following relations are satisfied for all $\xi, \eta \in \mathcal{H}$ and $a \in A$:

$$(1) \quad \langle \xi, \eta.a \rangle = \langle \xi, \eta \rangle a \qquad (2) \quad \langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$$

$$(3) \quad \langle \xi, \xi \rangle \geq 0 \qquad (4) \quad \langle \xi, \xi \rangle = 0 \Longleftrightarrow \xi = 0$$

A right pre-Hilbert A-module $\mathcal{H}$ is said to be a **right Hilbert A-module** if it is complete with respect to the norm $\|\xi\| = \| \langle \xi, \xi \rangle \|_A^{\frac{1}{2}}$. $\mathcal{H}$ is called a **Hilbert bimodule** over A if, in addition, there is a left action of A given by a $*$-homomorphism $\phi_{\mathcal{H}} : A \longrightarrow \mathcal{L}(\mathcal{H})$, where $\mathcal{L}(\mathcal{H})$ denotes the C*-algebra of all adjointable linear operators from $\mathcal{H}$ to $\mathcal{H}$.

***Examples* 3.1.2.**

(a) Every Hilbert space $\mathcal{H}$ is a Hilbert bimodule over $\mathbb{C}$.

(b) If A is a C*-algebra, we can form the trivial Hilbert bimodule $\mathcal{H} = {}_{\mathrm{id}}A_{\mathrm{id}}$ over A with inner product given by $\langle a, b \rangle = a^*b$. Here A acts from both sides by multiplication.

(c) If A is a C*-algebra and $\alpha \in \mathrm{Aut}(A)$, then replacing the left action of (b) by $\phi(a)(b) = \alpha(a)b$ yields a Hilbert bimodule $\mathcal{H} = {}_{\alpha}A_{\mathrm{id}}$. Pimsner showed in [Pim97] that this Hilbert bimodule serves as a model to construct the crossed product $A \rtimes_{\alpha} \mathbb{Z}$.

(d) Let X be a compact Hausdorff space, $\eta : X \longrightarrow X$ a regular surjective local homeomorphism for which the induced injective $*$-endomorphism of $C(X)$ is denoted by α. Then we can construct a Hilbert bimodule $\mathcal{H} = {}_{\mathrm{id}}C(X)_{\alpha}$ over $C(X)$ as follows: Starting with $C(X)$, we define an inner product $\langle f, g \rangle := L(\bar{f}g)$ for all $f, g \in C(X)$, where L denotes the transfer operator for α, see Example 1.3.6. It is clear that

$f \mapsto \|L(|f|^2)\|^{\frac{1}{2}}$ is actually a norm on $C(X)$. Due to [LR07, Lemma 3.3], this norm is equivalent to the standard norm $\|\cdot\|_\infty$. Hence, $C(X)$ is already complete with respect to this norm. The left action is given by multiplication whereas the right action is defined as $f.g = f\alpha(g)$ for $f, g \in C(X)$.

Definition 3.1.3. Let $\mathcal{H}$ be a right Hilbert module over A. For $\xi, \eta \in \mathcal{H}$, $\Theta_{\xi,\eta} \in \mathcal{L}(\mathcal{H})$, given by $\Theta_{\xi,\eta}(\zeta) = \xi.\langle \eta, \zeta \rangle$ for $\zeta \in \mathcal{H}$, is said to be a **generalized rank one operator**. The closed linear span of $(\Theta_{\xi,\eta})_{\xi,\eta\in\mathcal{H}}$ inside $\mathcal{L}(\mathcal{H})$ is called the **C*-algebra of generalized compact operators** $\mathcal{K}(\mathcal{H})$.

Lemma 3.1.4. *Let $\mathcal{H}$ be a right Hilbert module over A. $\mathcal{K}(\mathcal{H})$ is an ideal in $\mathcal{L}(\mathcal{H})$.*

Proof. Given $T \in \mathcal{L}(\mathcal{H})$ and $\xi, \eta \in \mathcal{H}$, one readily verifies

$$T\Theta_{\xi,\eta} = \Theta_{T(\xi),\eta} \text{ and } \Theta_{\xi,\eta}T = \Theta_{\xi,T^*(\eta)},$$

where we use the inner product $\langle \cdot, \cdot \rangle$ to observe that $T(\xi.a) = T(\xi).a$ holds for all $a \in A$. Since $\mathcal{K}(\mathcal{H})$ is the closed linear span of its generalized rank one operators, this concludes the proof. $\square$

The next lemma is a standard fact whose proof can be found in [Lan95, Proposition 4.5].

Lemma 3.1.5. *Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert bimodules over A whose left and right actions are denoted by ϕ_1, ϕ_2 and ρ_1, ρ_2, respectively. Then*

$$\langle [\xi_1 \otimes \xi_2], [\eta_1 \otimes \eta_2] \rangle_{\mathcal{H}_1 \otimes_A \mathcal{H}_2} = \langle \xi_2, \phi_2(\langle \xi_1, \eta_1 \rangle_1)\eta_2 \rangle_2$$

defines an inner product on $(\mathcal{H}_1 \odot \mathcal{H}_2)/\sim$, where $\xi_1 \otimes \xi_2 \sim \eta_1 \otimes \eta_2$ if there exists $a \in A$ such that $\xi_2 = \phi_2(a)\eta_2$ and $\eta_1 = \xi_1\rho_1(a)$. The completion of $(\mathcal{H}_1 \odot \mathcal{H}_2)/\sim$ with respect to the norm induced by this inner product can be equipped with left and right actions induced from ϕ_1 and ρ_2, respectively, yielding a Hilbert bimodule $\mathcal{H}_1 \otimes_A \mathcal{H}_2$.

This Hilbert bimodule is called the **balanced tensor product** of $\mathcal{H}_1$ and $\mathcal{H}_2$ over A.

Definition 3.1.6. Let A be a unital C*-algebra. A **product system of Hilbert bimodules** over P with coefficients in A is a monoid $\mathcal{X}$ together with a monoidal homomorphism $\rho : \mathcal{X} \longrightarrow P$ such that:

(1) $\mathcal{X}_p := \rho^{-1}(p)$ is a Hilbert bimodule over A for each $p \in P$,

(2) $\mathcal{X}_{1_P} \cong {}_{\mathrm{id}}A_{\mathrm{id}}$ as Hilbert bimodules and

(3) for all $p, q \in P$, we have $\mathcal{X}_p \otimes_A \mathcal{X}_q \cong \mathcal{X}_{pq}$ if $p \neq 1_P$, and $\mathcal{X}_{1_P} \otimes_A \mathcal{X}_q \cong \overline{\phi_q(A)\mathcal{X}_q}$.

***Remark* 3.1.7.**

a) Alternatively, one can describe a product system over P as a family of Hilbert bimodules $(\mathcal{X}_p)_{p\in P}$ such that $\mathcal{X}_{1_P} \cong {}_{\mathrm{id}}A_{\mathrm{id}}$ together with multiplication maps $M_{p,q} : \mathcal{X}_p \times \mathcal{X}_q \longrightarrow \mathcal{X}_{pq}$ for $p,q \in P$ satisfying several conditions forcing it to implement the isomorphism from Definition 3.1.6 (3). In the definition, these maps are given implicitly by the monoid structure of $\mathcal{X}$. Associativity of $\mathcal{X}$ can therefore also be expressed as

$$M_{pq,r} \circ (M_{p,q} \otimes Id_r) = M_{p,qr} \circ (Id_p \otimes M_{q,r}) \text{ for all } p,q,r \in P.$$

b) Note that the maps $M_{p,1_P}$ from a) are always isomorphisms of Hilbert bimodules whereas $M_{1_P,p}$ need not be one since its image equals $\overline{\phi_p(A)\mathcal{X}_p}$. But if $\overline{\phi_p(A)\mathcal{X}_p}$ is all of $\mathcal{X}_p$, $M_{1_P,p}$ is an isomorphism and $\mathcal{X}_p$ is said to be **essential**, see [Fow02]. This is for example the case, if $\mathcal{X}_p = {}_{\mathrm{id}}A_*$ since $AA = A$. More precisely, as we only deal with unital coefficient algebras, this is the case if and only if $\phi_p(1_A) = 1_{\mathcal{L}(\mathcal{X}_p)}$ holds.

c) The multiplicative structure of $\mathcal{X}$ yields $*$-homomorphisms

$$\begin{array}{ccc} \mathcal{L}(\mathcal{X}_p) & \xrightarrow{\iota_p^{pq}} & \mathcal{L}(\mathcal{X}_{pq}) \\ T & \mapsto & T \otimes id_{\mathcal{X}_q} \end{array}$$

for all $p,q \in P$, where we have identified $\mathcal{X}_p \otimes_A \mathcal{X}_q$ with $\mathcal{X}_{pq}$. According to b), ι_p^p is an isomorphism whereas $\iota_{1_P}^p$ is an isomorphism if and only if $\mathcal{X}_p$ is essential.

***Example* 3.1.8.** For every Hilbert bimodule $\mathcal{H}$ over A, there is a product system $\mathcal{X}$ of Hilbert bimodules over $\mathbb{N}$ given by $\mathcal{X}_0 = A$ and $\mathcal{X}_n = \mathcal{X}_{n-1} \otimes_A \mathcal{H}$ for $n \geq 1$.

***Example* 3.1.9.** The maps ι_p^{pq} introduced in 3.1.7 c) need not map generalized compact operators to generalized compact operators. Consider for example the trivial case of $A = \mathbb{C}$ acting by multiplication on the fibers $\mathcal{X}_p = H$, where H is a separable, infinite-dimensional Hilbert space (equipped with a suitable product structure obtained from bijections $\mathbb{N}^2 \longrightarrow \mathbb{N}$): $\iota_{1_P}^p$ is determined by the projection $\iota_{1_P}^p(1) = 1_{\mathcal{L}(\mathcal{X}_p)}$ which is infinite and hence non-compact.

There is a less restrictive requirement called compactly alignedness, which has been introduced for product systems over quasi-lattice ordered groups to avoid a certain pathology for the representation theory of product systems, see [Fow99, Example 1.3]. Recall that whenever two elements of the indexing semigroup have a (least) common upper bound, one can make sense of products of compact operators living on the two respective fibers using the maps ι_*^* from Remark 3.1.7. Compactly alignedness asks for these products to be compact again. This regularity property is commonly used as a standing hypothesis that can be transferred to requirements on the initial data in most situations studied so far, see for instance [Fow02, FPW13, HLS12, SY10].

Additionally, we would like to mention that there is a notion of compactly alignedness for topological k-graphs, see [Yee07]. It is shown in [CLSV11, Proposition 5.15] that the product systems naturally associated to topological k-graphs are compactly aligned if and only if the topological k-graphs are compactly aligned.

Definition 3.1.10. A product system of Hilbert bimodules $\mathcal{X}$ over P is called **compactly aligned**, if: For all $p, q \in P$ and $k_p \in \mathcal{K}(\mathcal{X}_p), k_q \in \mathcal{K}(\mathcal{X}_q)$, we have

$$\iota_p^{p\vee q}(k_p)\iota_q^{p\vee q}(k_q) = (k_p \otimes 1_{\mathcal{L}(\mathcal{X}_{(p\wedge q)^{-1}q})})(k_q \otimes 1_{\mathcal{L}(\mathcal{X}_{(p\wedge q)^{-1}p})}) \in \mathcal{K}(\mathcal{X}_{p\vee q}).$$

We will now proceed with stronger notions of regularity, namely the existence of a coherent system of finite Parseval frames or even a coherent system of orthonormal bases for product systems of Hilbert bimodules. This concept has been studied to some extent in [HLS12].

Definition 3.1.11. Let $\mathcal{H}$ be a Hilbert bimodule over A and $(\xi_i)_{i\in I} \subset \mathcal{H}$. Consider the following properties:

(1) $\langle \xi_i, \xi_j \rangle = \delta_{ij} 1_A$ for all $i, j \in I$.

(2) $\eta = \sum\limits_{i\in I} \xi_i \langle \xi_i, \eta \rangle$ for all $\eta \in \mathcal{H}$.

If the family $(\xi_i)_{i\in I}$ satisfies (2), it is called a **Parseval frame** for $\mathcal{H}$. It is said to be an **orthonormal basis** for $\mathcal{H}$, if (1) holds in addition to (2).

***Remark* 3.1.12.**

a) Equation 3.1.11 (2) is known as the reconstruction formula. Parseval frames play an important role in the theory of wavelet analysis, see for instance [LR07] and the references therein. As noted in [LR07, Section 4], many Hilbert modules admit Parseval frames without allowing for an orthonormal basis. In fact, this will be the generic case for the Hilbert bimodules arising from the dynamical systems of Section 1.3, as we will see in Section 3.3.

b) Every Hilbert bimodule which has a finite (countable) orthonormal basis is a finitely (countably) generated Hilbert bimodule.

***Example* 3.1.13.** In contrast to the case of orthonormal bases of a Hilbert space, the cardinality of an orthonormal basis of a Hilbert bimodule is not an invariant of the bimodule. As a toy example, take $A = C([-2,-1] \cup [1,2])$ and let $\mathcal{H} = {}_{id}A_{id}$ as in Example 3.1.2 (b). Then $\{1\}$ and $\{\chi_{[-2,-1]}, \chi_{[1,2]}\}$ are both orthonormal bases for $\mathcal{H}$.

Lemma 3.1.14. *Let $\mathcal{H}$ be a Hilbert bimodule. If $(\xi_i)_{i\in I} \subset \mathcal{H}$ satisfies 3.1.11 (1), then $\left(\Theta_{\xi_i,\xi_j}\right)_{i,j\in I}$ is a system of matrix units in $\mathcal{K}(\mathcal{H})$. If $(\xi_i)_{i\in I} \subset \mathcal{H}$ satisfies 3.1.11 (2) and I is finite, then $\sum_{i=1}^n \Theta_{\xi_i,\xi_i} = 1_{\mathcal{L}(\mathcal{H})}$ holds and hence $\mathcal{K}(\mathcal{H}) = \mathcal{L}(\mathcal{H})$.*

Proof. 3.1.11 (1) directly implies that $\left(\Theta_{\xi_i,\xi_j}\right)_{i,j\in I}$ is a system of matrix units. The reconstruction formula 3.1.11 (2) shows that $\left(\sum_{i\in F}\Theta_{\xi_i,\xi_i}\right)_{F\subset I \text{ finite}}$ converges strongly to $1_{\mathcal{L}(\mathcal{H})}$. Thus, if I is finite, we have $\sum_{i=1}^{n}\Theta_{\xi_i,\xi_i} = 1_{\mathcal{L}(\mathcal{H})}$ and the last claim follows since $\mathcal{K}(\mathcal{H})$ is an ideal in $\mathcal{L}(\mathcal{H})$ by Lemma 3.1.4. □

***Remark* 3.1.15.** A useful aspect of Parseval frames of Hilbert bimodules is that they are well-behaved with respect to the balanced tensor product: If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert bimodules over A with Parseval frames $(\xi_i)_{i\in I}$ and $(\eta_j)_{j\in J}$, respectively, then $(\xi_i\otimes\eta_j)_{(i,j)\in I\times J}$ is a Parseval frame for $\mathcal{H}_1\otimes_A\mathcal{H}_2$, where we refer to [LR07, Lemma 4.3] for a detailed proof. Therefore, a product system $\mathcal{X}$ of Hilbert bimodules over P is a product system with Parseval frames if and only if $\mathcal{X}_p$ admits a Parseval frame for each irreducible $p\in P$. Here $p\in P$ is said to be irreducible if $p=qr$ for $q,r\in P$ implies $q=1_P$ or $r=1_P$. The same statements hold for orthonormal bases instead of Parseval frames.

***Remark* 3.1.16.** Suppose X is a compact Hausdorff space and $\theta_1,\theta_2 : X \longrightarrow X$ are commuting regular surjective local homeomorphisms with $|\theta_1^{-1}(x)| = N_1$ and $|\theta_2^{-1}(x)| = N_2$ (where $x\in X$ is arbitrary). For $i=1,2$, denote by α_i the endomorphism of $C(X)$ given by $f\mapsto f\circ\theta_i$. As in Lemma 1.3.9, let us choose partitions of unity $(v_{1,i})_{i\in I_1}$ and $(v_{2,i})_{i\in I_2}$ subordinate to finite open covers $\mathcal{U}_1=(U_{1,i})_{i\in I_1}$ and $\mathcal{U}_2=(U_{2,i})_{i\in I_2}$ of X for θ_1 and θ_2, respectively. By Lemma 1.3.9, we know that each of these partitions of unity gives rise to a Parseval frame $(\nu_{j,i_j})_{i_j\in I_j}$ with $\nu_{j,i_j} := (N_j v_{j,i_j})^{\frac{1}{2}}$ of the Hilbert bimodule $C(X)_{\alpha_j}$, which is equipped with the inner product coming from the transfer operator L_j as constructed in Example 1.3.6. Taking into account [LR07, Lemma 4.3], it is no surprise that $(\nu_{1,i})_{i\in I_1}$ and $(\nu_{2,i})_{i\in I_2}$ yield a Parseval frame on the balanced tensor product of the two modules, i.e. on $C(X)_{\alpha_1\alpha_2}$. Interestingly, Lemma 1.3.10 indicates that this Parseval frame is again of the same form: We can construct a partition of unity $(v_{1,i_1}\alpha_1(v_{2,i_2}))_{i_1\in I_1,i_2\in I_2}$ for X from $(v_{1,i})_{i\in I_1}$ and $(v_{2,i})_{i\in I_2}$ which fits into the picture of Lemma 1.3.9 for $\theta_1\theta_2$.

For product systems of Hilbert bimodules, it may seem reasonable to ask for a system of (bilateral) orthonormal bases for the fibers that respect the semigroup structure on $\mathcal{X}$ inherited from P. This point of view is behind the definition of finite type systems given in [HLS12, Definition 3.5]. However, there is a problem arising from the commutativity of P: If we take $p,q\in P$, then $\mathcal{X}_p\otimes_A\mathcal{X}_q\cong\mathcal{X}_{pq}\cong\mathcal{X}_q\otimes_A\mathcal{X}_p$ implies the existence of an isomorphism of Hilbert bimodules $\tau_{p,q}:\mathcal{X}_p\otimes_A\mathcal{X}_q\longrightarrow\mathcal{X}_q\otimes_A\mathcal{X}_p$. If we fix orthonormal bases $(\xi_i)_{i\in I_p}$ and $(\eta_j)_{j\in I_q}$ of $\mathcal{X}_p$ and $\mathcal{X}_q$, respectively, we can either take the orthonormal basis $(\xi_i\eta_j)_{i\in I_p,j\in I_q}$ for $\mathcal{X}_{pq}$ coming from $M_{p,q}:\mathcal{X}_p\otimes_A\mathcal{X}_q\xrightarrow{\cong}\mathcal{X}_{pq}$ or $(\eta_j\xi_i)_{i\in I_p,j\in I_q}$ coming from $M_{q,p}:\mathcal{X}_q\otimes_A\mathcal{X}_p\xrightarrow{\cong}\mathcal{X}_{pq}$. But these two families need not match in general, as the following easy example shows.

***Example* 3.1.17.** Let $A=C^*(\mathbb{Z})=C^*((u_g)_{g\in\mathbb{Z}}), P=|2,3\rangle\subset\mathbb{N}^\times$ and $P\overset{\alpha}{\curvearrowright}A$ be given by $\alpha_p(u_g)=u_{pg}$. Then $\mathcal{X}_p:=A_{\alpha_p}$ with inner product $\langle u_g,u_h\rangle_p=\chi_{p\mathbb{Z}}(h-g)u_{p^{-1}(h-g)}$ defines a product system of Hilbert bimodules $\mathcal{X}$ over P. For fixed $p\in P$, an orthonormal basis of $\mathcal{X}_p$ is easily obtained by taking the unitaries corresponding to a transversal for $\mathbb{Z}/p\mathbb{Z}$, i.e. $(u_g)_{[g]\in\mathbb{Z}/p\mathbb{Z}}$. Let us consider $p=2$ and $q=3$. If we pick $\{0,1\}$ and $\{0,1,2\}$ as transversals for $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$, respectively, both $M_{2,3}$ and $M_{3,2}$ yield $\{0,\dots,5\}$ as the output transversal for $\mathcal{X}_{pq}$. However, if we choose $\{1,2,3\}$ instead of $\{0,1,2\}$ for q, we get $\{2,\dots,7\}$ and $\{1,\dots,6\}$. Thus, the two induced orthonormal bases of $\mathcal{X}_{pq}$ do not match.

In view of this example, the original definition for product systems of finite type proposed in [HLS12, Definition 3.5] seems to be too restrictive. Product systems arising from irreversible algebraic dynamical systems (G,P,θ), defined in Definition 1.1.5, will be tractable using this stronger notion if G contains a positive cone that is invariant under the action θ. This is the case in the previous example ($\mathbb{N}\subset\mathbb{Z}$) and we have seen that we do get a system of finite type in the sense of [HLS12, Definition 3.5] if we choose the representatives in a minimal way within the positive cone of G. But this rules out cases with mixed signs, e.g. $P=|-2,3\rangle$, and indicates that the choice of orthonormal bases on the fibers with irreducible index, if possible at all, has to be performed with care. That is why we will use a weaker notion of product systems of finite type:

Definition 3.1.18. A product system of Hilbert bimodules $\mathcal{X}$ over P with coefficients in a unital C*-algebra A is called a **product system of finite type** if there exists a finite Parseval frame for $\mathcal{X}_p$ for each irreducible $p\in P$.

***Remark* 3.1.19.** If $\mathcal{X}$ is a product system of finite type, then each fiber $\mathcal{X}_p$ has a finite Parseval frame by applying Remark 3.1.15 to a decomposition of p into irreducible elements (with multiplicities). Thus there exists a monoidal homomorphism $N: P\longrightarrow\mathbb{N}^\times$ which sends p to the cardinality N_p of the specified orthonormal basis for $\mathcal{X}_p$.

Obviously, Lemma 3.1.14 implies that $\mathcal{X}$ is compactly aligned. Fowler pointed out in [Fow99] that $\mathcal{X}$ is compactly aligned whenever all the fibers $\mathcal{X}_p$ are finite-dimensional.

3.2 Representations and C*-algebras for product systems

In this section, we recall some elementary facts about the representation theory for product systems of Hilbert bimodules in order to present the construction of the Cuntz-Nica-Pimsner algebra for compactly aligned product systems of Hilbert bimodules.

Definition 3.2.1. Let $\mathcal{X}$ be a product system over P and suppose B is a C*-algebra. A map $\mathcal{X}\xrightarrow{\varphi}B$, whose fiber maps $\mathcal{X}_p\longrightarrow B$ are denoted by φ_p, is called a **Toeplitz representation** of $\mathcal{X}$, if:

(1) φ_{1_P} is a $*$-homomorphism.

(2) φ_p is linear for all $p\in P$.

(3) $\varphi_p(\xi)^*\varphi_p(\eta)=\varphi_{1_P}(\langle\xi,\eta\rangle)$ for all $p\in P$ and $\xi,\eta\in\mathcal{X}_p$.

(4) $\varphi_p(\xi)\varphi_q(\eta) = \varphi_{pq}(\xi\eta)$ for all $p, q \in P$ and $\xi \in \mathcal{X}_p, \eta \in \mathcal{X}_q$.

A Toeplitz representation will be called a representation whenever there is no ambiguity.

***Remark* 3.2.2.** Let φ be a representation of $\mathcal{X}$ in B. For each $p \in P$, φ induces a $*$-homomorphism $\mathcal{K}(\mathcal{X}_p) \xrightarrow{\psi_{\varphi,p}} B$ given by $\Theta_{\xi,\eta} \mapsto \varphi_p(\xi)\varphi_p(\eta)^*$ for $\xi, \eta \in \mathcal{X}_p$.

Lemma 3.2.3. *A representation φ of $\mathcal{X}$ in B is contractive, i.e. $\|\varphi_p(\xi)\|_B \leq \|\xi\|_{\mathcal{X}_p}$ for all p and $\xi \in \mathcal{X}_p$. Moreover, φ is isometric if and only if φ_{1_P} is injective.*

Proof. Given $p \in P, \xi \in \mathcal{X}_p$, we get

$$\|\varphi_p(\xi)\|_B^2 = \|\varphi_p(\xi)^*\varphi(\xi)\|_B \overset{(3)}{=} \|\varphi_{1_P}(\langle \xi, \xi \rangle)\|_B \overset{(1)}{\leq} \| \langle \xi, \xi \rangle \|_A = \|\xi\|_{X_p}^2.$$

Since φ_{1_P} is a $*$-homomorphism, it is injective if and only if it is isometric. If this is the case, then the computation from above actually gives $\|\varphi_p(\xi)\|_B = \|\xi\|_{X_p}$. □

Definition 3.2.4. A representation φ of a compactly aligned product system $\mathcal{X}$ in B is **Nica covariant**, if

$$\psi_{\varphi,p}(k_p)\psi_{\varphi,q}(k_q) = \psi_{\varphi,p\vee q}\left(\iota_p^{p\vee q}(k_p)\iota_q^{p\vee q}(k_q)\right)$$

holds for all $p, q \in P$ and $k_p \in \mathcal{K}(\mathcal{X}_p), k_q \in \mathcal{K}(\mathcal{X}_q)$.

This is one particular instance where the property of being compactly aligned is needed to ensure that $\iota_p^{p\vee q}(k_p)\iota_q^{p\vee q}(k_q)$ is actually contained in the domain of $\psi_{\varphi,p\vee q}$. Before we proceed with additional covariance conditions that resemble the covariance condition originally used in [Pim97], let us look at the representation of a product system on its Fock space, compare [Fow02, Section 2].

***Example* 3.2.5.** Suppose $\mathcal{X}$ is a product system of Hilbert bimodules over P with coefficients in A. Let

$$\mathcal{F}(\mathcal{X}) = \overline{\bigoplus_{p\in P} \mathcal{X}_p} = \Big\{ (\eta_p)_{p\in P} \mid \sum_{p\in F} \langle \eta_p, \eta_p \rangle \text{ converges in } A \text{ as } F \nearrow P, F \text{ finite} \Big\}.$$

One can check that $\mathcal{F}(\mathcal{X})$ inherits the structure of a right Hilbert A-module from $\mathcal{X}$ so that the adjointable linear operators $\mathcal{L}(\mathcal{F}(\mathcal{X}))$ form a C*-algebra. For $p \in P$ and $\eta \in \mathcal{X}_p$, let e_p denote the element $(\delta_{pr})_{r\in P} \in \mathcal{F}(\mathcal{X})$ and $\eta e_p := (\eta\delta_{pr})_{r\in P}$. For $q \in P$ and $\xi \in X_q$, define $\varphi_{Fock,q}(\xi)$ to be the operator induced by multiplication in $\mathcal{X}$, i. e.

$$\varphi_{Fock,q}(\xi)(\eta e_p) = (\xi \cdot \eta)e_{qp}.$$

This defines a map $\varphi_{Fock} : \mathcal{X} \longrightarrow \mathcal{L}(\mathcal{F}(\mathcal{X}))$ and one readily verifies that φ_{Fock} is indeed a representation of the product system $\mathcal{X}$. Moreover, φ_{Fock} is isometric due to $\|\varphi_{Fock,1_P}(a)(a^*e_{1_P}\|_{\mathcal{F}(\mathcal{X})} = \|aa^*\|_A = \|a\|_A^2$ and Lemma 3.2.3.

The following fact is taken from [HLS12, Subsection 2.3].

Proposition 3.2.6. *The Fock representation φ_{Fock} of every compactly aligned product system $\mathcal{X}$ over P is Nica covariant.*

Proof. Let $p, q, r \in P$ and $\xi_p, \eta_p \in \mathcal{X}_p$, $\xi_q, \eta_q \in \mathcal{X}_q$ and $\zeta \in \mathcal{X}_r$. We have to show that

$$\psi_{\varphi_{Fock},p}(\Theta_{\xi_p,\eta_p})\psi_{\varphi_{Fock},q}(\Theta_{\xi_q,\eta_q})(\zeta e_r) = \psi_{\varphi,p\vee q}(\iota_p^{p\vee q}(\Theta_{\xi_p,\eta_p})\iota_q^{p\vee q}(\Theta_{\xi_q,\eta_q}))(\zeta e_r).$$

If $r \notin (p \vee q)P = pP \cap qP$, then both sides are 0 according to the definition of the representation φ_{Fock} in Example 3.2.5. So suppose we have $r \in (p \vee q)P$. Decomposing $\zeta = \zeta_q \zeta_{q^{-1}(p\vee q)} \zeta_{(p\vee q)^{-1}r}$ via $\mathcal{X}_r \cong \mathcal{X}_q \otimes_A \mathcal{X}_{q^{-1}(p\vee q)} \otimes_A \mathcal{X}_{(p\vee q)^{-1}r}$ and, similarly, $\Theta_{\xi_q,\eta_q}(\zeta_q)\zeta_{q^{-1}(p\vee q)} = \zeta'_p \zeta'_{p^{-1}(p\vee q)}$ yields

$$\begin{aligned}\psi_{\varphi_{Fock},p}(\Theta_{\xi_p,\eta_p})\psi_{\varphi_{Fock},q}(\Theta_{\xi_q,\eta_q})(\zeta e_r) &= \xi_p \left\langle \eta_p, \zeta'_p \right\rangle_{\mathcal{X}_p} \zeta'_{p^{-1}(p\vee q)} \zeta_{(p\vee q)^{-1}r} e_r \\ &= \psi_{\varphi,p\vee q}(\iota_p^{p\vee q}(\Theta_{\xi_p,\eta_p})\iota_q^{p\vee q}(\Theta_{\xi_q,\eta_q}))(\zeta e_r)\end{aligned}$$

This extends to all pairs of compact operators from $\mathcal{K}(\mathcal{X}_p)$ and $\mathcal{K}(\mathcal{X}_q)$. □

While Nica covariance is an outcome of having a product system instead of a single Hilbert bimodule and its form is rather straightforward, there have been different attempts to generalize the notion of Cuntz-Pimsner covariance from the case of a single Hilbert bimodule to general product systems. Let us recall the covariance condition introduced in [Pim97] using the product system picture provided in Example 3.1.8: Suppose $\mathcal{H}$ is a Hilbert bimodule over a C*-algebra A and (φ_0, φ_1) is a representation of $\mathcal{H}$. Then we can equally well study the induced representation φ of the product system $\mathcal{X}$ over $\mathbb{N}$ with fibers $\mathcal{X}_n = \mathcal{H}^{\otimes n}$, where $\mathcal{H}^{\otimes 0} = A$. (φ_0, φ_1) is said to be (Cuntz-Pimsner) covariant, if $\varphi_0(a) = \psi_{\varphi,n}(\phi_n(a))$ holds for all $a \in \phi_n^{-1}(\mathcal{K}(\mathcal{X}_n))$.

The intuitive approach to define a notion of Cuntz-Pimsner covariance for product systems by requiring Cuntz-Pimsner covariance on each fiber has been set up in [Fow02]. In [Kat04, Definition 3.4], Takeshi Katsura introduced a weaker version: Instead of $\phi_p^{-1}(\mathcal{K}(\mathcal{X}_p))$, only $\phi_p^{-1}(\mathcal{K}(\mathcal{X}_p)) \cap (\ker \phi_p)^\perp$ is taken into account. Since the left actions in our examples will always be injective, we will not discuss this aspect any further.

Several years later, a more involved approach of Aidan Sims and Trent Yeend led to a potentially different notion of Cuntz-Nica-Pimsner covariance, see [SY10, Section 3]. According to [SY10], their definition is motivated by the study of graph C*-algebras and was expected to be more suitable in the case of product systems where the left action ϕ need not be injective.

We will now present both covariance conditions for product systems and indicate what is currently known about their connections as well as their relation to Nica covariance. In order to avoid technicalities, we restrict ourselves to the case where the left action ϕ_s on each fiber $\mathcal{X}_s$ is injective. Therefore, we can neglect the inflation process from $\mathcal{X}$ to $\tilde{\mathcal{X}}$

taking place in [SY10, Section 3]. At this point, one may expect that the two notions ought to coincide. This is true at least to some extent, but non-trivial, see [SY10, Proposition 5.1 and Corollary 5.2].

Definition 3.2.7. Let B be a C*-algebra and suppose $\mathcal{X}$ is a compactly aligned product system of Hilbert bimodules over P with coefficients in A.

(CP_F) A representation $\mathcal{X} \xrightarrow{\varphi} B$ is called **Cuntz-Pimsner covariant** in the sense of [Fow02, Section 1], if it satisfies

$\psi_{\varphi,p}(\phi_p(a)) = \varphi_{1_P}(a)$ for all $p \in P$ and $a \in \phi_p^{-1}(\mathcal{K}(\mathcal{X}_p)) \subset A$.

(CP) A representation $\mathcal{X} \xrightarrow{\varphi} B$ is called **Cuntz-Pimsner covariant** in the sense of [SY10, Definition 3.9], if the following holds: Suppose $F \subset P$ is finite and we fix $k_p \in \mathcal{K}(\mathcal{X}_p)$ for each $p \in F$. If, for every $r \in P$, there is $s \geq r$ such that

$$\begin{array}{lcll} & \sum\limits_{p \in F} \iota_p^t(k_p) & = & 0 \qquad\qquad \text{holds for all } t \geq s, \\ \text{then} & \sum\limits_{p \in F} \psi_{\varphi,p}(k_p) & = & 0 \text{ holds true.} \end{array}$$

(CNP) A representation $\mathcal{X} \xrightarrow{\varphi} B$ is said to be **Cuntz-Nica-Pimsner covariant**, if it is Nica covariant and (CP)-covariant.

***Example* 3.2.8.** Although we have proven the Fock representation to be Nica covariant for compactly aligned product systems, there are lots of examples where it is far from being (CP_F)- or (CP)-covariant. Indeed, take $\mathcal{X}_p = A$ and $\phi_p(1_A) = 1_{\mathcal{L}(\mathcal{X}_p)} \in \mathcal{K}(\mathcal{X}_p)$. Then $E_p := \psi_{\varphi_{Fock},p}(\phi_p(1_A))$ defines a family of projections in $\mathcal{L}(\mathcal{F}(\mathcal{X}))$ with the following properties: $q \in pP$ implies $E_q \leq E_p$ and if, in addition, $p \notin qP$, then $E_p - E_q$ is a non-zero projection. Note that

$$E_{1_P} = \varphi_{Fock,1_P}(1_A) = 1_{\mathcal{L}(\mathcal{F}(\mathcal{X}))}.$$

If we assume that $\phi_p(A) \subset \mathcal{K}(\mathcal{X}_p) = \mathcal{L}(\mathcal{X}_p)$ holds for all $p \in P$, which is the case if we require $\phi_p(1_A) = 1_{\mathcal{L}(\mathcal{X}_p)}$ for all $p \in P$, we can obtain a (CP_F)-covariant representation from the Fock representation by modding out the ideal generated by the family $(1 - E_p)_{p \in P}$. Indeed, if we denote the corresponding quotient map by π and pick $a \in A$, then

$$\begin{aligned} \pi \circ \varphi_{Fock,1_P}(a) &= \pi(\varphi_{Fock,1_P}(a))\pi(1_{\mathcal{L}(\mathcal{F}(\mathcal{X}))}) \\ &= \pi(\varphi_{Fock,1_P}(a))\pi(E_p) \\ &= \psi_{\pi \circ \varphi_{Fock},p}(\phi_p(a)). \end{aligned}$$

When Aidan Sims and Trent Yeend introduced their notion of Cuntz-Pimsner covariance, they observed that their version is closely connected to the one proposed by Neal J. Fowler, compare [SY10, Proposition 5.1]:

Proposition 3.2.9. *Suppose $\mathcal{X}$ is a product system over P with coefficients in a unital C*-algebra A such that the left action ϕ_p on $\mathcal{X}_p$ is injective for all $p \in P$. If a representation φ of $\mathcal{X}$ is (CP_F)-covariant, then it is (CP)-covariant. If the left action $\phi_p(A)$ is by compacts for all $p \in P$, then the converse holds as well.*

In some instances, (CP_F)-covariance is known to imply Nica covariance. The result we are going to use is due to Fowler and we refer to [Fow02, Proposition 5.4] for a proof.

Proposition 3.2.10. *If $\mathcal{X}$ is a product system over P with coefficients in a unital C*-algebra A such that $\phi_p(1_A) = 1_{\mathcal{L}(\mathcal{X}_p)} \in \mathcal{K}(\mathcal{X}_p)$ for all $p \in P$, then $\mathcal{X}$ every (CP_F)-covariant representation is also Nica covariant.*

Corollary 3.2.11. *If $\mathcal{X}$ is a product system of finite type, then a representation φ of $\mathcal{X}$ is (CNP)-covariant if and only if it is (CP_F)-covariant.*

Proof. The result follows from Lemma 3.1.14 together with Proposition 3.2.9 and Proposition 3.2.10. □

We are now ready to associate three C*-algebras to $\mathcal{X}$ as universal objects corresponding to the different classes of representations.

Definition 3.2.12. For a compactly aligned product system $\mathcal{X}$ over P define $\mathcal{T}_\mathcal{X}$ to be the C*-algebra given by a Toeplitz representation $\iota_{\mathcal{T}_\mathcal{X}}$ of $\mathcal{X}$ that is universal for Toeplitz representations. In other words, if $\mathcal{X} \xrightarrow{\varphi} B$ is a Toeplitz representation, there is a $*$-homomorphism $\overline{\varphi} : \mathcal{T}_\mathcal{X} \longrightarrow C^*(\varphi)$ yielding a commutative diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, "\varphi"] \arrow[dr, "\iota_{\mathcal{T}_\mathcal{X}}"'] & & B \\
& \mathcal{T}_\mathcal{X} \arrow[ur, dotted, "\overline{\varphi}"'] &
\end{tikzcd}$$

Similarly, define $\mathcal{NT}_\mathcal{X}$ and $\mathcal{O}_\mathcal{X}$ to be the C*-algebras obtained from a universal Nica-covariant representation $\iota_{\mathcal{NT}_\mathcal{X}}$ and a universal Cuntz-Nica-Pimsner covariant representation $\iota_{\mathcal{O}_\mathcal{X}}$, respectively. $\mathcal{T}_\mathcal{X}$, $\mathcal{NT}_\mathcal{X}$ and $\mathcal{O}_\mathcal{X}$ are called the **Toeplitz algebra**, the **Nica-Toeplitz algebra**, and the **Cuntz-Nica-Pimsner algebra** associated to $\mathcal{X}$.

***Remark* 3.2.13.**

a) $\mathcal{NT}_\mathcal{X}$ is always non-trivial because Example 3.2.5 shows that the Fock representation is an isometric, Nica covariant representation. Additionally, following [HLS12, Remark 4.8], the homomorphism $\overline{\varphi}_{Fock} : \mathcal{NT}_\mathcal{X} \longrightarrow \mathcal{L}(F(\mathcal{X}))$ arising from the universal property of $\mathcal{NT}_\mathcal{X}$ is faithful if the left action on each fiber is by compacts. Hence, $\mathcal{NT}_\mathcal{X}$ is isomorphic to a C*-subalgebra of $\mathcal{L}(F(\mathcal{X}))$ in this case.

b) By their universal properties, $\mathcal{O}_{\mathcal{X}}$ is a quotient of $\mathcal{NT}_{\mathcal{X}}$, which in turn is a quotient of $\mathcal{T}_{\mathcal{X}}$.

c) This construction is a generalization of the original construction of Mihai Pimsner in [Pim97] as we can see by appealing to Example 3.1.8. It is not hard to show that we have $\mathcal{T}_{\mathcal{H}} \cong \mathcal{T}_{\mathcal{X}} \cong \mathcal{NT}_{\mathcal{X}}$ and $\mathcal{O}_{\mathcal{H}} \cong \mathcal{O}_{\mathcal{X}}$. Conversely, given a product system $\mathcal{X}$ over $\mathbb{N}$, we have $\mathcal{T}_{\mathcal{X}} \cong \mathcal{NT}_{\mathcal{X}} \cong \mathcal{T}_{X_1}$ and if the left action on each fiber is isometric or by compacts, then $\mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_{X_1}$ holds as well, see [Fow02, Proposition 2.11] for details. We note that all these isomorphisms are canonical.

d) In addition to c), the class of Cuntz-Nica-Pimsner algebras also includes crossed products of unital C*-algebras by more general groups than $\mathbb{Z}$. Moreover, given an action of an abelian monoid P on a non-unital C*-algebra A by endomorphisms that are extendible to its multiplier algebra $M(A)$, there is a suitable mean to obtain analogous objects, see [Lar10] for details.

The following lemma is a well-known fact resulting from [Fow02, Proposition 5.10]:

Lemma 3.2.14. *Suppose $\mathcal{X}$ is a compactly aligned product system over P. Then*

$$\begin{array}{rcl} \mathcal{T}_{\mathcal{X}} & = & \overline{span}\,\{\iota_{\mathcal{T}_{\mathcal{X}},p}(\xi)\iota_{\mathcal{T}_{\mathcal{X}},q}(\eta)^* \mid p,q \in P, \xi \in \mathcal{X}_p, \eta \in \mathcal{X}_q\}\,, \\ \mathcal{NT}_{\mathcal{X}} & = & \overline{span}\,\{\iota_{\mathcal{NT}_{\mathcal{X}},p}(\xi)\iota_{\mathcal{NT}_{\mathcal{X}},q}(\eta)^* \mid p,q \in P, \xi \in \mathcal{X}_p, \eta \in \mathcal{X}_q\}\,, \\ & \textit{and} & \\ \mathcal{O}_{\mathcal{X}} & = & \overline{span}\,\{\iota_{\mathcal{O}_{\mathcal{X}},p}(\xi)\iota_{\mathcal{O}_{\mathcal{X}},q}(\eta)^* \mid p,q \in P, \xi \in \mathcal{X}_p, \eta \in \mathcal{X}_q\} \end{array}$$

hold.

Proof. It suffices to prove the lemma for the Toeplitz algebra. In fact, we only have to show that the right hand side is multiplicatively closed since it is a self-adjoint, closed linear subspace and its elements generate $\mathcal{T}_{\mathcal{X}}$ as a C*-algebra. But this last part is provided by [Fow02, Proposition 5.10]. □

3.3 Applications for irreversible semigroup dynamical systems

This section is designed to build the bridge to the first two chapters by providing a product system of Hilbert bimodules for both irreversible algebraic dynamical systems (G,P,θ) and irreversible $*$-commutative dynamical systems (X,P,θ). The features of the dynamical systems (G,P,θ) and (X,P,θ) from Section 1.1 and Section 1.3, respectively, result in particularly well-behaved product systems $\mathcal{X}$. Therefore, it is possible to obtain a concrete presentation of $\mathcal{O}_{\mathcal{X}}$ in a natural way from the data of the dynamical system.

In the case of irreversible algebraic dynamical systems of finite type, this algebra is shown to be isomorphic to $\mathcal{O}[G,P,\theta]$ as introduced in Definition 2.2.1.

The corresponding result in the general case, that is, allowing for the presence of group endomorphisms θ_p of G with infinite index, requires a more involved argument. The reason is that the prerequisites for Corollary 3.2.11 are not met, so one has to deal with Nica covariance of representations. Here, we will only outline the strategy of the proof since this is more closely related to the Nica-Toeplitz algebra $\mathcal{NT}_{\mathcal{X}}$. Moreover, we will only need the results for finite type systems for the application in Chapter 4. In the case of irreversible $*$-commutative dynamical systems of finite type, we recover the C*-algebra $\mathcal{O}[X,P,\theta]$ from Section 2.4.

Proposition 3.3.1. *Suppose (G,P,θ) is an irreversible algebraic dynamical system. Let $(u_g)_{g\in G}$ denote the standard unitaries generating $C^*(G)$ and $P \overset{\alpha}{\curvearrowright} C^*(G)$ be the action induced by θ, i.e. $\alpha_p(u_g) = u_{\theta_p(g)}$ for $p \in P$ and $g \in G$. Then $\mathcal{X}_p := C^*(G)_{\alpha_p}$, with left action ϕ_p given by multiplication in $C^*(G)$ and inner product $\langle u_g, u_h\rangle_p = \chi_{\theta_p(G)}(g^{-1}h)u_{\theta_p^{-1}(g^{-1}h)}$ is an essential Hilbert bimodule. The union of all $\mathcal{X}_p$ forms a product system $\mathcal{X}$ over P with coefficients in $C^*(G)$. $\mathcal{X}$ is a product system with orthonormal bases. It is of finite type if (G,P,θ) is of finite type.*

Proof. It is straightforward to show that $\mathcal{X}$ defines a product system of essential Hilbert bimodules and we omit the details. For $p \in P$, we claim that every complete set of representatives $(g_i)_{i\in I}$ for $G/\theta_p(G)$ gives rise to an orthonormal basis of $\mathcal{X}_p$. Indeed, if we fix such a transversal $(g_i)_{i\in I}$ and pick $g \in G$, then $\langle u_{g_i}, u_g\rangle_p = \chi_{\theta_p(G)}(g_i^{-1}g)u_{\theta_p^{-1}(g_i^{-1}g)}$ equals 0 for all but one $j \in I$, namely the one representing the left-coset $[g]$ in $G/\theta_p(G)$. Thus, the family $(u_{g_i})_{i\in I} \subset \mathcal{X}_p$ consists of orthonormal elements with respect to $\langle\cdot,\cdot\rangle_p$, and

$$u_{g_i}\alpha_p\left(\langle u_{g_i}, u_g\rangle\right) = \delta_{ij}u_g,$$

so $(u_{g_i})_{i\in I}$ satisfies the right reconstruction formula from 3.1.11 (2). □

***Remark* 3.3.2.** If (G,P,θ) is an irreversible algebraic dynamical system and $\mathcal{X}$ denotes the associated product system from Proposition 3.3.1, then we have already seen in the proof of Proposition 3.3.1 that $\mathcal{X}_p$ has a finite orthonormal basis if $[G:\theta_p(G)]$ is finite. Since the left action is given by left multiplication, in other words, the elements of $C^*(G)$ act as diagonal operators, we have

$$\phi_p^{-1}(\mathcal{K}(\mathcal{X}_p)) = \begin{cases} C^*(G), & \text{if } [G:\theta_p(G)] \text{ is finite,} \\ 0, & \text{else.} \end{cases}$$

Lemma 3.3.3. *Suppose (G,P,θ) is an irreversible algebraic dynamical system and $\mathcal{X}$ denotes the associated product system from Proposition 3.3.1. Then the rank-one projection*

$\Theta_{u_g,u_g} \in \mathcal{K}(\mathcal{X}_p)$ *depends only on the equivalence class of* g *in* $G/\theta_p(G)$*. Moreover, if* φ *is a Nica covariant representation of* $\mathcal{X}$*, then*

$$\begin{aligned}&\psi_{\varphi,p}(\Theta_{u_{g_1},u_{g_1}})\psi_{\varphi,q}(\Theta_{u_{g_2},u_{g_2}})\\&= \begin{cases}\psi_{\varphi,p\vee q}(\Theta_{u_{g_3},u_{g_3}}) & \text{if } g_1^{-1}g_2 = \theta_p(g_3)\theta_q(g_4) \text{ for some } g_3, g_4 \in G,\\ 0 & \text{else.}\end{cases}\end{aligned}$$

holds for all $g_1, g_2 \in G$ *and* $p, q \in P$*.*

Proof. If $g_1 = g\theta_p(g_2)$ for some $g_2 \in G$, then

$$\Theta_{u_{g_1},u_{g_1}}(u_h) = \chi_{\theta_p(G)}(\theta_p(g_2^{-1})g^{-1}h)u_h = \chi_{\theta_p(G)}(g^{-1}h)u_h = \Theta_{u_g,u_g}(u_h)$$

for all $h \in G$ and hence $\Theta_{u_{g_1},u_{g_1}} = \Theta_{u_g,u_g}$. For the second claim, Nica covariance of $\iota_{\mathcal{O}_\mathcal{X}}$ implies

$$\psi_{\varphi,p}(\Theta_{u_{g_1},u_{g_1}})\psi_{\varphi,q}(\Theta_{u_{g_2},u_{g_2}}) = \psi_{\varphi,p}(\iota_p^{p\vee q}(\Theta_{u_{g_1},u_{g_1}})\iota_q^{p\vee q}(\Theta_{u_{g_2},u_{g_2}})).$$

If we denote $p' := (p \wedge q)^{-1}p$ and $q' := (p \wedge q)^{-1}q$, then

$$\iota_p^{p\vee q}(\Theta_{u_{g_1},u_{g_1}}) = \sum_{[g_3]\in G/\theta_{q'}(G)} \Theta_{u_{g_1\theta_p(g_3)},u_{g_1\theta_p(g_3)}} \in \mathcal{L}(\mathcal{X}_{p\vee q})$$

and

$$\iota_q^{p\vee q}(\Theta_{u_{g_2},u_{g_2}}) = \sum_{[g_4]\in G/\theta_{p'}(G)} \Theta_{u_{g_2\theta_q(g_4)},u_{g_2\theta_q(g_4)}} \in \mathcal{L}(\mathcal{X}_{p\vee q})$$

hold. We observe that

$$\Theta_{u_{g_1\theta_p(g_3)},u_{g_1\theta_p(g_3)}}\Theta_{u_{g_2\theta_q(g_4)},u_{g_2\theta_q(g_4)}}$$

is non-zero if and only if $[g_1\theta_p(g_3)] = [g_2\theta_q(g_4)] \in G/\theta_{p\vee q}(G)$. In particular, this is always zero if $g_1^{-1}g_2 \notin \theta_p(G)\theta_q(G)$. Let us assume that there are $g_3, \ldots, g_8 \in G$ such that

$$\begin{aligned}\theta_p(g_3^{-1})g_1^{-1}g_2\theta_q(g_4) &= \theta_{p\vee q}(g_7)\\ &\text{and}\\ \theta_p(g_5^{-1})g_1^{-1}g_2\theta_q(g_6) &= \theta_{p\vee q}(g_8).\end{aligned}$$

Rearranging the first equation to insert it into the second, we get

$$\theta_p(g_5^{-1}g_3)\theta_{p\vee q}(g_7)\theta_q(g_4^{-1}g_6) = \theta_{p\vee q}(g_8).$$

By injectivity of $\theta_{p\wedge q}$ this is equivalent to

$$\theta_{p'}(g_5^{-1}g_3)\theta_{(p\wedge q)^{-1}(p\vee q)}(g_7)\theta_{q'}(g_4^{-1}g_6) = \theta_{(p\wedge q)^{-1}(p\vee q)}(g_8).$$

From this equation we can easily deduce $g_5^{-1}g_3 \in \theta_{q'}(G)$ and $g_4^{-1}g_6 \in \theta_{p'}(G)$ from independence of $\theta_{p'}$ and $\theta_{q'}$, see Definition 1.1.5 (C). Thus, if there are $g_3, g_4 \in G$ such that $\theta_p(g_3^{-1})g_1^{-1}g_2\theta_q(g_4) \in \theta_{p\vee q}(G)$, then they are unique up to $\theta_{q'}(G)$ and $\theta_{p'}(G)$, respectively. This completes the proof. □

Theorem 3.3.4. *For an irreversible algebraic dynamical system of finite type (G, P, θ), let $\mathcal{X}$ be the product system of Hilbert bimodules over P with coefficients in $C^*(G)$ from Proposition 3.3.1. Then the map*

$$\begin{array}{ccc} \mathcal{O}[G,P,\theta] & \xrightarrow{\varphi} & \mathcal{O}_{\mathcal{X}} \\ u_g s_p & \mapsto & \iota_{\mathcal{O}_{\mathcal{X}},p}(u_g) \end{array}$$

is an isomorphism.

Proof. The idea is to exploit the respective universal property on both sides. We begin by showing that $(\iota_{\mathcal{O}_{\mathcal{X}},1_P}(u_g))_{g\in G}$ is a unitary representation of G and $(\iota_{\mathcal{O}_{\mathcal{X}},p}(1_{C^*(G)}))_{p\in P}$ is a representation of the monoid P by isometries satisfying (CNP 1)–(CNP 3), compare Definition 2.2.1. $\iota_{\mathcal{O}_{\mathcal{X}},1_P}$ is a $*$-homomorphism, so we get a unitary representation of G. In addition,

$$\begin{aligned} \iota_{\mathcal{O}_{\mathcal{X}},p}(1_{C^*(G)})^*\iota_{\mathcal{O}_{\mathcal{X}},p}(1_{C^*(G)}) &= \iota_{\mathcal{O}_{\mathcal{X}},1_P}(\langle 1_{C^*(G)}, 1_{C^*(G)}\rangle_p) \\ &= \iota_{\mathcal{O}_{\mathcal{X}},1_P}(1_{C^*(G)}) = 1_{\mathcal{O}_{\mathcal{X}}} \end{aligned}$$

and

$$\iota_{\mathcal{O}_{\mathcal{X}},p}(1_{C^*(G)})\iota_{\mathcal{O}_{\mathcal{X}},q}(1_{C^*(G)}) = \iota_{\mathcal{O}_{\mathcal{X}},pq}(1_{C^*(G)}\alpha_p(1_{C^*(G)})) = \iota_{\mathcal{O}_{\mathcal{X}},pq}(1_{C^*(G)})$$

show that we have a representation of P by isometries. (CNP 1) follows from

$$\iota_{\mathcal{O}_{\mathcal{X}},p}(1_{C^*(G)})\iota_{\mathcal{O}_{\mathcal{X}},1_P}(u_g) = \iota_{\mathcal{O}_{\mathcal{X}},p}(u_{\theta_p(g)}) = \iota_{\mathcal{O}_{\mathcal{X}},1_P}(u_{\theta_p(g)})\iota_{\mathcal{O}_{\mathcal{X}},p}(1_{C^*(G)}).$$

Let $p, q \in P$ and $g \in G$. Then (CNP 2) follows easily from applying Lemma 3.3.3 to

$$\begin{aligned} &\iota_{\mathcal{O}_{\mathcal{X}},p}(1_{C^*(G)})^*\iota_{\mathcal{O}_{\mathcal{X}},1_P}(u_g)\iota_{\mathcal{O}_{\mathcal{X}},q}(1_{C^*(G)}) \\ &\quad = \iota_{\mathcal{O}_{\mathcal{X}},p}(1_{C^*(G)})^*\psi_{\iota_{\mathcal{O}_{\mathcal{X}}},p}(\Theta_{1,1})\psi_{\iota_{\mathcal{O}_{\mathcal{X}}},q}(\Theta_{u_g,u_g})\iota_{\mathcal{O}_{\mathcal{X}},q}(u_g). \end{aligned}$$

Finally we observe that

$$\iota_{\mathcal{O}_{\mathcal{X}},1_P}(u_g)\iota_{\mathcal{O}_{\mathcal{X}},p}(1_{C^*(G)})\iota_{\mathcal{O}_{\mathcal{X}},p}(1_{C^*(G)})^*\iota_{\mathcal{O}_{\mathcal{X}},1_P}(u_g)^* = \psi_{\iota_{\mathcal{O}_{\mathcal{X}}},p}(\Theta_{u_g,u_g})$$

and the computation

$$\begin{aligned} \sum_{[g]\in G/\theta_p(G)} \psi_{\iota_{\mathcal{O}_{\mathcal{X}}},p}(\Theta_{u_g,u_g}) &= \psi_{\iota_{\mathcal{O}_{\mathcal{X}}},p}(1_{\mathcal{L}(\mathcal{X})}) = \psi_{\iota_{\mathcal{O}_{\mathcal{X}}},p}(\phi_p(1_{C^*(G)})) \\ &= \iota_{\mathcal{O}_{\mathcal{X}},1_P}(1_{C^*(G)}) = 1_{\mathcal{O}_{\mathcal{X}}} \end{aligned}$$

yield (CNP 3). Thus, $\varphi : \mathcal{O}[G,P,\theta] \longrightarrow \mathcal{O}_{\mathcal{X}}$ defines a surjective $*$-homomorphism. For the reverse direction, we want to show that

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\varphi_{CNP}} & \mathcal{O}[G,P,\theta] \\ \xi_{p,g} & \mapsto & u_g s_p \end{array}$$

defines a (CNP)-covariant representation of $\mathcal{X}$, where $\xi_{p,g}$ denotes the representative for u_g in $\mathcal{X}_p$. To do so, we have to verify (1)–(4) from Definition 3.2.1 and the (CNP)-covariance condition. (1) and (2) are obvious. Using (CNP 2) to compute

$$\begin{aligned}\varphi_{CNP,p}(\xi_{p,g_1})^*\varphi_{CNP,p}(\xi_{p,g_2}) &= s_p^* u_{g_1^{-1}g_2} s_p \\ &= \chi_{\theta_p(G)}(g_1^{-1}g_2) u_{\theta_p^{-1}(g_1^{-1}g_2)} \\ &= \varphi_{CNP,1_P}(\langle \xi_{p,g_1}, \xi_{p,g_2}\rangle),\end{aligned}$$

we get (3). (4) follows from (CNP 1) as

$$\begin{aligned}\varphi_{CNP,p}(\xi_{p,g_1})\varphi_{CNP,q}(\xi_{q,g_2}) &= u_{g_1} s_p u_{g_2} s_q \\ &= u_{g_1\theta_p(g_2)} s_{pq} \\ &= \varphi_{CNP,pq}(\xi_{p,g_1}\alpha_p(\xi_{q,g_2})).\end{aligned}$$

Thus, we are left with the (CNP)-covariance condition. But since $\mathcal{X}$ is a product system of finite type, see Proposition 3.3.1, we only have to show that φ_{CNP} is (CP_F)-covariant due to Corollary 3.2.11. Noting that $\varphi_p^{-1}(\mathcal{K}(\mathcal{X}_p)) = C^*(G)$ for all $p \in P$, we obtain

$$\begin{aligned}\psi_{\varphi_{CNP},p}(\phi_p(u_g)) &= \psi_{\varphi_{CNP},p}\left(\sum_{[h]\in G/\theta_p(G)} \Theta_{u_{gh},u_h}\right) \\ &= u_g \sum_{[h]\in G/\theta_p(G)} e_{h,p} \\ &= u_g = \varphi_{CNP,1_P}(\xi_{1_P,g}).\end{aligned}$$

Thus, φ_{CNP} is a (CNP)-covariant representation of $\mathcal{X}$. By the universal property of $\mathcal{O}_{\mathcal{X}}$, there exists a $*$-homomorphism $\overline{\varphi}_{CNP} : \mathcal{O}_{\mathcal{X}} \longrightarrow \mathcal{O}[G,P,\theta]$ such that $\overline{\varphi}_{CNP} \circ \iota_{\mathcal{O}_{\mathcal{X}}} = \varphi_{CNP}$. It is apparent that $\overline{\varphi}_{CNP}$ and φ are inverse to each other, so φ is an isomorphism. □

***Remark* 3.3.5.** For irreversible algebraic dynamical systems (G,P,θ) that are not of finite type, that is, there is some $p \in P$ such that $\theta_p(G)$ has infinite index in G, it is still true that $\mathcal{O}_{\mathcal{X}}$ and $\mathcal{O}[G,P,\theta]$ are canonically isomorphic. However, the proof requires more work. The reason is that Corollary 3.2.11 is not available in this situation. There is a proof which reveals a close connection between Nica covariance, (CNP 2), and independence built into (G,P,θ) by showing that Nica covariance boils down to its original form, see [Nic92]: A representation φ of the product system $\mathcal{X}$ for (G,P,θ) is Nica covariant if and only if $\varphi_p(1_{C^*(G)})$ and $\varphi_q(1_{C^*(G)})$ are doubly commuting isometries whenever p and q are relatively prime in P. As this is more closely related to $\mathcal{NT}_{\mathcal{X}}$ and will be addressed in a forthcoming project of the author in collaboration with Nathan Brownlowe and Nadia S.Larsen, we will not carry out the details here. Still, the interested reader may find further hints for a proof along these lines within the proof of Theorem 3.3.7.

In a similar fashion as in Proposition 3.3.1, we can construct a product system for a system of $*$-commuting transformations as presented in Section 1.3:

Proposition 3.3.6. *Suppose (X, P, θ) is an irreversible $*$-commutative dynamical system of finite type and $P \overset{\alpha}{\curvearrowright} C(X)$ is the action induced by θ, i.e. $\alpha_p(f) = f \circ \theta_p$ for $p \in P$ and $f \in C(X)$. Then $\mathcal{X}_p := C(X)_{\alpha_p}$, with left action ϕ_p given by multiplication in $C(X)$ and inner product $\langle f, g\rangle_p = L_p(\overline{f}g)$ is an essential Hilbert bimodule, where L_p is the natural transfer operator associated to α_p, see Example 1.3.6. The union of all $\mathcal{X}_p, p \in P$ forms a product system $\mathcal{X}$ of finite type over P with coefficients in $C(X)$.*

Proof. To see that $\mathcal{X}_p$ is an essential Hilbert bimodule, we recall that the transfer operator L_p, which is given by

$$L_p(f)(x) = \frac{1}{N_p} \sum_{y \in \theta_p^{-1}(x)} \overline{f}(y)$$

is a positive, linear map such that $L_p(f\alpha_p(g)) = L_p(f)g$ holds for all $f, g \in C(X)$. Thus, we can use [LR07, Lemma 3.3] to conclude that the seminorm $\|f\|_p := \langle f, f\rangle_p^{\frac{1}{2}}$ on $C(X)$ is equivalent to $\|\cdot\|_\infty$. Thus, $\langle\cdot,\cdot\rangle$ is positive definite on $C(X)$ and $C(X)$ is complete with respect to $\|\cdot\|_p$. The $\mathcal{X}_p$ form a product system since

$$\begin{array}{ccc} \mathcal{X}_p \otimes_{C(X)} \mathcal{X}_q & \xrightarrow{M_{p,q}} & \mathcal{X}_{pq} \\ f \otimes g & \mapsto & f\alpha_p(g) \end{array}$$

defines an isomorphism of Hilbert bimodules. Indeed, the left action is the same on both sides and

$$M_{p,q}((f \otimes g).h) = M_{p,q}(f \otimes g\alpha_q(h)) = f\alpha_p(g)\alpha_{pq}(h) = M_{p,q}(f \otimes g).h$$

shows that the right actions match. Finally, the inner products coincide as

$$\begin{aligned} \langle M_{p,q}(f \otimes g), M_{p,q}(f' \otimes g')\rangle_{pq} &= \langle f\alpha_p(g), f'\alpha_p(g')\rangle_{pq} \\ &= L_{pq}\left(\alpha_p(\overline{g})\overline{f}f'\alpha_p(g')\right) \\ &= L_q\left(\overline{g}L_p(\overline{f}f')g'\right) \\ &= \langle g, \phi_q\left(\langle f, f'\rangle_p\right) g'\rangle_q \\ &= \langle f \otimes g, f' \otimes g'\rangle_{\mathcal{X}_p \otimes_{C(X)} \mathcal{X}_q}. \end{aligned}$$

This shows that we have an injective morphism of Hilbert bimodules. Due to the structure of the balanced tensor product, $f \otimes g = f\alpha_p(g) \otimes 1_{C(X)}$ and $\alpha_p(1_{C(X)}) = 1_{C(X)}$, so $M_{p,q}$ is surjective as well. Thus, $\mathcal{X}$ is a product system over P with coefficients in $C(X)$. Lastly, $\mathcal{X}$ is seen to be of finite type by appealing to Lemma 1.3.9. □

Theorem 3.3.7. *Suppose (X, P, θ) is an irreversible $*$-commutative dynamical system of finite type, let $\mathcal{X}$ denote the product system constructed in Proposition 3.3.6. Then the map*

$$\begin{array}{ccc} \mathcal{O}[X,P,\theta] & \stackrel{\varphi}{\longrightarrow} & \mathcal{O}_{\mathcal{X}} \\ fs_p & \mapsto & \iota_{\mathcal{O}_{\mathcal{X}},p}(f) \end{array}$$

is an isomorphism.

Proof. The strategy is similar to the one for Theroem 3.3.4. We begin by showing that $(\iota_{\mathcal{O}_{\mathcal{X}},p}(1))_{p\in P}$ and $\iota_{\mathcal{O}_{\mathcal{X}},1_P}(C(X))$ induce φ. First of all, note that

$$\iota_{\mathcal{O}_{\mathcal{X}},p}(1)^*\iota_{\mathcal{O}_{\mathcal{X}},p}(1) = \iota_{\mathcal{O}_{\mathcal{X}},1_P}(\langle 1,1\rangle_p) = \iota_{\mathcal{O}_{\mathcal{X}},1_P}(L_p(1)) = 1_{\mathcal{O}_{\mathcal{X}}}$$

and $\iota_{\mathcal{O}_{\mathcal{X}},1_P}$ is a unital $*$-homomorphism. Conditions (I),(II) and (IV) are immediate:

(I) $\quad \iota_{\mathcal{O}_{\mathcal{X}},1_P}(\alpha_p(f))\iota_{\mathcal{O}_{\mathcal{X}},p}(1) = \iota_{\mathcal{O}_{\mathcal{X}},p}(\alpha_p(f)) = \iota_{\mathcal{O}_{\mathcal{X}},p}(1)\iota_{\mathcal{O}_{\mathcal{X}},1_P}(f)$

(II) $\iota_{\mathcal{O}_{\mathcal{X}},p}(1)^*\iota_{\mathcal{O}_{\mathcal{X}},1_P}(f)\iota_{\mathcal{O}_{\mathcal{X}},p}(1) = \iota_{\mathcal{O}_{\mathcal{X}},1_P}(\langle 1,f\rangle_p) = \iota_{\mathcal{O}_{\mathcal{X}},1_P}(L_p(f))$

(IV) Whenever $f_{i,j} \in C(X)$, where $i = 1,\ldots,n$ and $j = 1,2$, satisfy the reconstruction formula for $p \in P$, then

$$\sum_{1\leq i\leq n} \iota_{\mathcal{O}_{\mathcal{X}},1_P}(f_{i,1})\iota_{\mathcal{O}_{\mathcal{X}},p}(1)\iota_{\mathcal{O}_{\mathcal{X}},p}(1)^*\iota_{\mathcal{O}_{\mathcal{X}},1_P}(f_{i,2})^*$$
$$= \psi_{\iota_{\mathcal{O}_{\mathcal{X}}},p}\Big(\sum_{1\leq i\leq n}\Theta_{f_{i,1},f_{i,2}}\Big) = \psi_{\iota_{\mathcal{O}_{\mathcal{X}}},p}(\phi_p(1)) = \iota_{\mathcal{O}_{\mathcal{X}},1_P}(1) = 1_{\mathcal{O}_{\mathcal{X}}}$$

by (CP_F)-covariance of $\iota_{\mathcal{O}_{\mathcal{X}}}$, see Definition 3.2.7 and Corollary 3.2.11. Proving (III) is substantially harder. We need to show that the isometries corresponding to relatively prime $p,q \in P$ are doubly commuting. Since $\iota_{\mathcal{O}_{\mathcal{X}},p}(1)$ and $\iota_{\mathcal{O}_{\mathcal{X}},q}(1)$ are isometries, (III) is equivalent to

$$\psi_{\iota_{\mathcal{O}_{\mathcal{X}}},p}(\Theta_{1,1})\psi_{\iota_{\mathcal{O}_{\mathcal{X}}},q}(\Theta_{1,1}) = \psi_{\iota_{\mathcal{O}_{\mathcal{X}}},pq}(\Theta_{1,1}).$$

Nica covariance of $\iota_{\mathcal{O}_{\mathcal{X}}}$ implies that this is in turn the same as

$$\psi_{\iota_{\mathcal{O}_{\mathcal{X}}},pq}\left(\iota_p^{pq}(\Theta_{1,1})\iota_q^{pq}(\Theta_{1,1})\right) = \psi_{\iota_{\mathcal{O}_{\mathcal{X}}},pq}(\Theta_{1,1}),$$

which is reminiscent of the situation in Lemma 3.3.3. But this time we are only allowed to use Parseval frames instead of orthonormal bases for $\mathcal{X}_p$ and $\mathcal{X}_q$. So let us fix $(\nu_i)_{i\in I}$ with I finite for θ_p as in Lemma 1.3.9. In the same way, we choose $(\mu_j)_{j\in J}$ for θ_q. Then Lemma 1.3.9 says that these two families satisfy the reconstruction formula for p and q, respectively. Therefore, they fulfill $\sum_{i\in I}\Theta_{\nu_i,\nu_i} = 1_{\mathcal{L}(\mathcal{X}_p)}$ and $\sum_{j\in J}\Theta_{\mu_j,\mu_j} = 1_{\mathcal{L}(\mathcal{X}_q)}$. Next, we compute

$$\begin{aligned}
\psi_{\iota_{\mathcal{O}_{\mathcal{X}}},pq}\left(\iota_p^{pq}(\Theta_{1,1})\iota_q^{pq}(\Theta_{1,1})\right) &= \sum_{\substack{i\in I\\ j\in J}} \psi_{\iota_{\mathcal{O}_{\mathcal{X}}},pq}\left(\Theta_{\alpha_p(\mu_j),\alpha_p(\mu_j)}\Theta_{\alpha_q(\nu_i),\alpha_q(\nu_i)}\right)\\
&= \sum_{\substack{i\in I\\ j\in J}} \psi_{\iota_{\mathcal{O}_{\mathcal{X}}},pq}\left(\Theta_{\alpha_p(\mu_j\alpha_q(L_{pq}(\alpha_p(\mu_j)\alpha_q(\nu_i)))),\alpha_q(\nu_i)}\right)\\
&= \sum_{\substack{i\in I\\ j\in J}} \psi_{\iota_{\mathcal{O}_{\mathcal{X}}},pq}\left(\Theta_{\alpha_p(\mu_jE_q(\mu_jL_p(\alpha_q(\nu_i)))),\alpha_q(\nu_i)}\right)\\
&= \sum_{i\in I} \psi_{\iota_{\mathcal{O}_{\mathcal{X}}},pq}\left(\Theta_{E_p(\alpha_q(\nu_i)),\alpha_q(\nu_i)}\right),
\end{aligned}$$

where we used the (internal) reconstruction formula for $(\mu_j)_{j\in J}$ in the last step, compare Lemma 1.3.9. Since p and q are relatively prime, θ_p and θ_q $*$-commute by Definition 2.4.1. So Proposition 1.3.12 implies that $E_p(\alpha_q(f)) = \alpha_q(E_p(f))$ holds for all $f \in C(X)$. Therefore, we have shown that

$$\psi_{\iota_{\mathcal{O}_{\mathcal{X}}},p}(\Theta_{1,1})\psi_{\iota_{\mathcal{O}_{\mathcal{X}}},q}(\Theta_{1,1}) = \psi_{\iota_{\mathcal{O}_{\mathcal{X}}},pq}\left(\sum_{i\in I}\Theta_{\alpha_q(E_p(\nu_i)),\alpha_q(\nu_i)}\right).$$

Applying $\sum_{i\in I}\Theta_{\alpha_q(E_p(\nu_i)),\alpha_q(\nu_i)}$ to an element $f \in \mathcal{X}_{pq}$ takes the form

$$\sum_{i\in I}\alpha_q(E_p(\nu_i))\alpha_q(E_p(\nu_iL_q(f))) = \sum_{i\in I}\alpha_{pq}(L_p(\nu_i)L_p(\nu_iL_q(f))).$$

In view of Lemma 1.3.9 and $L_p(gE_p(h)) = L_p(g)L_p(h)$ for arbitrary $g,h \in C(X)$, see Definition 1.3.5, we deduce

$$\sum_{i\in I}\Theta_{\alpha_q(E_p(\nu_i)),\alpha_q(\nu_i)}(f) = E_{pq}(f) = \Theta_{1,1}(f) \text{ in } \mathcal{X}_{pq}.$$

Since f was arbitrary, we get

$$\sum_{i\in I}\Theta_{\alpha_q(E_p(\nu_i)),\alpha_q(\nu_i)} = \Theta_{1,1} \text{ in } \mathcal{L}(\mathcal{X}_{pq})$$

and hence (III) holds. This shows that the map φ is a $*$-homomorphism from $\mathcal{O}[X,P,\theta]$ onto $\mathcal{O}_{\mathcal{X}}$. For the reverse direction, we show that

$$\begin{array}{rcl}
\mathcal{X} & \overset{\varphi_{CNP}}{\longrightarrow} & \mathcal{O}[X,P,\theta]\\
\mathcal{X}_p \ni f & \mapsto & fs_p
\end{array}$$

defines a (CNP)-covariant representation of $\mathcal{X}$. Clearly, φ_{CNP} satisfies (1) and (2) from Definition 3.2.1. For (3), note that

$$\varphi_{CNP,p}(f)^*\varphi_{CNP,p}(g) = s_p^*\overline{f}gs_p \overset{(III)}{=} L_p(\overline{f}g) = \varphi_{CNP,1_P}(\langle f,g\rangle_p)$$

holds for all $p \in P$ and $f, g \in C(X)$. (4) follows from

$$\varphi_{CNP,p}(f)\varphi_{CNP,q}(g) = fs_p gs_q \overset{(II)}{=} f\alpha_p(g)s_{pq} = \varphi_{CNP,pq}(f\alpha_p(g)).$$

As in Theorem 3.3.4, we only have to show (CP_F)-covariance in order to get that $\tilde{\varphi}$ is (CNP)-covariant. To verify this, we fix $(\nu_i)_{I\in I} \subset C(X)$ with I finite for $p \in P$ as in Lemma 1.3.9 and obtain

$$\begin{array}{rclcl} \psi_{\varphi_{CNP},p}(\phi_p(f)) & = & \psi_{\varphi_{CNP},p}\left(\sum\limits_{i\in I}\Theta_{f\nu_i,\nu_i}\right) & = & f\sum\limits_{i\in I}\nu_i s_p s_p^*\nu_i \\ & \overset{(IV)}{=} & f & = & \varphi_{CNP,1_P}(f) \end{array}$$

for all $f \in C(X)$. Thus, φ_{CNP} is a (CNP)-covariant representation of $\mathcal{X}$. It is apparent that the induced $*$-homomorphism $\overline{\varphi}_{CNP} : \mathcal{O}_{\mathcal{X}} \longrightarrow \mathcal{O}[X, P, \theta]$ is the inverse of φ. $\square$

Chapter 4

Topological freeness for irreversible $*$-commutative dynamical systems

For this chapter, we will restrict our focus to irreversible $*$-commutative dynamical systems of finite type (X,P,θ). We examine in how far $\mathcal{O}[X,P,\theta]$ witnesses topological freeness and minimality of (X,P,θ). From the point of view of the interplay between topological dynamical systems and their associated C*-algebras, this is a fundamental question.

Classically, the first object to study in topological dynamical systems is a single homeomorphism σ of a compact Hausdorff space X, which induces an automorphism α of $C(X)$ via $\alpha(f) := f \circ \sigma$. The C*-algebra naturally associated to (X,σ) is the crossed product $C(X) \rtimes_\alpha \mathbb{Z}$ generated by a copy of $C(X)$ and a unitary u that implements α in the sense that $ufu^* = \alpha(f)$ holds for all $f \in C(X)$. We would like to mention that the crossed product is sometimes referred to as the transformation group C*-algebra for (X,σ).

It is well known that the crossed product is simple if and only if the topological dynamical system is minimal in the sense that the only closed, σ-invariant subsets of X are $\emptyset$ and X. Looking for a generalization of this result to the case of $\mathbb{Z}^d$-actions, minimality of $(X,\mathbb{Z}^d,\sigma)$ alone turned out to be insufficient for simplicity of $C(X) \rtimes_\alpha \mathbb{Z}^d$, unless the action is free. This is automatic in the case of a single, minimal homeomorphism on an infinite space X and means that σ_n has no fixed points for all $n \neq 0$. Soon it turned out that simplicity of the transformation group C*-algebra does not detect the combination of minimality and freeness on the nose. Instead, one has to weaken freeness to topological freeness, where the set of fixed points of σ_n is required to have empty interior for each $n \neq 0$, see [KT90, AS94].

Interestingly, the proof of this correspondence exhibits the less prominent intermediate result that topological freeness of $(X,\mathbb{Z}^d,\sigma)$ is characterized by the property that every non-zero ideal inside the transformation group C*-algebra intersects $C(X)$ non-trivially. This property is sometimes referred to as the *ideal intersection property* (of $C(X)$ in

$C(X) \rtimes_\alpha \mathbb{Z}^d$) and is actively studied for group crossed products, see for instance [Sie10, ST09, dJST12].

Additionally, building on [ZM68], it has been observed that the ideal intersection property is equivalent to $C(X)$ being a maximal abelian subalgebra in the transformation group C*-algebra for amenable discrete groups, see [KT90, Theorem 4.1 and Remark 4.2].

Doubtlessly, there is much more to say about the structure of group crossed products and we refer to [BO08] for an extensive and well-structured exposition. Instead, let us return to the case of a single transformation, which we now denote by θ, and drop the reversibility of the system. One way of doing this in a moderate fashion is to demand that $\theta : X \longrightarrow X$ be a surjective local homeomorphism. This has the convenient consequence that the induced map $\alpha : C(X) \longrightarrow C(X)$, given by $\alpha(f) = f \circ \theta$ is a unital, injective endomorphism. Moreover, θ is finite-to-one and the number of preimages $|\theta^{-1}(x)|$ of a singleton $x \in X$ is constant on the path-connected components of X. For simplicity, let us also assume that this number is the same for all path-connected components of X. Such transformations are called regular in Definition 1.3.4. Under these assumptions, there is a natural transfer operator L for α, see Example 1.3.6. In place of the group crossed product of $C(X)$ by $\mathbb{Z}$, it is reasonable to use the construction of a crossed product by an endomorphism $C(X) \rtimes_{\alpha,L} \mathbb{N}$ as introduced by Ruy Exel in [Exe03a].

For this setup, Ruy Exel and Anatoly Vershik showed that $C(X) \rtimes_{\alpha,L} \mathbb{N}$ is simple if and only if (X, θ) is minimal, see [EV06, Theorem 11.3]. Their argument shows that topological freeness implies that $C(X)$ intersects every non-zero ideal in $C(X) \rtimes_{\alpha,L} \mathbb{N}$ non-trivially. But to the best of the author's knowledge, it was not until the work of Toke Meier Carlsen and Sergei Silvestrov that the equivalence of these two conditions was established in the irreversible setting described in the preceding paragraph, see [CS09]. In fact, their approach partially used results from [EV06] and incorporated two additional equivalent formulations.

Briefly speaking, we will show that the results and most ideas from [CS09] are extendible to the realm of irreversible $*$-commutative dynamical systems of finite type:

Theorem 4.1.9. *Suppose (X, P, θ) is an irreversible $*$-commutative dynamical system of finite type. Then the following statements are equivalent:*

(1) The dynamical system (X, P, θ) is topologically free.

(2) Every non-zero ideal I in $\mathcal{O}[X, P, \theta]$ satisfies $I \cap C(X) \neq 0$.

(3) The representation φ of $\mathcal{O}[X, P, \theta]$ on $\ell^2(X)$ from Proposition 2.4.4 is faithful.

(4) $C(X)$ is a masa in $\mathcal{O}[X, P, \theta]$.

For this purpose we will employ several auxiliary results from Section 2.4. In addition, Theorem 3.3.7 will grant us access to the gauge-invariant uniqueness theorem for Cuntz-Nica-Pimsner algebras of product systems that was established in [CLSV11].

Once this is completed, it takes relatively little effort to characterize simplicity of $\mathcal{O}[X,P,\theta]$ by minimality of (X,P,θ), see Theorem 4.2.11. In view of the group case, this may seem a bit odd at first since topological freeness is not part of the characterization. But a modification of [EV06, Proposition 11.1] shows that minimal irreversible $*$-commutative dynamical systems of finite type are automatically topological free, see Proposition 4.2.10. As a corollary we deduce that commutative irreversible algebraic dynamical systems of finite type (G,P,θ) are minimal in the sense of Definition 1.1.5 if and only if their corresponding C*-algebra $\mathcal{O}[G,P,\theta]$ is simple.

Let us also mention that it is possible to derive a characterization of simplicity of $\mathcal{O}[X,P,\theta]$ by considering the transformation groupoid associated to (X,P,θ). This has been accomplished in greater generality by Jonathan H. Brown, Lisa Orloff Clark, Cynthia Farthing and Aidan Sims, see [BOCFS14, Theorem 5.1 and Corollary 7.8]. Moreover, one can deduce the equivalence of (1) and (2) out of [BOCFS14, Proposition 5.5 and Proposition 7.5]. Nevertheless, the methods used here differ substantially from the ones in [BOCFS14] and provide an account that is formulated entirely in the language of topological dynamical systems. Furthermore, the part involving conditions (3) and (4) is not covered by [BOCFS14].

For convenience, we recall the definition of an irreversible $*$-commutative dynamical system of finite type (X,P,θ) and its C*-algebra $\mathcal{O}[X,P,\theta]$:

Definition 1.3.13. An **irreversible $*$-commutative dynamical system of finite type** is a triple (X,P,η) consisting of

(A) a compact Hausdorff space X,

(B) a countably generated free abelian monoid P with unit 1_P and

(C) an action $P \overset{\eta}{\curvearrowright} X$ by regular surjective local homeomorphisms with the following property: η_p and η_q $*$-commute if and only if p and q are relatively prime in P.

Definition 2.4.1. $\mathcal{O}[X,P,\theta]$ is the universal C*-algebra generated by $C(X)$ and a representation of the monoid P by isometries $(s_p)_{p\in P}$ subject to the relations:

$$
\begin{array}{llcll}
(I) & s_pf & = & \alpha_p(f)s_p & \text{for all } f \in C(X), p \in P.\\
(II) & s_p^*fs_p & = & L_p(f) & \text{for all } f \in C(X), p \in P.\\
(III) & s_p^*s_q & = & s_qs_p^* & \text{if } p \text{ and } q \text{ are relatively prime in } P.
\end{array}
$$

(IV) If $p \in P$ and $f_{i,1}, f_{i,2} \in C(X), 1 \leq i \leq n$, satisfy the reconstruction formula

$$\sum_{1\leq i\leq n} f_{i,1}E_p(\bar{f}_{i,2}f) = f \qquad \text{for all } f \in C(X),$$

then $$\sum_{1\leq i\leq n} f_{i,1}s_ps_p^*\bar{f}_{i,2} = 1.$$

Throughout this chapter, let (X,P,θ) be an irreversible $*$-commutative dynamical system of finite type, unless specified otherwise.

4.1 C*-algebraic characterizations of topological freeness

In this section we establish an equivalence between topological freeness for irreversible $*$-commutative dynamical systems of finite type (X,P,θ) and three different C*-algebraic properties of $\mathcal{O}[X,P,\theta]$, see Theorem 4.1.9. The proof of this result essentially relies on Proposition 4.1.8, where we prove that topological freeness of (X,P,θ) gives the ideal intersection property for $C(X)$ in $\mathcal{O}[X,P,\theta]$, and the technical Lemma 4.1.4, which uses a faithful version $\tilde{\varphi}$ of the representation φ from Proposition 2.4.4, see Proposition 4.1.1 and Proposition 4.1.3. In fact, Lemma 4.1.4 is a straightforward generalization of [CS09, Lemma 5] to the setting of irreversible $*$-commutative dynamical systems of finite type.

Recall that P is an Ore semigroup with enveloping group $P^{-1}P$ denoted by H. In the following, $(\xi_{x,h})_{(x,h)\in X\times H}$ denotes the standard orthonormal basis of $\ell^2(X\times H)$.

Proposition 4.1.1. *Let* $\tilde{M}_f\xi_{x,h} := f(x)\xi_{x,h}$ *and* $\tilde{S}_p\xi_{x,h} = N_p^{-\frac{1}{2}} \sum\limits_{y\in\theta_p^{-1}(x)} e_{y,ph}$ *for* $f\in C(X)$ *and* $(x,h)\in X\times H$. *Then*

$$\begin{array}{ccc} \mathcal{O}[X,P,\theta] & \xrightarrow{\tilde{\varphi}} & \mathcal{L}\left(\ell^2(X\times H)\right) \\ fs_p & \mapsto & \tilde{M}_f\tilde{S}_p \end{array}$$

defines a $*$-homomorphism, which is faithful on $C(X)$.

Proof. As $\tilde{S}_p^*\xi_{x,h} = N_p^{-\frac{1}{2}}\xi_{\theta_p(x),p^{-1}h}$, the proof of Proposition 2.4.4 carries over verbatim. □

***Remark* 4.1.2.** As in [CS09, Proposition 4], we would like to show that $\tilde{\varphi}$ is faithful by using a gauge-invariant uniqueness theorem. For this purpose, let us recall that Theorem 3.3.7 asserts that $\mathcal{O}[X,P,\theta]$ is the Cuntz-Nica-Pimsner algebra for the product system of Hilbert bimodules associated to (X,P,θ) in Proposition 3.3.6. We intend to make use of [CLSV11, Corollary 4.12 (iv)] and remark that the terminology related to coactions can be phrased in terms of actions of the dual group of the discrete, abelian group $H=P^{-1}P$, which we denote by L. Under this transformation, the coaction δ in [CLSV11, Proposition 3.5] corresponds to the natural gauge action γ of L on $\mathcal{O}[X,P,\theta]$ given by

$$\gamma_\ell(f) = f \text{ and } \gamma_\ell(s_p) = \ell(p)s_p$$

for $f\in C(X), p\in P$ and $\ell\in L$. Thus [CLSV11, Definition 4.10] and [CLSV11, Corollary 4.12 (iv)] imply that $\mathcal{O}[X,P,\theta]$ has the following gauge-invariant uniqueness property:
A surjective $*$-homomorphism $\phi : \mathcal{O}[X,P,\theta] \longrightarrow B$ onto a C*-algebra B is injective if and only if the following two conditions hold:

a) There is an L-action β on B for which ϕ is (γ,β)-equivariant ($\beta_\ell\circ\phi = \phi\circ\gamma_\ell$).

b) ϕ is faithful on $C(X)$.

This enables us to prove the analogue of [CS09, Proposition 4]:

Proposition 4.1.3. *The representation $\tilde{\varphi}$ is faithful.*

Proof. Faithfulness of $\tilde{\varphi}$ on $C(X)$ has already been established. For $\ell \in L$ define $U_\ell \in \mathcal{L}\left(\ell^2(X \times H)\right)$ by $U_\ell \xi_{x,h} = \ell(h)\xi_{x,h}$. This gives a unitary representation of L and enables us to define an action β of L on $\mathcal{L}\left(\ell^2(X \times H)\right)$ via $\beta_\ell(T) = U_\ell T U_\ell^*$. We observe that, on $\tilde{\varphi}(\mathcal{O}[X, P, \theta])$, β is given by

$$\beta_\ell\left(\tilde{M}_f\right) = \tilde{M}_f \text{ and } \beta_\ell\left(\tilde{S}_p\right) = \ell(p)\tilde{S}_p$$

for all $f \in C(X)$ and $p \in P$. Thus $\tilde{\varphi}$ is (γ, β)-equivariant. According to the conclusion of Remark 4.1.2, $\tilde{\varphi}$ is faithful on all of $\mathcal{O}[X, P, \theta]$. □

Recall from Corollary 2.4.13, that the conditional expectation $G : \mathcal{O}[X, P, \theta] \longrightarrow C(X)$ is given by $G(f s_p s_q^* g) = \delta_{pq}\ N_p^{-1} f g$.

Lemma 4.1.4. *Let $\tilde{\varphi}$ be the representation from Proposition 4.1.1 and $a \in \mathcal{O}[X, P, \theta]$. Then the following statements hold:*

i) $\langle \tilde{\varphi}(a)\xi_{x,h}, \xi_{x,h}\rangle = G(a)(x)$ *for all* $(x, h) \in X \times H$.

ii) $a \in C(X)$ *if and only if* $\langle \tilde{\varphi}(a)\xi_{x_1,h_1}, \xi_{x_2,h_2}\rangle = 0$ *for all* $(x_1, h_1) \neq (x_2, h_2)$.

iii) If $(x_1, h_1), (x_2, h_2) \in X \times H$ *satisfy* $\langle \tilde{\varphi}(a)\xi_{x_1,h_1}, \xi_{x_2,h_2}\rangle \neq 0$, *there are* $p, q \in P$ *and open neighbourhoods* U_1 *of* x_1, U_2 *of* x_2 *with the following properties:*

  (a) $ph_1 = qh_2$.

  (b) $\theta_q(x_1) = \theta_p(x_2)$.

  (c) If $x_3 \in U_1$ *and* $x_4 \in U_2$ *satisfy* $\theta_q(x_3) = \theta_p(x_4)$, *then* $\langle \tilde{\varphi}(a)\xi_{x_3,h_1}, \xi_{x_4,h_2}\rangle \neq 0$.

Proof. Recall that the linear span of $\{f s_p s_q^* g \mid f, g \in C(X),\ p, q \in P\}$ is dense in $\mathcal{O}[X, P, \theta]$ according to Lemma 2.4.5. As both sides of the equation in i) are linear and continuous in a, it suffices to prove the equation for $a = f s_p s_q^* g$. This is achieved by

$$\begin{aligned}
\left\langle \xi_{x,h}, \tilde{\varphi}(f s_p s_q^* g)\xi_{x,h}\right\rangle &= \left\langle \tilde{\varphi}(s_p^* \overline{f})\xi_{x,h}, \tilde{\varphi}(s_q^* g)\xi_{x,h}\right\rangle \\
&= \delta_{pq}\ N_p^{-1}\ f(x)g(x) \\
&= G(f s_p s_q^* g)(x)
\end{aligned}$$

For ii), we note that $a \in C(X)$ certainly implies $\langle \tilde{\varphi}(a)\xi_{x_1,h_1}, \xi_{x_2,h_2}\rangle = 0$ for all $(x_1, h_1) \neq (x_2, h_2)$. Conversely, if $a \in \mathcal{O}[X, P, \theta]$ satisfies $\langle \tilde{\varphi}(a)\xi_{x_1,h_1}, \xi_{x_2,h_2}\rangle = 0$ whenever $(x_1, h_1) \neq (x_2, h_2)$, part i) implies $\tilde{\varphi}(a) = \tilde{\varphi}(G(a))$. As $\tilde{\varphi}$ is faithful, see Proposition 4.1.3, this shows $a = G(a) \in C(X)$.

In order to prove iii), suppose we have $(x_1,h_1),(x_2,h_2) \in X \times H$ such that $\varepsilon := |\langle\tilde{\varphi}(a)\xi_{x_1,h_1},\xi_{x_2,h_2}\rangle| > 0$. Using Lemma 2.4.5, we can choose $p_1,q_1,\dots,p_m,q_m \in P$ and $f_1,g_1,\dots,f_m,g_m \in C(X)$ such that

$$a_m := \sum_{i=1}^{m} f_i s_{p_i} s_{q_i}^* g_i \text{ satisfies } \|a - a_m\| < \frac{\varepsilon}{3}.$$

As

$$\tilde{\varphi}\left(f_i s_{p_i} s_{q_i}^* g_i\right)\xi_{x_1,h_1} = N_{p_i q_i}^{-\frac{1}{2}} \sum_{y\in\theta_{p_i}^{-1}\left(\theta_{q_i}(x_1)\right)} f_i(y)g_i(x_1)\xi_{y,p_i q_i^{-1}h_1},$$

we either get $\left\langle\tilde{\varphi}\left(f_i s_{p_i} s_{q_i}^* g_i\right)\xi_{x_1,h_1},\xi_{x_2,h_2}\right\rangle = 0$ or $x_2 \in \theta_{p_i}^{-1}\left(\theta_{q_i}(x_1)\right)$ and $p_i q_i^{-1}h_1 = h_2$. The latter conditions are equivalent to $\theta_{p_i}(x_2) = \theta_{q_i}(x_1)$ and $p_i h_1 = q_i h_2$ since P is commutative. Note that there is at least one i such that

$$\left\langle\tilde{\varphi}\left(f_i s_{p_i} s_{q_i}^* g_i\right)\xi_{x_1,h_1},\xi_{x_2,h_2}\right\rangle \neq 0$$

because $\tilde{\varphi}$ is contractive and $\|a-a_m\| < \frac{\varepsilon}{3}$. Therefore, possibly changing the enumeration, we can assume that there is $1 \leq n \leq m$ such that

$$\left\langle\tilde{\varphi}\left(f_i s_{p_i} s_{q_i}^* g_i\right)\xi_{x_1,h_1},\xi_{x_2,h_2}\right\rangle \neq 0 \text{ if and only if } 1 \leq i \leq n.$$

Let $a_n := \sum_{i=1}^{n} f_{i,1}s_{p_{i,1}}s_{p_{i,2}}^* f_{i,2}$. Since P is lattice ordered there is a unique element $p_0 := p_1 \vee \dots \vee p_n$. Additionally, set $q_0 := h_2^{-1}ph_1 \in H$ and note that $q_0 \in P$ since $h_2^{-1}p_i h_1 = q_i \in P$ for all $1 \leq i \leq n$. For each $1 \leq i \leq n$, there are open neighbourhoods $U'_{i,1}$ of x_1 and $U'_{i,2}$ of x_2 such that

- θ_{p_i} is injective on $U'_{i,1}$,
- θ_{q_i} is injective on $U'_{i,2}$, and
- $N_{p_i q_i}^{-\frac{1}{2}} |f_i(y_1)g_i(y_2) - f_i(x_1)g_i(x_2)| < \frac{\varepsilon}{3n}$ for all $y_1 \in U'_{i,1}, y_2 \in U'_{i,2}$.

This is always possible because the transformations $\theta_{p_i},\theta_{q_i}$ are local homeomorphisms and the function $X^2 \longrightarrow \mathbb{C}$ given by $(y_1,y_2) \mapsto f_{i,1}(y_1)f_{i,2}(y_2)$ is continuous since. Then

$$U_{i,1} := \theta_{p_i}^{-1}\left(\theta_{p_i}(U'_{i,1}) \cap \theta_{q_i}(U'_{i,2})\right)$$

defines an open neighbourhood of x_1 such that for each $y_1 \in U_{i,1}$ there is a unique $y_2 \in U_{i,2}$ with $\theta_{p_i}(y_2) = \theta_{q_i}(y_1)$. Accordingly, set

$$U_{i,2} := \theta_{q_i}^{-1}\left(\theta_{p_i}(U'_{i,1}) \cap \theta_{q_i}(U'_{i,2})\right).$$

and take $U_j := \bigcap_{i=1}^n U_{i,j}$ for $j = 1, 2$. Now suppose $x_3 \in U_1,\ x_4 \in U_2$ satisfy $\theta_q(x_3) = \theta_p(x_4)$. Using the triangle inequality for the first two steps, we get

$$\begin{aligned}
|\langle\tilde{\varphi}(a)\xi_{x_3,h_1}, \xi_{x_4,h_2}\rangle| \geq\ & \varepsilon - |\langle\tilde{\varphi}(a)\xi_{x_1,h_1}, \xi_{x_2,h_2}\rangle - \langle\tilde{\varphi}(a)\xi_{x_3,h_1}, \xi_{x_4,h_2}\rangle| \\
\geq\ & \varepsilon - |\langle\tilde{\varphi}(a)\xi_{x_1,h_1}, \xi_{x_2,h_2}\rangle - \langle\tilde{\varphi}(a_m)\xi_{x_1,h_1}, \xi_{x_2,h_2}\rangle| \\
& - |\langle\tilde{\varphi}(a_m)\xi_{x_1,h_1}, \xi_{x_2,h_2}\rangle - \langle\tilde{\varphi}(a_m)\xi_{x_3,h_1}, \xi_{x_4,h_2}\rangle| \\
& - |\langle\tilde{\varphi}(a_m)\xi_{x_3,h_1}, \xi_{x_4,h_2}\rangle - \langle\tilde{\varphi}(a)\xi_{x_3,h_1}, \xi_{x_4,h_2}\rangle| \\
=\ & \varepsilon - |\langle\tilde{\varphi}(a - a_m)\xi_{x_1,h_1}, \xi_{x_2,h_2}\rangle| \\
& - |\langle\tilde{\varphi}(a_n)\xi_{x_1,h_1}, \xi_{x_2,h_2}\rangle - \langle\tilde{\varphi}(a_n)\xi_{x_3,h_1}, \xi_{x_4,h_2}\rangle| \\
& - |\langle\tilde{\varphi}(a_m - a)\xi_{x_3,h_1}, \xi_{x_4,h_2}\rangle| \\
>\ & \varepsilon - \tfrac{\varepsilon}{3} - n\tfrac{\varepsilon}{3n} - \tfrac{\varepsilon}{3} = 0.
\end{aligned}$$

□

This marks the end of the first half of the preparations for Theorem 4.1.9. The second part will show that topological freeness of (X, P, θ) results in the ideal intersection property for $C(X)$ inside $\mathcal{O}[X, P, \theta]$, see Proposition 4.1.8.

Lemma 4.1.5. *If $x \in X,\ p, q \in P$ satisfy $\theta_p(x) \neq \theta_q(x)$, then there exists a positive contraction $h \in C(X)$ such that $h(x) = 1$ and $hs_ps_q^*h = 0$.*

Proof. The steps leading to a proof are:

a) There is an open neighbourhood U of x satisfying $U \cap \theta_q^{-1}(\theta_p(U)) = \emptyset$.

b) $\operatorname{supp} L_r(f) \subset \theta_p(\operatorname{supp} f)$ holds for all $f \in C(X)$ and $r \in P$.

c) There exists a positive contraction $h \in C(X)$ with $h(x) = 1$ and $\operatorname{supp} h \subset U$ for the U obtained in a). Every h of this form satisfies $h\alpha_q(L_p(h^2)) = 0$.

As X is Hausdorff, there are disjoint, open neighbourhoods V and W of $\theta_p(x)$ and $\theta_q(x)$, respectively. Hence $U := \theta_p^{-1}(V) \cap \theta_q^{-1}(W)$ is an open neighbourhood of x and $\theta_p(U) \cap \theta_q(U) \subset V \cap W = \emptyset$, so

$$U \cap \theta_q^{-1}(\theta_p(U)) \subset \theta_q^{-1}(\theta_q(U)) \cap \theta_q^{-1}(\theta_p(U)) = \theta_q^{-1}(\theta_p(U) \cap \theta_q(U)) = \emptyset$$

establishes a). Claim b) is straightforward. For the first claim from c), we note that such an h exists because U is an open neighbourhood of x and X is a normal space. Therefore we get

$$\operatorname{supp} h\alpha_q(L_p(h^2)) \subset U \cap \theta_q^{-1}(\operatorname{supp} L_p(h^2)) \overset{b)}{\subset} U \cap \theta_q^{-1}(\theta_p(\underbrace{\operatorname{supp} h^2}_{\subset U})) \overset{a)}{=} \emptyset$$

which proves $h\alpha_q(L_p(h^2)) = 0$. Combining these ingredients, we deduce

$$\|hs_ps_q^*h\|^2 = \|hs_qs_p^*h^2s_ps_q^*h\| = \|h\ \alpha_q(L_p(h^2))s_qs_q^*h\| \overset{c)}{=} 0.$$

□

***Remark* 4.1.6.** Observe that we can deduce from the proof of Lemma 4.1.5 that condition ii) is equivalent to $h\alpha_{p_1}(L_{p_2}(h)) = 0$ as well as to $h\alpha_{p_2}(L_{p_1}(h)) = 0$.

Before we reach the central result of this section, let us recall the notion of topological freeness for dynamical systems, where the transformations need not be reversible.

Definition 4.1.7. A topological dynamical system consisting of a topological space Y and a semigroup S together with an action $S \overset{\eta}{\curvearrowright} Y$ by continuous transformations is said to be **topologically free** if the set $\{y \in Y \mid \eta_s(y) = \eta_t(y)\}$ has empty interior for all $s, t \in S,\ s \neq t$.

Proposition 4.1.8. *If (X, P, θ) is topologically free, every non-zero ideal I in $\mathcal{O}[X, P, \theta]$ satisfies $I \cap C(X) \neq 0$.*

Proof. We will follow the strategy from [EV06, Theorem 10.3]. Suppose I is an ideal in $\mathcal{O}[X, P, \theta]$ satisfying $I \cap C(X) = 0$ and denote by π the corresponding quotient map. Then π is isometric on $C(X)$. We claim that $\|\pi(a)\| \geq \|G(a)\|$ holds for all positive $a \in \mathcal{O}[X, P, \theta]$. By continuity (of the norms, of π, and of G), it suffices to prove the above equation for

$$a = \sum_{j=1}^{n} f_j s_{p_j} s_{q_j} g_j, \text{ with } n \in \mathbb{N}, f_j, g_j \in C(X) \text{ and } p_j, q_j \in P.$$

Without loss of generality, we can assume that there is $1 \leq n_0 \leq n$ such that $p_j = q_j$ holds if and only if $j \leq n_0$. In fact, possibly inflating the elements $f_j s_{p_j} s_{q_j}^* g_j$ by $1 = \sum_i \nu_i s_p s_p^* \nu_i$ for $p \geq p_1 \vee \cdots \vee p_{n_0}$, see Lemma 2.4.2, we can assume that $p_j = q_j = p$ holds for all $1 \leq j \leq n_0$.

If $n_0 = n$, then we have $a \in (\mathcal{F}_p)_+$. In this case, set $h = 1$. For the case $n_0 < n$, note that since (X, P, θ) is topologically free, $\bigcap_{j=n_0+1}^{n} \{x \in X \mid \theta_{p_j}(x) \neq \theta_{q_j}(x)\}$ is dense in X. Thus, for each $\varepsilon \in (0, 1)$, there exists $x \in X$ satisfying

a) $G(a)(x) > (1 - \varepsilon)\ \|G(a)\|$, and

b) $\theta_{p_j}(x) \neq \theta_{q_j}(x)$ for all $n_0 < j \leq n$.

Applying Lemma 4.1.5 to each $n_0 < j \leq n$ yields functions $h_{n_0+1}, \ldots, h_n \in C(X)$, which we use to build $h := \prod_{j=n_0+1}^{n} h_j$. Then h satisfies

(a) $0 \leq h \leq 1$,

(b) $h(x) = 1$, and

(c) $hs_{p_j}s_{q_j}^*h = 0$ for all $n_0 < j \leq n$.

This results in

$$hah = \sum_{j=1}^{n} f_j hs_{p_j}s_{q_j}^* hg_j \overset{(c)}{=} \sum_{j=1}^{n_0} f_j hs_p s_p^* hg_j = hE_1(a)h,$$

where $E_1 : \mathcal{O}[X,P,\theta] \longrightarrow \mathcal{F}$ is the faithful conditional expectation from Lemma 2.4.7. Note that we have $E_1(a) = \sum_{j=1}^{n_0} f_j s_p s_p^* g_j \in (\mathcal{F}_p)_+$. Next, choose a partition of unity $(v_k)_{1\leq k\leq m}$ for X and θ_p as in Lemma 1.3.9 and, as before, let $\nu_k := (N_p v_k)^{\frac{1}{2}}$. Then we obtain

$$\begin{aligned} G(a) \;=\; G(E_1(a)) \;&=\; N_p^{-1} \sum_{j=1}^{n_0} f_j g_j \\ &=\; N_p^{-1} \sum_{j=1}^{n_0} f_j \left(\sum_{k=1}^{m} \nu_k s_p s_p^* \nu_k \right) g_j \\ &=\; \sum_{k=1}^{m} v_k^{\frac{1}{2}} E_1(a) v_k^{\frac{1}{2}}. \end{aligned}$$

Combining this with the fact that $\pi(a) \mapsto \sum_{k=1}^{m} \pi(v_k^{\frac{1}{2}})\, \pi(a)\, \pi(v_k^{\frac{1}{2}})$ is a unital completely positive map, hence contractive, we get

$$\begin{aligned} \|\pi(a)\| \;&\geq\; \|\pi\left(\sum_{k=1}^{m} v_k^{\frac{1}{2}} a v_k^{\frac{1}{2}} \right)\| \\ &\geq\; \|\pi\left(\sum_{k=1}^{m} v_k^{\frac{1}{2}} hah v_k^{\frac{1}{2}} \right)\| \\ &=\; \|\pi\left(h \sum_{k=1}^{m} v_k^{\frac{1}{2}} E_1(a) v_k^{\frac{1}{2}} h \right)\| \\ &=\; \|\pi(hG(a)h)\| \\ &=\; \|hG(a)h\| \end{aligned}$$

since π is isometric on $C(X)$. On the other hand,

$$\|hG(a)h\| \geq (hG(a)h)(x) = G(a)(x) > (1-\varepsilon)\|G(a)\|,$$

so $\|\pi(a)\| > (1-\varepsilon)\|G(a)\|$ for all $\varepsilon > 0$. This forces $\|\pi(a)\| \geq \|G(a)\|$.

So given $a \in \mathcal{O}[X,P,\theta]_+ \cap I$, we have $0 = \|\pi(bab^*)\| \geq \|G(bab^*)\|$ for all $b \in \mathcal{O}[X,P,\theta]$. In particular, $G(bab^*) = 0$ holds for all $b \in \mathcal{F}$. But according to Lemma 2.4.14, this implies $a = 0$ and hence $I = 0$. □

We are now ready for the main result of this section:

Theorem 4.1.9. *Suppose (X, P, θ) is an irreversible $*$-commutative dynamical system of finite type. Then the following statements are equivalent:*

(1) The dynamical system (X, P, θ) is topologically free.

(2) Every non-zero ideal I in $\mathcal{O}[X, P, \theta]$ satisfies $I \cap C(X) \neq 0$.

(3) The representation φ of $\mathcal{O}[X, P, \theta]$ on $\ell^2(X)$ from Proposition 2.4.4 is faithful.

(4) $C(X)$ is a masa in $\mathcal{O}[X, P, \theta]$.

Proof. The plan is as follows:

$$\begin{tikzcd} (1) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & (2) \arrow[d, Rightarrow] \\ (4) & (3) \arrow[ul, Rightarrow] \end{tikzcd}$$

The implication from (1) to (2) is precisely covered by Proposition 4.1.8 and (2) gives (3) because we have $\ker \varphi \cap C(X) = 0$, see Proposition 2.4.4. Next, we show that (3) or (4) implies (1), where we proceed by contraposition. If the system is not topologically free, there are $p, q \in P$ with $p \neq q$ such that $\{x \in X \mid \theta_p(x) = \theta_q(x)\}$ has non-empty interior. Since the maps θ_p and θ_q are local homeomorphisms, there exists a non-empty open $U \subset \{x \in X \mid \theta_p(x) = \theta_q(x)\}$ such that $\theta_p|_U = \theta_q|_U$ is injective. We fix $x_0 \in U$ and choose a positive $f \in C(X)$ satisfying $f(x_0) \neq 0$ and $\operatorname{supp} f \subset U$. By appealing to the existence of partitions of unity for open covers of compact Hausdorff spaces, we know that such a function f always exists. Let us point out that $fs_ps_q^*f$ does not belong to $C(X)$, which can formally be deduced from Lemma 4.1.4 ii), $p \neq q$, and

$$\left\langle \tilde{\varphi}(fs_ps_q^*f)\xi_{x_0,q}, \xi_{x_0,p} \right\rangle = N_{pq}^{-\frac{1}{2}} f(x_0)^2 \neq 0.$$

Then

$$\begin{aligned} \left\langle \varphi(fs_ps_q^*f)\xi_x, \xi_y \right\rangle &= \left\langle \varphi(s_q^*f)\xi_x, \varphi(s_p^*f)\xi_y \right\rangle \\ &= \delta_{\theta_q(x)\theta_p(y)}\ N_{pq}^{-\frac{1}{2}}\ f(x)f(y) \\ &= \delta_{xy}\ N_{pq}^{-\frac{1}{2}}\ f(x)^2 \end{aligned}$$

holds for all $x, y \in U$, where we used injectivity of $\theta_p|_U = \theta_q|_U$. Note that the expression vanishes whenever x or y is not contained in U due to $\operatorname{supp} f \subset U$. Hence we get $0 \neq fs_ps_q^*f - N_{pq}^{-\frac{1}{2}} f^2 \in \ker \varphi$, which shows that (3) implies (1).

In order to prove that (4) forces (1), it suffices to show that the function f from the last part satisfies $fs_ps_q^*f \in C(X)' \cap \mathcal{O}[X, P, \theta]$. Let us pick $(\nu_i)_{i \in I}$ for θ_q as in Lemma 1.3.9. We claim that

$$\alpha_p(L_q(fg\nu_i)) = N_q^{-1} gf\nu_i = g\alpha_p(L_q(f\nu_i))$$

holds for all $g \in C(X)$ and $i \in I$. Using the property that $\theta_q|_{\operatorname{supp}\nu_i}$ is injective, it is straightforward to check that the functions match on $X \setminus \operatorname{supp} f$, so let $x \in \operatorname{supp} f \subset U$. Then

$$\begin{aligned} \alpha_p(L_q(fg\nu_i))(x) &= N_q^{-1} \sum_{y \in \theta_q^{-1}(\theta_p(x))} g(y)f(y)\nu_i(y) \\ &= N_q^{-1} \sum_{y \in \theta_q^{-1}(\theta_q(x))} g(y)f(y)\nu_i(y) \\ &= N_q^{-1} g(x)f(x)\nu_i(x) \end{aligned}$$

holds, where we used $\theta_p|_U = \theta_q|_U$ and injectivity of $\theta_q|_U$. Similarly we get

$$\begin{aligned} g\alpha_p(L_q(f\nu_i))(x) &= g(x)N_q^{-1} \sum_{y \in \theta_q^{-1}(\theta_p(x))} f(y)\nu_i(y) \\ &= N_q^{-1} g(x)f(x)\nu_i(x). \end{aligned}$$

Thus

$$\alpha_p(L_q(fg\nu_i)) = N_q^{-1} gf\nu_i = g\alpha_p(L_q(f\nu_i))$$

is valid for all $g \in C(X)$ and $i \in I$. Using this equation, we deduce

$$\begin{aligned} fs_ps_q^*fg &= fs_ps_q^*fg \sum_{i \in I} \nu_i s_q s_q^* \nu_i \\ &= \sum_{i \in I} f\alpha_p(L_q(fg\nu_i))s_ps_q^*\nu_i \\ &= gfs_p \sum_{i \in I} L_q(f\nu_i)s_q^*\nu_i \\ &= gfs_ps_q^* \sum_{i \in I} \nu_i E_q(\nu_i f) \\ &= gfs_ps_q^*f. \end{aligned}$$

for arbitrary $g \in C(X)$. Thus, $fs_ps_q^*f \in \Big(C(X)' \cap \mathcal{O}[X,P,\theta]\Big) \setminus C(X)$, so $C(X)$ is not a masa in $\mathcal{O}[X,P,\theta]$.

In order to deduce (4) from (1), let $a \in C(X)' \cap \mathcal{O}[X,P,\theta]$. By Lemma 4.1.4 ii), $a \in C(X)$ follows provided that $\langle \tilde{\varphi}(a)\xi_{x_1,h_1}, \xi_{x_2,h_2}\rangle = 0$ holds for all $(x_1,h_1) \neq (x_2,h_2)$. In case $x_1 \neq x_2$, there is $f \in C(X)$ satisfying $f(x_1) \neq 0$ and $f(x_2) = 0$. Thus

$$\begin{aligned} f(x_1)\,\langle \tilde{\varphi}(a)\xi_{x_1,h_1}, \xi_{x_1,h_2}\rangle &= \langle \tilde{\varphi}(af)\xi_{x_1,h_1}, \xi_{x_2,h_2}\rangle \\ &= \langle \tilde{\varphi}(fa)\xi_{x_1,h_1}, \xi_{x_2,h_2}\rangle \\ &= f(x_2)\,\langle \tilde{\varphi}(a)\xi_{x_1,h_1}, \xi_{x_2,h_2}\rangle \\ &= 0 \end{aligned}$$

implies that $\langle \tilde{\varphi}(a)\xi_{x_1,h_1}, \xi_{x_1,h_2}\rangle = 0$. Now let $x_1 = x_2$ and $h_1 \neq h_2$ and we assume $\langle \tilde{\varphi}(a)\xi_{x_1,h_1}, \xi_{x_1,h_2}\rangle \neq 0$ in order to derive a contradiction: Part iii) from Lemma 4.1.4

states that there are $p, q \in P$ and open neighbourhoods U_1, U_2 of $x_1 = x_2$ with the properties (a)-(c). Note that $p \neq q$ due to (a) and $h_1 \neq h_2$. By passing to smaller neighbourhoods of x_1, if necessary, we may assume that for each $x_3 \in U_1$ there is a unique $x_4 \in U_2$ satisfying $\theta_q(x_3) = \theta_p(x_4)$ (and vice versa). In other words, the (a priori multivalued) maps $\theta_q^{-1}\theta_p : U_1 \longrightarrow U_2$ and $\theta_p^{-1}\theta_q : U_2 \longrightarrow U_1$ are homeomorphisms. This uses the standing assumption that θ_p and θ_q are local homeomorphisms. As (X, P, θ) is topologically free, the set $\{x \in U_1 \mid \theta_p(x) = \theta_q(x)\}$ has empty interior, so it cannot be all of U_1. Hence there are $x_3 \in U_1$ and $x_4 \in U_2$ such that $x_3 \neq x_4$ and $\theta_q(x_3) = \theta_p(x_4)$. Now Lemma 4.1.4 iii) implies $\langle \tilde{\varphi}(a)\xi_{x_3,h_1}, \xi_{x_4,h_2}\rangle \neq 0$. On the other hand, we observe that $\langle \tilde{\varphi}(a)\xi_{x_3,h_1}, \xi_{x_4,h_2}\rangle = 0$ follows from the consideration of the case $x_1 \neq x_2$ from before because $x_3 \neq x_4$. This reveals a contradiction and thus, $\langle \tilde{\varphi}(a)\xi_{x_1,h_1}, \xi_{x_2,h_2}\rangle = 0$ whenever $(x_1, h_1) \neq (x_2, h_2)$. According to Lemma 4.1.4 ii), this forces $a \in C(X)$, so $C(X)$ is a masa in $\mathcal{O}[X, P, \theta]$. □

***Remark* 4.1.10.** The representation φ is an analogue of the reduced representation for ordinary group crossed products, for if θ_p was a homeomorphism of X, then $S_p\xi_x = \xi_{\theta_p^{-1}(x)}$, see Proposition 2.4.4. Therefore condition (3) of Theorem 4.1.9 can be interpreted as an amenability property of the dynamical system (X, P, θ), compare [BO08, Theorem 4.3.4]. Interestingly, this property coincides with topological freeness for irreversible $*$-commutative dynamical systems of finite type as defined in Definition 1.3.13.

4.2 Simplicity of the C*-algebra

Let X be a compact Hausdorff space, G a discrete group, and α an action of G on $C(X)$. Then simplicity of the transformation group C*-algebra $C(X) \rtimes_\alpha G$ corresponds to minimality and topological freeness of the underlying topological dynamical system, given that the action α is amenable, see [AS94, Corollary following Theorem 2] or [BO08, Theorem 4.3.4 (1)]. An intermediate step for this result is to prove that every non-zero ideal I in the C*-algebra $C(X) \rtimes_\alpha G$ satisfies $I \cap C(X) \neq 0$ if the dynamical system is topologically free, see [AS94, Theorem 2]. In view of Proposition 4.1.8, the analogous statement for $\mathcal{O}[X, P, \theta]$ and (X, P, θ) has already been established. In fact, Theorem 4.1.9 revealed that these conditions are equivalent.

In contrast to the case of group actions, topological freeness is proven to be automatic for minimal irreversible $*$-commutative dynamical systems of finite type, see Proposition 4.2.10. The proof of this implication is an adaptation of [EV06, Proposition 11.1]. Once this is accomplished, we show that $\mathcal{O}[X, P, \theta]$ is simple if and only if (X, P, θ) is minimal, see Theorem 4.2.11. Hence we achieve a direct generalization of [EV06, Theorem 11.2], if we suppress the additional requirement that each θ_p is assumed to be regular, see Definition 1.3.13. We note that this extra condition is assumed in [EV06, Section 9] as well, but not in [EV06, Sections 8,10 and 11]. As an application of Theorem 4.2.11, we characterize simplicity of $\mathcal{O}[G,P,\theta]$ for commutative irreversible algebraic dynamical systems of finite type by minimality of (G,P,θ), see Corollary 4.2.12.

Unlike the case of group actions, there are different options available for the notion of an orbit of a point $x \in X$ under the action θ of P. For example, we can consider

$$\mathcal{O}^+(x) = \{\theta_p(x) \mid p \in P\} \qquad \text{or} \qquad \bigcup_{p \in P} \theta_p^{-1}(\theta_p(x)).$$

Although both versions have some striking features an orbit ought to have, they lack other serious features at the same time. In a sense, the two candidates are complementary. The first one does not necessarily yield an equivalence relation because there may be a lack of opportunities to get back to x via θ once we arrive at some $y \in \mathcal{O}^+(x)$. The second object need not contain any elements of the former object other than x itself. Moreover, both versions lack the feature of being invariant with respect to taking preimages. Nevertheless, these versions appear in the existing literature. The conclusion we want to draw from this is that one has to be cautious about which notion of an orbit is used in an exposition. We will take the following one, which seems best suited for working with structures resembling crossed products:

Definition 4.2.1. Let Z be a topological space, S a commutative semigroup and $S \overset{\eta}{\curvearrowright} Z$ a semigroup action by continuous maps. For $x \in Z$,

$$\mathcal{O}(x) := \{\eta_t^{-1}(\eta_s(x)) \mid s,t \in S\} \subset Z$$

is called the **orbit** of x under η. Two elements $x, y \in Z$ are called **orbit-equivalent**, denoted by $x \sim y$, if $\mathcal{O}(x) = \mathcal{O}(y)$.

***Remark* 4.2.2.** Note that x and y are orbit-equivalent if and only if there are $s,t \in S$ such that $\eta_s(x) = \eta_t(y)$. This definition is the natural generalization of trajectory-equivalence as defined in [EV06, Section 11]. $\sim$ is an equivalence relation because S is commutative. Indeed, reflexivity and symmetry are obvious. For transitivity, suppose $\eta_{s_1}(x) = \eta_{t_1}(y)$ and $\eta_{s_2}(y) = \eta_{t_2}(z)$ hold for $x,y,z \in Z$ and $s_i, t_i \in S$. Then $\eta_{s_2 s_1}(x) = \eta_{s_2 t_1}(y) = \eta_{t_1 s_2}(y) = \eta_{t_1 t_2}(z)$ shows $x \sim z$.

Definition 4.2.3. $Y \subset Z$ is called **invariant**, if $\eta_s^{-1}(Y) = Y$ for all $s \in S$.

Lemma 4.2.4. *Let Z be a topological space, S a commutative semigroup and $S \overset{\eta}{\curvearrowright} Z$ a semigroup action by continuous, surjective maps. Then $Y \subset Z$ is invariant if and only if $x \sim y \in Y$ implies $x \in Y$ for all $x \in Z$.*

Proof. Since each η_s is surjective, we have $\eta_s(\eta_s^{-1}(Y)) = Y$. So if Y is invariant, then $\eta_s(Y) = Y$ holds for all $s \in S$. Hence, if $x \sim y \in Y$, say $\eta_s(x) = \eta_t(y)$, then

$$x \in \eta_s^{-1}(\eta_t(Y)) = \theta_s^{-1}(Y) = Y$$

follows. Conversely, suppose $x \sim y \in Y$ implies $x \in Y$ for all $x \in Z$. Then $\eta_s^{-1}(Y) \subset Y$ holds for all $s \in S$ as $x \sim y$ for all $y \in Y$ and $x \in \eta_s^{-1}(y)$. On the other hand, $\eta_s(y) \sim y$ for all $y \in Y$ forces $\eta_s(Y) \subset Y$, which in turn implies $Y \subset \eta_s^{-1}(\eta_s(Y)) \subset \eta_s^{-1}(Y)$. Thus, Y is invariant. □

In the case of actions by homeomorphisms, it is well-known that invariance of a subset passes to its closure. This is not clear for general irreversible transformations, but it is true for actions by local homeomorphisms. This is certainly well-known, but not easy to find in the literature, so we include a proof for convenience.

Lemma 4.2.5. *Let Z be a topological space, S a commutative semigroup and $S \overset{\eta}{\curvearrowright} Z$ a semigroup action by local homeomorphisms. For every $Y \subset Z$ and $s \in S$, we have $\eta_s^{-1}(\overline{Y}) = \overline{\eta_s^{-1}(Y)}$.*

Proof. The map η_s is continuous, so $\eta_s^{-1}(\overline{Y})$ is a closed subset of Z containing $\eta_s^{-1}(Y)$ and hence $\overline{\eta_s^{-1}(Y)} \subset \eta_s^{-1}(\overline{Y})$. To prove the reverse inclusion, let $x \in \eta_s^{-1}(\overline{Y})$. Since η_s is a local homeomorphism, there is an open neighbourhood U of x such that $\eta_s|_U : U \longrightarrow \eta_s(U)$ is a homeomorphism. Due to $\eta_s(x) \in \overline{Y}$, there is a net $(y_\lambda)_{\lambda\in\Lambda} \subset Y$ such that $y_\lambda \overset{\lambda\to\infty}{\longrightarrow} \eta_s(x)$. Note that $\eta_s(U)$ is open and contains $\eta_s(x)$. Hence, we can assume $(y_\lambda)_{\lambda\in\Lambda} \subset Y \cap \eta_s(U)$ without loss of generality. Now, $x_\lambda := \eta_s|_U^{-1}(y_\lambda)$ defines a net $(x_\lambda)_{\lambda\in\Lambda} \subset \eta_s^{-1}(Y) \cap U$ and continuity of $\eta_s|_U^{-1}$ gives $x_\lambda \longrightarrow x$. Therefore, we have shown that $x \in \overline{\eta_s^{-1}(Y)}$. □

Corollary 4.2.6. *Let Z be a topological space, S a commutative semigroup and $S \overset{\eta}{\curvearrowright} Z$ a semigroup action by local homeomorphisms. If $Y \subset Z$ is invariant, then so is $\overline{Y}$. In particular, the closure of the orbit $\mathcal{O}(x)$ is invariant for every $x \in Z$.*

Proof. For every $s \in S$, we get $\eta_s^{-1}(\overline{Y}) = \overline{\eta_s^{-1}(Y)} = \overline{Y}$ from Lemma 4.2.5 and the invariance of Y. □

Definition 4.2.7. Let Z be a topological space, S a commutative semigroup and $S \overset{\eta}{\curvearrowright} Z$ a semigroup action by surjective local homeomorphisms. The dynamical system (Z, S, η) is said to be **minimal**, if $\emptyset$ and Z are the only open invariant subsets of Z.

***Remark* 4.2.8.** In the above definition, one can replace open by closed. In [EV06], this property is called irreducibility, possibly to avoid confusion with a notion of minimality apparently used for the groupoid picture.

Corollary 4.2.9. *A dynamical system (Z, S, η) as in Definition 4.2.7 is minimal if and only if $\mathcal{O}(x) \subset Z$ is dense for all $x \in Z$.*

Proof. This follows immediately from Corollary 4.2.6. □

With these preparations at hand, let us return to the study of irreversible $*$-commutative dynamical systems of finite type (X, P, θ). The next proposition is based on [EV06, Proposition 11.1].

Proposition 4.2.10. *If (X, P, θ) is minimal, then it is topologically free.*

Proof. Let us assume that (X, P, θ) is minimal, but not topologically free and derive a contradiction. Assume that there exist $p, q \in P$ with $p \neq q$ such that $\theta_{p|U} = \theta_{q|U}$ on some non-empty, open subset U of X. Clearly, $\bigcup_{s,t \in P} \theta_s^{-1}(\theta_t(U)) \subset X$ is invariant, non-empty and open. Since the dynamical system is minimal, this set is all of X. Since each $\theta_s^{-1}(\theta_t(U))$ is open and X is compact, we can shrink the open cover $(\theta_s^{-1}(\theta_t(U)))_{s,t \in P}$ to a finite, open cover of X given by $s_1, \ldots, s_n, t_1, \ldots, t_n$. Next, fix an arbitrary $x \in X$ and let i satisfy $x \in \theta_{s_i}^{-1}(\theta_{t_i}(U))$, i.e. there is $y \in U$ such that $\theta_{s_i}(x) = \theta_{t_i}(y)$. Then

$$\theta_{ps_i}(x) = \theta_{pt_i}(y) = \theta_{t_i p}(y) \overset{y \in U}{=} \theta_{t_i q}(y) = \theta_{qt_i}(y) = \theta_{qs_i}(x)$$

and if we take $s := \bigvee_{j=1}^{n} s_j$, we get

$$\theta_{ps}(x) = \theta_{s_i^{-1}s}\theta_{ps_i}(x) = \theta_{s_i^{-1}s}\theta_{qs_i}(x) = \theta_{qs}(x)$$

for all x in X. Hence, we have $\theta_{ps} = \theta_{qs}$. As $\theta_{(p \wedge q)s}$ is surjective and P is commutative, this implies $\theta_{(p\wedge q)^{-1}p} = \theta_{(p\wedge q)^{-1}q}$. Without loss of generality, we can assume $(p \wedge q)^{-1}p \neq 1_P$, since $p \neq q$ forces $(p \wedge q)^{-1}p \neq 1_P$ or $(p \wedge q)^{-1}q \neq 1_P$. Using $*$-commutativity for $\theta_{(p\wedge q)^{-1}p}$ and $\theta_{(p\wedge q)^{-1}q}$ in the form of Proposition 1.3.2 (iii) yields

$$\theta_{(p\wedge q)^{-1}p}^{-1}(\theta_{(p\wedge q)^{-1}q}(x)) = \theta_{(p\wedge q)^{-1}q}(\theta_{(p\wedge q)^{-1}p}^{-1}(x)) = \{x\}.$$

However, $(p \wedge q)^{-1}p \neq 1_P$ implies that the cardinality of the set on the left hand side is strictly larger than one, see Definition 1.3.13 (C). Thus, we obtain a contradiction. □

Theorem 4.2.11. *Let (X, P, θ) be an irreversible $*$-commutative dynamical system of finite type. Then the C*-algebra $\mathcal{O}[X, P, \theta]$ is simple if and only if (X, P, θ) is minimal.*

Proof. If we assume $\mathcal{O}[X, P, \theta]$ to be simple, then $C(X)$ intersects every non-zero ideal in $\mathcal{O}[X, P, \theta]$ non-trivially, so (X, P, θ) is topologically free by Theorem 4.1.9. Now suppose $\emptyset \neq U \subset X$ is invariant and open. Then

$$\begin{array}{rclclcl} \operatorname{supp} \alpha_p(f) & = & \theta_p^{-1}(\operatorname{supp} f) & \subset & \theta_p^{-1}(U) & = & U \\ \operatorname{supp} L_p(f) & \subset & \theta_p(\operatorname{supp} f) & \subset & \theta_p(U) & = & U \end{array}$$

holds for every $p \in P$ and $f \in C_0(U)$ because U is invariant. We infer from this that the ideal I in $\mathcal{O}[X, P, \theta]$ generated by $C_0(U)$ satisfies $I \cap C(X) \subset C_0(U)$. But as $\mathcal{O}[X, P, \theta]$ is simple and $U \neq \emptyset$, we have $I = \mathcal{O}[X, P, \theta]$ and hence $U = X$.

Conversely, if (X, P, θ) is minimal and $0 \neq I$ is an ideal in $\mathcal{O}[X, P, \theta]$, we have $I \cap C(X) = C_0(U)$ for some open $U \subset X$. Due to Proposition 4.2.10, (X, P, θ) is topologically free. Hence U is non-empty according to Proposition 4.1.8. We claim that U is invariant. To see why, let $x \sim y \in U$, i.e. there exist $p, q \in P$ such that $\theta_p(x) = \theta_q(y)$. Pick a non-negative function $f \in C_0(U)$ satisfying $f(y) > 0$ (such an f always exists as U is open and X is a normal space). Additionally, choose $(\nu_i)_{1 \leq i \leq n}$ for θ_p as in Lemma 1.3.9. Using the relations (I),(II) and (IV) for $\mathcal{O}[X, P, \theta]$, we get

$$\alpha_p(L_q(f)) = \sum_{1 \leq i \leq n} \nu_i s_p s_q^* f s_q s_p^* \nu_i \in I,$$

which shows $\alpha_p(L_q(f)) \in C_0(U)$. Moreover, we have

$$\alpha_p(L_q(f))(x) = L_q(f)(\theta_p(x)) \geq N_q^{-1} f(y) > 0$$

because f is non-negative and $y \in \theta_q^{-1}(\theta_p(x))$. Thus $x \in \operatorname{supp} \alpha_p(L_q(f)) \subset U$, so U is invariant by Lemma 4.2.4. Since U is a non-empty, invariant open subset of X, minimality forces $U = X$ and hence $I = \mathcal{O}[X, P, \theta]$. Hence $\mathcal{O}[X, P, \theta]$ is simple. □

Coming back to irreversible algebraic dynamical systems, we recall that we can only treat commutative irreversible algebraic dynamical systems of finite type within the framework of irreversible $*$-commutative dynamical systems of finite type, see Corollary 1.3.17:

Corollary 4.2.12. *A commutative irreversible algebraic dynamical system of finite type (G, P, θ) is minimal if and only if the C*-algebra $\mathcal{O}[G, P, \theta]$ is simple.*

Proof. By Corollary 1.3.17, we know that $(\hat{G}, P, \hat{\theta})$ is an irreversible $*$-commutative dynamical system of finite type. According to Proposition 2.4.3, $\mathcal{O}[G, P, \theta]$ is isomorphic to $\mathcal{O}[\hat{G}, P, \hat{\theta}]$. By Proposition 1.2.8, (G, P, θ) is minimal precisely if the union of the kernels $(\ker \theta_p)_{p \in P}$ is dense in $\hat{G}$. In other words, the orbit of $1_{\hat{G}}$ in the sense of Definition 4.2.1 is dense in $\hat{G}$. Since $\hat{G}$ is a group and we are dealing with group endomorphisms, this is equivalent to minimality of the topological dynamical system by Corollary 4.2.9. Now the claim follows directly from Theorem 4.2.11. □

In particular, this characterization applies to the commutative irreversible algebraic dynamical systems of finite type presented in Section 1.1. Note that Example 1.3.21 and Example 1.3.23 also belong to this class of examples. Hence the corresponding C*-algebras are simple. In fact, they are always UCT Kirchberg algebras by Corollary 2.2.28.

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