Mathematik

Gromov Hyperbolic Manifolds, Weighted Isoperimetry and Bubbles

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Zusammenfassung

In den späten 1980er Jahren erzielte A. Ancona weitreichende Ergebnisse in der Potentialtheorie schwach koerziver Operatoren auf Gromov-hyperbolischen Mannigfaltigkeiten beschränkter Geometrie. So konnte er Rand-Harnack-Ungleichungen beweisen und damit den Gromov-Rand mit dem potentialtheoretischen Martin-Rand identifizieren. Nach einer ausführlichen Präsentation der zentralen Resultate benutzen wir diese, um eine asymptotische Formel für positive harmonische Funktionen ausgedrückt durch ihr Martin-Maß und Greensche Funktionen anzugeben. Zudem untersuchen wir den Fall des Laplace-Operators genauer und erhalten gewichtete Friedrichs- und lineare isoperimetrische Ungleichungen. Diese sind nicht nur robuster unter kleinskaligen Störungen als ihre ungewichteten Analoga, sondern enthalten auch zusätzliche ortsaufgelöste Informationen.

Eine auf großen Skalen stabile Entsprechung von Minimierern des isoperimetrischen Quotienten sind (verallgemeinerte) Seifenblasen. Nach allgemeinen Betrachtungen zur Existenz und Regularität geben wir unter Einbeziehung gewichteter isoperimetrischer Ungleichungen ein Kriterium für die Existenz kompakter Seifenblasen in vollständigen Mannigfaltigkeiten, das heißt, ein Entkommen minimierender Folgen nach Unendlich wird ausgeschlossen.

Diese Ergebnisse zeigen ein transparentes Transformationsverhalten unter konformen Deformationen wie der hyperbolischen Entfaltung uniformer Räume. Das benutzen wir zur direkten Konstruktion von Metriken positiver oder verschwindender Skalarkrümmung, in denen flächenminimierende Folgen von Hyperflächen nicht nach Unendlich entkommen können. Anwendungen betreffen das singuläre Yamabe-Problem für verschwindende Skalarkrümmung und Smale-Hyperflächen, die einzige bekannte Beispielklasse für singuläre Minimierer, sofern die Dimension der Singularitäten nicht zu hoch ist. Die Existenz von minimalen Hyperflächen niedrigerer Dimension im regulären Teil minimaler Hyperflächen ist von Bedeutung, um eine Methode zum Auffinden von Obstruktionen gegen Metriken positiver Skalarkrümmung in höhere Dimensionen übertragen zu können.

Summary

In the late 1980s, A. Ancona obtained far-reaching results in the potential theory of weakly coercive operators on Gromov hyperbolic manifolds of bounded geometry. He was able to prove boundary Harnack inequalities and use them to identify the Gromov boundary with the potential theoretic Martin boundary. After carefully presenting the central results, we use them to give an asymptotic formula for positive harmonic functions expressed in terms of their Martin measure and Green's functions. Furthermore, we examine the case of the Laplace operator in more detail and obtain weighted Friedrichs and linear isoperimetric inequalities. These are not only more robust under small-scale perturbations than their unweighted analogues, but also contain additional spatial information.

A large-scale equivalent of minimisers of the isoperimetric quotient are (generalised) bubbles. Following existence and regularity considerations, we give a criterion for the existence of compact bubbles in complete manifolds assuming weighted isoperimetric inequalities, i.e., minimising sequences are prevented from escaping to infinity.

These results transparently transform under conformal deformations such as the hyperbolic unfolding of uniform spaces. We use this for the direct construction of metrics of positive or vanishing scalar curvature that prevent area-minimising sequences of hypersurfaces from escaping to infinity. Applications include the singular Yamabe problem for vanishing scalar curvature and Smale hypersurfaces, the only known example class for singular area-minimisers, provided the dimension of the singular set is not too large. The existence of lower-dimensional minimal hypersurfaces in the regular part of minimal hypersurfaces is important for transferring a method for finding obstructions against metrics of positive scalar curvature to higher dimensions.

Contents

Introduction

In the 1980s, M. Gromov popularised what we now call Gromov hyperbolic spaces [Gro87]: they can be characterised as geodesic metric spaces where every not unreasonably long path (a quasi-geodesic) between two points lies in a neighbourhood of a shortest path (geodesic) connecting these points.

Originally intended for the study of (Cayley graphs of) groups, Gromov hyperbolicity soon became a fascinating subject of its own. An intrinsic justification is already given by a wealth of seemingly different but equivalent definitions, involving thin triangles, quadrangles, divergence of geodesics, two-dimensional isoperimetric inequalities or the presence of valleys as described above [BH99, III.H.1–2]. The most basic examples are metric trees and classical hyperbolic space $\mathbb{H}^n$. Gromov hyperbolic metrics also arise as natural conformal deformations of uniform spaces (generalising Lipschitz domains in $\mathbb{R}^n$) and singular minimal hypersurfaces in Riemannian manifolds. We will return to that in a moment.

Gromov hyperbolic spaces have a natural boundary at infinity, called *Gromov boundary*, that encodes a large amount of information about the large-scale structure of the space. It comes with a (bi-Lipschitz equivalence class of) induced (quasi)metric(s). A clear indicator for the abundance of Gromov hyperbolic spaces is the fact that every bounded metric space is the Gromov boundary of some Gromov hyperbolic space, up to bi-Lipschitz equivalence. On the other hand, two visual Gromov hyperbolic spaces with bi-Lipschitz equivalent boundaries are already quasi-isometric. The metric boundaries of uniform spaces and the singular set of minimal hypersurfaces turn out to be the Gromov boundaries of the associated hyperbolic metrics, hence one can try to study these intricate boundaries (think of nasty fractals, even with spatially varying dimension) via the locally much more well-behaved Gromov hyperbolic spaces (in many cases, graphs or Riemannian manifolds of bounded geometry).

Soon after Gromov, there were many fruitful attempts to carry facts about hyperbolic spaces in a more classical sense over to Gromov hyperbolic spaces. Among those are the works of A. Ancona [Anc87, Anc90], who started with results of Anderson and Schoen on harmonic functions on Cartan–Hadamard manifolds [AS85] and vastly generalised them to a large class of second-order elliptic operators, containing the Laplacian and Schrödinger operators with bounded potential, on Gromov hyperbolic manifolds of bounded geometry. Under the additional condition of *weak coercivity*, which is equivalent to a positive principal eigenvalue for symmetric operators, Ancona gives explicit estimates for the Green's function, proves a boundary Harnack inequality and identifies the Gromov boundary with the potential theoretic *Martin boundary.* Its relevance is that the space of positive harmonic functions can be identified with the space of Radon measures on the Martin boundary.

In the case of the Laplacian, one has even more control over the Green's function. Along geodesic rays, it decays at the same rate as the *harmonic measure* (the Martin measure corresponding to the constant function 1) of balls of decreasing radius. We exploit this interplay between the Green's function and the harmonic measure on a layer-wise cover of the hyperbolic manifold, the *onion cover*, to prove:

Theorem A (Weighted Friedrichs and Linear Isoperimetric Inequalities) *(Corollary 3.12 and Corollary 3.14) On a Gromov hyperbolic visual manifold X of bounded geometry with positive principal eigenvalue of the Laplacian, we have for any function w decaying exponentially slower than the Laplacian's Green's function and $p \geqslant 1$*

$$\int_X |u|^p \, w \, \mathrm{d}V \preccurlyeq \int_X |\nabla u|^p \, w \, \mathrm{d}V$$

for every $u \in W^{1,p}_{\mathrm{loc}}(X)$ such that the left-hand side is finite and

$$\int_U w \, \mathrm{d}V \preccurlyeq \int_{\partial U} w \, \mathrm{d}A$$

for every Caccioppoli set $U \subset X$ such that the left-hand side is finite.

In particular, as the Green's function decays exponentially towards infinity, the $p = 2$ weighted Friedrichs inequality implies

$$\int_X |u|^2 \, \mathrm{d}V \preccurlyeq \int_X |\nabla u|^2 \, \mathrm{d}V \tag{$\star$}$$

for $u \in C^\infty_\mathrm{c}(X)$. But this inequality is exactly the variational characterisation of the positivity of the Laplacian's principal eigenvalue, hence it *self-improves*: if this inequality holds, it also holds with exponentially decaying weights. By work of Martínez-Pérez and Rodríguez [MR18], this is also equivalent to the Gromov boundary of X being *uniformly perfect*, i.e., it does not contain parts of (lower Assouad) dimension zero. The weight function can be interpreted as a large-scale analogue of the best constant in ($\star$), containing additional directional information.

Our interest in Gromov hyperbolic manifolds is motivated by the two natural classes of examples hinted at earlier:

- *Uniform* spaces can be conformally deformed with the inverse distance to the boundary (or a regularisation thereof). This yields complete Gromov hyperbolic spaces as shown by Bonk, Heinonen and Koskela [BHK01], analogous to the hyperbolic Poincaré metric on the unit disk in $\mathbb{C}$. Examples for uniform spaces include manifolds with Lipschitz boundary, but the (metric) boundary might as well be lower-dimensional or fractal, as in the Koch snowflake.

- Area-minimising hypersurfaces in an $(n+1)$-dimensional Riemannian manifold (and solutions of similar variational problems) are smooth submanifolds only outside a singular set of Hausdorff dimension up to $n - 7$. This is a major result of geometric measure theory from the 1960s due to De Giorgi, Reifenberg, Federer, Fleming, Almgren, Simons, and others. Since then, these singularities have remained rather mysterious. Recently, J. Lohkamp developed the theory of $\mathcal{S}$-structures to handle potential theory on the regular part of singular minimal hypersurfaces: the regular part is uniform with respect to the *$\mathcal{S}$-distance*, a generalised distance from the singular set incorporating the norm of the second fundamental form, and a conformal deformation with its inverse yields a complete Gromov hyperbolic metric of bounded geometry, the *hyperbolic unfolding* [Loh18].

We handle both these situations simultaneously using the notion of a *generalised distance function*.

Area-minimising hypersurfaces are an important tool in scalar curvature geometry: as first observed by Schoen and Yau [SY79], a smooth, closed, stably minimal hypersurface in a manifold of positive scalar curvature carries itself a conformal metric of positive scalar curvature. It is constructed by conformal deformation with the first eigenfunction of the *conformal Laplacian*. Using this argument inductively on area-minimisers of decreasing dimension in prescribed homology classes one can find obstructions to the existence of metrics of positive scalar curvature. E.g., assuming there is a metric of positive scalar curvature on the n-torus, one can inductively find $\mathrm{Scal} > 0$-metrics on minimal hypersurfaces of decreasing dimension with a torus component, until the Gauß–Bonnet formula yields a contradiction in dimension two.

To make this line of reasoning viable in dimensions higher than 7, one has to understand the potential theory of the conformal Laplacian on singular minimal hypersurfaces. In a $\mathrm{Scal} > 0$ ambience, a version of this operator is in fact weakly coercive on Lohkamp's hyperbolic unfolding and hence Ancona's potential theory applies. The singular set, the Gromov boundary of the hyperbolic unfolding and the Martin boundary classifying positive harmonic functions are all homeomorphic [Loh20a, Loh20b].

Now one suddenly has a wealth of positive eigenfunctions of the conformal Laplacian (as there are many Radon measures on the singular set)—we try to invest this freedom of choice in finding an eigenfunction such that the conformally deformed regular part of the hypersurface is *mean convex at infinity*, i.e., there is an exhaustion by compact mean convex domains. This is relevant for the induction step in Schoen-Yau dimensional descent because this condition guarantees that minimising hypersurfaces in such a conformally deformed hypersurface are blocked from a neighbourhood of the singular set. One goal of this thesis is to explore the possibility of a one-shot construction of such a metric by prescribing its Martin measure. We propose to use the harmonic measure of the Laplacian on the hyperbolic unfolding for this purpose.

In order to investigate mean convexity at infinity in a way that survives across conformal deformations, we will introduce *(β, ϕ)-bubbles* for usually positive functions β and ϕ on a manifold. These are (local) minimisers of the functional

$$\mathrm{bubb}_{\beta,\phi}(U) = \int_{\partial U} \beta \,\mathrm{d}A - \int_U \phi \,\mathrm{d}V$$

for Caccioppoli sets U. The boundaries of $(1, \phi)$-bubbles are also known as *hypersurfaces of prescribed mean curvature*, as they have mean curvature ϕ on their regular parts, and the boundaries of $(1,0)$-bubbles are area-minimisers. If a manifold is (β, ϕ)*–mean convex at infinity*, i.e., there is an exhaustion by compact outer-minimising (β, ϕ)-bubbles, it is also (β, ϕ')–mean convex at infinity for $\phi' \leqslant \phi$.

The connection to weighted isoperimetric inequalities is given by the following main result:

Theorem B (Mean Convexity at Infinity) *(Theorem 4.12) If a manifold (M, g) admits a w-weighted linear isoperimetric inequality and sufficiently many weighted nonlinear isoperimetric inequalities*[1] *and $(M, w^{\frac{2}{n-1}} g)$ is complete, then (M, g) is $(w, C \cdot w)$–mean convex at infinity, for some $C > 0$.*

Now if we can choose w in terms of an eigenfunction of the conformal Laplacian in such a way that there is a w-weighted isoperimetric inequality, conformal deformation with $w^{\frac{2}{n-1}}$ has positive scalar curvature and the other assumptions of Theorem B are satisfied, we end up with a metric of positive scalar curvature that is $(1, \phi)$–mean convex at infinity (and in particular $(1, 0)$–mean convex at infinity). We now describe a setting where this can be done.

N. Smale constructed a large class of singular homologically area-minimising hypersurfaces [Sma00, Theorem B]. Their singular set is a disjoint union of closed manifolds, and in a neighbourhood of each component, the hypersurface is isometric to a product of the singular set and a *regular* area-minimising cone, i.e., its only singularity is in the origin. We will call such minimisers *Smale hypersurfaces* (see Definition 5.5 for a more precise definition). As far as the author is aware, this is the most general known construction of singular homologically area-minimising hypersurfaces.

We will show:

Theorem C (Shielding Singularities) *(Theorem 5.8) The regular part of every Smale hypersurface H^n in a manifold M^{n+1} of positive scalar curvature can be conformally deformed to a metric of positive scalar curvature that is mean convex at infinity, provided the components Σ_i of the singular set have dimension*

$$1 \leqslant \dim \Sigma_i < \left(2\sqrt{3 + \frac{1}{n-1}} - 3\right)(n-1) - 1 \approx 0.46(n-1) - 1\,.$$

In particular, this is always true for $n \leqslant 11$ since the dimension of the singular set is at most $n - 7$.

As a warm-up for the case of minimal hypersurfaces, we will make a short excursion into the *singular Yamabe problem*. This concerns the question whether one can find complete metrics of constant scalar curvature that are conformally equivalent to the original metric on the complement of a certain set $\Sigma \subset M^n$ in a closed manifold M of positive scalar curvature. With the methods above, we can prove:

Theorem D (Singular Yamabe Problem for Zero Scalar Curvature) *(Corollary 5.4) If (M^n, g) is a closed manifold of positive scalar curvature and $\Sigma \subset M$*

[1] This will be made precise in the statement of Theorem 4.12.

a disjoint union of closed submanifolds of dimension at most $\frac{n-2}{2}$, then there is a complete scalar flat metric on $M \setminus \Sigma$ that is conformally equivalent to g and mean convex at infinity.

Apart from the mean convexity at infinity, this was already proven independently in [Del92] and [MM92] using different methods. Our result also applies to more general subsets Σ with *uniform* complement and a certain bound on the Laplacian of the distance to Σ, see Theorem 5.2 "metrics of zero scalar curvature".

This thesis is structured as follows:

In **chapter 1**, we discuss the geometric and analytic prerequisites to apply Ancona's potential theory for weakly coercive operators on Gromov hyperbolic manifolds of bounded geometry. We review known results about Gromov hyperbolic spaces and give examples and context for all assumptions. The last section of this chapter is focused on uniform spaces, their hyperbolic unfoldings, and how the conditions on hyperbolic manifolds translate into this setting.

In **chapter 2**, we give a mostly self-contained account of Ancona's potential theory on Gromov hyperbolic manifolds of bounded geometry. This is essentially the content of the survey [KL18]. We will present Ancona's theory with slight improvements, mainly keeping track of the involved constants, which will turn out to depend only on global constants in the assumptions, not on the particular space or operator. A new result presented at the end of this chapter is the *ray expansion* which expresses a positive harmonic function along a geodesic ray solely in terms of the Green's function along this ray and the associated Martin measure of balls around the endpoint of the ray in the Gromov/Martin boundary.

The subject of **chapter 3** are stronger results for the Laplace operator on Gromov hyperbolic manifolds of bounded geometry and a tree-like cover for such a space, which are combined to yield a weighted "mesoscale" Friedrichs inequality. This involves an averaged version of the gradient. Known Poincaré inequalities for different function spaces permit to upgrade this mesoscale inequality to the more usual weighted Friedrichs and linear isoperimetric inequalities in Theorem A.

Chapter 4 introduces (β, ϕ)-bubbles, their basic properties such as regularity and existence, and examples, as well as mean convexity at infinity and implications. The main result of this chapter is Theorem B "mean convexity at infinity" connecting weighted linear isoperimetric inequalities with mean convexity at infinity.

In **chapter 5**, we first give a general procedure to generate conformal metrics that are thick at infinity. Here all previous results come together. After applying this to the singular Yamabe problem, we conclude our journey through different areas of analysis and geometry in the realm of singular minimal hypersurfaces. We give an overview of some results from Lohkamp's potential theory on minimal hypersurfaces [Loh18, Loh20a, Loh20b] and then present the more involved application to Smale hypersurfaces in a positive scalar curvature ambience.

Danksagung

Notation

By "manifold" we mean a connected second-countable C^∞-smooth Riemannian manifold with generic name M^n, where $n \geqslant 1$ is the dimension. M always carries a C^∞-smooth Riemannian metric, usually called g. Additional data naturally constructed from g bear the following names: d is the induced distance function (infimal length of curves), ∇ is the Levi-Civita connection, $\exp_x : T_xM \to M$ the exponential map, $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ the Riemannian curvature tensor, Sec the sectional curvature, Ric the Ricci curvature and Scal the scalar curvature. Integration with respect to the volume measure is denoted by $\mathrm{d}V$, integration over a hypersurface (= codimension one submanifold) by $\mathrm{d}A$ (for area). If anything is induced by a different metric than g, this will be indicated by sub- or superscripts. λM is shorthand for the scaled manifold $(M, \lambda^2 g)$, for $\lambda > 0$.

Curves in a metric space X (where the distance function is generically called d) are often given as $\gamma : x \rightsquigarrow y$, meaning the curve starts in x and ends in y. *Geodesics* are always length-minimising, i.e., rectifiable curves $\gamma : x \rightsquigarrow y$ of length $d(x,y)$, even if the underlying space is a manifold. A *geodesic ray* is a curve $\gamma : [0,\infty) \to X$ such that $\gamma|_{[0,t]}$ is a geodesic for every $t > 0$. Geodesics and geodesic rays are always parameterised by arc length, starting with parameter 0. Sometimes we will not distinguish between a geodesic and its image. If a metric space X has no canonical superspace, $\overline{X}$ denotes the metric completion of X (e.g., equivalence classes of Cauchy sequences) and $\partial X = \overline{X} \setminus X$ is called the metric boundary. The distance from a point x to a set A in a metric space is written as $\operatorname{dist}(x, A)$.

For a topological space V, the notation $U \Subset V$ means U is an open relatively compact subset of V. A *domain* is a non-empty connected open subset.

A *Radon measure* on a locally compact topological space is an inner and outer regular locally finite positive Borel measure.

We will use the symbols $\preccurlyeq, \succcurlyeq, \asymp$ to mean $\leqslant, \geqslant, =$ up to a positive multiplicative constant. Usually this constant depends only on other natural constants that appear in the assumptions, if in doubt they are given as a subscript. E.g., the notation $A \preccurlyeq_\delta B$ stands for "there is a constant $C = C(\delta) > 0$, such that $A \leqslant C\,B$". $A \asymp B$ is shorthand for $A \preccurlyeq B$ and $A \succcurlyeq B$ and can be paraphrased as "there is a $C \geqslant 1$ such that $C^{-1}\,A \leqslant B \leqslant C\,A$". When these symbols are used, A and B are always positive numbers or functions.

Many of our statements are *quantitative* in the sense that constants in the assertions depend only on constants in the assumptions. If the dependence of constants is more intricate, we write it out explicitly.

Part I

Potential Theory on Hyperbolic Spaces

Chapter 1

Basic Concepts

In this chapter, we introduce the geometric and analytic setting for the chapters to follow. We will familiarise ourselves with Gromov hyperbolic manifolds of bounded geometry, introduce basic concepts from potential theory such as balayage and Martin boundary, and find large classes of examples in the form of hyperbolic unfoldings of uniform manifolds. The goal is to present basic definitions and standard results used in later chapters to the non-expert reader. We give examples and simple proofs, but for most proofs we refer to the indicated literature. There are no completely new and original results in this chapter, but some variations or abstractions, notably Theorem 1.34 "natural regularisation of generalised distance functions" was adapted for more general distance functions on manifolds. Moreover, the transformed operator in subsection 1.3.5 does not seem to have been used before.

This chapter is an extended version of parts of the preprint [KL18] by J. Lohkamp and the author.

1.1 Geometric Structures

One scope of this work is a class of usually noncompact, complete manifolds that have all of their interesting structure concentrated near infinity. This will be formalised in the following two geometric conditions.

1.1.1 Bounded Geometry

The **first condition** ensures that the manifold has no interesting local structure, it looks everywhere the same up to a uniformly controlled deviation.

Definition 1.1 (Bounded Geometry) A manifold M has $\boldsymbol{(\sigma,\ell)}$***–bounded geometry*** for global constants $\sigma > 0$, $\ell \geqslant 1$ if for every ball $B_\sigma(p) \subset M$ there is a smooth ℓ-bi-Lipschitz chart ϕ_p to an open subset U_p of $\mathbb{R}^n$ with its Euclidean metric.

Examples 1.2.

- Closed manifolds always have bounded geometry.
- This is also true for their covering spaces, i.e., complete manifolds with a cocompact isometric group action.

- A complete manifold with an absolute sectional curvature bound $|\,\mathrm{Sec}\,| \leqslant C$ and injectivity radius $\geqslant C^{-1}$ for some $C > 0$ has bounded geometry as this gives bounds on the metric tensor in normal coordinates by basic comparison estimates, see e.g. [Pet16, Theorem 6.27].

1.1.2 Gromov Hyperbolic Spaces

The **second condition** is complementary to the first in the sense that it has no local impact, but rather limits all efficient communication between two points to a region near the shortest path between them.

Definition 1.3 (Gromov Hyperbolicity) A Riemannian manifold, or more generally a geodesic metric space[1], is *Gromov hyperbolic* or, quantitatively, **δ-hyperbolic**, if there is a $\delta > 0$ such that each point on the edge of any geodesic triangle[2] is within δ-distance of one of the other two edges. See Figure 1.1.

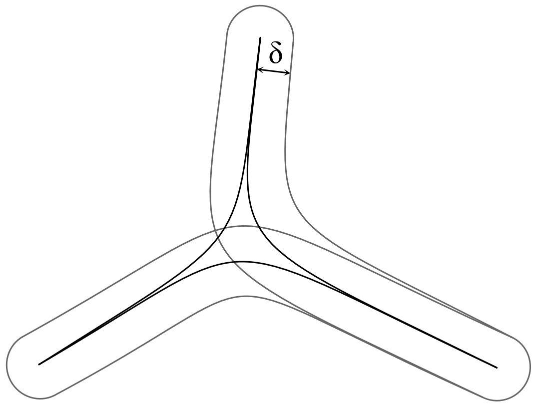


Figure 1.1: A geodesic triangle in a δ-hyperbolic space.

δ-hyperbolicity was introduced for Cayley graphs of finitely generated groups by Gromov in [Gro87]. Basic examples are:

Examples 1.4.

(i) Bounded spaces are trivially δ-hyperbolic for δ equal to the diameter.

(ii) Hyperbolic space $\mathbb{H}^n$ is δ-hyperbolic for $\delta = \ln 3$ [Gd90, 2.23]. More generally, CAT(-1)-spaces are $\ln 3$-hyperbolic, as directly seen from the definition [BH99, Proposition 1.2].

(iii) *Cartan–Hadamard manifolds*, i.e., complete simply-connected manifolds with sectional curvature bounded from above by a negative constant, are CAT(κ)-spaces for some $\kappa < 0$ and hence Gromov hyperbolic [Gd90, §3.2]. If the sectional curvature is additionally bounded from below, they have also bounded geometry.

[1] A metric space is *geodesic*, if each two points can be connected by a curve of length equal to the distance of the points. Such a curve is called a *geodesic* (in the metric space sense).

[2] Here, "geodesic" is meant in the metric space sense as "length-minimising geodesic".

(iv) Although we will mostly be concerned with manifolds, particularly simple examples that can illustrate many phenomena are graphs. By a ***graph*** we mean the geodesic metric space obtained by gluing intervals of a certain length between elements of a potentially infinite set of *vertices*, such that each vertex is adjacent to a finite number of these *edges*. Bounded geometry (and finite-dimensionality) of manifolds is analogous to a global upper bound on the number of adjacent edges for each vertex and a lower bound on their length.

Trees (simply connected graphs) are precisely the 0-hyperbolic graphs, hence for δ-hyperbolic graphs, δ can be seen as a measure of deviation from being a tree.

(v) A finitely generated group is called hyperbolic, if the Cayley graph with respect to some (and hence any, see [BH99, Examples I.8.17 (2) and (3)]) finite set of generators with edge length 1 is Gromov hyperbolic. Here, the easiest example are free groups (corresponding to trees, for canonical generators).

(vi) The universal covering of a closed manifold M of negative sectional curvature satisfies the requirements in (iii). By the Švarc-Milnor Lemma [BH99, Prop. I.8.19], any Cayley graph of the fundamental group $\pi_1(M)$ is quasi-isometric to M and hence also Gromov hyperbolic.

Here, a map $f : X \to Y$ between metric spaces (X, d_X), (Y, d_Y) is a (λ, S)-***quasi-isometry*** if there are constants $\lambda \geqslant 1$, $S \geqslant 0$ such that

$$\lambda^{-1} d_X(x, x') - S \leqslant d_Y(f(x), f(x')) \leqslant \lambda d_X(x, x') + S \quad \forall x, x' \in X$$

and every point $y \in Y$ has distance at most S to a point in the image of f. Without the last condition, f is a (λ, S)-***quasi-isometric embedding***. If such a quasi-isometric map exists, X and Y are *quasi-isometric*.

For invariance of Gromov hyperbolicity under quasi-isometries see e.g. [BS07, 1.3.1].

(vii) A non-example: Gromov hyperbolicity can be more demanding than constant negative sectional curvature alone. Consider a $\mathbb{Z}^2$-covering of a Riemann surface of genus $\geqslant 2$ equipped with a metric of constant negative sectional curvature, as in Figure 1.2. This is quasi-isometric to Euclidean $\mathbb{R}^2$ and hence not Gromov hyperbolic.

The idea alluded to above that there is no efficient path between two points that is far away from the shortest path can be formalised as follows:

Proposition 1.5 (Stability of Geodesics) *[BS07, Theorem 1.3.2], [Bon96] In a δ-hyperbolic geodesic metric space X, for every $\lambda \geqslant 1$ and $S > 0$ there is an $H = H(\delta, \lambda, S) > 0$ such that each two* (λ, S)-quasi-geodesics[3] *$x \rightsquigarrow y$ with same start- and endpoint $x, y \in X$ have Hausdorff distance at most H. In fact, this* geodesic stability *is equivalent to Gromov hyperbolicity.*

[3] (λ, S)-quasi-isometric embeddings of a compact interval

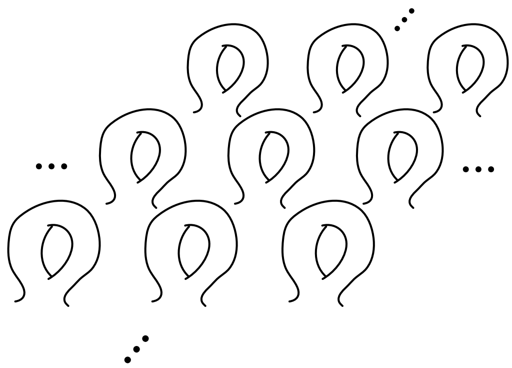


Figure 1.2: This 2-manifold has a metric with constant negative sectional curvature that is not δ-hyperbolic.

There are several definitions of Gromov hyperbolicity highlighting different aspects, see e.g. [BS07, Chapter 1], [BH99, Chapter III.H] or [Gd90, Chapitre 2]. We will only mention one other version using the ***Gromov product*** on X defined as

$$(x|y)_z := \frac{1}{2}\left(d(z,x) + d(z,y) - d(x,y)\right) \quad \text{for } x, y, z \in X.$$

This version works on arbitrary metric spaces.

Definition 1.6 (Hyperbolicity Via Gromov Product) A metric space X is *Gromov hyperbolic* if there is a $\delta' \geqslant 0$ such that for any four points $x, y, z, w \in X$,

$$(x|y)_z \geqslant \min\{(x|w)_z, (w|y)_z\} - \delta' . \tag{1.1}$$

This implies $4\delta'$-hyperbolicity and is implied by $\delta'/8$-hyperbolicity as defined above, see [Gd90, Proposition 2.21]. For convenience (and since the precise constant never really matters) we assume from now on every δ-hyperbolic space to satisfy (1.1) with $\delta' = \delta$.

An intuitive interpretation of the Gromov product in Gromov hyperbolic spaces is given by the following estimate (see also Figure 1.3):

Lemma 1.7 (Gromov Product as Distance to a Geodesic) *[Gd90, Lemme 2.17] In a δ-hyperbolic geodesic metric space X, let $\gamma : x \rightsquigarrow y$ be a geodesic and $z \in X$. Then*

$$(x|y)_z \leqslant \operatorname{dist}(z, \gamma) \leqslant (x|y)_z + 4\delta .$$

1.1.3 Gromov Boundary and Visuality

In Gromov hyperbolic spaces, the structure at infinity can be encoded in an ideal boundary. We outline the construction presented in [BS07, Section 2.2], where more details can be found.

Definition 1.8 (Gromov Boundary) In a Gromov hyperbolic metric space X with basepoint $o \in X$, a sequence (x_i) in x *converges at infinity* if $(x_i|x_j)_o \overset{i,j\to\infty}{\longrightarrow} \infty$.

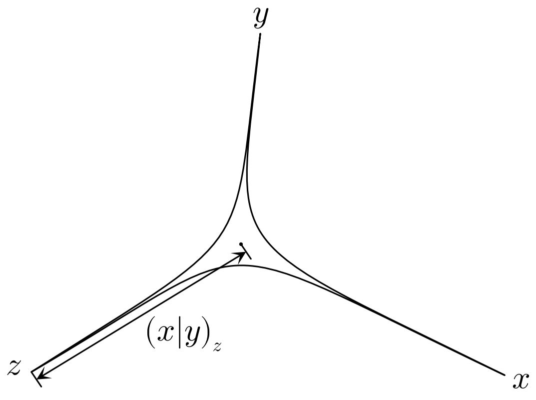


Figure 1.3: Approximate value of the Gromov product in a δ-hyperbolic space.

We say two sequences (x_i), (y_i) convergent at infinity are *equivalent* if $(x_i|y_i)_o \overset{i\to\infty}{\longrightarrow} \infty$. The ***Gromov boundary*** $\partial_{\mathrm{G}} X$ of X is defined as the set of equivalence classes of sequences convergent at infinity.

Note that $\partial_{\mathrm{G}} X$ is independent of the chosen basepoint because $|(x|y)_o - (x|y)_{o'}| \leqslant d(o, o')$ for a different basepoint $o' \in X$.

To define a topology on $\overline{X}^{\mathrm{G}} := X \dot{\cup} \partial_{\mathrm{G}} X$, we first extend the Gromov product to the boundary. We formally identify points in X with constant sequences. Then for $a, b \in \overline{X}^{\mathrm{G}}$ and $z \in X$, we can set

$$(a|b)_z := \inf_{\substack{(x_i)\in a \\ (y_i)\in b}} \liminf_{i\to\infty} (x_i|y_i)_z \, .$$

A topology on $\overline{X}^{\mathrm{G}}$ is defined by considering $X \hookrightarrow \overline{X}^{\mathrm{G}}$ as an embedding, choosing a basepoint $o \in X$ and declaring the sets

$$\mathcal{W}^o_\varrho(\xi) := \{a \in \overline{X}^{\mathrm{G}} \mid (a|\xi)_o > \varrho\} \quad \text{for } \varrho \geqslant 0$$

to be a neighbourhood basis for $\xi \in \partial_{\mathrm{G}} X$. As above, this topology is independent of o.

The sets $\mathcal{W}^o_\varrho(\xi) \cap \partial_{\mathrm{G}} X$ are in fact the open balls of radius $\mathrm{e}^{-\varrho}$ with respect to

$$d_o(\xi, \eta) := \mathrm{e}^{-(\xi|\eta)_o} \, ,$$

which is a *quasi-metric* on $\partial_{\mathrm{G}} X$, i.e., an ultrametric triangle inequality holds only up to a constant $Q = Q(\delta) = \mathrm{e}^\delta$,

$$d_o(\xi, \zeta) \leqslant Q \max\{d_o(\xi, \eta), d_o(\eta, \zeta)\} \text{ for } \xi, \eta, \zeta \in \partial_{\mathrm{G}} X,$$

while all other properties of metrics are still satisfied. This quasi-metric triangle inequality follows from an extension of (1.1) to the boundary [BS07, Lemma 2.2.2 (2)], while the other properties are obvious.

Remarks 1.9.

(i) If the Gromov hyperbolic space X is proper (balls are relatively compact) and geodesic (e.g., a complete manifold), it is sufficient to consider sequences on geodesic rays emanating from the basepoint o. One could also consider equivalence classes consisting of geodesic rays with finite Hausdorff distance [BS07, 2.4.2]. Hence any geodesic ray has a well-defined endpoint in $\partial_G X$ and we may use the notation $\gamma : x \rightsquigarrow \xi$ for a geodesic ray γ starting in $x \in X$ with endpoint $\xi \in \partial_G X$.

(ii) For a proper geodesic Gromov hyperbolic space X, $\overline{X}^G$ is in fact a compactification, i.e., $X \subset \overline{X}^G$ is open and dense and $\overline{X}^G$ is compact [BH99, Proposition III.H.3.7].

(iii) For quasi-metrics with constant $Q \leqslant 2$, there is a canonical procedure to construct a bi-Lipschitz equivalent metric [BS07, 2.2.2]. This can be applied to the quasi-metrics d_o^ε for sufficiently small ε to get a family of metrics on the boundary (called *visual metrics*), but we prefer to work with the canonical quasi-metric instead.

(iv) There are *many* hyperbolic spaces in the following sense: for each bounded metric space Z, there is a *hyperbolic approximation* of Z: this is a Gromov hyperbolic graph X with basepoint o such that the boundary $\partial_G X$ equipped with the quasi-metric d_o is bi-Lipschitz equivalent to Z [BS07, Theorem 6.4.1]. X is proper if and only if Z is compact [BS07, 6.4.3].

(v) The graph in the preceding remark can be approximated by a manifold of any dimension $n \geqslant 2$. To this end, represent each vertex by an n-sphere and for each edge connecting two vertices, form a connected sum of the representing spheres. If there is a global upper bound on the number of edges adjacent to any vertex, the metric on this manifold can be arranged to have bounded geometry.

For later use, we note that following (special case of a) Lemma of Bonk and Schramm, which also has a nice geometric interpretation:

Lemma 1.10 (Bonk–Schramm Lemma) *[BS00, 5.1] In a proper geodesic δ-hyperbolic space X, let $x, y \in X$ be points on geodesics $o \rightsquigarrow a \in \overline{X}^G$ and $o \rightsquigarrow b \in \overline{X}^G$ respectively. Then*

$$|(x|y)_o - \min\{(a|b)_o, d(o,x), d(o,y)\}| \leqslant 4\delta\,.$$

Proof. We can assume $d(o,x) \leqslant d(o,y)$. In the case $a, b \in X$, (1.1) shows

$$(a|b)_o \geqslant \min\{(a|x)_o, (x|y)_o, (y|b)_o\} - 2\delta\,.$$

Now $(a|x)_o = d(o,x)$, $(b|y)_o = d(o,y)$, and $(x|y)_o \leqslant \min\{d(o,x), d(o,y)\}$ by the triangle inequality, hence $\min\{(a|b)_o, d(o,x), d(o,y)\} \geqslant (x|y)_o - 2\delta$.

On the other hand, another application of (1.1) shows

$$\begin{aligned}(x|y)_o &\geqslant \min\{(x|a)_o, (a|b)_o, (b|y)_o\} - 2\delta \\ &= \min\{(a|b)_o, d(o,x), d(o,y)\} - 2\delta\,.\end{aligned}$$

This extends to $a, b \in \overline{X}^{\mathrm{G}}$ up to an additional constant of 2δ because

$$(a|b)_o = \inf_{\substack{(x_i)\in a \\ (y_i)\in b}} \liminf_{i\to\infty} (x_i|y_i)_o \geqslant \sup_{\substack{(x_i)\in a \\ (y_i)\in b}} \limsup_{i\to\infty} (x_i|y_i)_o - 2\delta$$

by [BS07, Lemma 2.2.2(1)]. □

The intuition behind this is that the geodesics stay close together up a point of divergence in approximate distance $(a|b)_o$ from o, and for points x, y beyond this point, $(x|y)_o$ is a good approximation of its distant counterpart $(a|b)_o$.

For some but not all applications, we will need another condition to ensure that the large-scale structure a Gromov hyperbolic space is completely encoded in the boundary:

Definition 1.11 (Visuality) A proper geodesic δ-hyperbolic space X is S-***visual*** for some $S > 0$ if there is a basepoint $o \in X$ such that every point $x \in X$ has distance at most S to a geodesic ray emanating from o.

Remarks 1.12.

- An S-visual δ-hyperbolic space X with basepoint o is S'-visual from any other basepoint $o' \in X$, where $S' = S'(S, o', \delta)$.
- The hyperbolic approximations from Remark 1.9(iv) are always visual.
- Two proper geodesic visual Gromov hyperbolic spaces with bi-Lipschitz equivalent boundaries are $(1, S)$-quasi-isometric for some $S \geqslant 0$ [BS07, Corollary 7.1.6].

1.1.4 Connecting Two Points: Harnack and Φ-Chains

Our two principal geometric assumptions each imply a way how two distant points can be linked. In the case of bounded geometry, the construction is rather trivial, but instructive by analogy to the more involved hyperbolic case.

Bounded Geometry In a complete manifold M of bounded geometry, one can link any two points $x, y \in M$ by a sequence of balls that are uniformly bi-Lipschitz equivalent to a Euclidean ball. Such configurations are known as *Harnack chains* [JK82] because they are mostly used to apply Harnack inequalities along greater distances as we will see later.

Definition 1.13 (Harnack Chains) For a fixed $r \in (0, \sigma)$, we call a sequence of balls $B_r(x_1), \dots, B_r(x_k)$ with

$$x_1 = x,\ x_k = y \text{ and } d(x_i, x_{i+1}) < r/2 \text{ for } i = 1, \dots, k-1$$

a ***Harnack chain*** of length k connecting x and y, see Figure 1.4.

Thus any point x_i is contained even in the smaller neighbouring balls $B_{r/2}(x_{i\pm1})$. By setting $x_i = \gamma(i \cdot r/3)$ on a shortest geodesic $\gamma : x \rightsquigarrow y$ parameterised by arc length, we get a Harnack chain connecting x and y of length proportional to $d(x, y)$.

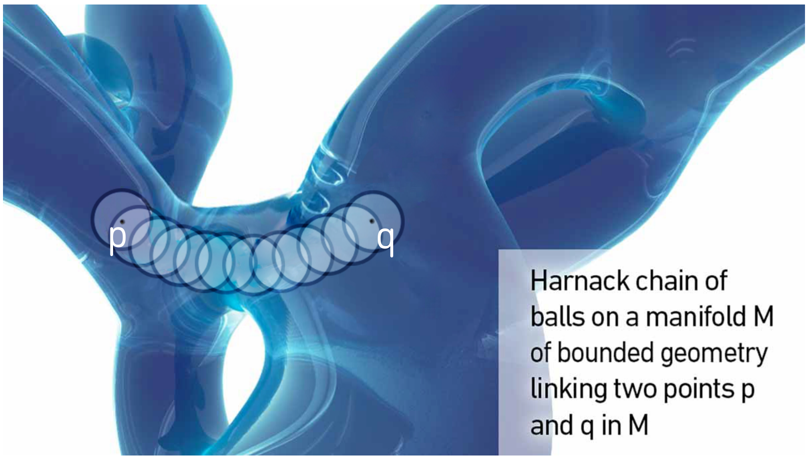


Figure 1.4: Harnack chains (image from [KL18]).

Gromov Hyperbolicity To exploit Gromov hyperbolicity analytically, Ancona introduced the concept of *Φ-chains*. As in the case of bounded geometry versus hyperbolicity, Harnack chains and Φ-chains act complementary. This is best seen in the discussion preceding Proposition 2.15 "Growth Recovery Along Φ-Chains" where Harnack chains give basic estimates which, however, weaken the overall control, while Φ-chains can be used to recover the apparently lost details.

Definition 1.14 (*Φ*-Chains) For a monotonically increasing function $\Phi : [0, \infty) \to (0, \infty)$ with $\Phi_0 := \Phi(0) > 0$ and $\Phi(d) \overset{d\to\infty}{\longrightarrow} \infty$, a ***Φ*-chain** on a proper geodesic metric space X is a finite or infinite sequence (U_i) of open subsets of X with $U_i \supset U_{i+1}$ together with a sequence of **track points** (x_i) such that

(i) $\Phi_0 \leqslant d(x_i, x_{i+1}) \leqslant 3\Phi_0$,

(ii) $x_i \in \partial U_i$,

(iii) $d(x, \partial U_{i\pm1}) \geqslant \Phi(d(x, x_i))$, for every $x \in \partial U_i$

for every i where applicable.[4] See Figure 1.5.

Note that a Φ-chain traversed backwards, i.e., with sets $\cdots \supset X \setminus \overline{U_i} \supset X \setminus \overline{U_{i-1}} \supset \cdots$, is again a Φ-chain with the same track points.

The existence of infinite Φ-chains can be considered as a partial hyperbolicity property of the underlying space. It is easy to see that neither Euclidean space nor asymptotically flat spaces admit any infinite Φ-chains.

A first non-trivial example can be created as follows: with the coordinates $(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$ we consider the metric $(1 + |y|^2)^2 \cdot g_{\mathbb{R}} + g_{\mathrm{Eucl}}$ on $\mathbb{R} \times \mathbb{R}^{n-1}$. Then the half-spaces $U_i := (i, \infty) \times \mathbb{R}^{n-1}$ form a Φ-chain with track points $x_i = (i, 0)$ for $\Phi(t) := 1 + t^2$. But as in the Euclidean case, the half-spaces $U_i[k] := \mathbb{R}^k \times (i, \infty) \times \mathbb{R}^{n-k-1}$, for $1 \leqslant k \leqslant n-1$, do not make up a Φ-chain.

[4] For notational convenience this is slightly different from Ancona's version in [Anc90, définitions V.5.1], but essentially the same.

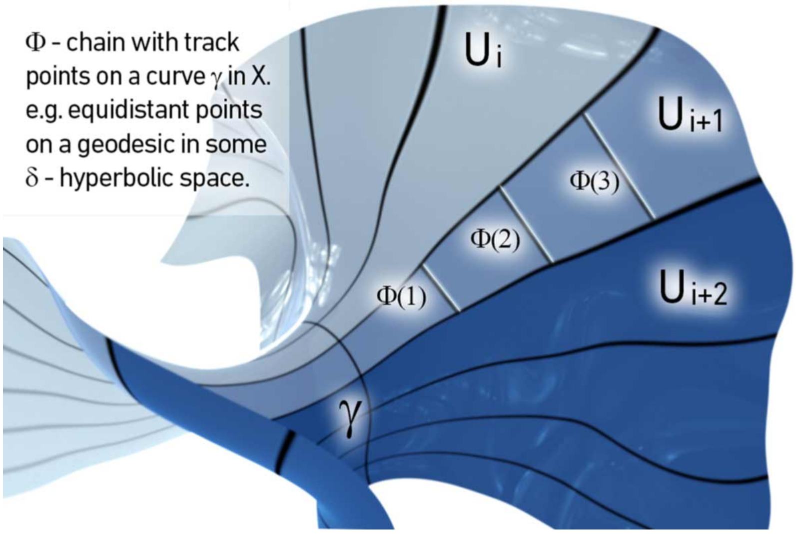


Figure 1.5: Φ-Chains (image from [KL18]).

The hyperbolic n-space $\mathbb{H}^n$ carries Φ-chains in all directions and for the same Φ. Namely, when we think of $\mathbb{H}^n$ as the upper half-space and $B_{1/2^i}(0)$ is the Euclidean ball of radius $1/2^i$, then $U_i = B_{1/2^i}(0)$, $i \geqslant 1$, form a Φ-chain for $\Phi(t) = \alpha \cdot t + \beta$ for suitable $\alpha, \beta > 0$, independent even of n. Due to the homogeneity of $\mathbb{H}^n$ this also gives a Φ-chain along any hyperbolic geodesic γ, i.e., with track points on γ.

This ubiquity of Φ-chains, which we see already from considering hyperbolic half-spaces relative to geodesics, extends to arbitrary non-homogenous Gromov hyperbolic spaces.

Theorem 1.15 (Φ-Chains on Hyperbolic Spaces) *[BHK01, Section 8] On a proper geodesic δ-hyperbolic space X, let $\gamma : [0, 4\delta k] \to X$ be a geodesic with $k \in \mathbb{Z}^+ \cup \{\infty\}$. Set $a = \gamma(0) \in X$ and $b = \gamma(4k\delta) \in \overline{X}^{\mathrm{G}}$. Then the sets*

$$U_i := \{x \in X \mid (x|b)_a > 4i\delta\}$$

form a Φ_δ-chain with track points $x_i = \gamma(4i\delta)$ for $\Phi_\delta(t) = \alpha \cdot t + \beta$, with constants $\alpha = \alpha(\delta) > 0$ and $\beta = \beta(\delta) > 0$.

Note that for a geodesic ray $o = a \rightsquigarrow b = \xi \in \partial_{\mathrm{G}} X$, the sets $U_i = \mathcal{W}^o_{4i\delta}(\xi) \cap X$ are the restriction of our usual neighbourhood basis for ξ to X.

1.2 Analytic Structures

Now we turn to the analytic side. On a complete Riemannian manifold M^n of bounded geometry, we consider an operator L with the following properties:

Definition 1.16 (Adaptedness) For constants $k \geqslant 1$ and $\beta \in (0, 1]$, a linear elliptic operator L of second order on M is called (k, β)-***adapted*** if relative to a bounded geometry chart ϕ_p for every point $p \in M$,

$$L(u) = -\sum_{i,j} a^{ij} \cdot \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_i b^i \cdot \frac{\partial u}{\partial x^i} + c \cdot u$$

for β-Hölder continuous coefficients $a^{ij}, b^i, c : U_p \to \mathbb{R}$, for $i, j = 1, \dots, n$, with

$$k^{-1} \sum_i \xi_i^2 \leqslant \sum_{i,j} a^{ij} \xi_i \xi_j \leqslant k \sum_i \xi_i^2$$

for any $\xi \in \mathbb{R}^n$ and

$$|a^{ij}|, |b^i|, |c| \leqslant k$$

and the same is true for the adjoint operator L^*, which is defined by the condition $\int_M u \cdot Lv \, \mathrm{d}V = \int_M L^* u \cdot v \, \mathrm{d}V$ on smooth functions $u, v \in C_c^\infty(M)$ of compact support.

Solutions $u \in C^{2,\alpha}(M)$ of the equation $Lu = 0$ are called ***L*-harmonic** functions.

The adaptedness assumption is a generalised form of uniform ellipticity. It is a compromise between generality and readability, trimmed to ensure that one can apply the usual maximum principles and all weak L^2_{loc} and strong solutions of $Lu = 0$ are classical $C^{2,\beta}$-regular solutions [BJS64, p. 136–138]. The uniform boundedness of coefficients will be used in global uniform estimates for L and for the adjoint operator L^*. For even more general conditions, see [Anc87].

1.2.1 Green's Functions

One main object of study will be a **Green's function** G for L. This is a function $G : M \times M \to (0, \infty]$ which is finite and $C^{2,\beta}$-regular outside the diagonal $\{(x, x) \mid x \in M\}$ and satisfies the equation $L\, G(\cdot, y) = \delta_y$ in a distributional sense, where δ_y is the Dirac delta function with basepoint y. This means that for any $y \in M$, $G(\cdot, y) > 0$ is a (singular) function so that for a given $f \in C_c^\infty(M)$, the function

$$u(x) = \int_M G(x, y)\, f(y) \,\mathrm{d}y$$

is a solution of $Lu = f$, where integration is with respect to the volume measure induced by the Riemannian metric on M. Note that a Green's function associated to L^* is given by $G^*(x, y) = G(y, x)$.

The sum of a Green's function $G(\cdot, y)$ and a positive solution of $Lu = 0$ is again a (different) Green's function, but there is always a uniquely determined *minimal* Green's function that does not admit a sum decomposition into another Green's function and a positive L-harmonic function. Unless stated differently, we henceforth always mean minimal Green's functions when referring to Green's functions.

A Green's function for an adapted operator L does not always exist. We will see criteria for its existence in subsection 1.2.3.

1.2.2 Potentials and Balayage

Later we will use several constructions that are best described in the language of potential theory. Here we will briefly describe central concepts. More background information and details can be found in standard literature on the subject such as [Bau66, Bre67, Hel69, CC72, BH86].

A basic building block for axiomatic potential theory is local solvability of the Dirichlet problem, i.e., one expects the existence of a topological basis of open sets, such that for every continuous function on the boundary, there is an L-harmonic function in the interior that is continuous on the closure. For an adapted operator L, this is true on small, smoothly bounded balls.

Any not necessarily minimal Green's function $G(\cdot, y)$ is an example of an ***L-superharmonic*** function, that is a lower semi-continuous function u with values in $(-\infty, \infty]$ such that u is finite on a dense set and $u \geqslant v_B$ for every ball B, where v_B is the Dirichlet solution of $Lv_B = 0$ on B with boundary conditions $v_B|_{\partial B} \equiv u|_{\partial B}$. On sufficiently regular parts, this is equivalent to $Lu \geqslant 0$.

An L-superharmonic function $p \geqslant 0$ such that there is *no* positive L-harmonic function h with $p \geqslant h > 0$, is called an ***L-potential***. *Minimal* Green's functions are not only examples for potentials, but also their basic building blocks:

Theorem 1.17 (Integral Representation of Potentials) *[Her62, 22.], [Hel69, Thm. 6.18] If L has a Green's function G, every L-potential p on M can be represented by a unique (positive) Radon measure μ_p as*

$$p(x) = G(\mu_p)(x) := \int_M G(x, y)\, \mathrm{d}\mu_p(y)\,.$$

The support of p (i.e., the complement of the largest open set where p is L-harmonic) equals the support of μ_p.

The fact that every positive L-superharmonic function is uniquely representable as the sum of an L-potential and a positive L-harmonic function is known as the *Riesz representation theorem.* Hence, to get an integral representation for positive *L-superharmonic* functions, only the L-harmonic part is left. We will see later that the corresponding measures are supported on the *Martin boundary* which is defined exactly for this purpose. Its identification in terms of more common geometric boundaries is one of the goals of chapter 2.

Later we want to control L-superharmonic functions along Φ-chains. Here we shift the part of the defining measure supported in $M \setminus U$ onto ∂U without changing the function on U, for an unbounded open set U. This strategy is called sweeping or, due to its French origin (Poincaré, Cartan), ***balayage***.

Concretely, for an L-superharmonic function $u \geqslant 0$ on M and a subset $A \subset M$ we define

$$\mathcal{R}_u^A := \inf\{v \geqslant 0 \mid v \text{ is } L\text{-superharmonic on } M \text{ with } v \geqslant u \text{ on } A\}\,.$$

This is called the ***reduit*** (reduced). It enjoys the following properties which we will need later:

- $\mathcal{R}_u^A$ is L-harmonic outside of $\bar{A}$ and equal to u on A.
- The reduit is always L-superharmonic.
- If A is relatively compact, $\mathcal{R}_u^A$ is an L-potential.
- $\mathcal{R}_{\lambda u}^A = \lambda \mathcal{R}_u^A$ for a constant $\lambda \geqslant 0$.
- $\mathcal{R}_{u+v}^A = \mathcal{R}_u^A + \mathcal{R}_v^A$ for functions u, v.

- $\mathcal{R}_u^{A\cup B} \leqslant \mathcal{R}_u^A + \mathcal{R}_u^B$ for sets $A, B \subset M$.
- Denoting the reduit with respect to the adjoint operator L^* of L by ${}^*\mathcal{R}$, we have $\mathcal{R}^A_{G(\cdot,y)}(x) = {}^*\mathcal{R}^A_{G(x,\cdot)}(y)$ [Anc90, I.5.1, p. 19].
- If $G(\mu)$ is an L-potential, $\mathcal{R}^A_{G(\mu)}(x) = \int_M \mathcal{R}^A_{G(\cdot,y)}(x)\,\mathrm{d}\mu(y)$ for any $x \notin \bar{A}$ [Her62, Théorème 22.4].

For general sets A, it may happen that $\mathcal{R}_u^A$ is not lower semi-continuous, but it always admits a canonical regularisation $\widehat{\mathcal{R}}_u^A$, the *balayée* (swept), defined as the maximal lower semi-continuous function $\leqslant \mathcal{R}_u^A$. For open sets A or in general outside of $\bar{A}$ the two concepts coincide. We can even recover the classical Perron solution u of the Dirichlet problem on a ball B with continuous positive boundary value f as

$$u(x) = \mathcal{R}_f^{\partial B}(x)\,.$$

Also useful in this context are global variants of the maximum principle.

Theorem 1.18 (Global Maximum Principle) *[Her62, p. 429]*

(i) If u is L-superharmonic on an open set $V \subset M$, $u \geqslant 0$ on ∂V, and there is an L-potential p such that $u \geqslant -p$, then $u \geqslant 0$ on V.

(ii) Let p an L-potential, L-harmonic on an open set V and locally upper bounded near every point of ∂V. If $u \geqslant p$ on ∂V for some positive L-superharmonic function u, then $u \geqslant p$ in all of V.

Proof. For (i), note that the function $\bar{u}$ defined as $\min(u, 0)$ on V and 0 on $M \setminus V$ is L-superharmonic and $\geqslant -p$. Now the supremum of the family $\{L\text{-subharmonic functions} \leqslant \bar{u}\}$ is L-harmonic, $\geqslant -p$, and $\leqslant 0$, hence by the definition of L-potentials it is 0 which implies $u \geqslant 0$.

(ii) follows from (i) by considering the function $u - p$. □

1.2.3 Weak Coercivity

The existence of a Green's function is not always granted. All relevant effects can already be seen when considering the Laplacian $-\Delta$ on flat $\mathbb{R}^n$. It is well-known that the "shifted" operator $-\Delta - \lambda$ admits a Green's function if and only if $\lambda \leqslant 0$. Moreover, we will see later on that there is a fine distinction between the cases $\lambda = 0$ and $\lambda < 0$. The following assumption means that we are focussing on the second case.

Definition 1.19 (Weak Coercivity) An adapted operator L is called *weakly coercive* if there is a $t > 0$ such that the operator $L_t := L - t$ admits a Green's function G^t.

Another way to express weak coercivity is to say that there is a positive $C^{2,\alpha}$-regular strictly L_t-superharmonic function (i.e., $Lu \geqslant 0$ and $Lu \not\equiv 0$) for some $t > 0$. This is equivalent to the existence of a Green's function G^t [Pin95, Proposition 4.2.3, p. 133]. Now such a function is $L_{t'}$-superharmonic for every $t' \leqslant t$ and this almost directly implies the following result:

Theorem 1.20 (Criticality) *[Pin95, sections 4.3 and 4.11] For an adapted operator L, there is a* **principal eigenvalue** $\tau \in [-\infty, \infty]$ *such that*

- *for $t < \tau$, the Green's function G^t exists and there are many positive L_t-superharmonic functions, and*
- *for $t > \tau$, there is no globally defined positive L_t-superharmonic function.*
- *In the borderline case $t = \tau$, for a finite τ, the Green's function G^τ might or might not exist, but in any case there is a globally defined positive L_τ-harmonic function, unique up to multiplication with positive constants.*

Hence weak coercivity of an adapted L amounts to $\tau > 0$ and the operators L_t, which are adapted (with shared constants for any bounded range of t-values), are for $t < \tau$ even weakly coercive.

In examples, we are mostly concerned with *Schrödinger operators* of the form $L = -\Delta + V$ for a real-valued function V. Such an operator is adapted as soon as V is Hölder-continuous and bounded. It is often comfortable to estimate the principal eigenvalue of a Schrödinger operator with variational methods:

Theorem 1.21 (Characterisation of the Principal Eigenvalue) *For an adapted Schrödinger operator $L = -\Delta + V$, the principal eigenvalue τ is given as*[5]

$$\tau = \inf_{f \in C_c^\infty(M)} \int_M \left(|\nabla f|^2 + V f^2\right) \mathrm{d}V \Big/ \int_M f^2 \, \mathrm{d}V \,.$$

Proof. We denote the infimum by μ.

For a bounded domain $D \Subset M$ with smooth boundary, let λ_L^D be the first Dirichlet eigenvalue of L on D and u_D the corresponding eigenfunction, positive on D and vanishing on ∂D. For a positive L_τ-harmonic function u on M, Green's identity shows

$$\begin{aligned}(\lambda_L^D - \tau) \int_D u \, u_D \, \mathrm{d}V &= \int_D \left(u(Lu_D) - u_D(Lu)\right) \mathrm{d}V \\ &= \int_{\partial D} \left(-u \frac{\partial u_D}{\partial \nu} + u_D \frac{\partial u}{\partial \nu}\right) \mathrm{d}A \\ &= -\int_{\partial D} u \frac{\partial u_D}{\partial \nu} \, \mathrm{d}A \geqslant 0 \,,\end{aligned}$$

where $\frac{\partial}{\partial \nu}$ is the outer normal derivative, and hence $\lambda_L^D \geqslant \tau$. From the variational characterisation of the first Dirichlet eigenvalue by the Rayleigh quotient,

$$\lambda_L^D = \inf_{\substack{f \in C^\infty(D) \\ f|_{\partial D} \equiv 0}} \int_D \left(|\nabla f|^2 + V f^2\right) \mathrm{d}V \Big/ \int_D f^2 \, \mathrm{d}V \,,$$

we see that $\mu = \inf_{D \Subset M} \lambda_L^D \geqslant \tau$.

[5] Here, V is the usual name for the "potential" in a Schrödinger operator and $\mathrm{d}V$ the usual name for the volume measure on M, *not* integration with respect to V. We hope this is not too confusing. Besides, V is usually called "potential", but it is no L-potential in the sense of potential theory, these concepts merely have similar origins in physics.

Conversely, the existence of an L_t-superharmonic function for $0 < t < \mu$ is commonly proven with Hilbert space methods [Anc87, Lemma 2], [KS80, section II.4]: the Dirichlet form

$$a_t(u,v) := \int_M (g(\nabla u, \nabla v) + (V - t)\, u\, v)\, \mathrm{d}V$$

associated to the operator L_t is a symmetric bilinear form which continuously extends to the closure $H_0^{1,2}(M)$ of $C_c^\infty(M)$ in the Sobolev space $H^{1,2}(M)$ of square-integrable functions on M with square-integrable first derivatives. $H_0^{1,2}(M)$ is a Hilbert space with scalar product $\langle u, v\rangle_{H^{1,2}} = \int_M (g(\nabla u, \nabla v) + u\, v)\, \mathrm{d}V$.

Now a_t is also a scalar product on $H_0^{1,2}(M)$ (it is *coercive*) for $0 < t < \mu$: because V is bounded from below, there is a $c > 0$ such that

$$a_t(f,f) \geqslant \|f\|^2_{H^{1,2}} - (c+t)\|f\|^2_{L^2}\,,$$

and from the definition of μ, we have $a_t(f,f) \geqslant (\mu - t)\|f\|^2_{L^2}$ and hence

$$\left(1 + \frac{c+t}{\mu - t}\right) a_t(f,f) \geqslant \|f\|^2_{H^{1,2}}\,.$$

Then we can apply the Riesz representation theorem to the functional $\psi \mapsto \int_M \phi\, \psi\, \mathrm{d}V$ for a fixed function $\phi \in H_0^{1,2}(M)$ with $\phi > 0$ to see that there is a $v \in H_0^{1,2}(M)$ with

$$a_t(v, \psi) = \int_M \phi\, \psi\, \mathrm{d}V \quad \text{for all } \psi \in H_0^{1,2}(M).$$

In particular, v is a weak solution of the equation $L_t\, v = \phi$ and hence (a version of v is) a classical solution by the remark after Definition 1.16 "adaptedness". It is nonnegative because for $v^- := \min\{0, v\}$,

$$0 \leqslant a_t(v^-, v^-) = a_t(v, v^-) = \int_M \phi\, v^-\, \mathrm{d}V \leqslant 0\,.$$

Since v satisfies $L_t\, v = \phi > 0$, there is a $H^{1,2}$-small perturbation of v that is strictly positive and still L_t-superharmonic. Because $t < \mu$ was arbitrary, this shows $\tau \geqslant \mu$. □

Hence a Schrödinger operator L is weakly coercive if and only if

$$\int_M f\, Lf\, \mathrm{d}V \succcurlyeq \int_M f^2\, \mathrm{d}V \quad \text{for every } f \in C_c^\infty(M).$$

1.2.4 Martin Boundary

In the same spirit as L-potentials can be represented by Green's functions integrated over Radon measures on M, there is a representation of positive L-harmonic functions by Radon measures on an abstractly associated space $\partial_{\mathrm{M}}(M, L)$, the *Martin boundary*. A priori, this is a purely potential theoretic object and there might be additional peculiarities in the representation, but as a central result of chapter 2 we will see that the Martin boundary for a weakly coercive adapted operator on a Gromov hyperbolic manifold of bounded geometry can be canonically identified with the Gromov boundary.

Here, we sketch the basic notions from Martin theory, see [Anc90], [Pin95, 7.1] or [Hel69, Ch. 12] for detailed proofs.

Definition 1.22 (Martin Boundary) For a non-compact Riemannian manifold M, a linear second order elliptic operator L on M with a minimal Green's function $G : M \times M \to (0, \infty]$ and a basepoint $o \in M$, we consider the set S of sequences s of points $x_i \in M$, $i = 1, 2, \ldots$, such that

- s has no accumulation point in M, and
- $K_{x_i} := \frac{G(\cdot, x_i)}{G(o, x_i)} \stackrel{i\to\infty}{\longrightarrow} K_s$ compactly, for some function $K_s : M \to (0, \infty)$.

As a set, the **Martin boundary** $\partial_{\mathrm{M}}(M, L)$ is the quotient of S modulo the relation $s \sim s'$ if $K_s \equiv K_{s'}$. For $\zeta \in \partial_{\mathrm{M}}(M, L)$, this function is written K_ζ and called **Martin function**.

The Martin boundary does not depend on the choice of the basepoint o. The Harnack inequality which we will prove in Theorem 2.5 and elliptic theory show that each $K_\zeta \in \partial_{\mathrm{M}}(M, L)$ is a positive L-harmonic function on M. This also shows that the convex set $S_L(M)$ of positive L-harmonic functions u on M with $u(o) = 1$ is compact in the topology of compact convergence. In turn, $\partial_{\mathrm{M}}(M, L)$ is a compact subset of $S_L(M)$.

We topologise the space $\overline{M}^{\mathrm{M}} := M \cup \partial_{\mathrm{M}}(M, L)$ using the topology of compact convergence on the space of associated Martin functions $\{K_y \mid y \in \overline{M}^{\mathrm{M}}\}$, which can be canonically identified with $\overline{M}^{\mathrm{M}}$. Then the usual topology is induced on $M \subset \overline{M}^{\mathrm{M}}$, $\partial_{\mathrm{M}}(M, L)$ is closed and $\overline{M}^{\mathrm{M}}$ is compact. The space $\overline{M}^{\mathrm{M}}$ is called the **Martin compactification**. It is easy to see that $\overline{M}^{\mathrm{M}}$ is metrisable, see [BJ06, Ch. I.7] or [Hel69, Ch. 12] for further details.

To motivate the idea of Martin integrals we recall the following classical result, see e.g. [Cho69, Ch. 6]:

Proposition 1.23 (Minkowski's Theorem) *Each point in a convex set $K \subset \mathbb{R}^n$ is a convex combination of the extremal points of K.*

The **Martin integral representation** is essentially an extension of this result to the case of the convex set $S_L(M)$. The extremal elements of $S_L(M)$ form a subset $\partial^0_{\mathrm{M}}(M, L) \subset \partial_{\mathrm{M}}(M, L)$ of the Martin boundary which can be seen as the vertices of an infinite dimensional simplex spanning $S_L(M)$. A positive solution u of $Lu = 0$ on M with $u(o) = 1$ is extremal if and only if u is a *minimal* solution in the following sense: for any other solution $v > 0$, $v \leqslant u$, there is a constant $c > 0$ such that $v \equiv c \cdot u$. Therefore $\partial^0_{\mathrm{M}}(M, L) \subset \partial_{\mathrm{M}}(M, L)$ is also called the **minimal Martin boundary**.

The Choquet integral representations in [Cho69, Ch. 6] give the following general version of the Martin representation theorem, see [Pin95, 7.1]:

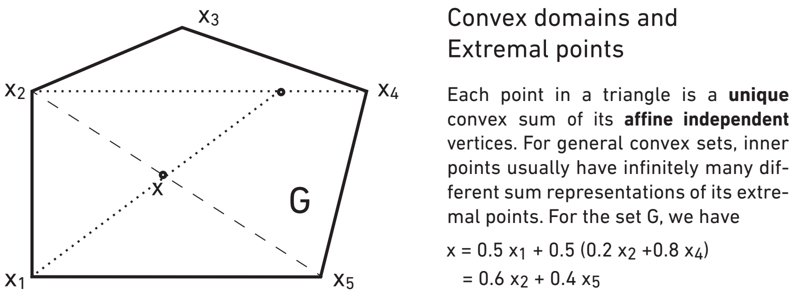


Figure 1.6: Minkowski's Theorem in $\mathbb{R}^2$ (image from [KL18]).

Proposition 1.24 (Martin Integral Representation) *For any positive solution u of $Lu = 0$ on M, there is a unique Radon measure μ_u on $\partial^0_{\mathrm{M}}(M, L)$, sometimes called* Martin measure *of u, so that*

$$u(x) = \int_{\partial^0_{\mathrm{M}}(M,L)} K_\zeta(x)\, \mathrm{d}\mu_u(\zeta)\,.$$

Conversely, for any Radon measure μ on $\partial^0_{\mathrm{M}}(M, L)$,

$$u_\mu(x) = \int_{\partial^0_{\mathrm{M}}(M,L)} K_\zeta(x)\, \mathrm{d}\mu(\zeta)$$

defines a positive solution of $Lu_\mu = 0$ on M.

Although this already looks like a classical contour integral, the result is not yet truly satisfactory. Unlike the classical case, the boundary $\partial^0_{\mathrm{M}}(M, L)$ depends not only on the underlying space, but also on the analysis of the operator L. A natural question is whether one could get rid of this dependence. In general the answer is no, as the by no means exotic Examples 2.29 show. However, we will see that in our case of weakly coercive adapted operators on Gromov hyperbolic spaces of bounded geometry, this is actually possible. This is a remarkable particularity not even valid for such simple spaces as in Example 2.27 "Ideal Boundaries of $\mathbb{H}^m \times \mathbb{H}^n$".

1.3 Hyperbolic Unfoldings

A large source of Gromov hyperbolic manifolds are *hyperbolic unfoldings* of *uniform spaces*. The basic idea is that a bounded subspace (a domain or the regular part of a minimal hypersurface) of a well-controlled space ($\mathbb{R}^n$ or a compact manifold) may degenerate towards a complicated boundary, but on the scale of the distance to this boundary, it might look less obscure. In this situation, one can conformally deform with the inverse of the distance to the boundary to push all difficulties towards infinity. For uniform spaces, this hyperbolic unfolding yields indeed a Gromov hyperbolic space.

In this section, we will explain how geometric and analytic conditions and results translate between uniform spaces and their hyperbolic unfoldings. This provides us with a rich source of spaces and operators where can apply the results of chapter 2.

1.3.1 Generalised Distance Functions

The main ingredient for passing from uniform to Gromov hyperbolic spaces is a function encoding the *distance to the boundary*. The most obvious candidate, the actual distance to the boundary, has analytical shortcomings: it is Lipschitz regular, but in general not even C^1, e.g. for a ball in $\mathbb{R}^n$. To facilitate the discussion of regularisations and to later be able to include information about curvature in such a function, we define:

Definition 1.25 (Generalised Distance Function) On an incomplete Riemannian manifold M with metric boundary $\partial M := \overline{M} \setminus M$, a function $đ : M \to (0, \infty)$ is called a *generalised distance function* if

(i) it is L-Lipschitz for some $L > 0$: $|đ(x) - đ(y)| \leqslant L\, d(x, y)$ for $x, y \in M$, and

(ii) $đ(x) \to 0$ for $x \to \partial M$.

A direct consequence of these properties is

$$đ(x) \leqslant L \operatorname{dist}(x, \partial M) \quad \text{for any } x \in M.$$

Examples 1.26.

- We can always take the distance to the boundary $đ(x) = \operatorname{dist}(x, \partial M)$. It is 1-Lipschitz by the triangle inequality.
- For area-minimising hypersurfaces $H^n \subset M^{n+1}$ with singular set Σ (or more generally, almost-minimisers with bounded generalised mean curvature), the $\mathcal{S}$*-distance* $\delta_{\langle A \rangle}$ is a generalised distance function on $H \setminus \Sigma$. There are various possible definitions, e.g.,

  $$\delta_{\langle A \rangle}(x) = \sup\{r > 0 \mid \sup_{B_r(x)} |A| \leqslant r^{-1}\},$$

  where $|A|$ is the norm of the second fundamental form on $H \setminus \Sigma$. Besides the distance to the boundary, it encodes information about the relative curvature of H, giving more control. For more details, see [Loh18].
- Regularisations of these functions are again generalised distance functions, see subsection 1.3.4.

1.3.2 Uniform Spaces and **đ**–Bounded Geometry

The path connectedness of an open set is a topological condition. For finer geometric and analytic investigations one seeks for a quantitative form of path connectedness. One of the nowadays central notions is described in the following definition. We refer to [Aik12] for an instructive overview and comparison of other regularity concepts in the context of Euclidean domains.

Definition 1.27 (Uniform Spaces) An incomplete Riemannian manifold M with metric boundary $\partial M := \overline{M} \setminus M$ and generalised distance function $đ$ is called **uniform** (with respect to $đ$), or, more precisely, c-uniform, if there is a $c \geqslant 1$ such that any two points $x, y \in M$ can be joined by a ***c*-uniform curve**, that is a rectifiable curve $\gamma : [a, b] \to M$, for some $a < b$, with $\gamma(a) = x$ and $\gamma(b) = y$, so that the following conditions are satisfied:

- ***Quasi-geodesic:*** $\mathrm{length}(\gamma) \leqslant c\, d(x, y)$.
- ***Twisted double cones:*** $\min\{\mathrm{length}(\gamma|_{[a,t]}), \mathrm{length}(\gamma|_{[t,b]})\} \leqslant c\, đ(\gamma(t))$ for any $t \in [a, b]$.

Note that being c-uniform is a scaling invariant condition: whenever M is c-uniform, $\lambda\, M$, for some $\lambda > 0$, is also c-uniform.

Examples 1.28. We start with some types of mostly Euclidean domains that are uniform with respect to the distance to the boundary, see also Figure 1.7. Note that the distance d is defined here as the infimum over the lengths of curves connecting two points; for non-convex domains this is different from the restriction of the Euclidean distance. In the context of Euclidean domains, uniformity in this metric is sometimes called *inner uniformity* [Aik04].

- Any bounded domain with smooth or at least Lipschitz boundary is uniform. More generally, the complement of finitely many compact embedded Lipschitz submanifolds of arbitrary dimension in a closed Riemannian manifold is uniform with respect to the distance to these submanifolds. This follows from [MS79, 2.14] (bi-Lipschitz invariance of uniformity) and [Väi88, Theorem 4.1] (uniformity of bounded domains is a local property of the boundary).
- Non-compact rotationally symmetric domains bounded with profile functions of at least linear growth are uniform. As an explicit example, choose $f(t) = c_1 \cdot t + c_2$, for constants $c_i > 0$ and let $F : \mathbb{R}^{n-1} \to \mathbb{R}$, $n \geqslant 3$, be given by $F(x) := f(|x|)$. Consider the domain

  $$D_f := \{(x_1, .., x_n) \in \mathbb{R}^n \,|\, |x_1| < F(x_2, .., x_n)\}.$$

  To explain the idea of how to define the quasi-geodesics and twisted double cones joining, e.g., the pairs of points $p_k^{\pm} = (0, ...0, \pm k)$, $k \geqslant 1$, we note that for $k = 1$ we can choose any such twisted double cone $\subset D_f$ along a half circle in the $(n-1, n)$-plane joining p_1^- and p_1^+. Then the quasi-geodesic and twisted double cone scaled by k serve for p_k^- and p_k^+. Here it is essential that f grows at least linearly to ensure that the twisted double cone remains in D_f.
- Bounded domains with certain types of fractal boundaries, like the Koch snowflake in $\mathbb{R}^2$ or the complement of the Sierpiński gasket in $\mathbb{R}^3$ [ALM03], are uniform domains.

However, even seemingly harmless domains can be non-uniform:

- The difference of the cube $(-1, 1)^n \subset \mathbb{R}^n$ and the unit ball in Euclidean space, $(-1, 1)^n \setminus B_1(0) \subset \mathbb{R}^n$, is not a uniform domain, since we cannot reach points arbitrarily near to the boundary point $(0, \dots, 0, 1) \in \partial\left((-1, 1)^n \setminus B_1(0)\right)$ by twisted cones, for a common constant $c > 0$.

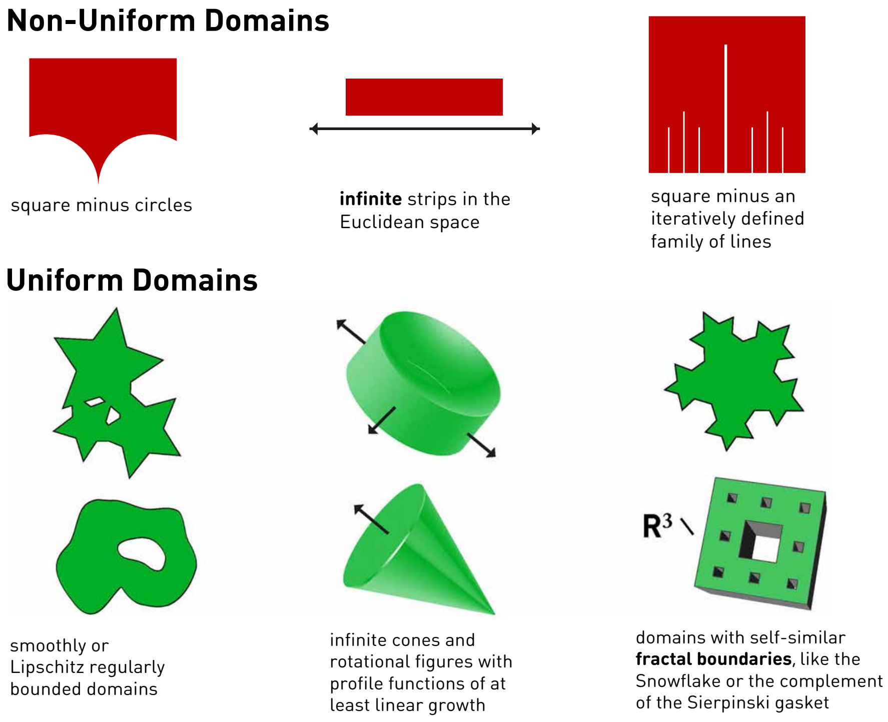


Figure 1.7: Typical (non-)uniform domains (image from [KL18]).

- The cylinder $B_1^{n-1}(0) \times \mathbb{R} \subset \mathbb{R}^n$ and similarly $B_1^k(0) \times \mathbb{R}^{n-k} \subset \mathbb{R}^n$, for $1 \leqslant k \leqslant n-1$, are non-uniform. For the pairs of points $p_i^{\pm} = (0, ...0, \pm i)$, $i \geqslant 1$, there are only twisted double cones reaching from p_i^+ to p_i^-, for $c = c(i) \leqslant 1/i$.
- Similarly, the domain $\mathbb{R}^3 \setminus \mathbb{R} \times \mathbb{Z}^2$ is not uniform.
- As in the earlier example of non-compact rotationally symmetric domains D_f, we can choose profile functions, but this time of sublinear growth like $f(t) = c_1 \cdot \sqrt{t} + c_2$, for constants $c_i > 0$. Then, as in the last counterexample, D_f is non-uniform.

For area-minimisers, we have:

- Every area-minimising hypersurface is c-uniform with respect to $\delta_{\langle A \rangle}$, for some $c > 0$, which depends only on the dimension n in the case of area-minimisers in $\mathbb{R}^n$ [Loh18, Proposition 2.7]. Blow-up limits (such as tangent cones) of c-uniform area-minimisers are also c-uniform [Loh18, Corollary 2.11].

As an analogue of bounded geometry for hyperbolic spaces we consider a version scaled by $\bar{d}$.

Definition 1.29 M has **$\bar{d}$–bounded geometry** if there are constants $\varepsilon > 0$, $K \geqslant 0$, such that for every $x \in M$, the injectivity radius of the exponential map at x is at least $\varepsilon \bar{d}(x)$ and on the rescaled ball $\frac{1}{\bar{d}(x)} B_{\varepsilon \bar{d}(x)}(x)$, $-K \leqslant \mathrm{Sec} \leqslant K$ (i.e., $-K \leqslant \bar{d}^2(x) \,\mathrm{Sec} \leqslant K$ in the original metric).

Examples 1.30.

- For domains in $\mathbb{R}^n$ this is trivially true with $\varepsilon = \frac{1}{2}$ and $K = 0$.
- In closed manifolds, there are global bounds on injectivity radius and sectional curvatures that also apply to subdomains.
- For area-minimisers H, curvature bounds on the surrounding space and the crucial property $|A| \leqslant 1/\delta_{\langle A\rangle}$ give bounds on the sectional curvatures of H using the Gauß equation. Regularity theory helps to get a bound on the injectivity radius. For details see the proof of Proposition 3.4 on page 111 and step 2.2 in the proof of Proposition B.1 on page 126 in [Loh18].

1.3.3 Quasi-Hyperbolic Geometry

On any incomplete manifold M with generalised distance function $đ$ we can define the ***quasi-hyperbolic metric***

$$k(x,y) := \inf\left\{\int_\gamma 1/đ \,\middle|\, \gamma : x \rightsquigarrow y \text{ rectifiable}\right\} \quad \text{for } x, y \in M.$$

This corresponds to a conformal deformation of g to the merely Lipschitz continuous Riemannian metric $đ^{-2}g$. Note that for $đ = \operatorname{dist}(\cdot, \partial M)$, the quasi-hyperbolic metric is tailored to make the resulting space *just* complete.

The quasi-hyperbolic metric can be interpreted as a generalisation of the Poincaré metric on the unit disc $B_1(0) \subset \mathbb{C}$, given by

$$g_{\text{hyp}} = \left(\frac{2}{1-r^2}\right)^2 g_{\text{Eucl}}$$

with $r = d(\cdot, 0)$. Asymptotically near the boundary, $\frac{2}{1-r^2} = \frac{2}{(1-r)(1+r)} \to \frac{1}{1-r} = \frac{1}{\operatorname{dist}(\cdot,\partial B_1(0))}$.

In general, the quasi-hyperbolic metric k does not have to be (Gromov) hyperbolic, but uniformity is a sufficient condition. In the case $đ = \operatorname{dist}(\cdot, \partial M)$, this was first proven by Gehring and Osgood for uniform domains in Euclidean space [GO79] and later generalised to locally compact, rectifiably connected incomplete metric spaces by Bonk, Heinonen and Koskela [BHK01, Theorem 3.6].

After rescaling with L, a generalised distance function always underestimates the distance to the boundary, $đ \leqslant L \operatorname{dist}(\cdot, \partial M)$. The resulting additional stretching when deforming with $đ$ instead of $\operatorname{dist}(\cdot, \partial M)$ might destroy hyperbolicity, see [Loh18, Example 3.6.C3], but uniformity with respect to $đ$ counteracts this phenomenon since a smaller $đ$ strengthens the condition. For area-minimising hypersurfaces with the $\mathcal{S}$-distance $\delta_{\langle A\rangle}$, Gromov hyperbolicity of the quasi-hyperbolic metric is proven in [Loh18, Section 3.2]. This proof uses only axiomatically stated properties of $\delta_{\langle A\rangle}$ that are satisfied for our generalised distance function, and inspection of the proof shows that indeed the following is true:

Theorem 1.31 (Hyperbolisation of Uniform Spaces) *If an incomplete manifold M with L-Lipschitz generalised distance function $đ$ is c-uniform with respect* *to $đ$, it is δ-hyperbolic in the quasi-hyperbolic metric, for some $\delta = \delta(L,c) \geqslant 0$. Moreover, if (M,d) is bounded*[6]*, the metric boundary $\partial(M,d)$ is naturally quasi-symmetrically equivalent*[7] *to the Gromov boundary $\partial_{\mathrm{G}}(M,k)$ equipped with a canonical quasi-metric.*

Note that the metric boundary $\partial(M,d)$ is determined using the metric completion of the inner metric on M, hence for domains in Euclidean space it may not be identical to the topological boundary. E.g., for $M = \mathbb{R}^2 \setminus \{(x,0) \,|\, x \leqslant 0\}$, the metric boundary is naturally isometric to two copies of $\{(x,0) \,|\, x \leqslant 0\}$ glued at $(0,0)$.

Our version of $đ$–bounded geometry implies bounded geometry in the quasi-hyperbolic metric:

Proposition 1.32 (Bounded Geometries) *If an incomplete manifold M with L-Lipschitz generalised distance function $đ$ has $đ$–bounded geometry with constants ε and K, the quasi-hyperbolic deformation (M,k) has (σ,ℓ)–bounded geometry in the sense of Definition 1.1 with constants $\sigma = \sigma(L,\varepsilon)$ and $\ell = \ell(K)$.*

Proof. We assume $L\varepsilon < \frac{1}{2}$, else make ε smaller. Then for a fixed $x \in M$, the map

$$\mathrm{Id} : (B^d_{\varepsilon đ(x)}(x), k) \to (B^d_{\varepsilon đ(x)}(x), đ^{-2}(x)g)$$

(with radii measured in the metric d) is 2-bi-Lipschitz since $(1 - L\varepsilon)đ(x) \leqslant đ \leqslant (1 + L\varepsilon)đ(x)$ on $B^d_{\varepsilon \mathrm{d}(x)}$. But on this ball, the exponential map $\exp_x$ at x is a diffeomorphism and the absolute sectional curvatures are bounded by K, hence $\exp_x$ is $\ell/2$-bi-Lipschitz for some $\ell = \ell(K)$ by basic comparison estimates [Pet16, Theorem 6.27]. For $\sigma = \varepsilon/2$, $B^k_\sigma(x) \subset B^d_{\varepsilon đ(x)}(x)$ and hence we have an ℓ-bi-Lipschitz chart for $B^k_\sigma(x)$ with constants $\sigma = \sigma(L,\varepsilon)$ and $\ell = \ell(K)$. □

Generalised distance functions that are better adapted to the geometry of the underlying space make it easier to obtain bounded geometry. E.g. in converging sequences of almost-minimisers, the $\mathcal{S}$-distance can detect singularities in the limit space even if every other space is non-singular. But this behaviour makes them prone to having regions with arbitrarily small $\delta_{\langle A \rangle}$ that are far away from the boundary. In the quasi-hyperbolic metric, this may yield regions that are "bubbling off" from shortest paths to infinity, the space is no longer visual.

This behaviour can be suppressed if a generalised distance function is not far away from being the distance to the boundary and the manifold is bounded:

Proposition 1.33 (Visual Unfolding) *A* bounded *uniform manifold M with generalised distance function $đ \succcurlyeq \mathrm{dist}(\cdot, \partial M)$ is visual in the quasi-hyperbolic metric, with constant depending only on universal constants for the input data.*

[6] For unbounded uniform spaces, there is a version of this result involving the one-point compactification [Loh18, Theorem 3.17], but we will only be concerned with bounded uniform spaces.

[7] A homeomorphism $f : (X,d) \to (Y,d')$ between quasi-metric spaces is a *quasi-symmetry*, if there is a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that

$$\frac{d'(f(x),f(y))}{d'(f(x),f(z))} \leqslant \eta\left(\frac{d(x,y)}{d(x,z)}\right) \quad \text{for distinct points } x,y,z \in X.$$

Proof. The proof from [BHK01, p. 23] applies directly with obvious modifications for $đ \asymp \operatorname{dist}(\cdot, \partial M)$ instead of $đ = \operatorname{dist}(\cdot, \partial M)$.

Alternatively, the quasi-hyperbolic metric corresponding to $đ$ can be reached from the $\operatorname{dist}(\cdot, \partial M)$-quasi-hyperbolic metric (which is visual [BHK01, Theorem 3.6]) by a conformal deformation with conformal factor $\asymp 1$. This is a quasi-isometry and maps geodesics to quasi-geodesics which are in bounded distance from geodesics by Proposition 1.5 "stability of geodesics". Hence it preserves visuality. □

Assumptions for an incomplete manifold and the resulting properties of the quasi-hyperbolic unfolding can be summarised as follows:

Incomplete Manifold	**Hyperbolic Unfolding**
$đ$–bounded geometry	bounded geometry
uniform	complete and Gromov hyperbolic
bounded and $đ \asymp \operatorname{dist}(\cdot, \partial M)$	visual

1.3.4 Natural Regularisation of Distance Functions

Since $đ$ is merely Lipschitz continuous, but in most cases not smooth, (M, k) is in general not a smooth Riemannian manifold. But this will be a problem when considering differential operators on (M, k), in particular we need uniform bounds on the coefficients of operators that involve second derivatives of $đ$. Thus we will work with a regularised version of $đ$.

For the distance to the boundary, there is a well-controlled regularisation developed by Whitney, see [Ste70, VI.2.1, p. 171]. This construction utilises a Besicovitch-style cover and was generalised to area-minimisers in [Loh18, Appendix B]. Unfortunately, such a cover involves many choices and does not behave naturally under limits or symmetries. With applications in blow-up limits and on highly symmetric tangent cones in mind, we will present a more natural regularisation for arbitrary generalised distance functions on manifolds of $đ$–bounded geometry. This approach utilises mollifiers instead of coverings and is based on [Lie85] considering domains in $\mathbb{R}^n$. As in Whitney's construction, the central idea to obtain the regularisation $\eth(x)$ is to average $đ$ over a neighbourhood of x that is small compared to $đ(x)$. To avoid the influence of the less regular parameter $đ(x)$, we use in a Münchhausen-like manner $\eth(x)$ in its stead.

The only choice is a non-increasing C^∞-smooth function $\phi : [0, 1] \to [0, 1]$ with $\phi \equiv \text{const}$ near 0, $\phi \equiv 0$ near 1 and $\int_{\mathbb{R}^n} \phi(|z|)\, \mathrm{d}z = 1$.

Theorem 1.34 (Natural Regularisation of Generalised Distance Functions) *Let M^n be an incomplete manifold with L-Lipschitz generalised distance function $đ$ and of $đ$–bounded geometry with constants ε and K. If we set*

$$I(x, \alpha) := \left(\frac{L}{\alpha}\right)^n \int_M đ(y)\, \phi(L\, d(x, y)/\alpha)\, \mathrm{d}V(y) \quad \text{for } \alpha > 0\,, \tag{1.2}$$

then there is an $\varepsilon_0 = \varepsilon_0(\varepsilon, K) > 0$ such that for every $0 < \hat{\varepsilon} < \varepsilon_0$, the equation $\eth(x) = I(x, \hat{\varepsilon}\eth(x))$ has a unique solution $\eth$ with the following properties:

(i) $\eth : M \to (0, \infty)$ is a C^∞-smooth Lipschitz function with $\eth \to 0$ at ∂M.

(ii) $\eth \asymp \dj$.

(iii) *The partial derivatives of* $\eth$ *in normal coordinates around* $x \in M$ *satisfy*

$$\left|\frac{\partial^\beta \eth}{\partial x^\beta}\right|(x) \preccurlyeq_{n,L,\phi,\hat{\varepsilon},\beta} \eth^{1-|\beta|}(x)$$

for any multiindex β.

For the proof, we need the following technical consequence of $\dj$–bounded geometry:

Lemma 1.35 (Volume Bounds) *On a manifold of* $\dj$*–bounded geometry with constants* ε *and* K*, there is a* $0 < \varepsilon'(K, \varepsilon) \leqslant \varepsilon$ *such that on every ball* $B_{\varepsilon' \dj(x)}(0) \subset T_x M$ *in normal coordinates around* x*, the function* $\sqrt{|\det g|}$ *is* $L'/\dj(x)$*-Lipschitz for an* $L' = L'(K) > 0$.

Proof. On the rescaled ball $\dj(x) \cdot B_{\varepsilon' \dj(x)}(x)$, sectional curvatures are absolutely bounded by K, hence we can apply basic comparison estimates, e.g. [Pet16, Chapter 6.5, Theorem 27], to get estimates on derivatives of g_{ij} in normal coordinates, and hence on derivatives of $\sqrt{|\det g|}$, on a sufficiently small ball. □

Proof of Theorem 1.34 "natural regularisation of generalised distance functions". We start by investigating the integral $I(x, \alpha)$. Note that the properties of $\dj$ imply $\operatorname{dist}(x, \partial M) \geqslant \dj(x)/L$. I is C^∞ on $\{(x, \alpha) \in M \times (0, \infty) \mid \alpha < \varepsilon' \dj(x)\}$ (with ε' from the preceding Lemma) because the integrand has support in $B_{\alpha/L}(x) \subset B_{\dj(x)/2L}(x) \subset B_{\operatorname{dist}(x,\partial M)/2}(x) \Subset M$, $\eth$ is continuous there, $d(x, y)$ is C^∞-smooth away from the diagonal $x = y$ and ϕ is constant near 0.

For a fixed $x \in M$, we can write in normal coordinates

$$\begin{aligned} I(x, \alpha) &= \left(\frac{L}{\alpha}\right)^n \int_{T_x M} \dj(\exp_x(y))\, \phi(L\,|y|/\alpha)\, \sqrt{|\det g(y)|}\, \mathrm{d}y \\ &= \int_{T_x M} \dj(\exp_x(z\alpha/L))\, \phi(|z|)\, \sqrt{|\det g(z\alpha/L)|}\, \mathrm{d}z \end{aligned}$$

with substitution $z = Ly/\alpha$. From this last expression, we can see $I(x, \alpha) \to \dj(x)$ for $\alpha \to 0$ and hence set $I(x, 0) := \dj(x)$.

We can assume for simpler notation that $L' = L$ for the Lipschitz constant from Lemma 1.35 "volume bounds", else we increase one of the constants. Then Lipschitz estimates on $\dj$ and from the Lemma imply

$$\begin{aligned} |I(x, \alpha) - I(x, \beta)| &\leqslant \int_{T_x M} L \frac{|z|}{L} |\alpha - \beta|\, \phi(|z|)\, \sqrt{|\det g(z\alpha/L)|}\, \mathrm{d}z \\ &\quad + \int_{T_x M} \dj(\exp_x(z\beta/L))\, \phi(|z|)\, \frac{L}{\dj(x)} \frac{|z|}{L} |\alpha - \beta|\, \mathrm{d}z \\ &\leqslant \int_{T_x M} |z|\, |\alpha - \beta|\, \phi(|z|)\, (1 + |z|\alpha)\, \mathrm{d}z \\ &\quad + \int_{T_x M} \left(1 + |z| \frac{\beta}{\dj(x)}\right) \phi(|z|)\, |z|\, |\alpha - \beta|\, \mathrm{d}z \\ &\leqslant 4|\alpha - \beta|\,. \end{aligned}$$

Hence for $0 < \hat{\varepsilon} < \min\{\varepsilon'/2, 1/8\}$, the map $\gamma \mapsto I(x, \hat{\varepsilon}\gamma)$ is a $\frac{1}{2}$-contraction for every x, i.e.,

$$|I(x,\hat{\varepsilon}\alpha) - I(x,\hat{\varepsilon}\beta)| \leqslant \frac{1}{2}|\alpha - \beta| \quad \text{for } \alpha, \beta \in [0, 2\eth(x)],$$

and it maps $[0, 2\eth(x)]$ to itself because

$$|I(x,\hat{\varepsilon}\gamma) - \eth(x)| = |I(x,\hat{\varepsilon}\gamma) - I(x,0)| \leqslant \frac{1}{2}\gamma \leqslant \eth(x) \quad \text{for } \gamma \in [0, \eth(x)].$$

Thus by the Banach fixed-point theorem, there is a unique $\partial(x) \in [0, 2\eth(x)]$ with $\partial(x) = I(x, \hat{\varepsilon}\partial(x))$. ∂ is smooth in x by the implicit function theorem.

From the contraction property, we also see

$$|\partial(x) - \eth(x)| = |I(x, \hat{\varepsilon}\partial(x)) - I(x, 0)| \leqslant \frac{1}{2}\partial(x)$$

and hence

$$\frac{1}{2}\partial(x) \leqslant \eth(x) \leqslant \frac{3}{2}\partial(x)\,.$$

Furthermore, from (1.2),

$$\begin{aligned}
|I(x,\alpha) - I(y,\alpha)| &\leqslant \left(\frac{L}{\alpha}\right)^n \int_M \eth(z)\,|\phi(L\,d(x,z)/\alpha) - \phi(L\,d(y,z)/\alpha)|\,\mathrm{d}V(z) \\
&\leqslant (\sup\phi')\frac{L}{\alpha}d(x,y)\left(\frac{L}{\alpha}\right)^n \int_{B_{\alpha/L}(x)\cup B_{\alpha/L}(y)} \eth(z)\,\mathrm{d}V(z) \\
&\leqslant 3\,\mathrm{Vol}(B_1(0) \subset \mathbb{R}^n)(\sup\phi')\frac{L}{\alpha}d(x,y)\max\{\eth(x), \eth(y)\}\,.
\end{aligned}$$

This implies

$$\begin{aligned}
|\partial(x) - \partial(y)| &\leqslant |I(x,\hat{\varepsilon}\partial(x)) - I(y,\hat{\varepsilon}\partial(x))| + |I(y,\hat{\varepsilon}\partial(x)) - I(y,\hat{\varepsilon}\partial(y))| \\
&\leqslant 3\,\mathrm{Vol}(B_1(0) \subset \mathbb{R}^n)(\sup\phi')\frac{L}{\hat{\varepsilon}\partial(x)}d(x,y)\max\{\eth(x), \eth(y)\} \\
&\quad + \frac{1}{2}|\partial(x) - \partial(y)|
\end{aligned}$$

and hence by switching the roles of x and y

$$|\partial(x) - \partial(y)| \preccurlyeq_{n,\phi,L,\hat{\varepsilon}} d(x,y)\,.$$

This implies the asserted derivative estimate for $|\beta| = 1$. For higher $|\beta|$ one proceeds similarly. Note that every differentiation of (1.2) produces an additional factor of $1/\alpha$ which determines the asymptotics $\partial^{1-|\beta|}(x)$. □

Remarks 1.36.

- By choosing smaller $\hat{\varepsilon}$, the quotient of ∂ and $\eth$ can pushed arbitrarily near to 1, but the derivative estimates become worse.
- On large scales, e.g. for $d(x,y) \geqslant \max\{\partial(x), \partial(y)\}$, comparison with $\eth$ shows that ∂ is $(L+1)$-Lipschitz, even for small $\hat{\varepsilon}$.

1.3.5 Operators on Uniform Spaces

Quasi-hyperbolic metrics can be used to understand the potential theory on uniform manifolds by transferring it to Gromov hyperbolic manifolds. Since there are examples of non-uniform domains which carry a rather complicated potential theory, this correspondence also gives us non-hyperbolic manifolds of bounded geometry with a much less transparent Martin theory than in the hyperbolic case.

In this section, we consider incomplete manifolds M with a *smooth* generalised distance function $\eth$ as produced by Theorem 1.34 "natural regularisation of generalised distance functions". In particular, we will make use of the bounds in point (iii) of the theorem.

A uniformly elliptic second-order operator L on (M, g) is in general not adapted with respect to the quasi-hyperbolic metric $g' = \eth^{-2} g$ because the coefficients measured in the new metric degenerate towards ∂M. This can be mitigated by considering a transformed operator L' with natural bijections between the spaces of L- and L'-harmonic functions. Ancona proposed the operator $\eth^2 L$, which has the same harmonic functions as L, but for symmetric operators L, $\eth^2 L$ it is not symmetric anymore. Ancona's requirements for L might be a little more general, but as our main examples are Schrödinger operators of the form $L = -\Delta + V$ for smooth functions V, we prefer to use a h-transform [Pin95, Section 4.1] of the operator $\eth^2 L$, given by

$$L'v = h^{-1} \eth^2 L(h \cdot v) = \eth^{\frac{n+2}{2}} L(\eth^{-\frac{n-2}{2}} v) \quad \text{with } h = \eth^{-\frac{n-2}{2}}.$$

A function u is L-harmonic if and only if $\eth^{\frac{n-2}{2}} u$ is L'-harmonic. For symmetric operators L, the operator L' is again symmetric with respect to the metric g' since

$$\int_M u \, L'v \, \mathrm{d}V' = \int_M u \, \eth^{\frac{n+2}{2}} L(\eth^{-\frac{n-2}{2}} v) \, \eth^{-n} \mathrm{d}V = \int_M L\left(\eth^{-\frac{n-2}{2}} u\right) \eth^{-\frac{n-2}{2}} v \, \mathrm{d}V$$

for test functions $u, v \in C^\infty_{\mathrm{c}}(M)$, where $\mathrm{d}V' = \eth^{-n} \mathrm{d}V$ is the volume measure associated to g'.

If G is a Green's function for L with respect to $\mathrm{d}V$, a Green's function of L' with respect to $\mathrm{d}V'$ is given by

$$G'(x, y) = \eth^{\frac{n-2}{2}}(x) \, \eth^{\frac{n-2}{2}}(y) \, G(x, y)$$

because then

$$\begin{aligned} L' \int_M G'(\cdot, y) \, u(y) \, \mathrm{d}V'(y) &= \eth^{\frac{n+2}{2}} L \int_M \eth^{\frac{n-2}{2}}(y) \, G(\cdot, y) \, u(y) \, \eth^{-n}(y) \, \mathrm{d}V(y) \\ &= \eth^{\frac{n+2}{2}} \, u \, \eth^{-\frac{n+2}{2}} = u \end{aligned}$$

for any $u \in C^\infty_c(M)$.

One could find precise conditions on L in order for L' to be an adapted operator by translating the requirements and using the bound (iii) from Theorem 1.34 "natural regularisation of generalised distance functions", but we will content ourselves with the following class of operators:

Lemma 1.37 (Schrödinger Operators) *If $L = -\Delta + V$ with $|V| \preccurlyeq \eth^{-2}$ is a Schrödinger operator on a manifold (M, g) of $\eth$–bounded geometry, then L' is a Schrödinger operator on $(M, g' = \eth^{-2} g)$ with bounded potential term and hence adapted.*

Proof. Under conformal deformation $g' = \eth^{-2} g$, the Laplacian transforms as [Bes87, Theorem 1.159 j)]

$$-\Delta_{g'} v = \eth^2(-\Delta_g v + (n-2) g(\nabla \ln \eth, \nabla v)) \,.$$

On the other hand, the transformed operator associated to $-\Delta_g$ is given by

$$\begin{aligned}
-\Delta' v &= -\eth^{\frac{n+2}{2}} \Delta_g (\eth^{-\frac{n-2}{2}} v) \\
&= -\eth^{\frac{n+2}{2}} \operatorname{div}_g \left(\eth^{-\frac{n-2}{2}} \nabla v + (\nabla \eth^{-\frac{n-2}{2}}) v \right) \\
&= -\eth^2 \Delta_g v - 2 \eth^{\frac{n+2}{2}} g(\nabla \eth^{-\frac{n-2}{2}}, \nabla v) - \eth^{\frac{n+2}{2}} (\Delta_g \eth^{-\frac{n-2}{2}}) v \\
&= -\eth^2 \Delta_g v + (n-2) \eth^2 g(\nabla \ln \eth, \nabla v) - \eth^{\frac{n+2}{2}} (\Delta_g \eth^{-\frac{n-2}{2}}) v \\
&= \left(-\Delta_{g'} - \eth^{\frac{n+2}{2}} (\Delta_g \eth^{-\frac{n-2}{2}}) \right) v \,.
\end{aligned}$$

Thus a Schrödinger operator $L = -\Delta_g + V$ is mapped to

$$L' = -\Delta_{g'} + \eth^2 \cdot \left(V - \eth^{\frac{n-2}{2}} (\Delta_g \eth^{-\frac{n-2}{2}}) \right) \,.$$

The potential term is bounded by the assumption on V and property (iii) from Theorem 1.34 "natural regularisation of generalised distance functions" with $|\beta| = 0, 1, 2$. ☐

A short calculation shows that the operator L' is weakly coercive if and only if the operator L satisfies the following condition:

Definition 1.38 (Strong Barrier) L admits a *$\eth$-strong barrier* if there are a function $s > 0$ on M and an $\varepsilon > 0$ such that

$$Ls \geqslant \varepsilon \, \eth^{-2} s \,.$$

Analogous to the unfolded case in subsection 1.2.3, the *weighted principal eigenvalue* $\lambda_L^{\eth}$ of L is the largest such ε and for Schrödinger operators we have the variational characterisation

$$\lambda_L^{\eth} = \inf_{f \in C_c^\infty(M)} \int_M \left(|\nabla f|^2 + V f^2 \right) \mathrm{d}V \Big/ \int_M f^2 \, \eth^{-2} \, \mathrm{d}V \,.$$

This follows from translating the characterisation of the principal eigenvalue 1.21. Hence for Schrödinger operators, strong barrier is equivalent to a so-called *Hardy inequality*

$$\int_M f \, Lf \, \mathrm{d}V \preccurlyeq \int_M f^2 \, \eth^{-2} \, \mathrm{d}V \quad \text{for every } f \in C_c^\infty(M).$$

Example 1.39 (regularity versus strong barrier). The Laplace operator on bounded domains with smooth or Lipschitz boundary always admits a strong barrier [Anc86, KK66]. For more general domains, the strong barrier condition is not a consequence of uniformity but of additional exterior conditions. An exterior twisted cone condition is sufficient [Aik12].

Chapter 2

Potential Theoretic Results

In this chapter, we present Ancona's potential theory for weakly coercive operators on Gromov hyperbolic manifolds of bounded geometry [Anc87, Anc90] with complete proofs. The main result is Ancona's boundary Harnack inequality 2.21 which will be used to identify the potential theoretic Martin boundary with the Gromov boundary. Ancona's original article [Anc87] was conceived as a generalisation of work of Anderson and Schoen on Cartan–Hadamard manifolds [AS85] and still primarily focuses on those while we will directly approach the more general case of Gromov hyperbolic manifolds.

Another valuable source are the comprehensive French lecture notes [Anc90]. They point out connections to heat kernels and stochastic processes such as Brownian motion and introduce potential theory on graphs. We try to give a more streamlined account of (nearly) everything needed to arrive at the central for later use. As a byproduct, we carefully keep track of all involved constants to show that the quantitative results depend only on a set of universal constants, not on the explicit manifold or operator under consideration. This is useful for blow-up arguments such as in [Loh20a, Loh20b] where these constant can be shared among sequences and limit spaces.

Apart from some details in the presentation, the only completely new result in this chapter is the ray expansion in section 2.4 which expresses a harmonic function along a geodesic ray in terms of the Green's function along this ray and the corresponding Martin measure of balls around the endpoint of this ray on the boundary at infinity. The sections 2.1 to 2.3 and 2.5 previously appeared in the preprint [KL18] by J. Lohkamp and the author.

2.1 Local Maximum Principles and Harnack Inequalities

We derive estimates for positive solutions and Green's functions of the shifted adapted operators $L_t = L - t$ on uniformly sized balls $B_R := B_{\sigma/\ell} := B_{\sigma/\ell}(0) \subset \mathbb{R}^n$ in bounded geometry charts. For notational convenience, in this section, all balls are measured in the Euclidean distance. The required assumptions on Locally defined, Adapted operators with a Green's function will be called

Assumption (LAG) We say that a second-order operator L on $B_R(0) \subset \mathbb{R}^n$ satisfies assumption (LAG) for $R > 0$, $k > 0$, $\beta \in (0, 1]$ and $n \geqslant 2$ if

- L is (k, β)-*adapted* on $B_R(0)$ (Definition 1.16) and
- there is a Green's function for L on $B_R(0)$ (subsection 1.2.1).

Later all results can be transferred to a manifold of bounded geometry using the bi-Lipschitz charts with only a loss in constants. Weak coercivity will not yet be used explicitly, but the results and estimates apply to all operators L_t with $0 \leqslant t < \tau$, where the constants now depend on τ as well (via k), but not on t.

2.1.1 Maximum Principles

Contrary to the usual statement, the following version of the weak maximum principle does not require positivity of the "potential" term c in L (see Definition 1.16 "adaptedness"). Instead the existence of a Green's function (or equivalently, a positive L-superharmonic function) is sufficient, but the result is weaker. It can easily be obtained from the version for $c \geqslant 0$ using the h-transform, see [Pin95], where the sign convention for L is different from ours.

Theorem 2.1 (Local Minimum-Zero Principle) *[Pin95, Theorem 3.2.2, p. 81] If L satisfies assumption (LAG), u is continuous on B_R, L-superharmonic on a domain $D \Subset B_R$ and $u|_{\partial D} \geqslant 0$, then $u \geqslant 0$ in all of D.*

With help of the uniform bounds on the coefficients, this can be upgraded to quantitative bounds in terms of the boundary values.

Lemma 2.2 (Local Almost-Maximum Principle) *If L satisfies (LAG), there are constants $0 < r_{\mathrm{am}} < R$ and $m > 0$ depending only on k and n, such that any L-subharmonic function u on B_r, $r < r_{\mathrm{am}}$, with $u|_{\partial B_r} \leqslant m$ satisfies $u \leqslant 1$ on B_r.*

Proof. In polar coordinates, L applied to a radial function $f(r)$ can be written as

$$Lf = -\alpha f''(r) + \frac{\alpha - \operatorname{tr} a}{r} f'(r) + bf'(r) + cf(r)$$

with $k^{-1} \leqslant \alpha := \frac{a_{ij}(x)x^i x^j}{r^2} \leqslant k$, $nk^{-1} \leqslant \operatorname{tr} a \leqslant nk$ and $|b|$, $|c|$ bounded by k.

Consider the function $f(r) = 1 - \beta r^2$ with a constant $\beta > 0$ to be fixed later. We want to have the following properties:

- $Lf \geqslant 0$ for $r < r_{\mathrm{am}}$,
- $f \leqslant 1$, and
- $f|_{\partial B_r} \geqslant f(r_{\mathrm{am}}) =: m$ for $r < r_{\mathrm{am}}$.

Only the first property is non-obvious and we need to tune the parameters r_{am} and β to achieve it. We estimate

$$Lf = 2\beta \operatorname{tr} a - 2b\,\beta\, r + c(1 - \beta r^2) \geqslant 2\beta n k^{-1} - 2\beta k r - k - k\beta r^2\,.$$

Hence we can find a large β and a small r_{am} only depending on k and n such that $Lf \geqslant 0$ and $f(r) > 0$ for $r \leqslant r_{\mathrm{am}}$.

Now we can apply the local minimum-zero principle 2.1 to $f-u$, because $L(f-u) \geqslant 0$ and $(f-u)|_{\partial B_r} \geqslant 0$. This yields $u \leqslant f \leqslant 1$ on B_r. □

With completely different methods we can see that Green's functions behave for small distances like the Euclidean Green's function

$$G_{\mathrm{Eucl}}(x,y) = \Phi(|x-y|) := \begin{cases} -\frac{1}{2\pi} \log|x-y|\,, & \text{if } n=2 \\ \frac{1}{(n-2)\omega_n} \frac{1}{|x-y|^{n-2}}\,, & \text{if } n \geqslant 3 \end{cases}$$

with $\omega_n = \operatorname{Vol} S^{n-1}$.

Lemma 2.3 (Local Bound for Some Green's Function) *For L satisfying (LAG) there are constants $r_{\mathrm{gb}} > 0$, $\tilde{q} \geqslant 1$ depending only on k and n and some (not necessarily minimal) Green's function $\tilde{G}(x,y)$ such that*

$$\tilde{q}^{-1}\, G_{\mathrm{eucl}}(x,y) \leqslant \tilde{G}(x,y) \leqslant \tilde{q}\, G_{\mathrm{eucl}}(x,y) \quad \textit{for } x,y \in B_{r_{\mathrm{gb}}}\,.$$

Proof. We sketch the argument for $a_{ij} = \delta_{ij}$ and $n \geqslant 3$. The general case is merely notationally more involved and can be found in [Mir70, pp. 61–63].

As distributions, we have $L_x G_{\mathrm{Eucl}}(x,y) = \delta(x-y) + R(x,y)$, where δ is the Dirac delta function and $R(x,y)$ has a singularity at $x=y$ of order $\mathcal{O}(r^{-(n-1)})$. This means that we can choose $r_{\mathrm{gb}} > 0$ so that $\int_{B_{r_{\mathrm{gb}}}} |R(x,y)|\,\mathrm{d}y < 1/2$ for $x \in B_{r_{\mathrm{gb}}}$ because the integral scales as $\mathcal{O}(r_{\mathrm{gb}})$ and all involved constants depend only on the bound k on the coefficients. This in turn implies the finiteness of all iterates R^i where the product of kernels S, T is defined as $(ST)(x,y) = \int_{B_{r_{\mathrm{gb}}}} S(x,z)\,T(z,y)\,\mathrm{d}z$. The bound ensures also the summability of

$$\tilde{G} := G_{\mathrm{Eucl}} \sum_{i=0}^{\infty} (-R)^i\,,$$

again as products of kernels. Application of L yields

$$L\tilde{G} = (\delta + R) \sum_{i=0}^{\infty} (-R)^i = \sum_{i=0}^{\infty} (-R)^i - \sum_{i=1}^{\infty} (-R)^i = \delta\,,$$

so that $\tilde{G}$ is indeed a Green's function. $\tilde{G} - G_{\mathrm{Eucl}}$ has a singularity at $x=y$ of quantitatively lower order than G_{Eucl}, hence the explicit bounds. □

Corollary 2.4 (Local Bound for the Minimal Green's Function) *If L fulfills (LAG), there are constants $q > 0$ and $0 < r_{\mathrm{mgbi}} < r_{\mathrm{mgbo}} := \min(r_{\mathrm{am}}, r_{\mathrm{gb}})$ depending only on k and n such that the* minimal *Green's function $g(x,y)$ of $B_{r_{\mathrm{mgbo}}}$ (i.e., vanishing on the boundary $\partial B_{r_{\mathrm{mgbo}}}$) satisfies*

$$q^{-1}\, G_{\mathrm{eucl}}(x,y) \leqslant g(x,y) \leqslant q\, G_{\mathrm{eucl}}(x,y) \quad \textit{for } x,y \in B_{r_{\mathrm{mgbi}}}\,.$$

Proof. For fixed $y \in B_{r_{\text{mgbo}}/2}$, g can be represented as $g(x,y) = \tilde{G}(x,y) - u(x)$ where u is the solution of the Dirichlet problem $Lu = 0$ in $B_{r_{\text{mgbo}}}$ and $u(x) = \tilde{G}(x,y)$ on $\partial B_{r_{\text{mgbo}}}$. By the local bound for some Green's function 2.3 we know

$$u(x) \leqslant \tilde{q}\,\Phi(|x-y|) \leqslant \tilde{q}\,\Phi(r_{\text{mgbo}}/2) \qquad \text{for } x \in \partial B_{r_{\text{mgbo}}}\,.$$

Thus we can apply the local almost-maximum principle 2.2 to get $u \leqslant \tilde{q}\,m^{-1}\Phi(r_{\text{mgbo}}/2)$ on $B_{r_{\text{mgbo}}}$, whence

$$g(x,y) \geqslant \tilde{q}^{-1} G_{\text{eucl}}(x,y) - \tilde{q}\,m^{-1}\Phi(r_{\text{mgbo}}/2) \quad \text{for } x \in B_{r_{\text{mgbo}}}\,.$$

From this equation, $q \geqslant \tilde{q}$ and r_{mgbi} can be determined to obtain the lower bound. For the upper bound, note that $\tilde{G}$ is positive and thus $g \leqslant \tilde{G}$ by the local minimum-zero principle 2.1 applied to u. □

2.1.2 Harnack Inequalities

This standard result can be found e.g. in [GT98, Theorem 8.20], but we include a proof here because it is an easy consequence of the previous results.

Theorem 2.5 (Harnack Inequalities) *For L satisfying (LAG), there is an $H(k,n,R) \geqslant 1$ such that if $u > 0$ is L-harmonic on $B_r(x_0) \subset B_R$, for some $0 < r \leqslant R$, then*

$$H^{-1}u(x_0) \leqslant u(x) \leqslant Hu(x_0) \quad \textit{for any } x \in B_{r/2}(x_0)\,.$$

Proof. We start with the case $r \leqslant r_{\text{mgbi}}$.

The L-potential $\mathcal{R}_u^{B_r(x_0)}$ (on the total space $B_{r_{\text{mgbo}}}$) can be represented as $\mathcal{R}_u^{B_r(x_0)} = g(\mu_u)$ for some positive measure μ_u with support in $\partial B_r(x_0)$. On $B_r(x_0)$, $\mathcal{R}_u^{B_r(x_0)}$ coincides with u and we have $u = g(\mu_u)$. Now for $y \in B_{r/2}(x_0)$, we can apply both estimates from Corollary 2.4 “local bound for the minimal Green's function” to get

$$\begin{aligned} u(y) = \int_{\partial B_r(x_0)} g(y,z)\,\mathrm{d}\mu_u(z) &\leqslant \int_{\partial B_r(x_0)} q\,\Phi(r/2)\,\mathrm{d}\mu_u(z) \\ &\leqslant 2^{n-2}q^2 \int_{\partial B_r(x_0)} g(x_0,z)\,\mathrm{d}\mu_u(z) = 2^{n-2}q^2\,u(x_0)\,. \end{aligned}$$

The other inequality is analogous.

For the case of larger r, we can employ the Harnack chains introduced in Definition 1.13 with radius r_{mgbi}. With every ball the estimate collects the constant for that case, but the total number of balls needed is bounded from above by a constant multiple of R/r_{mgbi}, so we can just use the resulting power of $2^{n-2}q^2$ as H in every case. □

The Harnack chain tactic just demonstrated will be used *very* frequently in the rest of this thesis, sometimes implicitly, to get estimates in controlled distances or interpolate known estimates for discrete larger distances. Applied alone for larger distances, the Harnack inequalities provide exponential bounds (with fixed constants) for the growth or decay of L-harmonic functions—that is not especially satisfying, but a good starting point for improvement.

This Harnack inequality always needs a little more space around the balls where it holds, but for instance on bounded domains, it would be useful to have a similar result near points on the boundary. A blueprint is the following classical result.

Theorem 2.6 (Boundary Harnack Inequality on a Disc) *[Kem72] There exist constants A, $C > 1$ such that for any point $\xi \in \partial B_1(0) \subset \mathbb{R}^2$ and $0 < R < 1$ the following is true: for any two harmonic functions u, $v > 0$ (with respect to the Laplacian) on $B_{A \cdot R}(\xi) \cap B_1(0)$ that vanish along $B_{A \cdot R}(\xi) \cap \partial B_1(0)$,*

$$\frac{u(x)}{v(x)} \leqslant C \frac{u(y)}{v(y)} \textit{ for all } x,\, y \in B_R(\xi) \cap B_1(0)\,.$$

In the non-boundary version, the appearance of another function v is obscured by the fact that the constant function 1 is (nearly) harmonic. Furthermore, the restriction to functions vanishing on the relevant part of the boundary is unavoidable, yet in combination with balayage techniques there are powerful applications as we will see after the proof of a much more general boundary Harnack inequality in subsection 2.3.2.

2.2 Global Results from Resolvent Equations and Bounded Geometry

Now we use that our manifold M has Bounded geometry and L is Adapted and weakly Coercive, more precisely:

Assumption (BAC) We say that the pair (M, L) satisfies assumption (BAC) for $\sigma > 0$, $\ell \geqslant 1$, $k > 0$, $\beta \in (0, 1]$, $n \geqslant 2$ and $\tau > 0$ if L is a differential operator on a connected complete noncompact Riemannian manifold M^n such that

- M is of (σ, ℓ)–*bounded geometry* (Definition 1.1),
- L is (k, β)-*adapted* in the bounded geometry charts (Definition 1.16), and
- L is *weakly coercive* with principal eigenvalue $\tau =: 2\theta$ (Definition 1.19).

This gives us global growth estimates for the minimal Green's function G of L. The basic idea is to combine the local estimates we derived in the last chapter with the resolvent equation for Green's functions viewed as operators.

While bounded geometry and adaptedness of L allow to carry over all results from section 2.1 with constants now depending on k, n, σ, ℓ and τ as explained in the beginning of that section, the weak coercivity assumption is the most important new ingredient in this section. Note that we do not yet make use of Gromov hyperbolicity.

2.2.1 Resolvent Equation

For a closed operator L defined on a dense set of a Banach space, the set $\varrho(L)$ of all $\lambda \in \mathbb{R}$ such that the *resolvent* $R_\lambda = (L - \lambda)^{-1}$ exists and is continuous is typically called the *resolvent set.* $\varrho(L)$ is known to be an open set. For $\lambda, \mu \in \varrho(L)$ we have the resolvent equation

$$R_\lambda \circ R_\mu = (\lambda - \mu)^{-1} \cdot (R_\lambda - R_\mu).$$

One of the main applications of this identity in the context of elliptic operators is the comparison of solutions of $R_\lambda w = 0$ with those of $R_\mu w = 0$. We will use this idea in the special case of the minimal Green's functions G^t corresponding to the operators $L^t = L - t$ on M, viewed as resolvents. We include the simple proof of the resolvent equation in our context and note some of its consequences.

Lemma 2.7 (Resolvent Equation) *Assume (M, L) satisfies assumption (BAC). For any $0 \leqslant t < \tau$, the minimal Green's function G^t, viewed as an operator on the space of positive Radon measures μ with $G^t\mu < \infty$, satisfies*

$$G^t = G + t \cdot G \circ G^t\,.$$

Since all involved operators are positive, this yields the inequalities

$$G \leqslant G^t \quad \textit{and} \tag{2.1}$$

$$G \circ G^t \leqslant \frac{1}{t} G^t\,. \tag{2.2}$$

In particular, these results hold in the cases of characteristic functions of bounded measurable sets or Dirac measures.

Proof. Consider an increasing sequence of relatively compact, smoothly bounded open sets (U_i) with $\bigcup U_i = M$. On each U_i, the corresponding Dirichlet Green's function G_i (i.e., $G_i(\cdot, y)|_{\partial U_i} \equiv 0$) satisfies the resolvent equation in the form

$$G_i(x,y) = G_i^t(x,y) - t \int_{U_i} G_i(x,z) G_i^t(z,y) \mathrm{d}z$$

because the right-hand side fulfills the properties of a Green's function (application of L yields δ_y) and has the correct boundary behaviour. To see that the integral is finite for $x \neq y$, notice that any Green's function is integrable near its pole by the same arguments as in the proof of Corollary 2.4 “local bound for the minimal Green's function”.

For $i \to \infty$, the G_i and G_i^t converge to G and G^t, respectively, uniformly on compact sets away from the pole because they are increasing and bounded from above by the minimal Green's function. The same is true if the first argument is fixed because then one has the Green's function for the adjoint operator. Hence the equation survives in the limit $i \to \infty$.

By integration and Fubini's theorem one obtains the resolvent equation for arbitrary Radon measures μ, as soon as $G^t\mu < \infty$ (and thereby $G\mu < \infty$). $\square$

Recall from section 1.2 that unlike L_t, for $t < \tau$, the operator L_τ is no longer weakly coercive. In turn, we observe from (2.2) that using resolvents loses its strength when t approaches 0. This suggests to work with L_θ, for

$$\theta := \tau/2.$$

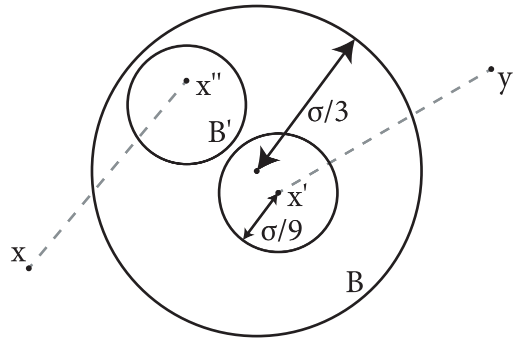


Figure 2.1: Constructions in the proof of Proposition 2.8 "bound for the Green's function". Along the dashed lines, we use Harnack chains.

2.2.2 Behaviour of Green's Functions

We combine the resolvent equation with Harnack inequalities 2.5 to derive growth estimates of minimal Green's functions.

Proposition 2.8 (Bound for the Green's Function) *Given (M, L) satisfying (BAC), there is a constant $c_1(\sigma, \ell, k, n, \tau) \geqslant 1$ such that for the minimal Green's function of L,*

$$c_1^{-1} \leqslant G(x,y) \qquad \text{if } d(x,y) \leqslant \sigma, \text{ and}$$
$$G(x,y) \leqslant c_1 \qquad \text{if } d(x,y) = \sigma\,.$$

Proof. The lower bound is directly obtained from the local bound for the minimal Green's function 2.4, because we have $G(\cdot, y) \geqslant g(\cdot, y)$ for any Green's function g on a smaller domain, and iterated application of the Harnack inequalities 2.5 along a Harnack chain of length depending only on σ, ℓ and r_{mgbi}.

For the upper bound, the strategy is to first obtain *any* localised bound and then transfer it to two given points using Harnack chains. To this end, we apply the resolvent equation 2.7 to the characteristic function χ_B of an arbitrary ball B with radius $\sigma/3$ (see Figure 2.1) and integrate the result over B to see

$$\begin{aligned} 0 \leqslant \int_M \chi_B G(\chi_B) &= \int_M \chi_B G^\theta(\chi_B) - \theta \int_M \chi_B G(G^\theta(\chi_B)) \\ &= \int_M \left[\chi_B - \theta G^*(\chi_B)\right] G^\theta(\chi_B) \end{aligned} \tag{2.3}$$

using Fubini's theorem. Recall that G^* denotes the Green's function for the adjoint operator L^*.

Note that all appearing integrals are in fact finite because for an arbitrary point z with $\operatorname{dist}(z, B) = \sigma/3$,

$$G^\theta(\chi_B) = \int_B G^\theta(\cdot, \zeta)\mathrm{d}\zeta \leqslant c_1 \int_M G^\theta(\cdot,\zeta) G(\zeta, z)\mathrm{d}\zeta \leqslant \frac{c_1}{\theta} G^\theta(\cdot, z)$$

by the lower bound $1 \leqslant c_1 \cdot G(\zeta, z)$ and (2.2), and $G^\theta(\cdot, z)$ is bounded on B.

From (2.3), we infer the existence of an $x' \in B$ with $(\chi_B - \theta G^*(\chi_B))\,(x') \geqslant 0$ and therefore

$$\int_B G(\zeta, x')\mathrm{d}\zeta = G^*(\chi_B)(x') \leqslant 1/\theta\,.$$

Using the triangle inequality, we can find another ball $B' \subset B$ of radius $\sigma/9$, such that $B' \cap B_{\sigma/9}(x') = \varnothing$. Obviously $\int_{B'} G(\zeta, x')\mathrm{d}\zeta \leqslant \int_B G(\zeta, x')\mathrm{d}\zeta \leqslant 1/\theta$ and hence there is an $x'' \in B'$ with

$$G(x'', x') \leqslant 1/(\theta \operatorname{Vol} B')\,.$$

By the bounded geometry assumption, $\operatorname{Vol} B'$ is bounded from below by a constant depending only on σ, ℓ and n. The result is a universal upper bound on $G(x'', x')$ for two points with distance at least $\sigma/9$ in an arbitrary ball B.

To get an upper bound for any two points $x, y \in M$ with $d(x, y) = \sigma$, we choose a ball B in between and connect x and x'' by a Harnack chain in sufficiently large distance from x', and analogously for y and x'. Along these chains we can apply Harnack inequalities for the operators L and L^* respectively. Bounded geometry shows that these chains can be chosen in such a way that the length depends only on σ and ℓ, thus we obtain an upper bound $G(x, y) \leqslant c_1$ with $c_1 = c_1(\sigma, \ell, k, n, \tau)$. □

Note that we could now easily get more explicit growth estimates for G near the pole by comparison with the local bound for the minimal Green's function 2.4, but for the following these coarse estimates are sufficient.

2.2.3 Relative Maximum Principles

We start with an elementary comparison result which already visualises the primary effect of weak coercivity: with the same boundary conditions, L-harmonic functions *sag* compared to L_θ-harmonic functions. Imagine a rope[1] or a rubber blanket where you apply less and less tension. To see this, we start with a paraboloid that fits in between.

Lemma 2.9 *Under the local assumptions (LAG), there is a constant $m(R = \sigma/\ell, k, n) > 0$ such that we can find a smooth function f defined on $B_{R/2}(0)$ with*

$$f(0) = -m\,, \quad f|_{\partial B_{R/4}(0)} \geqslant 0 \quad \textit{and} \quad Lf > -1\,.$$

Proof. In polar coordinates, L applied to a radial function $f(r)$ can be written as

$$Lf(r) = -af''(r) + \frac{b}{r}f'(r) + cf(r)$$

with functions a, b, c bounded by a constant depending only on k, n and R.

With the ansatz $f(r) = s(r^2 - \varrho)$ we have $Lf(r) = -2as + 2bs + cs(r^2 - \varrho)$ and first choose $\varrho > 0$ sufficiently small to assure $f \geqslant 0$ outside of $B_{R/4}(0)$ and then make $s > 0$ small to achieve $Lf \geqslant -1$. This only uses the bounds on coefficients such that $f(0) = -s\varrho =: -m < 0$ depends only on k, n and R. □

[1] This analogy is quite precise, a suspended rope forms a hyperbolic cosine, a solution of the shifted one-dimensional Laplace equation.

Proposition 2.10 (Relative Maximum Principle, Local Version) *Under assumption (LAG) for L and $L_\theta = L - \theta$, assume further we have two functions u and $\bar{u} > 0$ on $B_R = B_R(0)$, u L-subharmonic ($Lu \leqslant 0$) and $\bar{u}$ L_θ-harmonic on B_R with $\bar{u}|_{\partial B_{R/4}} \geqslant u|_{\partial B_{R/4}}$. Then there is a constant $\tilde{\eta} = \tilde{\eta}(R, k, n, \tau) \in (0, 1)$ such that*

$$u(0) \leqslant \tilde{\eta}\, \bar{u}(0)\,.$$

Proof. Consider the function $h(z) := \bar{u}(z) + \theta f(z) \inf_{w \in B_{R/4}} \bar{u}(w) - u(z)$ on $B_{R/4}$ where f is the function from the previous lemma. On $B_{R/4}$, we have $L\bar{u} = L_\theta \bar{u} + \theta \bar{u} \geqslant \theta \bar{u}$, $Lf > -1$ and therefore

$$Lh(z) > \theta \left(\bar{u}(z) - \inf_{w \in B_{R/4}} \bar{u}(w) \right) \geqslant 0\,.$$

The boundary condition together with $f|_{\partial B_{R/4}} \geqslant 0$ yields $h|_{\partial B_{R/4}} \geqslant 0$ and we can apply the local minimum-zero principle 2.1 for L to see $h \geqslant 0$ in the interior of $B_{R/4}$ and especially $h(0) \geqslant 0$, with $f(0) = -m$ we have

$$u(0) = \bar{u}(0) - m\theta \inf_{w \in B_{R/4}} \bar{u}(w) \leqslant (1 - m\theta H^{-1}) \bar{u}(0)\,.$$

The harmonicity on the full ball B_R was used only in the last step to apply the Harnack inequalities 2.5. ☐

Applied globally, this describes the relative growth of L-harmonic versus L_θ-harmonic functions.

Proposition 2.11 (Relative Maximum Principle, Global Version) *If (M, L) satisfies (BAC), there is a constant $\eta = \eta(\sigma, \ell, k, n, \tau) \in (0, 1)$ such that the following holds:*

Assume we have two functions u and $\bar{u} > 0$ defined on $B_{r+3}(x)$ for some $x \in M$ and $r \geqslant \sigma$, u L-subharmonic and $\bar{u}$ L_θ-harmonic on $B_{r+3}(x)$ and $\bar{u}|_{\partial B_r(x)} \geqslant u|_{\partial B_r(x)}$. Then

$$u(x) \leqslant \eta^r \bar{u}(x)\,.$$

Proof. For integer multiples r of $\sigma/4\ell$, this is proven by inductively applying the local version in adapted charts along a chain of intersecting balls of length proportional to $r/(\sigma/\ell)$. On each of them we may apply the same Harnack inequality as described in the proof of 2.8 above, so that we can choose η as a function of σ, ℓ, H and $\tilde{\eta}$. ☐

For Green's functions, we get the following variants. We do not use them in the following arguments but they are worth being mentioned since they give some non-trivial constraints on the Green's functions from our standard assumption (BAC).

Corollary 2.12 (Exponentially Stronger Decay) *Under assumptions (BAC), there are constants $A(\sigma, \ell, k, n, \tau) > 0$ and $\alpha_1(\sigma, \ell, k, n, \tau) > 0$ such that*

$$G(x, y) \leqslant A\, \mathrm{e}^{-\alpha_1 d(x,y)}\, G^\theta(x, y) \quad \forall x, y \in M\,.$$

From this we get the following growth estimate for Green's functions using the resolvent equation.

Proposition 2.13 (Exponential Decay) *Under assumptions (BAC), for suitable constants* $B(\sigma,\ell,k,n,\tau) > 0$ *and* $\alpha_2(\sigma,\ell,k,n,\tau) > 0$ *we have*

$$G(x,y)G(y,x) \leqslant B\,\mathrm{e}^{-\alpha_2 d(x,y)} \quad \textit{for } d(x,y) > 2\sigma\,.$$

Proof. Let $x' \in M$ such that $d(x,x') = \sigma$. Then employing the resolvent equation 2.7 we have

$$\begin{aligned} G(x,y)G(y,x) \leqslant G(x,y)\,H^3\,G(y,x') &\leqslant \frac{H^5}{\mathrm{Vol}(B_{\sigma/2}(y))}\int_{B_{\sigma/2}(y)} G(x,z)G(z,x')\mathrm{d}z \\ &\leqslant c\int_M G(x,z)G(z,x')\mathrm{d}z \overset{(2.1),(2.2)}{\leqslant} \frac{c}{\theta}G^\theta(x,x') \leqslant \frac{cc_1^\theta}{\theta} \end{aligned}$$

where we used the Harnack inequalities 2.5 for L^* and the bound for the Green's function 2.8 applied to L_θ which itself satisfies the assumptions, but it may lead to a weaker constant c_1^θ.

For the very same reason, we can do all of the above with G^θ instead of G to get the uniform boundedness of $G^\theta(x,y)G^\theta(y,x)$, again with slightly worse constants. Combined with Corollary 2.12 "exponentially stronger decay" we have proved the assertion. □

In the case of a symmetric operator $L = L^*$ the Green's function is also symmetric and the proposition says that it decays exponentially with the distance. This result does not use Gromov hyperbolicity and holds also e.g. in Euclidean space. One may remember that the familiar Euclidean Laplacian's Green's function only decays with $|x-y|^{-(n-2)}$, but this is not in conflict to the result above because the Euclidean Laplacian is not weakly coercive.

2.3 Hyperbolicity and Boundary Harnack Inequalities

Now we additionally invest the hyperbolicity of the underlying space. We employ the property that in Gromov hyperbolic spaces any two points can be connected by well-controlled Φ-chains as in Theorem 1.15 "Φ-Chains on Hyperbolic Spaces". We prove in fact a more general result which holds on a single Φ-chain, even if the space carries essentially only this one Φ-chain as in the example $(\mathbb{R}\times\mathbb{R}^{n-1}, (1+|y|^2)^2\cdot g_{\mathbb{R}} + g_{\mathrm{Eucl}})$. This can be seen as a directed form of hyperbolicity.

The technical main result of this section describes the behaviour of Green's function along Φ-chains, building on the results for bounded geometries and weakly coercive operators we derived in the last few sections. In the presence of sufficiently many Φ-chains, such as in Gromov hyperbolic spaces, we infer boundary Harnack inequalities and employ them to identify the Martin boundary with the Gromov boundary.

Our general assumptions (BAC) on the manifold M and the elliptic operator L remain the same as in the previous section: M is complete with bounded geometry and L is adapted and weakly coercive. The additional assumptions, that is, the presence of a Φ-chain (depending on the function Φ) or even of an underlying hyperbolic geometry (with constant δ and coming with a universal function $\Phi = \Phi_\delta$), are stated directly in the results.

2.3.1 Global Behaviour: Φ-Chains

The following result, sometimes called *3G inequality*, describes the major influence of Φ-chains on the behaviour of Green's functions.

Theorem 2.14 (Green's Functions Along Φ-Chains) *Under assumptions (BAC), there is a suitable constant $c(\sigma, \ell, k, n, \tau, \Phi) > 1$ such that for any Φ-chain with track points $x_1, \ldots, x_m$ (as in Definition 1.14), we have for the minimal Green's function G the estimate*

$$c^{-1} G(x_m, x_j)\, G(x_j, x_1) \leqslant G(x_m, x_1) \leqslant c\, G(x_m, x_j)\, G(x_j, x_1)\,, \quad j = 2, \ldots, m-1\,.$$

At the heart of the argument we employ the pairing of two at first sight entirely unrelated geometric and analytic properties: the existence of Φ-chains on M and the weak coercivity of L. The idea is that, on the one hand, Φ-chains allow to find balls of arbitrary large radii in $U_{i-1} \setminus U_{i+1}$ centered in ∂U_i within a uniformly upper bounded distance to the track points. On the other hand, the relative maximum principle 2.11 shows that on these balls we can improve crude estimates from a previous application of a Harnack inequality by investing weak coercivity. This makes the following result the main step in the proof of the Theorem.

Proposition 2.15 (Growth Recovery Along Φ-Chains) *With assumptions (BAC), for any given Φ-chain with track points $x_1, \ldots, x_j$, we have*

$$G(z, x_1) \leqslant c\, G^\theta(z, x_j)\, G(x_{j+1}, x_1) \quad \text{for } z \in \partial U_{j+1}, \tag{2.4}$$

for some constant $c(\sigma, \ell, k, n, \tau, \Phi) > 0$ independent of the length j of the Φ-chain.

Proof. The argument is by induction over the length j.

For $j = 1$, we can use $G^\theta \geqslant G$, inequality (2.1), the lower bound for the Green's function 2.8 $c_1\, G(x_2, x_1) \geqslant 1$ and Harnack inequalities to get a first guess for the constant $c = c_1$ and note that c_1 depends only on σ, ℓ, k, τ and Φ (via Harnack inequalities).

For the induction step we first assume we have proved the weaker assertion that there is a constant c_j so that (2.4) holds for any Φ-chain of length j. Then we can apply the Harnack inequalities for L and L^*_θ to get a constant $c'(\sigma, \ell, k, \tau, \Phi) \geqslant 1$, independent of j, such that

$$G(z, x_1) \leqslant c' c_j\, G^\theta(z, x_{j+1})\, G(x_{j+2}, x_1) \quad \text{for } z \in \partial U_{j+1}\,. \tag{2.5}$$

By the global maximum principle 1.18 this inequality extends to $z \in \bar{U}_{j+1}$. Now we invest the weak coercivity of L and the properties of the Φ-chains to improve this inequality.

Towards this end, we first apply the relative maximum principle 2.11 to the (L-superharmonic) function $G^\theta(\cdot, x_{j+1})$ and its greatest L-harmonic minorant u on some ball $B_R(x)$ which we can represent as $u = \mathcal{R}^{\partial B_R(x)}_{G^\theta(\cdot, x_{j+1})}$, the reduit always taken with respect to L (see subsection 1.2.2 for properties of the reduit employed here and below). For $R = \ln(1/c')/\ln\eta$ and $B_{R+3}(x) \subset U_{j+1}$, the relative maximum principle yields

$$u(x) \leqslant \frac{1}{c'} G^\theta(x, x_{j+1})\,.$$

In turn, the definition of a Φ-chain shows that there is some $\Delta(\Phi, R) > 0$ such that $B_{R+3}(x) \subset U_{j+1}$, for $x \in \partial U_{j+2}$, as soon as $d(x, x_{j+2}) \geqslant \Delta$. Then, we have from (2.5)

$$\begin{aligned} G(x, x_1) &= \mathcal{R}^{\partial B_R(x)}_{G(\cdot, x_1)}(x) \leqslant c' c_j \mathcal{R}^{\partial B_R(x)}_{G^\theta(\cdot, x_{j+1})}(x)\, G(x_{j+2}, x_1) \\ &= c' c_j u(x)\, G(x_{j+2}, x_1) \leqslant c_j G^\theta(x, x_{j+1})\, G(x_{j+2}, x_1). \end{aligned}$$

On the other hand, we get universal estimates for $x \in \partial U_{j+2}$ with $d(x, x_{j+2}) < \Delta$: since $d(x, x_{j+2}) < \Delta$, there are constants $c'', c''' \geqslant 1$ only depending on Δ, Φ, H and c_1, such that $G(x, x_1) \leqslant c''\, G(x_{j+2}, x_1)$ by Harnack inequalities and $G^\theta(x, x_{j+1}) \geqslant (c''')^{-1}$ by the bounds for the Green's function 2.8 and Harnack inequalities. The result is

$$G(x, x_1) \leqslant c'' c'''\, G^\theta(x, x_{j+1})\, G(x_{j+2}, x_1) \quad \text{for } x \in \partial U_{j+2} \text{ with } d(x, x_{j+2}) < \Delta\,.$$

Everything combined, we can choose $c = \max\{c_1, c'' c'''\}$ and outside a tube of radius Δ around the track points the constant can be kept in every induction step while on the inside we can always use the universal constant. □

Proof of Theorem 2.14 "Green's Functions Along Φ-Chains". The first inequality is rather easy: For $x \in \partial B_\sigma(x_j)$ we have

$$G(x, x_j) G(x_j, x_1) \leqslant c_1 G(x_j, x_1) \leqslant c_1 H\, G(x, x_1)$$

by Proposition 2.8 "bound for the Green's function" and the Harnack inequalities 2.5. Since the left-hand side is an L-potential and the right-hand side is L-superharmonic, this inequality extends to $M \setminus B_\sigma(x_j)$ and in particular to x_m by the global maximum principle 1.18.

For the second inequality, we use repeatedly Proposition 2.15 "Growth Recovery Along Φ-Chains" and the resolvent equation 2.7:

$$\begin{aligned} G(x_m, x_1) &= \mathcal{R}^{\partial U_j}_{G(\cdot, x_1)}(x_m) && \mid x_1 \notin U_j \\ &\overset{(2.4)}{\leqslant} c\, \mathcal{R}^{\partial U_j}_{G^\theta(\cdot, x_j)}(x_m)\, G(x_{j+1}, x_1) \\ &= c\, \mathcal{R}^{\partial U_j}_{G(\cdot, x_j) + \theta G(G^\theta(\cdot, x_j))}(x_m)\, G(x_{j+1}, x_1) && \mid \text{res.eq.} \\ &\leqslant c \left(G(x_m, x_j) + \theta \int_M \mathcal{R}^{\partial U_j}_{G(\cdot, z)}(x_m)\, G^\theta(z, x_j)\, \mathrm{d}z \right) G(x_{j+1}, x_1) && (2.6) \end{aligned}$$

At this point, we can again employ the first step (2.4), but now for the reversed Φ-chain $x_m, \ldots, x_1$ with $M \setminus U_m, \ldots, M \setminus U_1$ and for the adjoint operator L^*, namely

$$G(x_m, z) \leqslant c\, G^\theta(x_{j+2}, z)\, G(x_m, x_{j+1}) \quad \text{for } z \in M \setminus U_{j+1}\,.$$

This holds on all of $M \setminus U_{j+1}$ by the global maximum principle 1.18. Since $x_j \in M \setminus U_{j+1}$, this can be directly applied to $G(x_m, x_j)$. For the second summand in (2.6), we have $\mathcal{R}^{\partial U_j}_{G(\cdot, z)}(x_m) = {}^*\mathcal{R}^{\partial U_j}_{G(x_m, \cdot)}(z) \leqslant G(x_m, z)$ for $z \in \partial U_j \subset M \setminus U_{j+1}$ (denoting the reduit with respect to L^* by ${}^*\mathcal{R}$, see subsection 1.2.2), but the upper bound ${}^{*}\!\mathcal{R}^{\partial U_j}_{G(x_m,\cdot)}(z) \leqslant c\, G^\theta(x_{j+2}, z)\, G(x_m, x_{j+1})$ is valid for *all* $z \in M$ by definition of the reduit since the right-hand side is positive and L^*-superharmonic in z. Thus,

$$G(x_m, x_1) \leqslant c^2\, G(x_m, x_{j+1})\, G(x_{j+1}, x_1) \left(G^\theta(x_{j+2}, x_j) + \theta \int_M G^\theta(x_{j+2}, z)\, G^\theta(z, x_j)\, \mathrm{d}z \right) .$$

The large bracket is universally bounded from above by Proposition 2.8 "bound for the Green's function", the Harnack inequalities 2.5, and the inequalities (2.1) and (2.2) following from the resolvent equation 2.7 for $t = \frac{3}{2}\theta$. □

Now we assume that M is a δ-hyperbolic space, then we can choose $\Phi = \Phi_\delta$ and recall that there are Φ-chains along geodesics in M. Since Φ_δ is determined from δ, the Φ-dependence of the estimates now reduces to a δ-dependence.

Corollary 2.16 (Green's Function Along Hyperbolic Geodesics) *Assuming (BAC) for (M, L), M δ-hyperbolic, let $x, y, z \in M$ such that y lies on geodesic connecting x and z with $d(x, y), d(y, z) > 22\delta$. Then there is a constant $c(\sigma, \ell, k, n, \tau, \delta) > 1$ such that*

$$c^{-1} G(x, y) G(y, z) \leqslant G(x, z) \leqslant c\, G(x, y) G(y, z) .$$

Remark 2.17. This estimate for the Green's function can be interpreted stochastically and algebraically:

- Stochastically, the Green's function $G(x, y)$ is a density for the expected number of times an L-Brownian motion starting at y reaches x, see [Pin95] or [Anc90]. Now the estimate for the Green's function above states that on a hyperbolic geodesic $x \rightsquigarrow y \rightsquigarrow z$, up to a constant multiple there are as many Brownian particles travelling directly from x to z as there are particles travelling from x to y and then to z. This is in line with the geometric "valley" interpretation of Gromov hyperbolic spaces, as seen in Proposition 1.5 "stability of geodesics".

- Algebraically, we can examine the function

  $$d_G(x, y) := -\ln G(x, y) \quad \text{for } x, y \in M.$$

  For this *Green metric*, the estimate above says

  $$d_G(x, z) = d_G(x, y) + d_G(y, z) \pm \ln c$$

  along a geodesic $x \rightsquigarrow y \rightsquigarrow z$, if the points are sufficiently far apart. Note that the estimate "$\leqslant$" holds not only along a geodesic, but for general points x, y, z with large mutual distance—this is a rough triangle inequality for d_G. The other direction suggests that the large-scale geodesic structure of d_G is comparable to that of d. In the context of hyperbolic groups, this has been explored further in [BB07, BHM11], where the Green metric turns out to be Gromov hyperbolic and quasi-isometric to the word metric for non-amenable groups and certain operators.

2.3.2 Boundary Harnack Inequality

Using Theorem 2.6 "boundary Harnack inequality on a disc" as a blueprint, we want to formulate a boundary Harnack inequality near points on the Gromov boundary of a δ-hyperbolic space.

As a replacement for balls in the classical version of the boundary Harnack inequality, we need some characterisation of neighbourhoods of a point at infinity. This is made precise by the notion of Φ-neighbourhood bases.

Definition 2.18 (Φ-Neighbourhood Basis) Two nonempty open subsets $V \supset W$ of M are called **Φ-neighbourhoods with hub** $h \in M$, if $\overline{W} \subset V$, $B_{\Phi_0}(h) \subset V \setminus \overline{W}$ and any two points $p \in \partial V$ and $q \in \partial W$ can be joined by a Φ-chain that has h as a track point. We call an infinite family of nonempty open sets $\mathcal{N}_i \subset M$, $i = 1, 2, 3, \ldots$, with $\bigcap_i \mathcal{N}_i = \varnothing$ a **Φ-neighbourhood basis**, if $\mathcal{N}_i$ and $\mathcal{N}_{i+1}$ are Φ-neighbourhoods with hub o_i, for every i.

Just as metric balls are, besides their role as neighbourhood bases, the basic playground for Harnack inequalities, these Φ-neighbourhoods are their counterpart in *boundary* Harnack inequalities.

In δ-hyperbolic spaces, every point in the Gromov boundary has a canonical neighbourhood basis that is also a Φ_δ-neighbourhood basis. Namely, as shown in [BHK01, Proposition 8.10][2], we have:

Lemma 2.19 (Φ_δ-Neighbourhood Basis) *If M is δ-hyperbolic and $o \in M$ a basepoint, there is a constant $c_\delta > 0$ only depending on δ such that for any $\xi \in \partial_{\mathrm{G}} M \subset \overline{M}^{\mathrm{G}}$, the open sets*

$$\mathcal{N}_i^\delta(\xi) := \mathcal{W}^o_{c_\delta i}(\xi) \cap M = \{x \in M \mid (x|\xi)_o > c_\delta i\} \quad \text{for } i = 1, 2, \ldots$$

are a Φ_δ-neighbourhood basis. Recall that their closures $\overline{\mathcal{N}_i^\delta(\xi)} = \overline{\mathcal{W}^o_{c_\delta i}(\xi)} \subset \overline{M}^{\mathrm{G}}$ in the Gromov compactification form a neighbourhood basis of $\xi \in \partial_{\mathrm{G}} M$.

For non-symmetric operators it is not always possible to find L-harmonic functions that vanish at infinity, because even minimal Green's functions might diverge. Hence we introduce a more general notion which can be thought of as a minimal growth condition. This will be further explained in Proposition 2.26 "L-vanishing and Martin boundary" below.

Definition/Proposition 2.20 (L-Vanishing) *We say that a positive L-superharmonic function u* **L-vanishes** *on an open set $V \subset M$ if one of the following equivalent conditions is satisfied:*

(i) There is a positive L-superharmonic function w, such that $u/w \to 0$ at infinity, i.e., for every $\varepsilon > 0$ there is a compact set $K \subset M$ with $u/w < \varepsilon$ on $V \setminus K$.

(ii) There is an L-potential p such that $p \geqslant u$ on V.

[2]They define the neighbourhood basis with references to distances from geodesics, but the two concepts can easily be translated into each other using Lemma 1.7 "Gromov product as distance to a geodesic".

(iii) The reduit $\mathcal{R}_u^V$ is an L-potential (see subsection 1.2.2).[3]

Proof. (i) $\Rightarrow$ (iii) Assume there is a positive L-harmonic function h, such that $\mathcal{R}_u^V \geqslant h > 0$ on M. For some fixed $\varepsilon > 0$, choose a compact set K with $u < \varepsilon w$ on $V \setminus K$. By the properties of the reduit, we even have

$$\varepsilon w \geqslant \mathcal{R}_u^{V\setminus K} \geqslant \mathcal{R}_u^V - \mathcal{R}_u^{V\cap K} \geqslant h - \mathcal{R}_u^{V\cap K}$$

on all of M. Now $\mathcal{R}_u^{V\cap K}$ is an L-potential since $V \cap K$ is relatively compact in M, $\varepsilon w - h$ is L-superharmonic and $\varepsilon w \geqslant u = \mathcal{R}_u^V \geqslant h$ on $\partial(V\setminus K) = \partial(M\setminus(V\setminus K)) \subset M$, thus we can apply the global maximum principle 1.18 to see $\varepsilon w \geqslant h$ on all of M. Since ε was arbitrary, $h = 0$.

(iii) $\Rightarrow$ (ii) Choose $p = \mathcal{R}_u^V$.

(ii) $\Rightarrow$ (i) It suffices to show that an L-potential p satisfies the condition everywhere. Towards this end, consider the functions $\mathcal{R}_p^{M\setminus\overline{B_R(o)}}$ on balls around an arbitrary basepoint $o \in M$. They converge to zero for $R \to \infty$ because the limit is L-harmonic and $\leqslant p$. Let (x_j) be a countable dense set in M. We may choose a sequence (R_i) such that $\mathcal{R}_p^{M\setminus\overline{B_{R_i}(o)}}(x_j) \leqslant 2^{-i}$ for all $j \leqslant i$. Then the function $w = \sum_i \mathcal{R}_p^{M\setminus\overline{B_{R_i}(o)}}$ is finite on a dense set, L-superharmonic and $p/w \leqslant 1/i$ outside of $B_{R_i}(o)$. □

Note that all L-potentials such as the minimal Green's function are L-vanishing on $V = M$ and hence on all open sets because the property of L-vanishing is conserved on subsets as can be easily seen using condition (i).

On bounded sets, L-vanishing *at infinity* is trivially true for any L-superharmonic function. The concept is also useless for unbounded sets shrinking too quickly when approaching infinity. On Gromov hyperbolic manifolds, it becomes significant for sets $V = W \cap M$, where $W \subset \overline{M}^{\mathrm{G}}$ is open with non-empty intersection $W \cap \partial_{\mathrm{G}} M$. For more elaborate criteria in the context of Martin theory see Proposition 2.26 "L-vanishing and Martin boundary".

On Φ-neighbourhoods we can now formulate the following central result.

Theorem 2.21 (Boundary Harnack Inequality) *Assume (M, L) satisfies assumptions (BAC). Let $V \supset W$ be Φ-neighbourhoods with hub h and u, v two positive L-superharmonic functions that are L-harmonic and L-vanishing on V, then there is a constant $H_B = H_B(\sigma, \ell, k, n, \tau, \Phi)$ such that*

$$\frac{u(x)}{u(y)} \leqslant H_B \frac{v(x)}{v(y)} \quad \textit{for any } x, y \in W\,.$$

Proof. By Definition/Proposition 2.20 "L-vanishing", the reduit $\mathcal{R}_u^V$ is an L-potential and therefore admits a representation as

$$\mathcal{R}_u^V = \int_{\partial V} G(\cdot, z)\,\mathrm{d}\nu(z)$$

[3] [Anc87] uses the first definition, but it is easier to employ the last.

for some Radon measure ν. On $\overline{W}$, $\mathcal{R}_u^V$ agrees with u and therefore we have

$$u(x) = \int_{\partial V} G(x,z)\,\mathrm{d}\nu(z) \leqslant c \int_{\partial V} G(x,h)\,G(h,z)\,\mathrm{d}\nu(z) = c\,G(x,h)\,u(h) \text{ for } x \in \partial W$$

using the assumption that $x \in \partial W$ and $z \in \partial V$ can be connected by a Φ-chain through h and Theorem 2.14 "Green's Functions Along Φ-Chains". The other inequality from there gives

$$v(x) \geqslant c^{-1} G(x,h)\,v(h) \quad \text{for } x \in \partial W$$

and both inequalities extend to W by the global maximum principle 1.18 because $\mathcal{R}_u^V$ and $G(\cdot,h)$ respectively are L-potentials. We can combine them to obtain

$$\frac{u(x)}{v(x)} \leqslant c^2 \frac{u(h)}{v(h)} \quad \text{for } x \in W\,.$$

Interchanging the roles of u and v yields the result with $H_B = c^4$. □

In most applications, u and v are either globally L-harmonic functions or minimal Green's functions with pole outside of V.

In the case of a δ-hyperbolic manifold, the size of the smaller neighbourhood can be explicitly quantified using Lemma 2.19 "Φ_δ-neighbourhood basis". We get

Corollary 2.22 (Hyperbolic Boundary Harnack Inequality) *If (M,L) satisfies (BAC) and M is δ-hyperbolic, there is some positive constant $H_B(\sigma,\ell,k,n,\tau,\delta) > 1$ such that two positive L-superharmonic functions u, v that are L-harmonic and L-vanishing on a Φ_δ-neighbourhood $\mathcal{N}_i^\delta(\xi)$ of $\xi \in \partial_{\mathrm{G}} M$ satisfy*

$$\frac{u(x)}{u(y)} \leqslant H_B \frac{v(x)}{v(y)} \quad \textit{for any } x,y \in \mathcal{N}_{i+1}^\delta(\xi)\,.$$

2.3.3 The Hyperbolic Martin Boundary

We now turn our attention towards the *Martin boundary* as introduced in subsection 1.2.4. In the situation at hand, Φ-neighbourhood bases are essentially neighbourhood bases of minimal Martin boundary points.

Theorem 2.23 (Characterisation of Minimal Martin Points) *Assume (BAC) holds and $o \in M$ is a basepoint. Let $(\mathcal{N}_i)$ be a Φ-neighbourhood basis with hubs h_i and assume $o \notin \overline{\mathcal{N}_1}$. Denoting the interior of the closure of $\mathcal{N}_i \subset M \subset \overline{M}^{\mathrm{M}}$ in the Martin compactification $\overline{M}^{\mathrm{M}}$ of M by $\widetilde{\mathcal{N}_i}$, there is exactly one Martin boundary point ζ in $\bigcap \widetilde{\mathcal{N}_i}$. The resulting Martin function K_ζ is characterised as the only positive L-harmonic function L-vanishing on every $M \setminus \mathcal{N}_i$, up to scalar multiples. In particular, this Martin point ζ is minimal.*

Proof. By the Harnack inequalities, the sequence $K_{h_i} = \frac{G(\cdot,h_i)}{G(o,h_i)}$ has a subsequence compactly converging to some L-harmonic function K_ζ representing a Martin boundary point ζ. K_ζ is L-vanishing on every $M \setminus \mathcal{N}_i$ because by the boundary Harnack inequality 2.21, every K_{h_j} for $j \geqslant i$, and hence every limit, is upper bounded by the L-potential $H_B K_{h_i}$ on $M \setminus \mathcal{N}_i$.

Now assume there is another positive L-harmonic function u that is L-vanishing on $M \setminus \overline{\mathcal{N}}_i$ for every i, w.l.o.g. $u(o) = 1$. Applying the boundary Harnack inequality 2.21 we see $H_B^{-1} K_\zeta \leqslant u \leqslant H_B K_\zeta$ on M. Thus $\eta := \inf u/K_\zeta \leqslant 1$ is positive. By the strong maximum principle [Pin95, Theorem 3.2.6, p. 84], the L-harmonic function $u - \eta K_\zeta \geqslant 0$ has to be positive everywhere, else it would be identical zero. In the former case we can again apply the boundary Harnack inequality 2.21 to K_ζ and $u - \eta K_\zeta$ to get $(\eta + (1-\eta) H_B^{-1}) K_\zeta \leqslant u$ which contradicts the definition of η, unless $\eta = 1$ and $u = K_\zeta$. □

This yields the following characterisation of the Martin boundary in case we have enough Φ-neighbourhoods:

Corollary 2.24 (Identifying the Martin Boundary) *If under assumptions (BAC) in a given compactification $\overline{M}$ of M (i.e., $\overline{M}$ is compact and $M \subset \overline{M}$ open and dense) every boundary point admits a neighbourhood basis of the form $\overline{\mathcal{N}}_i \subset \overline{M}$ for some Φ-neighbourhood basis $(\mathcal{N}_i)$, it is canonically homeomorphic to the Martin compactification $\overline{M}^{\mathrm{M}}$.*

Proof. Theorem 2.23 "characterisation of minimal Martin points" yields an injective map from $\overline{M}$ to $\overline{M}^{\mathrm{M}}$. It is continuous because for every sequence (y_i) in $\overline{M}$ converging to $\zeta \in \overline{M} \setminus M$ the corresponding Martin functions K_{y_i} converge to the unique Martin function that is L-vanishing on all $M \setminus \mathcal{N}_i$ for some Φ-neighbourhood basis $(\mathcal{N}_i)$ of ζ, that is K_ζ. Thus, by elementary properties of compactifications [Mun00, §38], it is already a homeomorphism. Note that in particular all Martin boundary points are minimal. □

From this and Lemma 2.19 "Φ_δ-neighbourhood basis" we get the following principal potential theoretic result on Gromov hyperbolic manifolds.

Corollary 2.25 (Gromov Boundary and Martin Boundary) *Assume that M is Gromov hyperbolic and (BAC) holds. Then the Gromov and Martin boundaries of M are canonically homeomorphic and every Martin boundary point is already minimal,*

$$\partial_{\mathrm{G}} M \cong \partial_{\mathrm{M}}(M, L) \cong \partial^0_{\mathrm{M}}(M, L).$$

This means that a positive function $u > 0$ on M is L-harmonic if and only if there is a (unique) Radon measure μ_u on $\partial_{\mathrm{G}} M$ such that

$$u(x) = \int_{\partial_{\mathrm{G}} M} K_\zeta(x) \, \mathrm{d}\mu_u(\zeta).$$

Now that we have the right notion for a potential theoretic boundary at infinity, we can update the notion of L-vanishing. Note that the following characterisations are slightly different from the formulation in Definition/Proposition 2.20 "L-vanishing" because there, without explicit references to a boundary, it was only possible to refer to L-vanishing *at infinity of open sets V in M*, i.e., on the Martin boundary points in $\overline{V} \cap \partial_{\mathrm{M}} M \subset \overline{M}^{\mathrm{M}}$.

Proposition 2.26 (*L*-Vanishing and Martin Boundary) *Assume (BAC) holds and every Martin boundary point has a Φ-neighbourhood basis, e.g., M is Gromov hyperbolic. For an open subset $\Xi \subset \partial_{\mathrm{M}} M$ of the Martin boundary and a positive L-harmonic function u on M the following are equivalent:*

(i) *u is L-vanishing on any open set $V \subset M$ with $\overline{V} \cap \partial_{\mathrm{M}} M \subset \Xi$ in the Martin compactification.*

(ii) *On any open set $V \subset M$ with $\overline{V} \cap \partial_{\mathrm{M}} M \subset \Xi$ in the Martin compactification, the following property holds: each positive L-harmonic function v on V with $v \geqslant u$ on $\partial V \cap M$ satisfies $v \geqslant u$ on V.*

(iii) *The Martin measure μ_u associated to u is supported outside Ξ, i.e., $\mu_u(\Xi) = 0$.*

Proof. (i) ⇒ (ii) If u is L-vanishing on V, $\mathcal{R}_u^V$ is an L-potential and $\mathcal{R}_u^V = u$ on V. Then the global maximum principle 1.18 gives exactly the desired property.

(iii) ⇒ (i) Each Martin function K_ζ is L-vanishing outside ζ and, therefore, the Martin integral representing u L-vanishes outside the support of μ_u.

(ii) ⇒ (iii) Assume $\mu_u(\Xi) \neq 0$, then there is a compact $K \subset \partial_{\mathrm{M}} M$ and an open set $W \subset \overline{M}^{\mathrm{M}}$ such that $K \subset W \cap \partial_{\mathrm{M}} M \Subset \Xi$, $V := W \cap M$ satisfies the condition in (ii), and $\mu_u(K) \neq 0$ (since the Radon measure μ_u is inner regular). Therefore it is enough to consider the case where $u \equiv u_K := \int_K K_\zeta \, \mathrm{d}\mu_u(\zeta)$.

We compare u_K with the minimal Green's function $G(\cdot, p)$ with pole $p \in M \setminus V$. Recalling the argument of (iii) ⇒ (i) we know that u_K is L-vanishing on an open neighbourhood N of $M \setminus V$. By compactness of $\overline{M \setminus V}$ in the Martin compactification, $M \setminus V$ can be covered by finitely many Φ-neighbourhoods contained in N and a compact subset of M. Then the boundary Harnack inequality 2.21 shows that there is a $C > 0$ such that $C \cdot G(\cdot, p) \geqslant u_K$ on $M \setminus V$ and especially on ∂V. But then from (ii) it follows that $C \cdot G(\cdot, p) \geqslant u_K$ on V, hence on all of M, hence $u_K \equiv 0$ because $G(\cdot, p)$ is an L-potential. □

2.3.4 Sharpness

The validity of such a simple identification of the Martin boundary with a geometric boundary, which one may possibly expect from a naive guess, actually is a rare exception. If we only slightly violate the hyperbolicity constraint we can get a completely different and rather inscrutable outcome with many non-minimal Martin boundary points.

Example 2.27 (Ideal Boundaries of $\mathbb{H}^m \times \mathbb{H}^n$). The product space of two classical hyperbolic spaces $\mathbb{H}^m \times \mathbb{H}^n$, $m, n \geqslant 2$, has bounded geometry, but it is no longer Gromov hyperbolic since it contains flat planes obtained as products of pairs of geodesics in the two factors. In [GW93] we find a thorough discussion of the case of the Laplace operator on $\mathbb{H}^m \times \mathbb{H}^n$. There exist positive functions s with $-\Delta s = \lambda \cdot s$ if and only if $\lambda \leqslant \lambda_0$, where λ_0 is the principal eigenvalue. For $\lambda < \lambda_0$, the operator $-\Delta - \lambda$ is adapted and weakly coercive. The boundary at infinity (a natural generalisation of Gromov boundary) is homeomorphic to S^{n+m-1} [BH99, II.8.11(6), p. 266]. In turn, for the minimal Martin boundary of $-\Delta-\lambda$, for $\lambda<\lambda_0$, we have

$$\partial^0_{\mathrm{M}}(\mathbb{H}^m\times\mathbb{H}^n,-\Delta-\lambda)=S^{m-1}\times S^{n-1}\times I_\lambda\,,$$

where I_λ is a closed interval with a natural parametrisation which depends on λ and degenerates to a single point when $\lambda\to\lambda_0$ [GW93, p. 21]. The full Martin boundary contains two additional pieces,

$$\partial_{\mathrm{M}}(\mathbb{H}^m\times\mathbb{H}^n,-\Delta-\lambda)=(S^{m-1}\times\mathbb{H}^n)\cup(S^{m-1}\times S^{n-1}\times I_\lambda)\cup(\mathbb{H}^m\times S^{n-1})/\sim\ . \tag{2.7}$$

The gluing maps for $\sim$ are described in [GW93, p. 27]. We observe that not only the boundary at infinity does not coincide with $\partial^0_{\mathrm{M}}(\mathbb{H}^m\times\mathbb{H}^n,-\Delta-\lambda)$, but even the details of the partition of the full Martin boundary (2.7) depend on λ.

We will see more counterexamples in section 2.5 where we transfer Ancona's theory to uniform manifolds.

2.4 Ray Expansion of Harmonic Functions

In this short section, we assume that the operator L and manifold M with basepoint o satisfy (BAC) and that M is δ-hyperbolic. We will apply Ancona's results to obtain a representation of an arbitrary positive L-harmonic function along a geodesic ray in terms of the Green's function along that ray and the Martin measure of balls around the endpoint of the ray.

For a fixed boundary point $\zeta\in\partial_{\mathrm{G}}M$ and a geodesic ray $\gamma: o\rightsquigarrow\zeta$, the boundary can be partitioned into sets $U_i:=U_{\geqslant i}\setminus U_{\geqslant i+1}$, $i=0,1,\dots$, with

$$U_{\geqslant i}:=\{\eta\in\partial_{\mathrm{G}}M\mid(\eta|\zeta)_o\geqslant i\}\,.$$

Not that these sets are closed balls in the quasi-metric on the boundary.

The U_i are Borel sets and a positive L-harmonic function u with Martin measure μ (with respect to the basepoint o) can be written as

$$u(x)=\int_{\partial_{\mathrm{G}}M}K_\eta(x)\,\mathrm{d}\mu(\eta)=\sum_{i=0}^{\infty}\int_{U_i}K_\eta(x)\,\mathrm{d}\mu(\eta)\,.$$

For points $x_k=\gamma(k)$, $k=1,2,\dots$ along the geodesic ray γ, this representation can be improved because $K_\cdot(x_k)$ is almost constant on U_i:

- For $0<i<k$ and $\eta\in U_i$, $x_i=\gamma(i)$ has universally bounded distance from geodesic rays $o\rightsquigarrow\eta$ and $x_k\rightsquigarrow\eta$. For $o\rightsquigarrow\eta$,

  $$(o|\eta)_{x_i}=d(o,x_i)-(x_i|\eta)_o\leqslant i-i+4\delta=4\delta$$

  by the Bonk–Schramm Lemma 1.10 and hence the distance from x_i to the geodesic ray is bounded from above by Lemma 1.7 "Gromov product as distance to a geodesic". One argues similarly for $x_k\rightsquigarrow\eta$.

  Hence we have by Corollary 2.16 "Green's function along hyperbolic geodesics" and Harnack inequalities

  $$K_\eta(x_k)=\lim_{y\to\eta}\frac{G(x_k,y)}{G(o,y)}\asymp\lim_{y\to\eta}\frac{G(x_k,x_i)G(x_i,y)}{G(o,x_i)G(x_i,y)}=\frac{G(x_k,x_i)}{G(o,x_i)}$$

independent of the particular point $\eta \in U_i$, for y on a geodesic ray $o \rightsquigarrow \eta$ and far away from x_i.

- Now for the boundary cases: For $\eta \in U_0$, $K_\eta(x_k) \asymp G(x_k, o)$ by the hyperbolic boundary Harnack inequality 2.22 and ordinary Harnack inequalities since both functions are $\asymp 1$ in a point near o and L-vanishing on $U_{\geqslant 1}$.
- Analogously, $K_\eta(x_k) \asymp K_\zeta(x_k)$ for $\eta \in U_{\geqslant k}$ by the hyperbolic boundary Harnack inequality 2.22 on $M \setminus U_{\geqslant k-1}$ and more Harnack inequalities.

Putting all together, we have the *ray expansion* of u,

$$\begin{aligned} u(x_k) &= \sum_{i=0}^{\infty} \int_{U_i} K_\eta(x_k)\,\mathrm{d}\mu(\eta) \\ &\asymp G(x_k, o)\mu(U_0) + \sum_{i=0}^{k-1} \frac{G(x_k, x_i)}{G(o, x_i)} \mu(U_i) + K_\zeta(x_k)\mu(U_{\geqslant k}) \\ &\asymp G(x_k, o) \left(\mu(U_0) + \sum_{i=0}^{k-1} \frac{\mu(U_i)}{G(o, x_i)G(x_i, o)} + \frac{\mu(U_{\geqslant k})}{G(o, x_k)G(x_k, o)} \right) \\ &\asymp K_\zeta(x_k) \left(G(o, x_k)G(x_k, o)\mu(U_0) + \sum_{i=0}^{k-1} G(x_i, x_k)G(x_k, x_i)\mu(U_i) + \mu(U_{\geqslant k}) \right) . \end{aligned}$$

In the last two steps, we used again Corollary 2.16 "Green's function along hyperbolic geodesics" and the direct consequence

$$K_\zeta(x_k) \asymp \frac{1}{G(o, x_k)} .$$

2.5 Application to Uniform Spaces

Using the hyperbolisation presented in section 1.3, it is now easy to translate Ancona's results to the setting of uniform spaces. In the statement of a boundary Harnack inequality, the sets $\mathcal{N}_i^\delta(\xi)$ from Lemma 2.19 "Φ_δ-neighbourhood basis", defined relative to the hyperbolised metric, now act as a replacement for the concentric balls of Theorem 2.6 "boundary Harnack inequality on a disc" in the classical Euclidean setup.

Corollary 2.28 (Boundary Harnack Inequalities and Martin Theory on Uniform Manifolds) *Let $L = -\Delta + V$ be a Schrödinger operator with $|V| \leqslant a \cdot đ^{-2}$ satisfying a strong barrier condition (i.e., $\lambda_L^{đ} > 0$) on a bounded c-uniform manifold (M^n, g) of đ–bounded geometry with constants ε and K, with respect to an ℓ-Lipschitz generalised distance function đ, for constants $a, c, \varepsilon, K, \ell > 0$. Then there is a $C = C(a, c, \varepsilon, K, \ell, \lambda_L^{đ}, n) \geqslant 1$ such that for any two L-harmonic functions u, $v > 0$ on $\mathcal{N}_i^\delta(\xi)$ for some $\xi \in \partial M$ both L-vanishing[4] on $\mathcal{N}_i^\delta(\xi)$,*

$$\frac{u(x)}{v(x)} \leqslant C \frac{u(y)}{v(y)} \text{ for any two points } x, y \in \mathcal{N}_{i+1}^\delta(\xi) .$$

[4]The definition of L-vanishing on $\mathcal{N}_i^\delta(\xi)$ is the same as in Definition/Proposition 2.20 "L-vanishing" and Proposition 2.26 "L-vanishing and Martin boundary" where we merely had an ideal boundary of M.

Furthermore, the topological and the Martin boundary are homeomorphic and every Martin boundary point is minimal: $\partial M \cong \partial^0_{\mathrm{M}}(M, L) \cong \partial_{\mathrm{M}}(M, L)$*.*

Proof. We start by smoothing $đ$ to $ð$ using the natural regularisation of generalised distance functions 1.34. Then we apply Theorem 1.31 "hyperbolisation of uniform spaces" to M with generalised distance function $ð$. We get a δ-hyperbolic manifold $(M, g' = ð^{-2}g)$ for δ depending on c, ℓ and the bounded geometry constants (via the regularisation). (M, g') is of (σ, ℓ')–bounded geometry with σ and ℓ' depending on ℓ, ε and K by Proposition 1.32 "bounded geometries". For the boundary, we have

$$\partial M \cong \partial_{\mathrm{G}}(M, g')\,.$$

As argued in subsection 1.3.5, the operator L' is weakly coercive and adapted on (M, g') with universal constants. Hence we can apply the hyperbolic boundary Harnack inequality 2.22 to the L'-harmonic functions $ð^{\frac{n-2}{2}}u$ and $ð^{\frac{n-2}{2}}v$ which are directly seen to be L'-vanishing on $\mathcal{N}^{\delta}_i(\xi)$. The additional factors $ð^{\frac{n-2}{2}}$ cancel and we obtain the assertion.

Using Corollary 2.25 "Gromov boundary and Martin boundary" and the transformation behaviour of harmonic functions and Martin functions, we further conclude that

$$\partial M \cong \partial_{\mathrm{G}}(M, g') \cong \partial_{\mathrm{M}}(M, L') \cong \partial_{\mathrm{M}}(M, L)$$

and every Martin boundary point is minimal. □

On Euclidean domains which are uniform with respect to the distance to the boundary, the constants ε, K and ℓ can be dropped. In this case, Aikawa has used a somewhat different approach to prove some remarkable refinements underlining the sharpness of these potential theoretic result.

- In [Aik01], he derives boundary Harnack inequalities for the Laplacian on uniform Euclidean domains *without* imposing a strong barrier condition. However, in his result the Harnack constant depends on the domain D, whereas in the preceding result for the Laplacian it only depends on the parameters $\lambda^d_{-\Delta}$ and c and the dimension.
- In [Aik04], Aikawa even proves that $D \subset \mathbb{R}^n$ is a uniform domain *if and only if* the Laplacian satisfies boundary Harnack inequalities, as long as D is a *John domain* and the *capacity density condition* holds. Similar to Example 1.39 "regularity versus strong barrier", an exterior twisted cone condition is sufficient for the capacity density condition [Aik12].

Examples 2.29*.* We mention two instructive examples where non-uniformity of a domain destroys the existence of boundary Harnack inequalities and where we concretely see how far the topological boundary may deviate from the Martin boundary of the Laplacian.

- In [Anc12], Ancona describes non-uniform Euclidean cones with only one topological point at infinity but with uncountably many minimal Martin boundary points at infinity.

- In [IP94], Ioffe and Pinsky prove that for the non-uniform rotationally symmetric domains $D_f \subset \mathbb{R}^n$ from Examples 1.28 the set of Martin boundary points at infinity is homeomorphic to S^{n-2}.

Finally we notice that this Martin theory on uniform domains reproves classical contour integral formulae for harmonic functions. The Herglotz theorem [Her11, Dur83] states that a function $f > 0$ on the Euclidean unit disk (D, g_{Eucl}) is harmonic, $-\Delta f = 0$, if and only if there is a Radon measure μ_f on S^1 such that

$$f(x) = \int_{S^1} \frac{1 - |x|^2}{|x - y|^2} \, \mathrm{d}\mu_f(y) \, .$$

A direct computation shows that the Martin function for Δ is

$$K_\zeta(x) = \lim_{z \to \zeta} \frac{G(x,z)}{G(0,z)} \sim \frac{1 - |x|^2}{|x - \zeta|^2} \text{ for } x \in D \text{ and } \zeta \in S^1.$$

Hence the Herglotz theorem is a direct consequence of the unique Martin integral representation

$$f(x) = \int_{S^1} K_\zeta(x) \, \mathrm{d}\mu_f(\zeta)$$

on the Martin boundary $\partial_{\mathrm{M}}(D, -\Delta)$ which is in this case canonically homeomorphic to $\partial D = S^1$ by Corollary 2.28 “boundary Harnack inequalities and Martin theory on uniform manifolds”.

Part II

Isoperimetry and Bubbles

Chapter 3

Weighted Linear Isoperimetric Inequalities

The purpose of this chapter is to show Friedrichs and isoperimetric inequalities weighted with a function that decays exponentially slower than the Green's function of the Laplacian. This can be accomplished on a complete Gromov hyperbolic manifold M of bounded geometry as soon as two additional conditions are satisfied:

- M is *visual*, see Definition 1.11. This ensures that there can not be sequences of larger and larger "bubbles" that violate isoperimetric inequalities.
- The Laplacian $-\Delta$ has to be weakly coercive in order to apply Ancona's theory.

The last condition has a well-understood geometric meaning: if we denote by $\lambda_1(M)$ the principal eigenvalue of the Laplacian on M and by $h(M)$ the *Cheeger isoperimetric constant*

$$h(M) = \inf_{A \Subset M} \frac{\text{Area}(\partial A)}{\text{Vol}(A)},$$

then the following are equivalent:

(i) $h(M) > 0$.

(ii) $-\Delta$ is weakly coercive, i.e., $\lambda_1(M) > 0$.

(iii) $\partial_{\mathrm{G}} M$ is uniformly perfect[1].

(i) $\Rightarrow$ (ii) follows from Cheeger's inequality

$$\lambda_1(M) \geqslant \frac{1}{4} h(M)$$

for arbitrary complete manifolds of infinite volume [Che70], the reverse (ii) $\Rightarrow$ (i) was proven by Buser for complete manifolds with a condition implied by bounded geometry [Bus82]. Martínez-Pérez and Rodríguez [MR18] proved the equivalence of (i) and (iii), extending work of Cao [Cao00].

[1] A (quasi)metric space Y is *uniformly perfect*, if there is a constant $S \geqslant 1$ such that the annulus $B_R(y) \setminus B_{R/S}(y)$ is nonempty for every $R > 0$ and $y \in Y$, provided $B_R(y) \neq Y$, see e.g. [Hei01, 11.1] for further details. This is equivalent to the *lower Assouad dimension* of Y being strictly positive.

After sharp characterisations of harmonic measure and Green's function for $-\Delta$, a hierarchically organised cover will enable us to geometrically prove the weighted mesoscale Friedrichs inequality 3.8. This intermediate result features a "mesoscale" approximation of the gradient on balls of a certain size. In section 3.4, we will locally use Poincaré inequalities (implied by bounded geometry) to arrive at several more customary versions of weighted Friedrichs and isoperimetric inequalities.

In section 3.5, we give examples for weight functions and hint at generalisations. Some of these effectively reprove several implications in the equivalence above. We conclude this chapter with a mechanism to change the weight function for Sobolev inequalities in the presence of weighted Friedrichs inequalities for later usage.

Notation In this chapter, (M^n, g) is a δ-hyperbolic, S-visual, complete Riemannian manifold of (σ, ℓ)–bounded geometry. Without loss of generality, since S-visual spaces are S'-visual for any $S' \geqslant S$, we can and will assume

$$S > 4\delta\,.$$

Furthermore, we suppose that the Laplacian $-\Delta$ on M with Green's function G is weakly coercive or, equivalently, $\partial_{\mathrm{G}} M$ is uniformly perfect. The *harmonic measure*, i.e., the Martin measure of the harmonic function 1, with respect to a variable basepoint $x \in M$, will be denoted by μ_x. We fix a basepoint $o \in M$. All constants in this chapter depend only on δ, S, σ, ℓ and the principal eigenvalue of the Laplacian, $\lambda_1(M)$.

3.1 Laplacian and Harmonic Measure

We consider balls in the quasi-metric on the Gromov boundary of M as in subsection 1.1.3,

$$\mathcal{V}_\varrho(\xi) := \mathcal{W}^o_\varrho(\xi) \cap \partial_{\mathrm{G}} M = \{\eta \in \partial_{\mathrm{G}} M \mid (\xi|\eta)_o > \varrho\} \text{ for } \xi \in \partial_{\mathrm{G}} M \text{ and } \varrho \geqslant 0.$$

The main result of this section is that the harmonic measure of $\mathcal{V}_\varrho(\xi)$ can be estimated by the Green's function $G(\cdot, o)$ of the Laplacian in distance ϱ from the basepoint on a geodesic ray towards $\xi \in \partial_{\mathrm{G}} M$. This allows us to interpret results for the Green's function in terms of the harmonic measure and vice versa. Easy facts such as the Harnack inequality and additivity of measures will turn out to be powerful tools in the other picture.

By definition, the harmonic measure satisfies $\mu_x(\partial_{\mathrm{G}} M) = 1$ for any basepoint $x \in M$ and hence $\mu_x(V) \leqslant 1$ for any $V \subset \partial_{\mathrm{G}} M$. On the other hand, a significant contribution to the total measure is already given by the boundary points "nearest to x", or more precisely:

Lemma 3.1 (Lower Bound for Harmonic Measure) *[Pet15, Lemma 5.3] For any point $x = \gamma(t_x)$ on a geodesic ray $\gamma : o \rightsquigarrow \xi \in \partial_{\mathrm{G}} M$,*

$$\mu_x(\mathcal{V}_{t_x}(\xi)) \succcurlyeq 1\,.$$

Proof. First note that for any Borel set $U \subset \partial_{\mathrm{G}} M$, the restricted measure $\mu_o \llcorner U$ represents a positive harmonic function $u_{\mu_o \llcorner U}$ which $-\Delta$-vanishes on $\partial_{\mathrm{G}} M \setminus \bar{U}$ by Proposition 2.26 "L-vanishing and Martin boundary" and satisfies

$$u_{\mu_o \llcorner U}(x) = \int_U K^o_\eta(x) \, \mathrm{d}\mu_o(\eta) = \int_U K^x_\eta(x) \, \mathrm{d}\mu_x(\eta) = \mu_x(U) \, .$$

Here K^o and K^x denote the Martin functions with respect to the basepoint o and x, respectively.

Set $V = \mathcal{V}_{t_x}(\xi)$. Then,

$$\mu_{\gamma(t)}(\partial_{\mathrm{G}} M \setminus V) \asymp G(\gamma(t), x) \, \mu_x(\partial_{\mathrm{G}} M \setminus V) \leqslant G(\gamma(t), x)$$

by the hyperbolic boundary Harnack inequality 2.22 for $t > t_x$ sufficiently large to form Φ_δ-neighbourhoods where $\mu_\cdot(\partial_{\mathrm{G}} M \setminus V)$ and $G(\cdot, x) \, \mu_x(\partial_{\mathrm{G}} M \setminus V)$, which are comparable near x, $-\Delta$-vanish.

By the quantitative bounds on G from Proposition 2.13 "exponential decay", there is a $t_{1/2} > t_x$ with $t_{1/2} - t_x$ only depending on universal constants such that

$$\mu_{\gamma(t_{1/2})}(\partial_{\mathrm{G}} M \setminus V) \leqslant 1/2 \text{ and hence } \mu_{\gamma(t_{1/2})}(V) = 1 - \mu_{\gamma(t_{1/2})}(\partial_{\mathrm{G}} M \setminus V) \geqslant 1/2 \, .$$

Now we can apply the Harnack inequalities 2.5 to the harmonic function $u_{\mu_o \llcorner V} = \mu_\cdot(V)$ along a Harnack chain of controlled length along γ to see

$$\mu_x(V) \succcurlyeq \mu_{\gamma(t_{1/2})}(V) \succcurlyeq 1 \, .$$

□

Theorem 3.2 (Green's Function and Harmonic Measure) *For any geodesic ray $\gamma : o \rightsquigarrow \xi$ and $t \geqslant \sigma$,*

$$\mu_o(\mathcal{V}_t(\xi)) \asymp G(\gamma(t), o) \, .$$

Proof. Fix γ, $x = \gamma(t)$ and $V = \mathcal{V}_t(\xi)$. We have

$$\mu_o(V) = u_{\mu_o \llcorner V}(o) \asymp G(o, x) \, u_{\mu_o \llcorner V}(x) \leqslant G(o, x)$$

by Martin representation, the hyperbolic boundary Harnack inequality 2.22, Proposition 2.8 "bound for the Green's function" and $u_{\mu_o \llcorner V} \leqslant 1$.

On the other hand,

$$u_{\mu_o \llcorner V}(x) = \mu_x(V) \succcurlyeq 1$$

and thus $\mu_o(V) \asymp G(o, x)$ by Lemma 3.1 "lower bound for harmonic measure". □

Using this representation of the harmonic measure in terms of the Green's function, we directly get from Harnack inequalities for G:

Corollary 3.3 (Harnack Property of the Harmonic Measure)

$$\mu_o(\mathcal{V}_t(\xi)) \asymp \mu_o(\mathcal{V}_s(\eta)) \textit{ for } |t - s| \leqslant \sigma \textit{ and } (\xi|\eta)_o \geqslant t \geqslant 2\sigma.$$

Proof. Given $(\xi|\eta)_o \geqslant t$, the Bonk–Schramm Lemma 1.10 implies that $\gamma(t)$ and $\gamma'(t)$ are in distance at most 4δ from each other, for geodesic rays $\gamma : o \rightsquigarrow \xi$, $\gamma' : o \rightsquigarrow \eta$. Hence we can apply the Harnack inequality to G and translate $G(\gamma(t), o) \asymp G(\gamma'(s), o)$ according to Theorem 3.2 "Green's function and harmonic measure". □

This in turn implies an additivity property of μ_o. For the statement we recall from subsection 1.1.3 that $d_o(\xi,\eta) = \mathrm{e}^{-(\xi|\eta)_o}$ is a quasi-metric on $\partial_{\mathrm{G}} M$, i.e., an ultrametric triangle inequality holds only up to a constant $Q = Q(\delta) = \mathrm{e}^{\delta}$,

$$d(\xi,\zeta) \leqslant Q \max\{d(\xi,\eta), d(\eta,\zeta)\} \text{ for } \xi,\eta,\zeta \in \partial_{\mathrm{G}} M,$$

while all other properties of metrics are still satisfied. The balls $B_r(\xi)$ in this quasi-metric are precisely the sets $\mathcal{V}_{-\ln r}(\xi)$. Note that here $-\ln r$ is usually positive because $\partial_{\mathrm{G}} M$ has diameter $\leqslant 1$ in the quasi-metric and we can restrict our attention to the case $r \leqslant 1$.

Corollary 3.4 (Quasi-Additivity of the Harmonic Measure) *Let $\lambda \geqslant 1$, $B_R(\xi)$ a ball of radius $0 < R \leqslant 1$ in $\partial_{\mathrm{G}} M$ in the quasi-metric described above, and (η_i) a (necessarily finite) family of points in $B_{\lambda R}(\xi)$ with mutual distance at least r for a fixed $r \in (0,R)$ such that the balls $B_{\lambda r}(\eta_i)$ cover $B_{R/\lambda}(\xi)$. Then*

$$\mu_o(B_R(\xi)) \asymp_{\delta,\sigma,\lambda} \sum_i \mu_o(B_r(\eta_i))\,.$$

Proof. First, note that when plugged into μ_o we can ignore factors of λ or Q in the radii of balls by the Harnack property of the harmonic measure 3.3.

The balls $B_{\lambda r}(\eta_i)$ cover all of $B_{R/\lambda}(\xi)$, hence "$\preccurlyeq$" is clear from subadditivity of μ_o.

On the other hand, the balls $B_{r/Q}(\eta_i)$ are disjoint because points in $B_{r/Q}(\eta_i)$ have distance larger than r/Q to any other point η_j by the quasi–triangle inequality. Each such ball is completely contained in $B_{Q\lambda R}(\xi)$, hence

$$\sum_i \mu_o(B_{r/Q}(\eta_i)) \leqslant \mu_o(B_{Q\lambda R}(\xi))\,.$$

The assertion follows from (iterated) application of the Harnack property of the harmonic measure 3.3. □

Remarks 3.5.

- The Harnack property of the harmonic measure 3.3 basically says that the measure μ_o is *doubling*, i.e., $0 < \mu_o(B_{2r}(\xi)) \preccurlyeq \mu_o(B_r(\xi))$ for every ball. Corollary 3.4 "quasi-additivity of the harmonic measure" holds for every doubling measure, with constant only depending on the doubling constant, the quasi-metric constant and λ.

- There is a quantitative version of the doubling property, the *upper regularity dimension* of a measure, and the related *lower regularity dimension*, see e.g. [KLV13, section 3]. The above shows that they are equal to the best constants $\overline{\alpha} \geqslant \underline{\alpha} > 0$ such that

  $$\mathrm{e}^{-\overline{\alpha}\, d(x,y)} \preccurlyeq G(x,y) \preccurlyeq \mathrm{e}^{-\underline{\alpha}\, d(x,y)} \quad \text{for all } x,y \in M \text{ with } d(x,y) \geqslant \sigma$$

  respectively.

- Since only uniformly perfect spaces can carry measures of strictly positive lower regularity dimension (see *loc. cit.*), this implicitly shows that a uniform perfect boundary is necessary for the Laplacian to be weakly coercive.

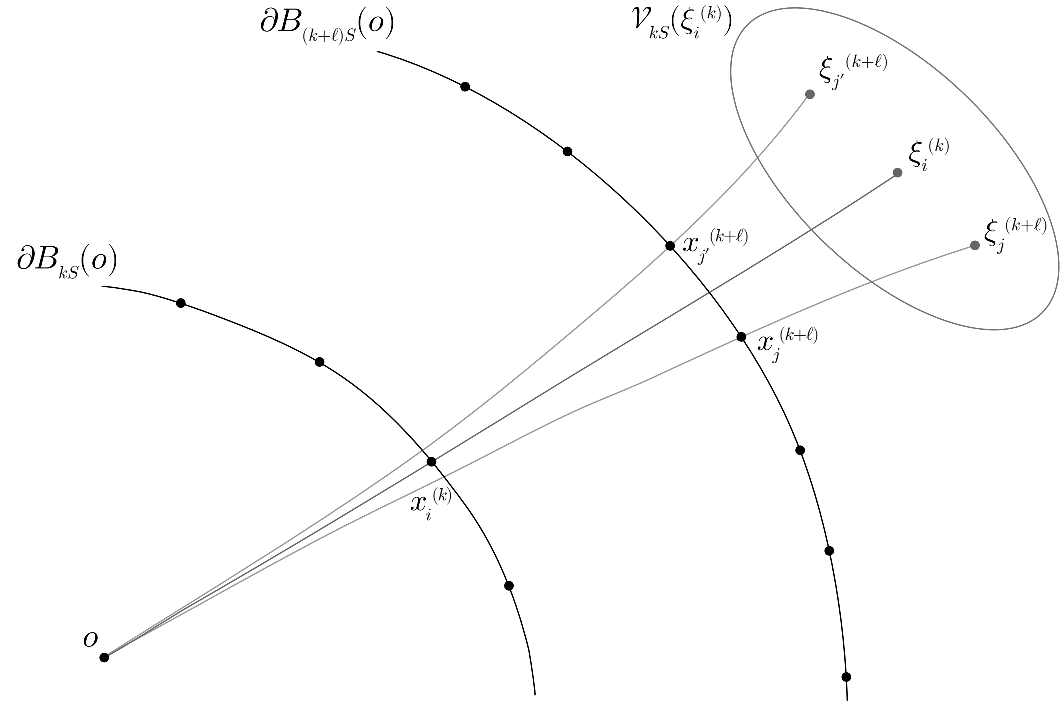


Figure 3.1: Two layers in the onion cover, with $j, j' \in I_{k,\ell,i}$.

3.2 Onion Cover

In this section, we construct a hierarchical cover of M, organised in *layers* like an onion. This cover is the main technical input for the proof of Theorem 3.8 "weighted mesoscale Friedrichs inequality".

We recall the assumption

$$S > 4\delta\,.$$

Theorem 3.6 (Onion Cover) *There is a cover of M by balls $B_i^{(k)} = B_{3S}(x_i^{(k)})$ with $k = 0, 1, 2, \ldots$ enumerating the* layers *and i from finite sets I_k, such that the following properties are satisfied:*

- *the centers of distinct balls have distance at least $S/2$,*
- *$d(o, x_i^{(k)}) = k \cdot S$,*
- *For any pair of indices $k < k + \ell$, there is a partition $I_{k+\ell} = \dot{\bigcup}_{i \in I_k} I_{k,\ell,i}$ such that for any $i \in I_k$,*

$$\sum_{j \in I_{k,\ell,i}} G(x_j^{(k+\ell)}, x_i^{(k)}) \asymp 1$$

and $d(x_i^{(k)}, x_j^{(k+\ell)}) < (\ell + 1)S$ for every $j \in I_{k,\ell,i}$.

Proof. The *layers* $B^{(k)} := \{B_i^{(k)} \mid i \in I_k\}$ are constructed as covers of thickened distance spheres around the basepoint o. More precisely, set $B^{(0)} = \{B_{3S}(o)\}$ and for every $k \geqslant 1$, choose an $S/2$-*net* $\{x_i^{(k)} | i \in I_k\}$ (which is finite since M is proper) in $\partial B^*_{kS}(o) := \partial B_{kS}(o) \cap \{\text{points on geodesic rays } o \rightsquigarrow \infty\}$, i.e., the points have mutual distance at least $S/2$ and $\{B_{S/2}(x_i^{(k)}) \mid i \in I_k\}$ is a cover of ∂B^*_{kS}. Then the $3S$-balls in M around these points cover the annulus $B_{(k+1/2)S}(o) \setminus B_{(k-1/2)S}(o)$: each point in this layer has distance at most S to a geodesic ray (since M is S-visual), the nearest point on this geodesic ray has distance at most $\frac{3}{2}S$ to $\partial B_{kS}(o)$ by the triangle inequality, and from there it is less than another $S/2$ to a point $x_i^{(k)}$. This adds up to a distance less than $3S$.

To each point $x_i^{(k)}$, we assign the endpoint $\xi_i^{(k)} \in \partial_{\mathrm{G}} M$ of a geodesic ray emanating from o and passing through $x_i^{(k)}$, for every k and $i \in I_k$. To construct the partition of $I_{k+\ell}$, we associate each point $\xi_j^{(k+\ell)}$ to a nearest point $\xi_i^{(k)}$, as seen in Figure 3.1, i.e., we assign each index $j \in I_{k+\ell}$ to a set $I_{k,\ell,i}$ such that

$$i \in \operatorname{argmin}_{i' \in I_k} \left\{ \mathrm{e}^{-(\xi_j^{(k+\ell)} | \xi_{i'}^{(k)})_o} \right\} .$$

We check the prerequisites of Corollary 3.4 "quasi-additivity of the harmonic measure":

- For any k, the family $(\xi_i^{(k)})$ is e^{-kS}-separated, i.e., the points have mutual distance at least e^{-kS}: for two points $x_i^{(k)}$ and $x_j^{(k)}$ in the same layer, we can apply the Bonk–Schramm Lemma 1.10. Since $d(x_i^{(k)}, o) = d(x_j^{(k)}, o) = kS$, $d(x_i^{(k)}, x_j^{(k)}) \geqslant S/2 > 2\delta$ and hence $kS - (x_i^{(k)} | x_j^{(k)})_o = d(x_i^{(k)}, x_j^{(k)})/2 > \delta$, the minimum in the Bonk–Schramm Lemma must be attained in $(\xi_i^{(k)} | \xi_j^{(k)})_o$, i.e., $\mathrm{e}^{-(\xi_i^{(k)} | \xi_j^{(k)})_o} > \mathrm{e}^{-kS}$.
- For any k, the family $(\mathcal{V}_{kS-S/2}(\xi_i^{(k)}))$ covers ∂M: for any point $\eta \in \partial_{\mathrm{G}} M$, choose a geodesic ray $\gamma : o \rightsquigarrow \eta$. Then $\gamma(kS)$ is contained in $B_{S/2}(x_i^{(k)})$ for some $i \in I_k$. By the Bonk–Schramm Lemma 1.10, either $(\xi_i^{(k)} | \eta)_o > kS$ (and we are done) or $(\xi_i^{(k)} | \eta)_o \geqslant kS - S/4 - \delta > kS - S/2$.
- For $j \in I_{k,\ell,i}$, the point $\xi_j^{(k+\ell)}$ is contained in $\mathcal{V}_{kS-S/2}(\xi_i^{(k)})$: this is true by the construction of the layers and the previous step.
- $\{\mathcal{V}_{(k+\ell)S-S/2}(\xi_j^{(k+\ell)}) \mid j \in I_{k,\ell,i}\}$ covers $\mathcal{V}_{kS+S/2}(\xi_i^{(k)})$: for every point η in $\mathcal{V}_{kS+S/2}(\xi_i^{(k)})$ there is a set $\mathcal{V}_{(k+\ell)S-S/2}(\xi_j^{(k+\ell)})$ with $j \in I_{k+\ell}$ containing it, but we still need to see $j \in I_{k,\ell,i}$. By the quasi–triangle inequality, $\mathrm{e}^{-(\xi_i^{(k)} | \xi_j^{(k+\ell)})_o} \leqslant \mathrm{e}^{\delta}\, \mathrm{e}^{-(kS+S/2)} < \mathrm{e}^{-\delta}\, \mathrm{e}^{-kS}$ (recall $S > 4\delta$). As the points in ∂M corresponding to the kth layer are e^{-kS}-separated, another application of the quasi–triangle inequality shows that any other of these points has larger distance from $\xi_j^{(k+\ell)}$ and hence $j \in I_{k,\ell,i}$.

Now we are in the position to apply Corollary 3.4 "quasi-additivity of the harmonic measure" with $R = \mathrm{e}^{-kS}$, $r = \mathrm{e}^{-(k+\ell)S}$ and $\lambda = \mathrm{e}^{S/2}$. This yields

$$\mu_o\left(\mathcal{V}_{kS}(\xi_i^{(k)})\right) \asymp_{\delta,S} \sum_{j\in I_{k,\ell,i}} \mu_o\left(\mathcal{V}_{(k+\ell)S}(\xi_j^{(k+\ell)})\right) .$$

With Theorem 3.2 "Green's function and harmonic measure", this is equivalent to

$$G(x_i^{(k)}, o) \asymp_{\delta,S} \sum_{j\in I_{k,\ell,i}} G(x_j^{(k+\ell)}, o) . \tag{3.1}$$

Furthermore, all $j \in I_{k,\ell,i}$ satisfy $(\xi_j^{(k+\ell)}|\xi_i^{(k)})_o > kS - S/2$ and hence $(x_j^{(k+\ell)}|x_i^{(k)})_o > kS - S/2 - 2\delta > (k-1)S$ by the Bonk–Schramm Lemma 1.10. On the one hand, this implies our last assertion $d(x_j^{(k+\ell)}, x_i^{(k)}) < (\ell+1)S$. On the other hand, $(x_j^{(k+\ell)}|o)_{x_i^{(k)}} = d(x_i^{(k)}, o) - (x_j^{(k+\ell)}|x_i^{(k)})_o < S$ and hence $x_i^{(k)}$ lies in controlled distance from a geodesic ray $o \rightsquigarrow x_j^{(k+\ell)}$ by Lemma 1.7 "Gromov product as distance to a geodesic". Together with Harnack inequalities, this permits us to apply Corollary 2.16 "Green's function along hyperbolic geodesics",

$$G(x_j^{(k+\ell)}, o) \asymp G(x_j^{(k+\ell)}, x_i^{(k)})\, G(x_i^{(k)}, o) ,$$

and we can write (3.1) as

$$\sum_{j\in I_{k,\ell,i}} G(x_j^{(k+\ell)}, x_i^{(k)}) \asymp 1 .$$

□

3.3 A Weighted Mesoscale Friedrichs Inequality

Our next step on the road to a weighted isoperimetric inequality is a weighted Friedrichs inequality with an averaged gradient. Since the integral

$$\int_{B_R(x)} |u(x) - u(y)| \,\mathrm{d}V(y)$$

is an approximation of the gradient $|\nabla u|$ near x on scale R, where R is typically significantly larger than S but still finite, we will call this a *mesoscale* gradient. For an elaboration on this idea in the context of Sobolev inequalities on metric measure spaces at different scales cf. [Tes08].

Definition 3.7 (Good Weight Function) On an S-visual δ-hyperbolic Riemannian manifold M of (σ,ℓ)–bounded geometry with basepoint o and uniformly perfect boundary we say that a function $w : M \to (0,\infty)$ is a *good weight function* if

- w is a *Harnack function*, i.e., there is a $H \geqslant 1$ such that

$$H^{-1} \leqslant \frac{w(x)}{w(y)} \leqslant H \quad \text{for } d(x,y) < \sigma\text{, and}$$

- w decays exponentially slower than the Laplacian's Green's function G: there are constants $C, \alpha > 0$ such that $w(x) \geqslant C\, \mathrm{e}^{\alpha d(x,y)}\, G(x,y)\, w(y)$ for any x and y on a geodesic ray emanating from the basepoint o with $d(o,x) \geqslant d(o,y) + \sigma$.

Using the scaffold provided by the onion cover, the proof of a weighted mesoscale Friedrichs inequality is now relatively straightforward. The method of proof is modelled after [DV14, Theorem 5], where an analogue on not necessarily uniform open sets in metric measure spaces is proven which is similar to our result after transition to the quasi-hyperbolic metric, see the remarks in section 3.5.

Theorem 3.8 (Weighted Mesoscale Friedrichs Inequality) *Let w be a good weight function. Then for any $p > 0$, there is an $R > 0$ such that for any measurable function u,*

$$\int_M |u(x)|^p\, w(x)\, \mathrm{d}V(x) \preccurlyeq \int_M \int_{B_R(x)} |u(x) - u(y)|^p\, w(x)\, \mathrm{d}V(y)\, \mathrm{d}V(x)$$

as soon as the left-hand side is finite.

Proof. First note that for any Harnack function v and any ball $B_{3S}(x)$,

$$\int_{B_{3S}(x)} v(y)\, \mathrm{d}V(y) \asymp v(x)$$

because the Harnack property bounds $v(y)$ and bounded geometry gives bounds for the volume. We will use this implicitly several times for $v = w$ and $v = G(\cdot, z)$ for z far from x.

In the onion cover 3.6, we consider two layers k and $k + \ell$. In the following, the constants depend only on the universal constants, the constants for the good weight function and p, but neither on k nor on ℓ. Fix $i \in I_k$, then

$$\begin{aligned}
&\int_{B_i^{(k)}} |u(x)|^p\, \mathrm{d}x \\
&\asymp \int_{B_i^{(k)}} |u(x)|^p\, \mathrm{d}x \underbrace{\sum_{j \in I_{k,\ell,i}} \int_{B_j^{(k+\ell)}} G(y, x_i^{(k)})\, \mathrm{d}y}_{\asymp 1} \\
&\preccurlyeq \int_{B_i^{(k)}} \sum_{j \in I_{k,\ell,i}} \int_{B_j^{(k+\ell)}} \left(|u(x) - u(y)|^p + |u(y)|^p\right) G(y, x_i^{(k)})\, \mathrm{d}y\, \mathrm{d}x \\
&\asymp \int_{B_i^{(k)}} \sum_{j \in I_{k,\ell,i}} \int_{B_j^{(k+\ell)}} |u(x) - u(y)|^p\, G(y, x_i^{(k)})\, \mathrm{d}y\, \mathrm{d}x \\
&\quad + \sum_{j \in I_{k,\ell,i}} \int_{B_j^{(k+\ell)}} |u(y)|^p\, G(y, x_i^{(k)})\, \mathrm{d}y \underbrace{\int_{B_i^{(k)}} \mathrm{d}x}_{\asymp 1} \\
&\asymp \int_{B_i^{(k)}} \sum_{j \in I_{k,\ell,i}} \int_{B_j^{(k+\ell)}} |u(x) - u(y)|^p\, G(y, x_i^{(k)})\, \mathrm{d}y\, \mathrm{d}x \\
&\quad + \sum_{j \in I_{k,\ell,i}} \int_{B_j^{(k+\ell)}} |u(y)|^p\, G(y, x_i^{(k)})\, \mathrm{d}y\,,
\end{aligned}$$

where in the second step, we used the elementary inequalities $|a|^p \leqslant |a-b|^p + |b|^p$ for $0 < p \leqslant 1$ or $|a|^p \leqslant 2^{p-1}\left(|a-b|^p + |b|^p\right)$ for $p > 1$, respectively. Multiplying this with $w(x_i^{(k)})$, summing over all k and i and using the explicit exponential bound $G(x_j^{(k+\ell)}, x_i^{(k)})\, w(x_i^{(k)}) \leqslant \mathrm{e}^{-\alpha(\ell+1)S}\, w(x_j^{(k+\ell)})$, we can make the contribution of the second term on the right-hand side arbitrarily small in relation to the term on the left-hand side and all involved constants by choosing ℓ sufficiently large. Subtracting this small term yields

$$\begin{aligned}
&\int_M |u(x)|^p\, w(x)\,\mathrm{d}x \\
&\preccurlyeq \sum_k \sum_{i\in I_k} \int_{B_i^{(k)}} \sum_{j\in I_{k,\ell,i}} \int_{B_j^{(k+\ell)}} |u(x)-u(y)|^p\, w(x)\,\mathrm{d}y\,\mathrm{d}x \\
&\preccurlyeq \int_M \int_{B_R(x)} |u(x)-u(y)|^p\, w(x)\,\mathrm{d}y\,\mathrm{d}x
\end{aligned}$$

with $R = (\ell+1)S$. □

3.4 Infinitesimal Friedrichs and Isoperimetric Inequalities

The weighted mesoscale Friedrichs inequality 3.8 implies an "infinitesimal" Friedrichs inequality for $W^{1,p}_{\mathrm{loc}}$-functions with the usual gradient on the right-hand side. This can be seen with the help of Poincaré inequalities.

On balls B in Euclidean space $\mathbb{R}^n$, we have the Poincaré inequality

$$\int_B |u-u_B|^p\,\mathrm{d}x \preccurlyeq_{n,p} (\operatorname{diam} B)^p \int_B |\nabla u|^p\,\mathrm{d}x \text{ for } u \in W^{1,p}(B)$$

with $1 \leqslant p < n$ and $u_B := \fint_B u(x)\,\mathrm{d}x := \mathrm{Vol}(B)^{-1}\int_B u(x)\,\mathrm{d}x$ [Hei01, (4.4)].

On balls in a manifold M^n of (σ,ℓ)–bounded geometry, this carries over in the form

$$\int_{B_r(x)} |u-u_{B_r(x)}|^p\,\mathrm{d}V \preccurlyeq_{n,p,\ell} r^p \int_{B_{\ell^2 r}(x)} |\nabla u|^p\,\mathrm{d}V \text{ for } u \in W^{1,p}_{\mathrm{loc}}(M)$$

for $x \in M$ and $0 < r < \sigma/\ell^2$, see [BB11, Proposition 4.16] for the short proof.

To apply this to the weighted mesoscale Friedrichs inequality 3.8, we need Poincaré inequalities on larger balls of radius R. To this end, we use the following extension principle:

Lemma 3.9 (Extension of Poincaré Inequalities) *[BB18, Lemma 4.11] If $A, E \subset M$ are open sets with $\mathrm{Vol}(A\cap E) \geqslant \theta\,\mathrm{Vol}(E)$ for some $\theta > 0$, u is a measurable function and there is a $Q \geqslant 0$ such that*

$$\int_A |u-u_A|^p\,\mathrm{d}x \leqslant Q \quad \text{and} \quad \int_E |u-u_E|^p\,\mathrm{d}x \leqslant Q\,,$$

then

$$\int_{A\cup E} |u-u_{A\cup E}|^p\,\mathrm{d}x \leqslant 4^p\left(1+\theta^{-1/p}\right)^p Q\,.$$

This has to be applied only a finite number of times because bounded geometry implies bounds on the number of smaller balls needed to cover a ball of larger radius.

Lemma 3.10 (Bounded Growth From Bounded Geometry) *[Koi17, Lemma 2] On a complete manifold M^n of (σ, ℓ)–bounded geometry, for every $\varrho \leqslant \sigma/2$ there is a number $\beta = \beta(n, \ell, \varrho/\sigma) \geqslant 1$ such that every ball of radius $k\sigma$ for can be covered by at most β^k balls of radius ϱ.*

Proof. By bounded geometry, there is a number $N = N(n, \ell, \varrho/\sigma)$ such that every ball of radius σ can be covered by at most N balls of radius ϱ. Cover a fixed ball $B_\sigma(x)$ with N such balls $B_\varrho(x_1), B_\varrho(x_2), \ldots, B_\varrho(x_N)$. Then the balls $B_\varrho(x_1), B_\varrho(x_2), \ldots, B_\varrho(x_N)$ already cover $B_{2\sigma-\varrho}(x)$: for every $y \in B_{2\sigma-\varrho}(x)$ there is a $y' \in B_\sigma(x)$ on a geodesic $x \rightsquigarrow y$ with $d(y, y') < \sigma - \varrho$. Then for an x_i with $d(x_i, y') < \varrho$ (which exists by the cover property) we have $d(x_i, y) \leqslant d(x_i, y) + d(y, y') < \sigma$.

As each of these balls $B_\sigma(x_i)$ can be covered by N balls of radius ϱ, we need at most N^2 balls of radius ϱ to cover $B_{2\sigma-\varrho}(x)$.

Iteration of the argument shows that every ball of radius $\varrho + k(\sigma - \varrho)$ can be covered by N^k balls of radius ϱ and we can choose $\beta := N^{\frac{\sigma}{\sigma-\varrho}}$. □

These are all ingredients to get Poincaré inequalities on a larger scale:

Theorem 3.11 (Poincaré Inequality) *On a complete manifold M^n of (σ, ℓ)–bounded geometry, for any $1 \leqslant p < n$ and $R > 0$ there is a Poincaré inequality*

$$\int_{B_R(x)} |u - u_{B_R(x)}|^p \, \mathrm{d}x \preccurlyeq_{n,p,\ell,R/\sigma} \int_{B_{\ell^2 R}(x)} |\nabla u|^p \, \mathrm{d}x \text{ for } u \in W^{1,p}(B_{\ell^2 R}(x))$$

for every $x \in M$.

Proof. In a given ball $B_R(x)$, we choose a $\varrho := \sigma/(2\ell^2)$-net (x_i), i.e., the points x_i have mutual distance at least ϱ and $(B_\varrho(x_i))$ is a cover of $B_R(x)$. By bounded growth from bounded geometry 3.10, there are at most $\preccurlyeq_{n,\ell,R/\sigma} 1$ many elements in the net. Because M is connected, we can assume they are arranged as $x_1, x_2, \ldots$ in such a way that each point has distance less than 2ϱ from one of its predecessors.

Starting with $B_{2\varrho}(x_1)$, we inductively apply the extension of Poincaré inequalities 3.9, where in the jth step, $A = \bigcup_{i<j} B_{2\varrho}(x_i)$, $E = B_{2\varrho}(x_j)$, and for Q we can take a constant multiple of $\int_{B_{\ell^2 R}(x)} |\nabla u|^p \, \mathrm{d}V$. This constant becomes worse in every step, but there is only a controlled number of steps and in every step there is only a controlled factor: x_j has distance less than 2ϱ from a predecessor $x_{i(j)}$ and there is a point x'_j on a geodesic $x_j \rightsquigarrow x_{i(j)}$ such that the ball $B_\varrho(x'_j)$ is contained in $A \cup E$. Hence bounded geometry gives a lower bound on θ, depending only on ℓ. In the end, we have the asserted Poincaré inequality. □

Now we fix an $R > 0$ and choose a $2R$-net (x_i) in M. Using the elementary inequality

$$|u(x) - u(y)|^p \leqslant 2^{p-1} \left(|u(x) - u_{B_{2R}(x_i)}|^p + |u(y) - u_{B_{2R}(x_i)}|^p \right),$$

we have for any Harnack-function w with constant H

$$
\begin{aligned}
&\int_M \int_{B_R(x)} |u(x) - u(y)|^p \, w(x) \, \mathrm{d}V(y) \, \mathrm{d}V(x) \\
&\preccurlyeq_{H,R/\sigma} \sum_i \int_{B_{2R}(x_i)} \int_{B_R(x)} |u(x) - u(y)|^p \, \mathrm{d}V(y) \, \mathrm{d}V(x) \, w(x_i) \\
&\leqslant \sum_i \int_{B_{2R}(x_i)} \int_{B_{2R}(x_i)} |u(x) - u(y)|^p \, \mathrm{d}V(y) \, \mathrm{d}V(x) \, w(x_i) \\
&\leqslant \sum_i 2^p \operatorname{Vol}(B_{2R}(x_i)) \int_{B_{2R}(x_i)} |u(x) - u_{B_{2R}(x_i)}|^p \, \mathrm{d}V(x) \, w(x_i) \\
&\preccurlyeq_{n,p,\ell,R/\sigma} \sum_i \int_{B_{2\ell^2 R}(x_i)} |\nabla u|^p \, \mathrm{d}V \, w(x_i) \\
&\preccurlyeq_{n,\ell,R/\sigma} \int_M |\nabla u|^p \, w \, \mathrm{d}V
\end{aligned}
$$

for $u \in W^{1,p}_{\mathrm{loc}}(M)$, where we used again bounded growth from bounded geometry 3.10 in the last step.

Together with the weighted mesoscale Friedrichs inequality 3.8, this yields:

Corollary 3.12 (Weighted Friedrichs Inequality) *On a δ-hyperbolic visual manifold M^n of bounded geometry with uniformly perfect boundary, we have for any good weight function*[2] *w and $1 \leqslant p < n$*

$$\int_M |u|^p \, w \, \mathrm{d}V \preccurlyeq \int_M |\nabla u|^p \, w \, \mathrm{d}V$$

for every $u \in W^{1,p}_{\mathrm{loc}}(M)$ such that the left-hand side is finite.

The term *Friedrichs inequality* for an inequality of this form is used e.g. in [EO93] and [LV16]. In common usage, it differs from a *Poincaré inequalities* in that these contain an averaged term on the left-hand side and the constant depends on the size of the domain in question. A *Sobolev inequality* usually has a different exponent $q > p$ on the left-hand side, see section 3.6.

BV functions, like the characteristic functions of Caccioppoli sets, are more general than $W^{1,1}_{\mathrm{loc}}$ functions, but there is an 1-Poincaré inequality for them in $\mathbb{R}^n$ as well in the form

$$\int_B |u - u_B| \, \mathrm{d}V \preccurlyeq_{n,\operatorname{Vol} B} \int_B |Du| \text{ for } u \in BV(B)$$

with the total variation of u on the right-hand side [EG15, Theorem 5.10(ii)]. For definitions and more details on BV functions and Caccioppoli sets we refer to section A.1. Either with the same reasoning as above for $W^{1,p}_{\mathrm{loc}}$-functions or by approximation with smooth functions and the $p = 1$ weighted Friedrichs inequality, we have:

[2]See Definition 3.7.

Corollary 3.13 (Weighted Friedrichs Inequality for BV-Functions) *On a complete δ-hyperbolic visual manifold M of bounded geometry with uniformly perfect boundary, we have for any good weight function w*

$$\int_M |u| \, w \, \mathrm{d}V \preccurlyeq \int_M w \, |Du|$$

for every $u \in BV_{\mathrm{loc}}(M)$ such that the left-hand side is finite.

Applied to characteristic functions of Caccioppoli sets, this is:

Corollary 3.14 (Weighted Linear Isoperimetric Inequality) *On a complete δ-hyperbolic visual manifold M of bounded geometry with uniformly perfect boundary, we have for any good weight function w*

$$\int_U w \, \mathrm{d}V \preccurlyeq P_w(U)$$

for every Caccioppoli set $U \subset M$ such that the left-hand side is finite.

3.5 Examples for Weight Functions and Generalisations

Here we will list some examples for good weight functions that can be used in the inequalities in the previous section and hint at possible generalisations.

- The simplest choice is the constant function 1. It is a good weight function because the Green's function decays exponentially. The resulting (unweighted) linear isoperimetric inequality

  $$\operatorname{Vol} U \preccurlyeq \operatorname{Area} \partial U \quad \text{for every } U \text{ of finite volume}$$

  shows that the Cheeger isoperimetric constant $h(M)$ is positive, reproving the implication (ii) $\Rightarrow$ (i) mentioned in the beginning of this chapter.

- It is really necessary that M is non-compact, visual and the boundary uniformly perfect because on a complete Gromov hyperbolic manifold of bounded geometry these conditions are *equivalent* to the unweighted linear isoperimetric inequality, as shown in a recent note by Martínez-Pérez and Rodríguez [MR20, Theorem 5.2].

- A sharper choice of weight function is $G(x, o)^{1-\varepsilon}$ for an arbitrarily small $\varepsilon > 0$, smoothed out in a neighbourhood of o.

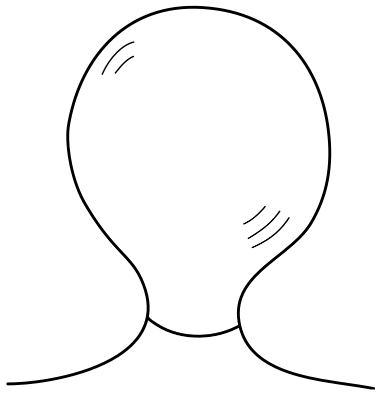
- Any $-\Delta - \varepsilon$-superharmonic Harnack function for $\varepsilon > 0$ is a good weight function: this follows directly from the global maximum principle 1.18 (ii) and Corollary 2.12 "exponentially stronger decay".

- The Harnack property in the previous example is automatically satisfied for positive solutions of $-\Delta w + V w = 0$ for $V \leqslant -\varepsilon$ by the Harnack inequalities 2.5.

- Note that we can substitute any other symmetric adapted operator L with $L1 = 0$ for the Laplacian since we never used any other properties. Since this corresponds to a choice of the second order coefficients, this amounts to taking the Laplacian with respect to a metric g' comparable to g ($g'(X,X) \asymp g(X,X)$ for any tangent vector X).

- The best constant in the $p = 1$ Friedrichs inequality,

$$F_1(M) := \inf_{f \in C_c^\infty(x)} \frac{\int_M |\nabla f| \, \mathrm{d}V}{\int_M |f| \, \mathrm{d}V} ,$$



is equal to the best constant in the linear isoperimetric inequality, the Cheeger constant $h(M)$ [Cha06, Theorem VIII.3.2]. This constant is sensitive to compactly supported perturbations, it can be made arbitrarily small by “bubbling off” as depicted here while preserving the large scale structure and even bounded geometry (but on a smaller scale).

On a manifold with positive $F_1(M)$, set $v = u \, \mathrm{e}^{-\mu d(\cdot,o)}$ for a basepoint $o \in M$, $\mu > 0$ and $u \in C_c^\infty(M)$, then $\nabla v = (\nabla u) \, \mathrm{e}^{-\mu d(\cdot,o)} - u \, \mathrm{e}^{-\mu d(\cdot,o)}$ almost everywhere and hence

$$\begin{aligned} F_1(M) \int_M |u| \, \mathrm{e}^{-\mu d(\cdot,o)} \, \mathrm{d}V = F_1(M) \int_M |v| \, \mathrm{d}V \leqslant \int_M |\nabla v| \, \mathrm{d}V \\ \leqslant \int_M |\nabla u| \, \mathrm{e}^{-\mu d(\cdot,o)} \, \mathrm{d}V + \mu \int_M |u| \, \mathrm{e}^{-\mu d(\cdot,o)} \, \mathrm{d}V \end{aligned}$$

Subtracting the second term on the right-hand side, we see that M admits a $\mathrm{e}^{-\mu d(\cdot,o)}$-weighted Friedrichs (and hence linear isoperimetric) inequality for $0 < \mu < F_1(M)$.

This shows that weight functions for weighted Friedrichs inequalities are a more robust large-scale generalisation of the Cheeger constant. Additionally, the weights can take into account spatial differences, e.g., the weight function can differ in the direction of boundary components with different dimension.

- Much more general, for an arbitrary doubling measure ν on $\partial_{\mathrm{G}} M$, we could replace $G(\gamma(t), o)$ with $\nu(\mathcal{V}_t(\xi))$, for any $t > 0$ and geodesic ray $\gamma : o \rightsquigarrow \xi$. Then a weight function w would have to satisfy

$$\frac{w(\gamma(T))}{w(\gamma(t))} \succcurlyeq \mathrm{e}^{\varepsilon(T-t)} \frac{\nu(\mathcal{V}_T(\xi))}{\nu(\mathcal{V}_t(\xi))} \quad \text{for } T > t > \sigma$$

along every such ray, for a fixed $\varepsilon > 0$, to yield a w-weighted Friedrichs inequality. If $\underline{\dim}_{\mathrm{reg}}(\mu)$ denotes the lower regularity dimension of μ, this is satisfied by the function

$$w(x) = \mathrm{e}^{-(\underline{\dim}_{\mathrm{reg}}(\mu) - \varepsilon) \, d(x,o)} .$$

The lower regularity dimension of a measure never exceeds the *lower Assouad dimension* $\underline{\dim}_{\mathrm{As}}(\partial_{\mathrm{G}} M)$, defined as the supremum of all $d \geqslant 0$ such that there is a $C_d > 0$ such that one needs at least $C_d (R/r)^d$ balls of radius r to cover a ball of radius R in $\partial_{\mathrm{G}} M$, for any $0 < r < R < \operatorname{diam}(\partial_{\mathrm{G}} M)$, see [Fra14] for a detailed exposition. There are doubling measures whose lower regularity dimension comes arbitrarily close to $\underline{\dim}_{\mathrm{As}}(\partial_{\mathrm{G}} M)$ [BG00], hence we can choose

$$w(x) = \mathrm{e}^{-(\underline{\dim}_{\mathrm{As}}(\partial_{\mathrm{G}} M) - \varepsilon)\, d(x,o)}$$

for some $\varepsilon > 0$ as a geometrically motivated weight function. For boundaries with spatially varying dimension, all these concepts can be localised.

Note that this reproves the result of [MR18] that every uniformly perfect space has positive Cheeger constant since uniform perfectness is equivalent to positive lower Assouad dimension and for spaces with such a Gromov boundary we get in particular an unweighted linear isoperimetric inequality.

- The previous step can even be applied if the Gromov boundary is *not* uniformly perfect, i.e., $\underline{\dim}_{\mathrm{As}}(\partial_{\mathrm{G}} M) = 0$. Then Friedrichs and linear isoperimetric inequalities hold with weight function

$$w(x) = \mathrm{e}^{+\varepsilon\, d(x,o)}$$

for any fixed $\varepsilon > 0$.

- On uniform manifolds M with generalised distance function $\eth$, the results in this section can be applied as soon as
  - the hyperbolic unfolding is visual, which is e.g. guaranteed by $\eth \asymp \operatorname{dist}(\cdot, \partial M)$, and
  - the Gromov boundary is uniformly perfect, which is equivalent to the metric boundary ∂M being uniformly perfect, since quasi-symmetric equivalences preserve uniform perfectness [Hei01, Exercise 11.2].

Then translation of the involved quantities under conformal deformation shows in the uniform metric for $p \geqslant 1$ and any good weight function w, e.g., $w = 1$, the weighted Friedrichs inequality

$$\int_M |u|^p\, \eth^{-n}\, w \,\mathrm{d}V \preccurlyeq \int_M |\nabla u|^p \eth^{p-n}\, w \,\mathrm{d}V$$

for every $u \in W^{1,p}_{\mathrm{loc}}(M)$ such that the left-hand side is finite.

- Combining the last few points, one could hope that $\eth^{\underline{\dim}_{\mathrm{As}}(\partial M)}$ is a good weight function. However, the estimates on quasi-symmetry exponents in [BHK01, (3.19)] are not sufficiently sharp to see this directly. This would yield so-called $(p, p-n+\gamma)$-Hardy inequalities in the form

$$\int_M |u|^p\, \eth^{\gamma-n} \,\mathrm{d}V \preccurlyeq \int_M |\nabla u|^p \eth^{p-n+\gamma} \,\mathrm{d}V$$

for every $u \in W^{1,p}_{\mathrm{loc}}(M)$ such that the left-hand side is finite. These inequalities have in fact been studied in the case $1 < p < \infty$ and $\gamma < \min\{\underline{\dim}_{\mathrm{As}}(\partial M), n-1\}$ and under various other conditions that are partly more, partly less restrictive than ours, in [Leh17] and [DV14], where the latter uses the framework of *fractional* Hardy inequalities, an analogue to our mesoscale inequality in section 3.3.

3.6 Local Sobolev Inequalities from Friedrichs Inequalities

Usually, a Friedrichs inequality of type

$$\int |u|^p \preccurlyeq \int |\nabla u|^p \quad \text{for } u \in C^\infty_{\mathrm{c}}$$

for $p \geqslant 1$ is obtained from a Sobolev inequality of type

$$\left(\int |u|^q\right)^{\frac{1}{q}} \preccurlyeq \left(\int |\nabla u|^p\right)^{\frac{1}{p}} \quad \text{for } u \in C^\infty_{\mathrm{c}}$$

for some $q \geqslant p \geqslant 1$ (e.g., $q = \frac{np}{n-p}$ in $\mathbb{R}^n$) by an application of Hölder's inequality, but then the constant in the Friedrichs inequality depends on the volume of the support of u.

In our situation with a global w-weighted Friedrichs inequality, we just need a Sobolev inequality with *some* weight function to go in the other direction and get a w-weighted Sobolev inequality. The proof is modelled after an analogous situation before conformal hyperbolisation [LV16, Theorem 2.1], where a (weighted) Hardy inequality is improved to a (weighted) Hardy–Sobolev inequality in the presence of a Sobolev inequality.

Theorem 3.15 (Weight Transfer) *Assume on a complete manifold M of bounded geometry there are Harnack functions $w, \tilde{w} > 0$ and numbers $p, q \geqslant 1$ such that there are a w-weighted Friedrichs inequality*

$$\int_M |u|^p \, w \, \mathrm{d}V \preccurlyeq \int_M |\nabla u|^p \, w \, \mathrm{d}V \quad \text{for } u \in C^\infty_c(M)$$

and a $\tilde{w}$-weighted Sobolev inequality on an open subset $\Omega \subset M$,

$$\left(\int_\Omega |u|^q \, \tilde{w}^{\frac{q}{p}} \, \mathrm{d}V\right)^{\frac{1}{q}} \preccurlyeq \left(\int_\Omega |\nabla u|^p \, \tilde{w} \, \mathrm{d}V\right)^{\frac{1}{p}} \quad \text{for } u \in C^\infty_c(\Omega)\,.$$

Then there is also a w-weighted Sobolev inequality on Ω,

$$\left(\int_\Omega |u|^q \, w^{\frac{q}{p}} \, \mathrm{d}V\right)^{\frac{1}{q}} \preccurlyeq \left(\int_\Omega |\nabla u|^p \, w \, \mathrm{d}V\right)^{\frac{1}{p}} \quad \text{for } u \in C^\infty_c(\Omega)\,,$$

with constant depending only on the constants of the other inequalities, the Harnack functions, and bounded geometry.

Proof. First we observe that on a manifold of bounded geometry, for any Harnack function W we can find a smooth Harnack function $\mathcal{W} \asymp W$ with $|\nabla \mathcal{W}| \preccurlyeq \mathcal{W}$ by convolution with mollifiers as in the proof of Theorem 1.34 "natural regularisation of generalised distance functions". Without loss of generality we can assume this property for w, $\tilde{w}$ and all their products and powers because they are Harnack functions as well.

Now set $v = u\left(\frac{w}{\tilde{w}}\right)^{1/p}$ for an $u \in C_c^\infty(\Omega)$. Then

$$|\nabla v| \leqslant |\nabla u| \left(\frac{w}{\tilde{w}}\right)^{1/p} + |u| \left|\nabla \left(\frac{w}{\tilde{w}}\right)^{1/p}\right| \preccurlyeq |\nabla u| \left(\frac{w}{\tilde{w}}\right)^{1/p} + |u| \left(\frac{w}{\tilde{w}}\right)^{1/p}$$

and using (in this order) the weighted Sobolev inequality, Minkowski's inequality and the weighted Friedrichs inequality, we have

$$\begin{aligned}
\left(\int_\Omega |u|^q \, w^{\frac{q}{p}} \, \mathrm{d}V\right)^{\frac{1}{q}} &= \left(\int_\Omega |v|^q \tilde{w}^{\frac{q}{p}} \, \mathrm{d}V\right)^{\frac{1}{q}} \\
&\preccurlyeq \left(\int_\Omega |\nabla v|^p \, \tilde{w} \, \mathrm{d}V\right)^{\frac{1}{p}} \\
&\preccurlyeq \left(\int_\Omega |\nabla u|^p \, w \, \mathrm{d}V\right)^{\frac{1}{p}} + \left(\int_\Omega |u|^p \, |\nabla W|^p \tilde{w} \, \mathrm{d}V\right)^{\frac{1}{p}} \\
&\preccurlyeq \left(\int_\Omega |\nabla u|^p \, w \, \mathrm{d}V\right)^{\frac{1}{p}} .
\end{aligned}$$

□

By approximation, these inequalities can be extended to appropriately weighted Sobolev spaces. By classical arguments, see e.g. [BDGM69, Lemma 1], the $p = 1$ w-weighted Sobolev inequality is equivalent to the isoperimetric inequality

$$\left(\int_U w^q \, \mathrm{d}V\right)^{\frac{1}{q}} \preccurlyeq P_w(U) \quad \text{for } U \Subset M \text{ Caccioppoli.}$$

There are several situations where a Sobolev (or isoperimetric) inequality such as in the assumptions of the preceding theorem arises naturally:

Examples 3.16.

- By bounded geometry, we have an unweighted isoperimetric inequality with $q = \frac{n}{n-1}$ on small balls by comparison with Euclidean space, but it is obvious that this generalises to arbitrary Harnack weight functions.

More interesting examples arise for hyperbolic unfoldings of uniform manifolds. Here we typically have Sobolev (or equivalently isoperimetric) inequalities on sets Ω overlapping with the boundary, hence they are unbounded in the hyperbolic unfolding.

- In a closed Riemannian manifold M^n, an (unweighted) isoperimetric inequality with $q = \frac{n}{n-1}$ holds locally (by bounded geometry), hence on the hyperbolic unfolding $(M \setminus \Sigma, \eth^{-2}g)$ of a dense uniform domain $M \setminus \Sigma$, there are $\eth^{n-1}$-weighted isoperimetric inequalities

$$\left(\int_U \eth^n \mathrm{d}V\right)^{\frac{n}{n-1}} \preccurlyeq P_{\eth^{n-1}}(U) \quad \text{for } U \Subset \Omega \text{ Caccioppoli},$$

where Ω is the image of a sufficiently small ball in (M^n, g) and might be unbounded in $(M \setminus \Sigma, \eth^{-2}g)$.

- On a minimal hypersurface $H^n \subset M^{n+1}$, an isoperimetric inequality with $q = \frac{n}{n-1}$ holds locally [HS74], or even globally, if there is no closed minimal hypersurface in the ambient space [Whi09]. This is true even at the singular set, hence we get the same isoperimetric inequalities as in the previous example.

In the presence of a w-weighted Friedrichs inequality, we can now replace the indicated weights with w.

Chapter 4

Bubbles and Mean Convex Exhaustion

Let (M^n, g) be a not necessarily complete Riemannian manifold without boundary—this could be either of the types of manifolds discussed before, an incomplete uniform manifold or a complete Gromov hyperbolic manifold. In this chapter, we will be concerned with non-empty Caccioppoli sets U that are local minimisers of the functional

$$\mathrm{bubb}_{\beta,\phi}(U) := P_\beta(U) - \int_U \phi \, \mathrm{d}V$$

for given measurable functions $\beta : M \to (0, \infty)$ and $\phi : M \to \mathbb{R}$, i.e., each $x \in \partial U$ has a neighbourhood $\varOmega \subset M$ such that

$$\begin{aligned} \mathrm{bubb}_{\beta,\phi}(U, \varOmega) &:= P_\beta(U, \varOmega) - \int_{U \cap \varOmega} \phi \, \mathrm{d}V \\ &= \inf\{\mathrm{bubb}_{\beta,\phi}(V, \varOmega) \mid V \text{ Caccioppoli}, V \Delta U \Subset \varOmega\} \end{aligned}$$

where $V \Delta U$ is the symmetric difference $(V \setminus U) \cup (U \setminus V)$. Such a set U will be called a **(β, ϕ)-bubble**. Here,

$$P_\beta(U, \varOmega) := \sup\left\{ \int_U \operatorname{div} X \, \mathrm{d}V \;\middle|\; X \in C^1_{\mathrm{c}}(\varOmega, T\varOmega), |X| \leqslant \beta \right\}$$

is the weighted perimeter and for short we write $P_\beta(U) := P_\beta(U, M)$. For a brief summary of definitions and results for Caccioppoli sets and functions of bounded variation, see section A.1.

$(1, \phi)$-bubbles are also known as *hypersurfaces of prescribed mean curvature* since ϕ is essentially the mean curvature of the boundary as we will see soon. Their existence and regularity were first investigated by Massari and Miranda [Mas74, Mir73], but Allard's more general results for varifolds of bounded variation [All72] from the same era apply as well. The term "bubble" is from [Gro96, § $5\frac{5}{6}$]. Very recently, our generalised (β, ϕ)-bubbles appeared independently in [CL20].

We will make use of the extra flexibility provided by β to transport results across conformal deformations of (M, g), especially in the case of uniform manifolds and their hyperbolic unfoldings.

4.1 Regularity of Bubbles

We notice that under a conformal deformation $g' = f^2 g$, bubb transforms as

$$\operatorname{bubb}^g_{\beta,\phi}(U) = P^{g'}_{\beta f^{-(n-1)}}(U) - \int_U \phi f^{-n} \mathrm{d}V^{g'} = \operatorname{bubb}^{g'}_{(\beta f^{-(n-1)}, \phi f^{-n})}(U)$$

and (β,ϕ)-bubbles with respect to the metric g are $(\beta f^{-(n-1)}, \phi f^{-n})$-bubbles in the deformed metric g'. In particular, when we have a (β,ϕ)-bubble, we can deform with $f = \beta^{1/(n-1)}$ to obtain a $(1,\phi')$-bubble in the deformed metric, with $\phi' = \beta^{-n/(n-1)}\phi$.

Hence we only need to consider $(1,\phi)$-bubbles for regularity theory because regularity is independent of the metric. Conveniently, they are *almost-minimisers* for quite general ϕ and the regularity of almost-minimisers A.5 applies:

Proposition 4.1 (Bubbles Are Almost-Minimisers) *For a locally bounded measurable function $\phi : M \to \mathbb{R}$, a $(1,\phi)$-bubble U is locally an $(K,1)$-almost-minimiser: for every point in ∂U there are neighbourhood Ω and a constant $K > 0$ (depending on Ω and ϕ) such that*

$$\begin{aligned}\psi(U,x,\varrho) &:= P(U, B_\varrho(x)) - \inf\{P(V, B_\varrho(x)) \mid V \text{ Caccioppoli, } V\Delta U \Subset B_\varrho(x)\} \\ &\leqslant K\varrho^{n-1+1}\end{aligned}$$

for every $x \in \partial U \cap \Omega$ and $\varrho > 0$ with $B_\varrho(x) \subset \Omega$.

Proof. Using the elementary inequality $\inf(A+B) \geqslant \inf A + \inf B$, we have on balls $B_\varrho(x)$ where U is $\operatorname{bubb}_{1,\phi}$-minimising

$$\begin{aligned}\psi(U,x,\varrho) &= \operatorname{bubb}_{1,\phi}(U, B_\varrho(x)) + \int_{U\cap B_\varrho(x)} \phi \,\mathrm{d}V \\ &\quad - \inf_{V\Delta U \Subset B_\varrho(x)} \left(\operatorname{bubb}_{1,\phi}(V, B_\varrho(x)) + \int_{V\cap B_\varrho(x)} \phi\,\mathrm{d}V \right) \\ &\leqslant \operatorname{bubb}_{1,\phi}(U, B_\varrho(x)) - \inf_{V\Delta U \Subset B_\varrho(x)} \operatorname{bubb}_{1,\phi}(V, B_\varrho(x)) \\ &\quad + \int_{B_\varrho(x)} \phi^+ \,\mathrm{d}V - \int_{B_\varrho(x)} \phi^- \,\mathrm{d}V \\ &= \int_{B_\varrho(x)} |\phi| \,\mathrm{d}V \leqslant \sup_\Omega |\phi| \cdot \operatorname{Vol}(B_\varrho(x)),\end{aligned}$$

where $\phi^+ = \max\{\phi, 0\} \geqslant 0$ and $\phi^- = \min\{\phi, 0\} \leqslant 0$.

On sufficiently small Ω, U is $\operatorname{bubb}_{1,\phi}$-minimising and there is an estimate $\operatorname{Vol}(B_\varrho(x)) \leqslant C\varrho^n$ for every ball $B_\varrho(x) \subset \Omega$ so that we can choose $K = C\sup_\Omega |\phi|$. ☐

Remark 4.2. If ϕ is only assumed to be a L^p_{loc} function for some $p > n$, we can estimate the last line as

$$\ldots \leqslant \|\phi\|_{L^p(\Omega)} \operatorname{Vol}(B_\varrho(x))^{1-1/p}$$

by Hölder's inequality and we get (K,λ)-almost minimisers with $\lambda = 1 - n/p$.

Combined with the partial regularity theory for almost-minimisers, this directly gives us the general result:

Theorem 4.3 (Regularity of Bubbles) *For a C^1 function $\beta : M \to (0,\infty)$ and a locally bounded measurable function $\phi : M \to \mathbb{R}$, the boundaries of (β,ϕ)-bubbles are $C^{1,1/2}$-regular submanifolds outside a singular set of Hausdorff dimension at most $n-8$.*

Proof. By conformal deformation as indicated above we get an $(1,\phi')$-bubble with $\phi' = \beta^{-n/(n-1)}\phi$ in the new metric. This metric is only C^1 regular, but that is sufficient for our purposes. By the preceding Proposition, we can locally apply the regularity of almost-minimisers A.5 with $\lambda = 1$. ☐

Remarks 4.4.

- On smooth parts of ∂U and for sufficiently regular (say, smooth) β and ϕ, we can apply the first variation formula for a variational vector field $f \cdot \nu$ where ν is the exterior normal of ∂U and H the scalar-valued mean curvature of ∂U with respect to $-\nu$ (e.g. $H > 0$ for the boundary of Euclidean balls as seen from the interior):

$$0 = \mathrm{bubb}'_{\beta,\phi}(U)(f) = \int_{\partial U} f \cdot (\nabla_\nu \beta + \beta H - \phi)\,\mathrm{d}A\,.$$

  This implies $H = \beta^{-1}(\phi - \nabla_\nu\beta)$ on smooth parts of ∂U, or $H = \phi$ in the special case $\beta \equiv 1$, explaining the term *hypersurface of prescribed mean curvature*.

- On their regular part, $(1,\phi)$-bubbles can locally be written as solutions of the quasi-linear elliptic *prescribed mean curvature equation*. This shows even higher regularity as soon as ϕ is sufficiently regular, e.g., the regular part is smooth if β and ϕ are smooth, see [GT98, Chapter 16].

4.2 Mean Convexity at Infinity

We say a Caccioppoli set U is an *outer-minimising (β,ϕ)-bubble* if it minimises $\mathrm{bubb}_{\beta,\phi}$ among all relatively compact Caccioppoli sets containing U.

For smooth outer-minimising (β,ϕ)-bubbles, we have only the variational characterisation $\mathrm{bubb}'_{\beta,\phi}(U)(f) \geqslant 0$ for positive f, hence the mean curvature is $H \geqslant \beta^{-1}(\phi - \nabla_\nu\beta)$. This indicates that outer-minimising bubbles are a generalisation of *mean convex* sets, i.e., smoothly bounded open sets with positive mean curvature of the boundary. This motivates the following definition.

Definition 4.5 (Mean Convexity at Infinity) We call M *(β,ϕ)–mean convex at infinity* if $\beta : M \to (0,\infty)$ is C^1, $\phi : M \to \mathbb{R}$ locally bounded and measurable and there is an exhaustion of M by compact outer-minimising (β,ϕ)-bubbles U_i, i.e.,

$$\mathrm{bubb}_{\beta,\phi}(U_i) = \inf_{U_i \subset V \Subset M} \mathrm{bubb}_{\beta,\phi}(V), \quad U_i \subset U_{i+1} \quad \text{and} \quad \bigcup_i U_i = M\,.$$

"M is mean convex at infinity" is short for "M is $(1,0)$–mean convex at infinity".

Example 4.6. In Euclidean $\mathbb{R}^n$, $\mathrm{bubb}_{1,r^{-k}}(B_\varrho(0)) \sim \varrho^{n-1} - \int_0^\varrho r^{n-1-k}\,\mathrm{d}r = \varrho^{n-1} - \frac{1}{n-k}\varrho^{n-k}$. For $k > 1$, this has a strict minimum at $\varrho = (n-1)^{-1/(k-1)}$ and in fact we will see in Examples 4.11 that $\mathbb{R}^n$ is $(1, r^{-k})$–mean convex at infinity for any $k > 1$.

The significance of mean convexity at infinity is illustrated by the following consequence:

Proposition 4.7 (Existence of Bubbles) *If M^n is (β,ϕ)–mean convex at infinity and $\psi \leqslant \phi$, there is a solution for the following minimisation problem:*

Given an open set $\Omega \subset M$ with $M \setminus \Omega$ compact and a compact Caccioppoli set $L \subset M$, find a compact Caccioppoli set $E \subset M$ that minimises $\mathrm{bubb}_{\beta,\psi}$ among all compact Caccioppoli sets coinciding with L outside Ω, i.e., $E \setminus \Omega = L \setminus \Omega$ and

$$\mathrm{bubb}_{\beta,\psi}(E) \leqslant \mathrm{bubb}_{\beta,\psi}(F)$$

for every compact Caccioppoli set F with $F \setminus \Omega = L \setminus \Omega$.

Proof. Take an outer-minimising (β,ϕ)-bubble $U \Subset M$ containing $M \setminus \Omega$ and consider a $\mathrm{bubb}_{\beta,\psi}$-minimising sequence of Caccioppoli sets (F_i) with $F_i \setminus \Omega = L \setminus \Omega$. Since U is $\mathrm{bubb}_{\beta,\phi}$-outer-minimising, we have

$$\mathrm{bubb}_{\beta,\phi}(U \cup F_i) \geqslant \mathrm{bubb}_{\beta,\phi}(U)$$

and since $\psi \leqslant \phi$,

$$\begin{aligned}\mathrm{bubb}_{\beta,\psi}(U \cup F_i) &- \mathrm{bubb}_{\beta,\psi}(U)\\ &= \mathrm{bubb}_{\beta,\phi}(U \cup F_i) - \mathrm{bubb}_{\beta,\phi}(U) + \int_{F_i \setminus U} (\phi - \psi)\mathrm{d}V \geqslant 0\,.\end{aligned}$$

If $F_i \not\subseteq U$, we will replace F_i with $\tilde{F}_i := F_i \cap U$. Then subadditivity of the perimeter A.3 and additivity of volume integrals on Borel sets imply

$$\mathrm{bubb}_{\beta,\psi}(\tilde{F}_i) \leqslant \mathrm{bubb}_{\beta,\psi}(F_i) + \mathrm{bubb}_{\beta,\psi}(U) - \mathrm{bubb}_{\beta,\psi}(U \cup F_i) \leqslant \mathrm{bubb}_{\beta,\psi}(F_i)$$

and hence the sequence $(\tilde{F}_i)$ is minimising as well. Because $\bar{U}$ is compact, $\int_U \psi\,\mathrm{d}V$ is finite and hence the sequence $P(\tilde{F}_i) \leqslant P_\beta(\tilde{F}_i)/\min_{\bar{U}} \beta$ is bounded and we can apply compactness of Caccioppoli sets A.2 to find a subsequence converging to a relatively compact $\mathrm{bubb}_{\beta,\psi}$-minimising limit set. ☐

The sets Ω and L in the proposition are used to prescribe a boundary condition for bubbles. This can be done more comfortably in the language of (integral) *currents*, which are in our case essentially (possibly infinite) formal sums of integer-weighted Caccioppoli sets, see section A.2 for statements of results we use. A reader unfamiliar with currents may simply skip forward to Definition 4.9 "calibrations".

The additional difficulty when trying to prove an existence result in the setting of currents is that multiplicities or sums might become infinite. If there were a Caccioppoli set E with $\mathrm{bubb}_{\beta,\psi}(E) < 0$, we could just include more copies of this set to make the total (current analogue of) $\mathrm{bubb}_{\beta,\psi}$ arbitrarily small, hence $\mathrm{bubb}_{\beta,\psi}(E) \geqslant 0$ for all relatively compact Caccioppoli sets E is a necessary additional condition. We will see that in combination with (β,ψ)–mean convexity at infinity a slightly stronger condition is also sufficient:

Theorem 4.8 (Existence of Minimising Currents) *If M^n is (β, ϕ)–mean convex at infinity, $\psi \leqslant \phi$,* $\mathrm{bubb}_{\beta,\psi}(E) \geqslant 0$ *for all relatively compact Caccioppoli sets $E \subset M$, and $\alpha \in (0,1)$, there are* compactly supported *solutions for the* **Homological Plateau Problem** *for $(\beta, \alpha\psi)$-bubbles: given a compactly supported integral $(n-1)$-current R, find a compactly supported integral n-current T with minimal*

$$\mathbf{M}_\beta(R + \partial T) - \mathbf{M}_{\alpha\psi}(T)\,.$$

Proof. Take an outer-minimising (β, ϕ)-bubble $U \Subset M$ containing the support of R and consider a minimising sequence of integral n-currents T_i for the functional

$$\mathbf{M}_\beta(R + \partial T_i) - \mathbf{M}_{\alpha\psi}(T_i)\,.$$

Each T_i is a formal sum of Caccioppoli sets by the decomposition of currents A.6 and we can apply the procedure in the proof of the existence of bubbles 4.7 to each component of each T_i. This shows that we can assume that the T_i are supported in $\bar{U}$. Furthermore, we can assume $\mathbf{M}_\beta(R + \partial T_i) - \mathbf{M}_{\alpha\psi}(T_i) < \mathbf{M}_\beta(R)$, else $T = 0$ is our minimiser. Additionally, $\mathbf{M}_\beta(\partial T_i) \leqslant \mathbf{M}_\beta(R + \partial T_i) + \mathbf{M}_\beta(R)$ by subadditivity of mass A.7 and hence

$$\mathbf{M}_\beta(\partial T_i) - \mathbf{M}_{\alpha\psi}(T_i) < 2\mathbf{M}_\beta(R)\,.$$

The assumption $\mathrm{bubb}_{\beta,\psi} \geqslant 0$ carries over to currents (note that every series involved converges absolutely) and hence

$$(1-\alpha)\mathbf{M}_\beta(\partial T_i) \leqslant \mathbf{M}_\beta(\partial T_i) - \mathbf{M}_{\alpha\psi}(T_i) < 2\mathbf{M}_\beta(R)\,.$$

This yields a universal bound

$$\mathbf{M}(\partial T_i) \leqslant 2\mathbf{M}_\beta(R)/(1-\alpha)/\min_U \beta\,.$$

Linear isoperimetric inequalities[1] applied to each component give a similar bound on $\mathbf{M}(T_i)$, hence compactness of currents A.8 produces a minimising limit current T. □

Notable special cases are:

- In the case $\partial R = 0$, the boundary component $R + \partial T$ is homologous to R.
- For $\phi \geqslant 0$ and $\psi \equiv 0$, the results are β-weighted mass–minimising currents.
- For $\psi \equiv 0$ and $\partial R = 0$, this yields a β-weighted mass–minimising current $R + \partial T$ in the same homology class as R.

[1] We could not localise the appropriate statement in the literature, but here is a quick proof by on-board means: take a smoothly bounded open set $X \Subset M$ with $U \Subset X$, then the methods in the preceding chapter yield a (unweighted) linear isoperimetric inequality for Caccioppoli sets in the hyperbolic unfolding of X with respect to a regularised distance to the boundary $\eth$. On the compact subset $\overline{U}$, this can be upgraded to a linear isoperimetric inequality in the original metric because the (continuous) function $\eth$ is bounded and bounded away from zero.

- For $\psi \equiv 0$ and $\partial R \neq 0$, this is the β-weighted **Plateau Problem** (find a minimal current S spanning a given boundary $\partial S = \partial R$) with the additional restriction that $S = R + \partial T$ is homologous to R. But for $\psi \equiv 0$, $\mathbf{M}_\beta(R+\partial T) - \mathbf{M}_0(T)$ is just the β-weighted mass of S, T is no longer needed and we can drop this restriction while the proof still goes through with minor modifications.

- The classical situation is $\psi \equiv 0$ and $\beta \equiv 1$, where the ordinary mass is minimised.

A useful tool to construct smooth minimising surfaces are *calibrations*, with a famous one-line proof of minimality. One-sided minimality can be proven by so-called sub-/supercalibrations, see [DPP09], where minimality of Simons' cone is proven in an elegant way by sandwiching it between sub-/supercalibrated smooth hypersurfaces. It is surprisingly easy to modify the definition to accommodate (outer-minimising) (β, ϕ)-bubbles:

Definition 4.9 (Calibrations) A (β, ϕ)-*supercalibration* of a Caccioppoli set $U \subset M$ in $\Omega \subset M$ for measurable $\beta : M \to (0, \infty)$ and $\phi : M \to \mathbb{R}$ with $\log \beta$ and ϕ locally bounded is a C^1-vector field ξ on Ω such that

(i) $\xi / \beta = \nu$ is the (exterior) normal vector field on ∂U,

(ii) $\operatorname{div} \xi \geqslant \phi$ on $\Omega \setminus U$, and

(iii) $|\xi| \leqslant \beta$ on Ω.

Lemma 4.10 *A (β, ϕ)-supercalibrated Caccioppoli set U in Ω is an outer-minimising (β, ϕ)-bubble in Ω.*

Proof. $\operatorname{bubb}_{\beta,\phi}(V, \Omega) - \operatorname{bubb}_{\beta,\phi}(U, \Omega) \geqslant \int_{V \setminus U} (\operatorname{div} \xi - \phi) \, \mathrm{d}V \geqslant 0$ for $V \supset U$. □

Examples 4.11.

- On $\Omega = \mathbb{R}^n \setminus \{0\}$ with Euclidean metric, the balls around 0 are (β, ϕ)-supercalibrated by the radial vector field $\xi = \beta e_r$, for $\phi \leqslant \operatorname{div} \xi = \partial_r \beta + \frac{n-1}{r} \beta$.

- One solution of this is $\beta = 1$ and $\phi = \frac{n-1}{r}$, hence Euclidean $\mathbb{R}^n$ is $(1, (n-1)/r)$–mean convex at infinity, and $(1, r^{-k})$–mean convex at infinity for any $k > 1$.

- Another solution on $\Omega = B_1(0) \setminus \{0\}$ is $\beta = \left(\frac{2}{1-r^2}\right)^{n-1}$ and $\phi = \beta \frac{n-1}{r} \frac{1+r^2}{1-r^2}$. In combination with the change under conformal deformation, this shows that the hyperbolic Poincaré ball with metric $\left(\frac{2}{1-r^2}\right)^2 g_{\mathrm{Eucl}}$ is $(1, \phi')$–mean convex at infinity with $\phi' \leqslant \frac{n-1}{2r}(1+r^2)$. Hence hyperbolic space $\mathbb{H}^n$ is $(1, n-1)$–mean convex at infinity.

4.3 Mean Convexity from Isoperimetric Inequalities

The main result of this chapter is that a weighted linear isoperimetric inequality implies weighted mean convexity at infinity as soon as there are sufficiently many *nonlinear* isoperimetric inequalities around and a certain conformal deformation is complete.

Theorem 4.12 (Mean Convexity at Infinity) *Assume M is a manifold with a w-weighted* linear *isoperimetric inequality for a smooth function $w > 0$: there is a $C > 0$ such that*

$$P_w(U) \geqslant C \int_U w \, \mathrm{d}V \tag{4.1}$$

for any Caccioppoli set U such that the right-hand side is finite. Further assume that there is a cover M by open sets (Ω_i) on each of which a w-weighted nonlinear *isoperimetric inequality*

$$P_w(U) \succcurlyeq \left(\int_U w^{\frac{n}{n-1}} \mathrm{d}V \right)^{\frac{n-1}{n}} \qquad \text{for } U \Subset \Omega_i$$

holds, that the cover (Ω_i) has positive Lebesgue number in the conformally deformed metric $\tilde{g} := w^{\frac{2}{n-1}} g$, and that $(M, \tilde{g})$ is complete.

Then M is $(w, \alpha C w)$–mean convex at infinity, for any $\alpha \in (0, 1)$.

Proof. Assume there is *no* exhaustion of M by outer-minimising $(w, \alpha C w)$-bubbles, for a fixed α. Then there is a ball B (assumed smoothly bounded) that is not contained in an outer-minimising $(w, \alpha C w)$-bubble and we can find a sequence of relatively compact Caccioppoli sets (V_i) with $V_{i+1} \supset V_i \supset B$ and

$$\mathrm{bubb}_{w,\alpha C w}(B) \geqslant \mathrm{bubb}_{w,\alpha C w}(V_i) \to \inf_{\substack{V \supset B \\ V \Subset M}} \mathrm{bubb}_{w,\alpha C w}(V) \geqslant 0$$

strictly decreasing. Here we use the linear isoperimetric inequality (4.1) in the form

$$\mathrm{bubb}_{w,\alpha C w}(U) = P_w(U) - \alpha C \int_U w \, \mathrm{d}V \geqslant (1 - \alpha) \, C \, P_w(U) \geqslant 0$$

for any Caccioppoli set $U \Subset M$.

Another application of (4.1) shows

$$\mathrm{bubb}_{w,\alpha C w}(B) \geqslant \mathrm{bubb}_{w,\alpha C w}(V_i) \geqslant (1 - \alpha) C \int_{V_i} w \, \mathrm{d}V \,,$$

hence (V_i) cannot converge to the whole space M, because (4.1) yields in particular $\int_M w \, \mathrm{d}V = \infty$ (for $U = M$).

Neither can the sequence stay bounded, because that would imply the existence of an outer-minimising compact limit set by compactness of Caccioppoli sets A.2, contradicting the assumption.

The last resort for the V_i is to grow tentacles reaching for infinity, but we still get a limit set V_∞ from compactness of Caccioppoli sets A.2 that is $\mathrm{bubb}_{w,\alpha C w}$-minimising on every $\Omega \Subset M \setminus B$. Uniform upper bounds for weighted perimeter and volume carry over.

Using the subsequent lower bound for bubbles 4.13, we can find an infinite sequence of disjoint balls (B_i) (measured with respect to the complete metric $w^{\frac{2}{n-1}} g$ and with radius equal to the Lebesgue number of the cover (Ω_i)) such that $P_w(V_\infty, B_i) \succcurlyeq 1$. This contradicts the upper bound on $P_w(V_\infty)$. $\square$

The following lemma is a generalisation of [Giu84, Prop. 5.14] where an unweighted isoperimetric inequality $P(U) \succcurlyeq \operatorname{Vol}(U)^{\frac{n-1}{n}}$ is applied to area-minimising hypersurfaces in Euclidean space.

Lemma 4.13 (Lower Bound for Bubbles) *In the setting of Theorem 4.12 "mean convexity at infinity", assume U is a minimising $(w, \alpha C w)$-bubble in $\Omega \Subset M$. Then in the metric $\tilde{g} = w^{\frac{2}{n-1}} g$,*

$$\widetilde{P}(U, \widetilde{B}_r(x)) \succcurlyeq r^{n-1}$$

for any $x \in \partial U$ and $r > 0$ such that $\widetilde{B}_r(x) \Subset \Omega_i \cap \Omega$ for some i, with constant only depending on the constants of the isoperimetric inequalities.

Proof. For any $0 < \varrho < r$, the minimiser U satisfies

$$\operatorname{bubb}_{w,\alpha C w}(U, \Omega) \leqslant \operatorname{bubb}_{w,\alpha C w}(U \setminus \widetilde{B}_\varrho(x), \Omega)$$

and hence

$$P_w(U, \widetilde{B}_\varrho(x)) - \alpha C \int_{\widetilde{B}_\varrho(x) \cap U} w \, \mathrm{d}V \leqslant P_w(\widetilde{B}_\varrho(x), U) \,.$$

Using this,

$$\begin{aligned} \operatorname{bubb}_{w,\alpha C w}(\widetilde{B}_\varrho(x) \cap U, \Omega) &= P_w(U, \widetilde{B}_\varrho(x)) + P_w(\widetilde{B}_\varrho(x), U) - \alpha C \int_{\widetilde{B}_\varrho(x) \cap U} w \, \mathrm{d}V \\ &\leqslant 2 P_w(\widetilde{B}_\varrho(x), U) \,. \end{aligned}$$

On the other hand, by the *linear* weighted isoperimetric inequality, we have

$$\operatorname{bubb}_{w,\alpha C w}(\widetilde{B}_\varrho(x) \cap U, \Omega) \geqslant (1 - \alpha) C \, P_w(\widetilde{B}_\varrho(x) \cap U) \tag{4.2}$$

and therefore

$$P_w(\widetilde{B}_\varrho(x), U) \succcurlyeq P_w(\widetilde{B}_\varrho(x) \cap U) \,.$$

If we translate this to the metric $\tilde{g}$ where the perimeter is $\widetilde{P} = P_w$ and the volume $\widetilde{\operatorname{Vol}} = \int w^{\frac{n}{n-1}} \mathrm{d}V$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\varrho} \widetilde{\operatorname{Vol}}(\widetilde{B}_\varrho(x) \cap U) = \widetilde{P}(\widetilde{B}_\varrho(x), U) \succcurlyeq \widetilde{P}(\widetilde{B}_\varrho(x) \cap U) \succcurlyeq \widetilde{\operatorname{Vol}}(\widetilde{B}_\varrho(x) \cap U)^{\frac{n-1}{n}} \,,$$

for almost every ϱ, where the first step is the coarea formula [Giu84, Theorem 1.23] and the last step the *nonlinear* isoperimetric inequality, both in the metric $\tilde{g}$. Integration yields

$$\widetilde{\operatorname{Vol}}(\widetilde{B}_r(x) \cap U) \succcurlyeq r^n \,.$$

Now we can apply the same line of reasoning to the $\operatorname{bubb}_{w,-\alpha C w}$-minimising set $M \setminus U$. The only step that differs is (4.2) where we only need $w \geqslant 0$ (or can apply the linear isoperimetric inequality to get a constant $(1 + \alpha) C$ on the right-hand side). We end up with

$$\widetilde{\operatorname{Vol}}(\widetilde{B}_r(x) \setminus U) \succcurlyeq r^n \,.$$

Now we can apply a standard consequence of the Euclidean-style isoperimetric inequality, see [Giu84, Corollary 1.29],

$$\widetilde{P}(U, \widetilde{B}_r(x)) \succcurlyeq \min \left\{ \widetilde{B}_r(x) \cap U, \widetilde{B}_r(x) \setminus U \right\}^{\frac{n-1}{n}} \,,$$

to conclude

$$P_w(U, \widetilde{B}_r(x)) = \widetilde{P}(U, \widetilde{B}_r(x)) \succcurlyeq r^{n-1}\,.$$

□

Remarks 4.14.

(i) The assumptions in Theorem 4.12 "mean convexity at infinity" are sufficient to guarantee the existence of solutions for the Homological Plateau Problem for $(w, \alpha C w)$-bubbles as described in Theorem 4.8 "existence of minimising currents".

(ii) The proof still works for a different weight function on the right-hand side of (4.1), as long as it is nonnegative and somewhere strictly positive. But in that situation one does not get the appropriate *nonlinear* isoperimetric inequalities from weight transfer 3.15 so easily.

(iii) The trivial "(1,0)-weighted isoperimetric inequality" $P(U) \geqslant 0$ is not sufficient to get $(1,0)$–mean convexity as there are complete manifolds, even of bounded geometry and hence with local nonlinear isoperimetric inequalities, that are not mean convex at infinity. Take for example an infinite tube with diameter shrinking towards infinity, but always larger than a positive constant.

(iv) The previous example also shows that our notion of mean convexity at infinity is sometimes stronger than Gromov's *thickness at infinity* [Gro14, 2.1] which excludes *noncompact* locally area-minimising solutions of the Plateau problem with compact prescribed boundary and is satisfied for all complete manifolds of bounded geometry.

(v) On the other hand, the interior of a compact n-manifold with mean convex boundary is mean convex at infinity, but not thick at infinity (move a small $(n-2)$-sphere close to the boundary).

(vi) Complete visual Gromov hyperbolic manifolds of bounded geometry with uniformly perfect boundary are always $(1, C)$–mean convex at infinity for some $C > 0$ since they admit a $(1, 1)$-weighted linear isoperimetric inequality (see the introduction of chapter 3) and the nonlinear isoperimetric inequalities are provided by bounded geometry.

(vii) If a complete manifold (M, g) is $(w, \alpha C w)$–mean convex at infinity, the conformal deformation $(M, w^{\frac{2}{n-1}} g)$ is $(1, \alpha C w^{-\frac{1}{n-1}})$–mean convex at infinity because outer-minimising $(w, \alpha C w)$-bubbles in (M, g) become outer-minimising $(1, \alpha C w^{-\frac{1}{n-1}})$-bubbles in $(M, w^{\frac{2}{n-1}} g)$. This is exactly what we wish to show for weights w resulting from a conformal deformation of a minimal hypersurface with an eigenfunction of the conformal Laplacian.

Chapter 5

Application to the Conformal Laplacian

In this chapter, we will apply the previous results in the area of scalar curvature geometry. Here, the *conformal Laplacian* on an n-dimensional manifold (M^n, g) with $n \geqslant 3$,

$$L = -\Delta + \frac{n-2}{4(n-1)} \operatorname{Scal},$$

plays a major role because the scalar curvature of the conformally deformed metric $\tilde{g} = u^{\frac{4}{n-2}} g$, for a positive function $u \in C^\infty(M)$, is given by [Bes87, (1.161a)]

$$\operatorname{Scal}^{\tilde{g}} = 4\frac{n-1}{n-2} u^{-\frac{n+2}{n-2}} L^g u\,. \tag{5.1}$$

This fact can also be expressed in form of the transformation rule for the conformal Laplacian,

$$L^{\tilde{g}} v = u^{-\frac{n+2}{n-2}} L^g (u \cdot v) \quad \text{for } v \in C^\infty(M). \tag{5.2}$$

We want to conformally deform certain metrics on uniform manifolds with an eigenfunction of the conformal Laplacian in such a way that we obtain a metric of positive or vanishing scalar curvature that is mean convex at infinity. For that purpose, we start with a general recipe incorporating many of the results of the previous chapters. We first apply this in the easily accessible setting of the singular Yamabe problem for vanishing scalar curvature and then on Smale hypersurfaces, a class of examples for singular area-minimisers.

5.1 Putting It All Together

For the applications, we want to conformally deform a manifold (M^n, g), which is uniform with respect to a generalised distance function $\eth$, to $\tilde{g} = u^{\frac{4}{n-2}} g$, where u is a solution of the equation $Lu = 0$ for a natural Schrödinger operator L, typically the (shifted) conformal Laplacian. By the transformation rules in subsection 1.3.5, this is equivalent to $v = u\, \eth^{\frac{n-2}{2}}$ being a solution of $L'\, v = 0$, where the transformed operator L' is given by

$$L' v = \eth^{\frac{n+2}{2}} L(\eth^{-\frac{n-2}{2}} v)\,.$$

Note in passing that if L is the conformal Laplacian for g, L' is exactly the conformal Laplacian for the conformally deformed quasi-hyperbolic metric $g' = \eth^{-2} g$ by (5.2).

Seen from the perspective of the metric g',

$$\tilde{g} = u^{\frac{4}{n-2}} g = v^{\frac{4}{n-2}} g' .$$

Comparing this with Remark 4.14(vii), a w-weighted linear isoperimetric inequality for $w^{\frac{2}{n-1}} = v^{\frac{4}{n-2}} \Leftrightarrow w = v^{2\frac{n-1}{n-2}}$ in the quasi-hyperbolic metric g' is the first step to proving that $(M, \tilde{g})$ is mean convex at infinity using Theorem 4.12 "mean convexity at infinity".

Step 1: Linear Isoperimetric Inequality Corollary 3.14 "weighted linear isoperimetric inequality" provides us with this linear isoperimetric inequality, given (M, g') we can show that w is a good weight function, i.e., decays at a quantitatively slower rate towards infinity than the Green's function $G_0'(\cdot, o)$ of the Laplacian L_0' on the hyperbolic unfolding (M, g'), where o is an arbitrary fixed basepoint. Note that this also requires that (M, g') is Gromov hyperbolic, visual, of bounded geometry and that L_0' is weakly coercive, i.e., $\partial_{\mathrm{G}}(M, g') \cong \partial M$ is uniformly perfect, as mentioned in the introduction of chapter 3.

Let μ be the Martin measure of the L_0'-harmonic constant function 1 on (M, g'), with respect to the basepoint o. Then we can set v as the unique normalised L'-harmonic function with Martin measure μ. From Theorem 3.2 "Green's function and harmonic measure",

$$\mu(\mathcal{V}_t(\xi)) \asymp G_0'(\gamma(t), o)$$

along a geodesic ray $\gamma : o \rightsquigarrow \xi \in \partial_{\mathrm{G}}(M, g')$, for t sufficiently large. Hence the ray expansion in section 2.4 reads as

$$v(x_k) \asymp G'(x_k, o) \left(\mu(U_0) + \sum_{i=0}^{k-1} \frac{\mu(U_i)}{G'(x_i, o)^2} + \frac{G_0'(x_k, o)}{G'(x_k, o)^2} \right)$$

with $\mu(U_0) \leqslant 1$ and $\mu(U_i) \preccurlyeq G_0'(x_i, o)$ for $1 < i < k$.

Now assume we have the equation

$$G_0'(\gamma(T), \gamma(t)) \succcurlyeq G'(\gamma(T), \gamma(t))^{2\frac{n-1}{n} - \varepsilon'} \quad \text{for } T > t > \sigma, \tag{$\star'$}$$

for some small $\varepsilon' > 0$ and with constant independent of γ. By Corollary 2.16 "Green's function along hyperbolic geodesics", this implies

$$\frac{G_0'(\gamma(T), o)}{G'(\gamma(T), o)^2} G'(\gamma(T), \gamma(t))^{\varepsilon'} \succcurlyeq \frac{G_0'(\gamma(t), o)}{G'(\gamma(t), o)^2} \quad \text{for } T > t > \sigma$$

and using Proposition 2.13 "exponential decay", all summands in the ray expansion are exponentially suppressed in comparison to the last one. Hence we have

$$v(\gamma(t)) \asymp \frac{G_0'(\gamma(t), o)}{G'(\gamma(t), o)} \quad \text{for } t > \sigma$$

or, with another application of Corollary 2.16 "Green's function along hyperbolic geodesics",

$$\frac{v(\gamma(T))}{v(\gamma(t))} \asymp \frac{G_0'(\gamma(T), \gamma(t))}{G'(\gamma(T), \gamma(t))} \quad \text{for } T > t > \sigma.$$

Now our assumption $(\star')$ shows that

$$\left(\frac{v(\gamma(T))}{v(\gamma(t))}\right)^{2\frac{n-1}{n-2}} \asymp \left(\frac{G_0'(\gamma(T),\gamma(t))}{G'(\gamma(T),\gamma(t))}\right)^{2\frac{n-1}{n-2}} \succcurlyeq G_0'(\gamma(T),\gamma(t))^{1-\varepsilon^*} \quad \text{for } T > t > \sigma.$$

for a small $\varepsilon^* = \varepsilon^*(\varepsilon', n) > 0$, hence $w = v^{2\frac{n-1}{n-2}}$ is a good weight function in the sense of Definition 3.7 "good weight function" and the weighted linear isoperimetric inequality 3.14 applies.

Back to Uniform Manifolds We can translate this condition back to operators on (M, g). Using the formulae in subsection 1.3.5, $(\star')$ is equivalent to

$$G(x,y)^{2\frac{n-1}{n}-\varepsilon} \preccurlyeq (\eth(x)\eth(y))^{-\frac{(n-2)^2}{2n}} G_0(x,y) \tag{$\star$}$$

along quasi-hyperbolic geodesic rays, for some $\varepsilon > 0$, where G is the Green's function of L on (M, g) and G_0 the Green's function for the operator

$$L_0 = -\Delta_g + \eth^{\frac{n-2}{2}}(\Delta_g \eth^{-\frac{n-2}{2}})\,.$$

By the formula in the proof of Lemma 1.37 "Schrödinger operators", this operator is mapped to the hyperbolic Laplacian L_0'.

Step 2: Completeness of $\tilde{g}$ $(M, \tilde{g})$ is complete if $v^{\frac{2}{n-2}}$ is not integrable along hyperbolic geodesic rays emanating from the basepoint o. The ray expansion in the first step shows that this is the case if along each geodesic ray γ,

$$G_0'(\gamma(T),\gamma(t)) \succcurlyeq G'(\gamma(T),\gamma(t)) \quad \text{for } T > t > \sigma. \tag{$\star\star'$}$$

Because G' decays exponentially by Proposition 2.13 "exponential decay", this condition already implies $(\star')$. In the uniform picture, this translates as the condition

$$G_0(\gamma(T),\gamma(t)) \succcurlyeq G(\gamma(T),\gamma(t)) \quad \text{for } T > t > \sigma. \tag{$\star\star$}$$

Step 3: Nonlinear Isoperimetric Inequalities Typically our original manifold (M, g) is part of a larger but compact space $\overline{M} = M \dot\cup \partial M$ that is regular enough to support (unweighted) Sobolev inequalities

$$\left(\int_\Omega |u|^{\frac{n}{n-1}}\,\mathrm{d}V\right)^{\frac{n-1}{n}} \preccurlyeq \int_\Omega |\nabla u|\,\mathrm{d}V \quad \text{for } u \in C_c^\infty(\Omega_i)\,. \tag{5.3}$$

on balls Ω_i around points of ∂M. This is the case if $\overline{M}$ is itself a compact manifold, but also if $\overline{M}$ is a singular minimal hypersurface as we will see later. In the metric g', (5.3) is a $\eth^{n-1}$-weighted Sobolev inequality (see Examples 3.16) and $\eth^{n-1}$ is a Harnack function, hence we can use the w-weighted Friedrichs inequality 3.12 that we obtain alongside the linear isoperimetric inequality in Step 1 and weight transfer 3.15 to get the proper w-weighted Sobolev and hence nonlinear isoperimetric inequality.

For compact $\overline{M}$, we can choose a finite subcover of these sets (Ω_i) with positive Lebesgue number. The Lebesgue number is still positive in the metric $\tilde{g}$ because the assumptions in the previous step imply that the conformal factor $u^{\frac{2}{n-2}} = \eth^{-2} v^{\frac{2}{n-2}}$ explodes towards ∂M on all quasi-hyperbolic geodesic rays and hence is bounded from below.

This proves the following intermediate technical result:

Theorem 5.1 (From Green's Function Bounds to Mean Convexity at Infinity) *On a uniform manifold (M, g) of $\eth$–bounded geometry with uniformly perfect boundary and visual hyperbolic unfolding, assume that equation $(\star\star)$ holds along quasi-hyperbolic geodesic rays emanating from a basepoint o, where G and G_0 are the Green's functions of an adapted Schrödinger operator L admitting a strong barrier and of the operator L_0 above, respectively. Further assume that there is a cover with positive Lebesgue number of M such that Sobolev inequalities (5.3) hold on every set in this cover.*

Then there is a solution u of $Lu = 0$ on M such that the metric $u^{\frac{4}{n-2}} g$ is complete and mean convex at infinity.

The remainder of this chapter is devoted to proving these assumptions in two sorts of applications.

5.2 The Singular Yamabe Problem

For a compact manifold (M^n, g) and a closed subset $\Sigma \subset M$, we might ask: Is there a complete metric of constant scalar curvature on $M \setminus \Sigma$ that is conformally equivalent to g? Is there such a metric that is mean convex at infinity?

In general, this amounts to solving the nonlinear equation (5.1) for a prescribed constant $\mathrm{Scal}^{\tilde{g}}$. We will content ourselves with the *linear* case $\mathrm{Scal}^{\tilde{g}} \equiv 0$. This was studied before in [Del92] and [MM92] for submanifolds Σ of dimension at most $\frac{n-2}{2}$, but without reference to mean convexity. To get a complete metric of vanishing scalar curvature on $M \setminus \Sigma$, it is necessary that (M, g) has positive Yamabe invariant or, equivalently, is conformally equivalent to a metric of positive scalar curvature [Del92, Theorem 3]. We refer to these articles for more context.

With the methods from the preceding section, we can prove the following general result:

Theorem 5.2 (Metrics of Zero Scalar Curvature) *If a closed manifold (M^n, g) has positive scalar curvature, $\Sigma \subset M$ is uniformly perfect, $M \setminus \Sigma$ is uniform with respect to a smoothing $\eth$ of $đ = \mathrm{dist}(\cdot, \Sigma)$, and $\max\left\{\eth^{\frac{n-2}{2}}(\Delta \eth^{-\frac{n-2}{2}}), 0\right\} \to 0$ uniformly towards Σ, then g is conformally equivalent to a complete metric of zero scalar curvature on $M \setminus \Sigma$ that is mean convex at infinity.*

Proof. First we show that the conformal Laplacian $L = -\Delta + V := -\Delta + \frac{n-2}{4(n-1)} \mathrm{Scal}$ has a $\eth$-strong barrier on $M \setminus \Sigma$. To this end, we compare it with the operator $L_0 := -\Delta + E := -\Delta + \eth^{\frac{n-2}{2}}(\Delta \eth^{-\frac{n-2}{2}})$ corresponding to the Laplacian on the hyperbolic unfolding. L_0 has a strong barrier since Σ is uniformly perfect (see the introduction of chapter 3), hence by the characterisation after Definition 1.38 “strong barrier”, there is an $\varepsilon > 0$ such that we have the Hardy inequality

$$\int_{M\setminus\Sigma} \left(|\nabla u|^2 + E\, u^2\right) \mathrm{d}V \geqslant \varepsilon \int_{M\setminus\Sigma} u^2\, \eth^{-2}\, \mathrm{d}V \quad \text{for every } u \in C_c^\infty(M\setminus\Sigma)\,.$$

Because V is bounded from below by a positive constant while E is bounded from above, there is an $0 < \alpha \leqslant 1$ such that $\alpha E \leqslant V$ and

$$\int_{M\setminus\Sigma} \left(|\nabla u|^2 + V\, u^2\right) \mathrm{d}V \geqslant \int_{M\setminus\Sigma} \left(|\nabla u|^2 + \alpha\, E\, u^2\right) \mathrm{d}V \geqslant \alpha\varepsilon \int_{M\setminus\Sigma} u^2\, \eth^{-2}\, \mathrm{d}V$$

for every $u \in C_c^\infty(M\setminus\Sigma)$. This shows that L has a strong barrier.

We proceed with the other assumptions in Theorem 5.1 “from Green’s function bounds to mean convexity at infinity”:

Bound for Green’s Functions There is a neighbourhood U of Σ such that $V > E$ on U, hence the Green’s function G_0 of L_0 is L-superharmonic on U. The global maximum principle 1.18 (ii) together with local bounds on Green’s functions directly implies the condition ($\star\star$).

Sobolev Inequalities On (M, g), there are local Sobolev inequalities by bounded geometry (which is a triviality for compact manifolds) as required.

Hence Theorem 5.1 “from Green’s function bounds to mean convexity at infinity” can be applied to the conformal Laplacian L and we get a conformally equivalent complete metric of zero scalar curvature on $M\setminus\Sigma$ that is mean convex at infinity. □

Remark 5.3. To better understand the condition $\max\left\{\eth^{\frac{n-2}{2}}(\Delta\eth^{-\frac{n-2}{2}}), 0\right\} \to 0$, we calculate

$$\begin{aligned}\eth^{\frac{n-2}{2}}\left(\Delta\eth^{-\frac{n-2}{2}}\right) &= -\frac{n-2}{2}\eth^{\frac{n-2}{2}}\,\mathrm{div}\left(\eth^{-\frac{n+2}{2}}\nabla\eth\right)\\ &= \frac{n-2}{2}\left(-\eth^{-1}\Delta\eth + \frac{n+2}{2}\eth^{-2}|\nabla\eth|^2\right)\,.\end{aligned}$$

In the case $\eth = đ = \mathrm{dist}(\cdot, \Sigma)$, we have $|\nabla đ| = 1$ almost everywhere and a simple calculation shows that $-\Delta đ$ is the mean curvature H of the superlevel sets of $đ$, given they are sufficiently regular. Hence our condition can be interpreted as $H \leqslant -\frac{n+2}{2}đ^{-1}$ asymptotically towards Σ. For $\Sigma = \mathbb{R}^s \subset \mathbb{R}^n$, the mean curvature of the $đ$–superlevel sets is $H = -(n-s)đ^{-1}$ and the condition translates into $s \leqslant \frac{n-2}{2}$. In the following section we will see more formally that this reformulation works for submanifolds as well.

5.2.1 Submanifolds

Let $\Sigma^s \subset M^n$ be a closed embedded submanifold of a complete manifold M. Then we can consider *Fermi coordinates* around Σ: there is a map $\exp^\Sigma : N\Sigma \to M$, sending a neighbourhood of the zero section in the normal bundle $N\Sigma$ diffeomorphically to a neighbourhood of $\Sigma \subset M$ by following geodesic rays perpendicular to Σ. For a fixed point $\xi \subset \Sigma$, we can upgrade this to a map

$$\exp_\xi^\Sigma : T_\xi\Sigma \times N_\xi\Sigma \cong T_\xi M \to M$$

which is a diffeomorphism on a neighbourhood of 0 and maps $T_\xi\Sigma$ to Σ, see e.g. [Cha06, §III.6] for details.

For blow-ups $g^t := t^2 g$ for $t \geqslant 1$, we can consider the pull-back $\tilde{g}^t$ of g^t with the map

$$\phi_t : \mathbb{R}^s \times \mathbb{R}^k \cong (T_\xi\Sigma \times N_\xi\Sigma, g_\xi^t) \overset{\exp_\xi^{g^t,\Sigma}}{\longrightarrow} (M, g^t)\,.$$

This compactly converges to the Euclidean metric in C^3 norm.

The pullback of the distance to the boundary $đ = \operatorname{dist}(\cdot, \Sigma)$ with ϕ_t *is equal* to the distance function $đ = \operatorname{dist}(\cdot, \mathbb{R}^s)$ by construction of Fermi coordinates. Now we could just use $đ$ which is smooth on a neighbourhood of Σ and regularise it only far away from Σ, but also the natural regularisation $\eth$ from Theorem 1.34 "natural regularisation of generalised distance functions" is sufficiently well-behaved for our requirements: it compactly C^3-converges under blowup limits (because the smoothing kernel does) and $\eth = C \cdot đ$ on $\mathbb{R}^n \setminus \mathbb{R}^s$ for some C near to 1 because the regularisation process commutes with all isometries and scalings of $\mathbb{R}^n \setminus \mathbb{R}^s$.

Hence the function $\eth^{\frac{n-2}{2}}\left(\Delta \eth^{-\frac{n-2}{2}}\right)$ on $M \setminus \Sigma$ is *natural* in the sense that its pullback with ϕ_t compactly C^1-converges to its counterpart on $\mathbb{R}^n \setminus \mathbb{R}^s$. Now the important point is that this convergence is *uniform* in $\xi \in \Sigma$ because M and Σ are compact and this yields uniform bounds on metric and Christoffel symbols in Fermi coordinates.

In $\mathbb{R}^n \setminus \mathbb{R}^s$, we have

$$\eth^{\frac{n-2}{2}}(\Delta \eth^{-\frac{n-2}{2}}) = \frac{n-2}{2}\left(s - \frac{n-2}{2}\right)\eth^{-2}\,.$$

This is $\leqslant 0$ for $s \leqslant \frac{n-2}{2}$, hence the uniform convergence shows that for every $\varepsilon > 0$ we can find a neighbourhood U_ε of Σ in M such that $\eth^{\frac{n-2}{2}}(\Delta \eth^{-\frac{n-2}{2}}) < \varepsilon$ on $U_\varepsilon \setminus \Sigma$. Thus we can apply Theorem 5.2 "metrics of zero scalar curvature" to see:

Corollary 5.4 (Zero Scalar Curvature on Complements of Submanifolds) *Let (M^n, g) be a closed manifold with* Scal > 0 *and $\Sigma = \bigcup \Sigma_i$ a disjoint union of finitely many closed submanifolds of M with* $\dim \Sigma_i = s_i$. *If $1 < s_i \leqslant \frac{n-2}{2}$, there is a complete scalar flat metric on $M \setminus \Sigma$ that conformally equivalent to g and mean convex at infinity.*

5.3 Minimal Hypersurfaces

Looking for obstructions against metrics of positive scalar curvature, Schoen and Yau discovered that on any stably minimal (regular) closed hypersurface H^n in a manifold M^{n+1} of positive scalar curvature, the conformal Laplacian has positive principal eigenvalue and hence conformal deformation with its first eigenfunction produces a metric of positive scalar curvature on H.

In dimension $n = 7$ or higher, area-minimising hypersurface still exist (e.g. in prescribed homology classes, or for prescribed boundary in the Plateau problem) as *integral currents*, see section A.2, but they are smooth submanifolds only on the complement of a *singular set* Σ of Hausdorff dimension at most $n-7$. On the complement $H \setminus \Sigma$, there are now *many* positive eigenfunctions of the conformal Laplacian, more precisely they are classified by the Martin theory developed in chapter 2 because the conformal Laplacian has a $\delta_{\langle A\rangle}$-strong barrier with respect to Lohkamp's $\mathcal{S}$-distance $\delta_{\langle A\rangle}$ as a generalised distance function, see [Loh20b, Theorem 2]. For dimensional descent strategies, it is desirable to construct a metric on the complement of the singular set that is mean convex at infinity to inductively find compact area-minimising hypersurfaces of decreasing dimension.

The regular part of a singular minimal hypersurface is a much more intricate object than the complement of submanifolds in a smooth manifold considered in the preceding section. But not all is lost, e.g., it is uniform with respect to $\delta_{\langle A\rangle}$ and there are Sobolev inequalities. We will indicate special cases in which the "automatic" methods from section 5.1 still work, and point out limitations.

5.3.1 Potential Theory on Minimal Hypersurfaces

In this section, we will sketch results from Lohkamp's potential theory on minimal hypersurfaces [Loh18, Loh20a, Loh20b].

Let M^{n+1} be a closed manifold and $H^n \subset M$ (the support of) a compact area-minimising integral current in M. We assume that H has multiplicity one. Then there is a singular set $\Sigma \subset H$ of Hausdorff dimension at most $n-7$ such that $H \setminus \Sigma$ is a smooth embedded submanifold. This follows essentially from regularity of almost-minimisers A.5. We assume that Σ is not empty. The induced Riemannian metric on $H \setminus \Sigma$ is denoted by g.

The $\mathcal{S}$-distance $\delta_{\langle A\rangle}$ introduced in [Loh18] is a generalised distance function on $H \setminus \Sigma$ with the additional property $1/\delta_{\langle A\rangle} \geqslant |A|$, where $|A|$ is the norm of the second fundamental form of $H \setminus \Sigma \subset M$. It is *natural* in the sense of [Loh18, section 1.3], meaning that it behaves like the distance function under scalings, $\delta_{\langle A\rangle_{\lambda H}} = \lambda \delta_{\langle A\rangle_H}$ for $\lambda > 0$, and is preserved under certain limits of minimising hypersurfaces.

$H \setminus \Sigma$ is connected and uniform with respect to $\delta_{\langle A\rangle}$ [Loh18, Theorem 1.8]. As sketched in Examples 1.30, $H \setminus \Sigma$ has $\delta_{\langle A\rangle}$–bounded geometry. The natural regularisation of generalised distance functions 1.34 yields a smooth version $\eth_{\langle A\rangle}$ of $\delta_{\langle A\rangle}$. In this situation, the results reported in subsection 1.3.3 apply: $H \setminus \Sigma$ is Gromov hyperbolic in the quasi-hyperbolic metric $\eth_{\langle A\rangle}^{-2} g$, has bounded geometry, and the Gromov boundary is canonically homeomorphic (even quasi-symmetrically equivalent) to Σ. This concludes the geometric part of the prerequisites for Ancona's theory.

Note that $(H \setminus \Sigma, \eth_{\langle A\rangle}^{-2} g)$ might not be visual. There are converging sequences of minimisers where only the limit set has a singularity at a certain position, but this is already detected by the value of $\delta_{\langle A\rangle}$ on the sequence of minimisers which becomes smaller and smaller in anticipation of the limit singularity. In the quasi-hyperbolic metric, this phenomenon manifests as a protruding finger, possibly ruining visuality, or only a uniform bound on the visuality constant. As far as the author is aware, there are no explicit examples.

For the analytic part, we consider three *natural* Schrödinger operators (in the sense of [Loh20b, section 3.2]) on $H \setminus \Sigma$,

$$\text{the Jacobi field operator:} \qquad L^{\mathrm{J}} := -\Delta - |A|^2 - \operatorname{Ric}_M(\nu,\nu)\,,$$
$$\text{the conformal Laplacian:} \qquad L := -\Delta + V := -\Delta + \frac{n-2}{4(n-1)} \operatorname{Scal}_{H\setminus\Sigma}\,,$$
$$\text{the hyperbolic Laplacian:} \qquad L_0 := -\Delta + E := -\Delta + \eth_{\langle A\rangle}^{\frac{n-2}{2}} (\Delta \eth_{\langle A\rangle}^{-\frac{n-2}{2}})\,.$$

These operators are all adapted and L^{J} has a $\eth_{\langle A\rangle}$-strong barrier, while this is true for L if M has nonnegative scalar curvature [Loh20b, Theorem 2.8], and for L_0 if and only if Σ is uniformly perfect, as seen in section 5.1. In these situations, Ancona's theory from chapter 2 can be applied.

Even for minimal hypersurfaces in manifolds M of positive scalar curvature, the scalar curvature of $H \setminus \Sigma$ does not stay bounded near Σ, it even diverges towards $-\infty$ as $\operatorname{Scal}_H \sim -|A|^2$, which follows from taking traces of the Gauß equation,

$$\operatorname{Scal}_M = \operatorname{Scal}_H + 2\operatorname{Ric}_M(\nu,\nu) + |A|^2 - (\operatorname{tr} A)^2\,.$$

Furthermore, the term E is not well controlled near Σ, even in case Σ is a manifold. Hence we need methods different from those in subsection 5.2.1 using Fermi coordinates.

5.3.2 Smale Hypersurfaces

We work on a special class of area-minimising hypersurfaces that have manifold singularities and product form near these singularities, where the (unique) tangent cones are products of Euclidean space and *regular* tangent cones, i.e., area-minimising hypercones with singular set $\{0\}$.

Definition 5.5 (Smale Hypersurfaces) $H^n \subset M^{n+1}$ is called a *Smale hypersurface* if it is a compact homologically area-minimising hypersurface with singular set $\Sigma = \dot\cup_i \Sigma_i$ for finitely many closed orientable Riemannian manifolds Σ_i of dimension $s_i \leqslant n-7$, such that the pair $H \subset M$ is in a neighbourhood of Σ_i isometric to a neighbourhood of $\Sigma_i \times \{0\}$ in the product $\Sigma_i \times \tilde C_i \subset \Sigma_i \times \mathbb{R}^{k_i+1}$, where each $\tilde C_i^{k_i} \subset \mathbb{R}^{k_i+1}$ is a regular area-minimising cone of dimension $k_i = n - s_i$.

In [Sma00, Theorem B], Smale constructs singular homologically area-minimising hypersurfaces of this form for prescribed Σ_i and $\tilde C_i$. He requires that the cones are strictly stable and strictly minimising, but we do not need this. As far as the author is aware, this is the most general known construction of singular homologically area-minimising hypersurfaces, all known area-minimising hypercones are products of $\mathbb{R}^s$ with regular ones, and all known regular area-minimising hypercones are strictly stable and strictly minimising.

To construct metrics of positive scalar curvature that are mean convex at infinity on Smale hypersurfaces in manifolds of positive scalar curvature, we proceed analogously to the singular Yamabe problem. Nearly every step requires some modifications. The ideas are sketched here and then (i) and (iii) are made precise in the following Lemmas, while all points reappear in the proof of Theorem 5.8 "positive scalar curvature on Smale hypersurfaces".

(i) As a substitute for Fermi coordinates, we can use normal coordinates on Σ_i and the product structure of H near Σ_i.

(ii) A strong barrier for the conformal Laplacian can be found from stability of the minimal hypersurface, as in the classical Schoen-Yau argument for closed minimal hypersurfaces. This heavily uses the properties of $\mathcal{S}$-structures and works for general stably minimal hypersurfaces.

(iii) To show that the hyperbolic Laplacian's Green's function is L-superharmonic, we exploit the product structure of Smale hypersurfaces near the singular set. Similar to the singular Yamabe problem, there is a dimensional restriction, and in order to keep its influence as small as possible, we introduce a *new, optimised generalised distance function* $\eth$ that is constructed with help of the product structure. This does not interfere with results for the $\mathcal{S}$-distance $\delta_{\langle A\rangle}$ or $\operatorname{dist}(\cdot,\Sigma)$ because in our situation, they are all comparable: $\eth \asymp \delta_{\langle A\rangle} \asymp \operatorname{dist}(\cdot,\Sigma)$.

(iv) Local Sobolev inequalities hold on minimal hypersurfaces as already explained in Examples 3.16.

On Smale hypersurfaces, we have the following improved version of the tangent cone approximation in [Loh20a, 4.1] which is uniform in Σ_i.

Lemma 5.6 (Geometrical Freezing) *In the notation of Definition 5.5 "Smale hypersurfaces", for each component Σ_i, open set $U \Subset C_i \setminus \Sigma_i := \mathbb{R}^{s_i} \times (\tilde{C}_i \setminus \{0\})$ with $(\{0\} \times \tilde{C}_i) \cap U \neq \varnothing$ and $\varepsilon > 0$ there is a constant $\tau_{\mathcal{G}} = \tau_{\mathcal{G}}(H, \Sigma_i, U, \varepsilon) \geqslant 1$ such that:*

For every $\tau \geqslant \tau_{\mathcal{G}}$ and $\xi \in \Sigma_i$ there are canonical maps $\iota_\tau^\xi : U_i \to \tau H \setminus \Sigma$ with $\Phi_\tau(\tau(\{\xi\} \times \tilde{C}_i)) \cap \iota_\tau^\xi(U_i) \neq \varnothing$, where Φ_τ is the isometry identifying neighbourhoods of $\tau\Sigma_i$ in τC_i and τH, such that

$$\|(\iota_\tau^\xi)^*(\tau^2 g_{H\setminus\Sigma}) - g_{C_i}\|_{C^k(U_i)} < \varepsilon \quad \textit{and} \quad \|(\iota_\tau^\xi)^*(A_{\tau H\setminus\Sigma}) - A_{C_i}\|_{C^k(U_i)} < \varepsilon$$

for some $k \geqslant 5$ Here $A_{\tau H\setminus\Sigma}$ and A_{C_i} denote the second fundamental form of $\tau H \setminus \Sigma \subset \tau M$ and $C_i \subset \mathbb{R}^n$, respectively.

Proof. This Lemma is just a complicated reformulation of the fact that the pullback of $g_{\tau\Sigma_i} = \tau^2 g_{\Sigma_i}$ with the exponential map $\exp_\xi^\tau : \mathbb{R}^{s_i} \cong \tau T_\xi \Sigma_i \to \tau\Sigma_i$ converges uniformly in ξ to the Euclidean metric in compact C^{k+1} norm for $\tau \to \infty$ because Σ_i is compact. Indeed, we can set

$$\iota_\tau^\xi = \Phi_\tau \circ (\exp_\xi^\tau, \operatorname{Id}_{C_i}/\tau)$$

and the induced metric converges uniformly in C^k norm on U_i while convergence of the second fundamental form uses one derivative more. Uniform bounds on sectional curvatures and derivatives of the curvature tensor ensure uniform convergence in $\xi \in \Sigma_i$. The constant $\tau_{\mathcal{G}}$ depends additionally on H because the neighbourhood of $\Sigma_i \subset H$ that is isometric to a product might be arbitrarily small. ☐

Lemma 5.7 (Customised Distance Function) *On the regular part of a Smale hypersurface $H^n \subset M^{n+1}$ with singular set Σ in a manifold M of positive scalar curvature, there is a generalised distance function $\eth$ and an $\varepsilon > 0$ such that $\eth \asymp \delta_{\langle A \rangle} \asymp \operatorname{dist}(\cdot, \Sigma)$ and*

$$V > E + \varepsilon \eth^{-2} \quad \textit{in a neighbourhood of } \Sigma$$

for the Schrödinger operator potentials

$$V = \frac{n-2}{4(n-1)} \operatorname{Scal}_{H \setminus \Sigma} \quad \textit{and} \quad E = \eth^{\frac{n-2}{2}} (\Delta \eth^{-\frac{n-2}{2}})$$

of the conformal and the hyperbolic Laplacian, as soon as all components Σ_i of the singular set satisfy

$$1 \leqslant \dim(\Sigma_i) < \left(2\sqrt{3 + \frac{1}{n-1}} - 3 \right) (n-1) - 1 \, .$$

Proof. We will construct the function $\eth$ first on tangent cones and then on the total hypersurface. Subscripts denote where the currently constructed version lives.

Step 1: On Tangent Cones Let $\tilde{C}^k \subset \mathbb{R}^{k+1}$ be a regular area-minimising hypercone over $\tilde{S} = \tilde{C} \cap S^k \subset \mathbb{R}^{k+1}$, necessarily with $k \geqslant 7$. Then the cone $C^n := \tilde{C}^k \times \mathbb{R}^s \subset \mathbb{R}^{k+s+1}$ is an area-minimising hypercone in $\mathbb{R}^{n+1}$, $n := k + s$, with singular set $\sigma \cong \mathbb{R}^s$. We want to construct a customised distance function on C.

We denote the distance on C from σ (i.e., the radial coordinate in $\tilde{C}$) by ϱ, the $\tilde{S}$-coordinate by ω and the $\mathbb{R}^s$-coordinate by z. Then the metric on C is

$$g_C = d\varrho^2 + \varrho^2 g_{\tilde{S}} + dz^2 \, .$$

For a general hypersurface $H^n \subset M^{n+1}$ with normal vector ν and second fundamental form A, taking traces of the Gauß equation shows

$$\operatorname{Scal}_M = \operatorname{Scal}_H + 2 \operatorname{Ric}_M(\nu, \nu) + |A|^2 - (\operatorname{tr} A)^2 \, . \tag{5.4}$$

Exploiting the symmetries of C, namely invariance under scalings in ϱ-direction and the action of $O(s) \ltimes \mathbb{R}^s$ on the $\mathbb{R}^s$-factor, we have the following factorisations:

- $|A_C| = |A_{\tilde{C}}| = |A_{\tilde{S}}| \varrho^{-1}$, where $A_{\tilde{S}}$ is the second fundamental form of $\tilde{S}^{k-1} \subset S^k$ (*not* as a hypersurface in C^k).
- As C is stationary ($\operatorname{tr} A_C = 0$), (5.4) shows $\operatorname{Scal}_C = -|A_C|^2 = -|A_{\tilde{S}}|^2 \varrho^{-2}$.

For product solutions $u(\varrho, \omega, z) = u_\varrho(\varrho) u_\omega(\omega) u_z(z)$, the Laplacian factorises as

$$-\Delta_C u = -\left[\left(u_\varrho'' + \frac{k-1}{\varrho} u_\varrho' \right) / u_\varrho + \frac{1}{\varrho^2} (\Delta_{\tilde{S}} u_\omega) / u_\omega + (\Delta_{\mathbb{R}^s} u_z) / u_z \right] u \, . \tag{5.5}$$

To satisfy symmetry requirements, we set $\eth_C = \varrho \cdot \eth_{\tilde{S}}(\omega)$ for a function $\eth_{\tilde{S}}$ on $\tilde{S}$. Putting this together with (5.5), we have

$$\eth_C^{\frac{n-2}{2}} (\Delta_C \eth_C^{-\frac{n-2}{2}}) = \left(\frac{n(n-2)}{4} - (k-1) \frac{n-2}{2} + \eth_{\tilde{S}}^{\frac{n-2}{2}}(\omega) \Delta_{\tilde{S}} \eth_{\tilde{S}}^{-\frac{n-2}{2}}(\omega) \right) \varrho^{-2} \, .$$

Then the condition $V > E$ reads as

$$-\frac{n-2}{4(n-1)}|A_{\tilde S}|^2 > \frac{n-2}{2}\left(s-\frac{n-2}{2}\right) + \eth_{\tilde S}^{\frac{n-2}{2}}(\omega)\Delta_{\tilde S}\eth_{\tilde S}^{-\frac{n-2}{2}}(\omega)\,. \tag{5.6}$$

We want to optimise $\eth_{\tilde S}$ in such a way that this equation holds for s as large as possible. To this end, we define $\eth_{\tilde S}^{-\frac{n-2}{2}}(\omega) := f$, where f is the (positive, normalised) principal eigenfunction of the operator $L^n_{\tilde S} = -\Delta_{\tilde S} - \frac{n-2}{4(n-1)}|A_{\tilde S}|^2$ on the closed manifold $\tilde S$.

As a direct consequence of the stability of the cones C and $\tilde C$, the Jacobi field operator $L^{\mathrm J}_{\tilde C}$ on $\tilde C$ has a nonnegative principal eigenvalue. Using the factorisations above, this shows that the Jacobi field operator $L^{\mathrm J}_{\tilde S}$ on $\tilde S^{k-1} \subset S^k$ has principal eigenvalue $\geqslant -\left(\frac{k-2}{2}\right)^2$, see also [Loh20b, Theorem 4.5]. The variational characterisation of the principal eigenvalue of $L^{\mathrm J}_{\tilde S}$ is

$$\int_{\tilde S} f\, L^{\mathrm J}_{\tilde S} f \,\mathrm dV = \int_{\tilde S}\left(|\nabla f|^2 - |A_{\tilde S}|^2 f^2\right)\mathrm dV \geqslant -\left(\frac{k-2}{2}\right)^2\int_{\tilde S} f^2\,\mathrm dV \text{ for } f\in C^\infty_{\mathrm c}(\tilde S)$$

and hence we have for the operator $L^n_{\tilde S}$

$$\begin{aligned}\int_{\tilde S} f\, L^n_{\tilde S} f\,\mathrm dV &= \int_{\tilde S}\left(|\nabla f|^2 - \frac{n-2}{4(n-1)}|A_{\tilde S}|^2 f^2\right)\mathrm dV\\ &= \int_{\tilde S}\frac{3n-2}{4(n-1)}|\nabla f|^2\,\mathrm dV + \frac{n-2}{4(n-1)}\int_{\tilde S} f\,\tilde J^n_{\tilde S} f\,\mathrm dV\\ &\geqslant -\frac{n-2}{4(n-1)}\left(\frac{k-2}{2}\right)^2\int_{\tilde S} f^2\,\mathrm dV \quad \text{for any } f\in C^\infty_{\mathrm c}(\tilde S).\end{aligned}$$

This shows for the principal eigenvalue $\lambda^n_{\tilde S}$ of $L^n_{\tilde S}$

$$\lambda^n_{\tilde S} \geqslant -\frac{n-2}{4(n-1)}\left(\frac{k-2}{2}\right)^2.$$

Hence for our choice of $\eth_{\tilde S}$, equation (5.6) is satisfied if

$$\begin{aligned} &\lambda^n_{\tilde S} > \frac{n-2}{2}\left(s-\frac{n-2}{2}\right)\\ \Leftarrow\quad & s < \frac{n-2}{2} - \frac{1}{2(n-1)}\left(\frac{k-2}{2}\right)^2\\ \Leftrightarrow\quad & s < \left(2\sqrt{3+\frac{1}{n-1}}-3\right)(n-1)-1 \approx 0.46(n-1)-1\end{aligned}$$

where the approximation holds for n large. Because we assumed strict inequality, there is even enough space to smuggle in an $\varepsilon' > 0$ to get

$$V > E + \varepsilon'\varrho^{-2}\,.$$

Step 2: On the Full Hypersurface Near each component Σ_i, where H is isometric to a product $\Sigma_i \times \tilde{C}_i$, we can just set $\eth_H := \varrho \cdot \eth_{\tilde{S}_i}(\omega)$, where ϱ is the radial coordinate on $\tilde{C}_i$ from the previous step, and in the product metric, $\varrho = \operatorname{dist}(\cdot, \Sigma_i)$. As the continuous function $\eth_{\tilde{S}_i}$ is bounded on the closed manifold $\tilde{S}_i$, this shows $\eth_H \asymp \operatorname{dist}(\cdot, \Sigma)$ near Σ. Far away from Σ, *any* positive smooth continuation of $\eth_H$ will do fine. By geometrical freezing 5.6, the estimates from step 1 carry over in a sufficiently small neighbourhood of Σ, where ε' has to be replaced with an appropriate $\varepsilon > 0$, which is possible because $\varrho \asymp \eth$.

Comparison by geometrical freezing 5.6 and naturality of $\delta_{\langle A \rangle}$ show that also $\eth_H \asymp \delta_{\langle A \rangle}$. Furthermore, note that $\eth_H$ is Lipschitz continuous because $\nabla \eth_H$ is bounded. □

Theorem 5.8 (Positive Scalar Curvature on Smale Hypersurfaces) *Let $H^n \subset M^{n+1}$ be a Smale hypersurfaces, and suppose that M has positive scalar curvature and the singular set $\Sigma = \bigcup \Sigma_i$ a disjoint union of finitely many closed submanifolds of dimension*

$$1 \leqslant \dim \Sigma_i = s_i < \left(2\sqrt{3 + \frac{1}{n-1}} - 3\right)(n-1) - 1 \approx 0.46(n-1) - 1\,.$$

Then there is a complete metric of positive scalar curvature on $H \setminus \Sigma$ that is mean convex at infinity and conformally equivalent to the induced metric.

Here the approximation holds for n large. In the case $n \leqslant 11$, the dimensional condition is automatically satisfied since the singular set has Hausdorff dimension at most $n - 7$.

Proof of Theorem 5.8. We proceed analogously to the proof of Theorem 5.2 "metrics of zero scalar curvature". During this proof, we will exclusively use the customised distance function 5.7, denoted by $\eth$. Note that $\eth \asymp \delta_{\langle A \rangle} \asymp \operatorname{dist}(\cdot, \Sigma)$ ensures that all nice properties for these functions carry over to $\eth$ (with only a loss in constants): $H \setminus \Sigma$ is automatically uniform with respect to $\eth$ and has $\eth$–bounded geometry (because it is uniform with respect to $\delta_{\langle A \rangle}$ and has $\delta_{\langle A \rangle}$–bounded geometry, see subsection 5.3.1), the hyperbolic unfolding is visual (since this is always true for the generalised distance functions comparable to $\operatorname{dist}(\cdot, \Sigma)$, see Proposition 1.33 "visual unfolding"), and the conformal Laplacian has a $\eth$-strong barrier (because it has a $\delta_{\langle A \rangle}$-strong barrier, subsection 5.3.1). Additionally, Σ is uniformly perfect since it is assumed to have positive dimension.

Bound for Green's Functions By the construction of the customised distance function 5.7, we have $V - \varepsilon \eth^{-2} > E$ on a neighbourhood U of Σ, for some $\varepsilon > 0$. We might choose this ε so small that $L^\varepsilon := L - \varepsilon \eth^{-2}$ has a strong barrier. Then the Green's function G_0 of the hyperbolic Laplacian L_0 is L^ε-superharmonic on U and the global maximum principle 1.18 (ii) together with local bounds on Green's functions directly implies the condition ($\star\star$).

Sobolev Inequalities As indicated in Examples 3.16, we get local Sobolev inequalities on H.

Positive Scalar Curvature Now we have checked all the assumptions of Theorem 5.1 “from Green’s function bounds to mean convexity at infinity” and can apply this to the shifted conformal Laplacian L^ε. This yields a solution u of $L^\varepsilon u = 0$ and a metric $\tilde{g} = u^{\frac{4}{n-2}} g$ on $H \setminus \Sigma$ that is complete, mean convex at infinity and has by (5.1) scalar curvature

$$\mathrm{Scal}^{\tilde{g}} = 4\frac{n-1}{n-2} u^{-\frac{n+2}{n-2}} Lu = 4\frac{n-1}{n-2} \varepsilon \eth^{-2} u^{-\frac{4}{n-2}} > 0\,.$$

□

5.3.3 Future Prospects

We hope that the modular structure of this thesis makes it possible to reuse some results and methods in future applications. On the way to a metric of positive scalar curvature that is mean convex at infinity on *arbitrary* minimising hypersurfaces in a Scal > 0 ambience, there are a few obstacles to overcome. Here we list some of them together with suggestions.

- The dimensional restriction in Theorem 5.8 “positive scalar curvature on Smale hypersurfaces” can be traced back to the completeness requirement in Theorem 4.12 “mean convexity at infinity”. If it were possible to remove this restriction and effectively construct $(1, \phi)$-bubbles in incomplete spaces given the provided isoperimetric inequalities (which works at least in highly symmetric examples), we would only need condition $(\star)$ instead of $(\star\star)$ in section 5.1. Explicit calculations show that $(\star)$ holds on minimising tangent cones that are products of regular cones with $\mathbb{R}^s$.

- To prove this condition on more general minimal hypersurfaces than cones, one could employ tangent cone approximations such as the $\mathcal{S}$-freezing in [Loh20b, 3.2] combined with minimal growth stability [Loh20b, 3.3]. This might work for Smale hypersurfaces, but in general, a major problem for a global approach is that these approximations might not be uniform.

- Related to the preceding point is the possibility that the hyperbolic unfolding of a general minimal hypersurface might not be visual, as discussed in subsection 5.3.1, but this condition is necessary even for unweighted linear isoperimetric inequalities to hold, see the second point in section 3.5.

Appendix A

Generalised Hypersurfaces

Here we will summarise some general results for the main two models of generalised hypersurface that appear in this thesis, *Caccioppoli sets* and *currents*.

A.1 Perimeter and Caccioppoli Sets

The theory of Caccioppoli sets was developed by de Giorgi for subsets of Euclidean space. The standard reference [Giu84] only covers this case, but the results hold more generally in Riemannian manifolds [Sim84, §37]. Here we state the more general results from [Sim84] in the simpler language of [Giu84] adapted to manifolds.

In this section (M^n, g) is always a (connected) Riemannian manifold without boundary.

Definition A.1 (BV Functions, Perimeter and Caccioppoli Sets) For a function $f \in L^1_{\mathrm{loc}}(M)$ and a relatively compact open set $\Omega \Subset M$, we define

$$\int_\Omega |Df| := \sup\left\{ \int_\Omega f \operatorname{div} X \,\mathrm{d}V \;\middle|\; X \in C^1_{\mathrm{c}}(\Omega, T\Omega), |X| \leqslant 1 \right\}$$

and say that f has *(locally) bounded variation* if $\int_\Omega |Df| < \infty$ for every $\Omega \Subset M$. The space of functions of locally bounded variation on M is called $BV_{\mathrm{loc}}(M)$.

The *perimeter* of a Borel set $U \subset M$ in $\Omega \Subset M$ is

$$P(U, \Omega) := \int_\Omega |D\chi_U|$$

and Borel sets U of locally finite perimeter (i.e., χ_U has bounded variation) are called *Caccioppoli sets*.

Notice that the localised properties "bounded variation" and "finite perimeter" are independent of the metric, in particular we can locally choose an Euclidean metric and qualitative results from $\mathbb{R}^n$ carry over. An example is the following:

Theorem A.2 (Compactness of Caccioppoli Sets) *Let (U_i) be a sequence of Caccioppoli sets in M such that $P(U_i, \Omega)$ is bounded for every $\Omega \Subset M$. Then a subsequence converges in L^1_{loc} to a Caccioppoli set U with*

$$P(U, \Omega) \leqslant \liminf_{i\to\infty} P(U_i, \Omega) \quad \textit{for every } \Omega \Subset M.$$

Proof. On small, sufficiently regular balls one can apply the Euclidean BV compactness theorem [Giu84, Theorem 1.19]. Using a countable cover of M by such balls and a diagonal argument one gets a subsequence with the required properties. Semicontinuity of the perimeter is proven in [Giu84, Theorem 1.9], the proof applies verbatim on Riemannian manifolds. $\square$

We will also be interested in the weighted perimeter

$$P_\beta(U,\Omega) := \int_\Omega \beta\,|D\chi_U| := \sup\left\{ \int_U \operatorname{div} X\,\mathrm{d}V \;\middle|\; X \in C^1_{\mathrm{c}}(\Omega, T\Omega), |X| \leqslant \beta \right\}$$

for a positive function $\beta \in C^\infty(M)$. Completely analogous to the unweighted case in Euclidean space [Mag12, Lemma 12.22] one can show:

Lemma A.3 (Subadditivity of the Perimeter) *For any positive $\beta \in C^\infty(M)$, Caccioppoli sets E and F, and $\Omega \Subset M$,*

$$P_\beta(E \cup F, \Omega) + P_\beta(E \cap F, \Omega) \leqslant P_\beta(E,\Omega) + P_\beta(F,\Omega)\,.$$

A very flexible approach to regularity of Caccioppoli sets minimising a certain function is via *almost-minimisers*. They include area-minimising Caccioppoli sets, (smooth) submanifolds, and bubbles, as seen in detail in section 4.1.

Definition A.4 (Almost-Minimisers) A Caccioppoli set $U \subset M$ is (K,λ)-*almost-minimising* in $\Omega \subset M$ for constants $K, \lambda > 0$, if

$$\begin{aligned}\psi(U,x,\varrho) &:= P(U, B_\varrho(x)) - \inf\{P(V, B_\varrho(x)) \mid V \text{ Caccioppoli, } V\Delta U \Subset B_\varrho(x)\} \\ &\leqslant K\varrho^{n-1+\lambda} \quad \text{for every } x \in \partial U \cap \Omega \text{ and } \varrho > 0 \text{ with } B_\varrho(x) \subset \Omega.\end{aligned}$$

Here, $V\Delta U$ is the symmetric difference $(V \setminus U) \cup (U \setminus V)$.

The partial regularity theorem for almost-minimisers is a deep result due to Tamanini, extending the work of de Giorgi, Federer, Simons and others on actual minimisers ($K = 0$).

Theorem A.5 (Regularity of Almost-Minimisers) *[Tam82, Theorem 1], [MM84, Theorem 2 in 2.5.4, Theorem in 2.6.4] If a Caccioppoli set $U \subset M^n$ is (K,λ)-almost-minimising in $\Omega \subset M$, the boundary $\partial U \cap \Omega$ is a $C^{1,\lambda/2}$-regular submanifold outside a singular set of Hausdorff dimension at most $n-8$.*

If the boundary can locally be written as a graph of a solution of a quasi-elliptical equation, as in the case of area-minimisers or bubbles, the regularity can be improved further, see [GT98, Chapter 16].

A.2 Currents

Currents are a more general method to represent singular submanifolds. This is a deep subject and we do not even try to give an introduction, but simply state a few results that we need in the only place in this thesis where they occur, Theorem 4.8 "existence of minimising currents". For a thorough introduction, see [Mor09] or [Sim84].

Proposition A.6 (Decomposition of Currents) *In an oriented manifold M^n, each locally integral n-current T can be decomposed into Caccioppoli sets $(U_i)_{i\in\mathbb{Z}}$ with $U_i \supset U_{i+1}$ such that*

$$T = \sum_{i=1}^{\infty} [\![U_i]\!] - \sum_{i=0}^{\infty} [\![M \setminus U_{-i}]\!] \qquad \mathbf{M}_\Omega(T) = \sum_{i=1}^{\infty} \mathrm{Vol}(U_i \cap \Omega) + \sum_{i=0}^{\infty} \mathrm{Vol}(\Omega \setminus U_{-i})$$

$$\partial T = \sum_{i=-\infty}^{\infty} \partial[\![U_i]\!] \qquad \mathbf{M}_\Omega(\partial T) = \sum_{i=-\infty}^{\infty} P(U_i, \Omega) \quad \textit{for all } \Omega \Subset M\,.$$

Proof. There is an integer-valued function θ of locally bounded variation such that T can be written as $T(\omega) = \int_M \theta\,\omega$ for n-forms ω. Then we can set $U_i = \{x \in M | \theta \geqslant i\}$ to obtain the desired Caccioppoli sets. Since M is orientable and we assumed ∂T to be a boundary, this decomposition exists globally. The claimed properties can be proven locally, see [Sim84, 27.6–8] for details. □

We can use this decomposition to define integration of scalar-valued functions over locally integral n-currents and their boundaries: in the setting of the decomposition of currents A.6 and for measurable functions $\beta > 0$ and ϕ on M, we define the *weighted mass*

$$\mathbf{M}_\phi(T) = \sum_{i=-\infty}^{\infty} \int_{U_i} \phi \, \mathrm{d}V \quad \text{and} \quad \mathbf{M}_\beta(\partial T) = \sum_{i=-\infty}^{\infty} \int_M \beta \, |D\chi_{U_i}|\,.$$

Locally integral $(n-1)$-currents which are no global boundaries are at least locally boundaries, see [Sim84, 27.8], and we can modify the definitions above appropriately. One immediately gets:

Lemma A.7 (Subadditivity of Mass) *For locally integral $(n-1)$-currents S and T in M^n,*

$$\mathbf{M}_\beta(S+T) \leqslant \mathbf{M}_\beta(S) + \mathbf{M}_\beta(T)\,.$$

As for Caccioppoli sets, existence of currents is usually proven with a compactness theorem.

Theorem A.8 (Compactness of Currents) *[Sim84, Theorem 27.3] Let (T_i) be a sequence of integral n-currents in M^n such that $\mathbf{M}_\Omega(T_i) + \mathbf{M}_\Omega(\partial T_i)$ is bounded for every $\Omega \Subset M$. Then a subsequence (T_{i_j}) converges in flat norm to an integral current T with*

$$\mathbf{M}_\Omega(T) = \lim_{j\to\infty} \mathbf{M}_\Omega(T_{i_j}) \textit{ and } \mathbf{M}_\Omega(\partial T) \leqslant \liminf_{j\to\infty} \mathbf{M}_\Omega(\partial T_{i_j}) \textit{ for every } \Omega \Subset M\,.$$

Bibliography

[Aik01] H. Aikawa, Boundary Harnack principle and Martin boundary for a uniform domain, *J. Math. Soc. Japan* **53** (2001), 119–145. MR 1800526.

[Aik04] H. Aikawa, Potential-theoretic characterizations of nonsmooth domains, *Bull. London Math. Soc.* **36** (2004), 469–482. MR 2069009. DOI 10.1112/S002460930400308X.

[Aik12] H. Aikawa, Potential analysis on nonsmooth domains—Martin boundary and boundary Harnack principle, in *Complex Analysis and Potential Theory*, *CRM Proc. Lecture Notes* **55**, Amer. Math. Soc., Providence, RI, 2012, pp. 235–253. MR 2986906.

[ALM03] H. Aikawa, T. Lundh, and T. Mizutani, Martin boundary of a fractal domain, *Potential Anal.* **18** (2003), 311–357. MR 1953266.

[All72] W. K. Allard, On the first variation of a varifold, *Ann. of Math.* **95** (1972), 417–491. MR 307015. DOI 10.2307/1970868.

[Anc90] A. Ancona, Théorie du potentiel sur les graphes et les variétés, in *École d'été de Probabilités de Saint-Flour XVIII—1988*, *Lecture Notes in Math.* **1427**, Springer, Berlin, 1990, pp. 1–112. MR 1100282. DOI 10.1007/BFb0103041.

[Anc86] A. Ancona, On strong barriers and an inequality of Hardy for domains in $\mathbb{R}^n$, *J. London Math. Soc. (2)* **34** (1986), 274–290. MR 856511. DOI 10.1112/jlms/s2-34.2.274.

[Anc87] A. Ancona, Negatively curved manifolds, elliptic operators, and the Martin boundary, *Ann. of Math.* **125** (1987), 495–536. MR 890161. DOI 10.2307/1971409.

[Anc12] A. Ancona, On positive harmonic functions in cones and cylinders, *Rev. Mat. Iberoam.* **28** (2012), 201–230. MR 2904138. DOI 10.4171/RMI/674.

[AS85] M. T. Anderson and R. Schoen, Positive harmonic functions on complete manifolds of negative curvature, *Ann. of Math.* **121** (1985), 429–461. MR 794369. DOI 10.2307/1971181.

[Bau66] H. Bauer, *Harmonische Räume und ihre Potentialtheorie*, *Lecture Notes in Mathematics, Vol. 22*, Springer, Berlin-New York, 1966. MR 0210916.

[BJS64] L. Bers, F. John, and M. Schechter, *Partial Differential Equations*, *Lectures in Applied Mathematics, Vol. III*, Wiley, New York-London-Sydney, 1964. MR 0163043.

[Bes87] A. L. Besse, *Einstein Manifolds*, *Ergebnisse der Mathematik und ihrer Grenzgebiete* **10**, Springer-Verlag, Berlin, 1987. MR 0867684. DOI 10.1007/978-3-540-74311-8.

[BB11] A. Björn and J. Björn, *Nonlinear Potential Theory on Metric Spaces*, *EMS Tracts in Mathematics* **17**, European Mathematical Society, Zürich, 2011. MR 2867756. DOI 10.4171/099.

[BB18] A. Björn and J. Björn, Local and semilocal Poincaré inequalities on metric spaces, *J. Math. Pures Appl. (9)* **119** (2018), 158–192. MR 3862146. DOI 10.1016/j.matpur.2018.05.005. arXiv 1703.00752.

[BB07] S. Blachère and S. Brofferio, Internal diffusion limited aggregation on discrete groups having exponential growth, *Probab. Theory Related Fields* **137** (2007), 323–343. MR 2278460. DOI 10.1007/s00440-006-0009-2. arXiv math/0507582.

[BHM11] S. Blachère, P. Haïssinsky, and P. Mathieu, Harmonic measures versus quasiconformal measures for hyperbolic groups, *Ann. Sci. Éc. Norm. Supér. (4)* **44** (2011), 683–721. MR 2919980. DOI 10.24033/asens.2153. arXiv 0806.3915.

[BH86] J. Bliedtner and W. Hansen, *Potential Theory*, *Universitext*, Springer, Berlin, Heidelberg, 1986. MR 850715. DOI 10.1007/978-3-642-71131-2.

[BDGM69] E. Bombieri, E. De Giorgi, and M. Miranda, Una maggiorazione a priori relativa alle ipersuperfici minimali non parametriche, *Arch. Rational Mech. Anal.* **32** (1969), 255–267. MR 248647. DOI 10.1007/BF00281503.

[BS00] M. Bonk and O. Schramm, Embeddings of Gromov hyperbolic spaces, *Geom. Funct. Anal.* **10** (2000), 266–306. MR 1771428. DOI 10.1007/s000390050009.

[Bon96] M. Bonk, Quasi-geodesic segments and Gromov hyperbolic spaces, *Geom. Dedicata* **62** (1996), 281–298. MR 1406442. DOI 10.1007/BF00181569.

[BHK01] M. Bonk, J. Heinonen, and P. Koskela, *Uniformizing Gromov Hyperbolic Spaces*, *Astérisque* **270**, 2001. MR 1829896.

[BJ06] A. Borel and L. Ji, *Compactifications of Symmetric and Locally Symmetric Spaces*, *Mathematics: Theory & Applications*, Birkhäuser, Boston, 2006. MR 2189882.

[Bre67] M. Brelot, *Lectures on Potential Theory*, second ed., *Tata Institute of Fundamental Research Lectures on Mathematics* **19**, Tata Institute of Fundamental Research, Bombay, 1967. MR 0259146.

[BH99] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, *Grundlehren der Mathematischen Wissenschaften* **319**, Springer, Berlin, 1999. MR 1744486. DOI 10.1007/978-3-662-12494-9.

[Bus82] P. Buser, A note on the isoperimetric constant, *Ann. Sci. École Norm. Sup. (4)* **15** (1982), 213–230. MR 683635.

[BS07] S. Buyalo and V. Schroeder, *Elements of Asymptotic Geometry*, *EMS Monographs in Mathematics*, European Mathematical Society, Zürich, 2007. MR 2327160. DOI 10.4171/036.

[BG00] P. Bylund and J. Gudayol, On the existence of doubling measures with certain regularity properties, *Proc. Amer. Math. Soc.* **128** (2000), 3317–3327. MR 1676303. DOI 10.1090/S0002-9939-00-05405-8.

[Cao00] J. Cao, Cheeger isoperimetric constants of Gromov-hyperbolic spaces with quasi-poles, *Commun. Contemp. Math.* **2** (2000), 511–533. MR 1806945. DOI 10.1142/S0219199700000232.

[Cha06] I. Chavel, *Riemannian Geometry - a Modern Introduction*, second ed., *Cambridge Studies in Advanced Mathematics* **98**, Cambridge University Press, Cambridge, 2006. MR 2229062. DOI 10.1017/CBO9780511616822.

[Che70] J. Cheeger, A lower bound for the smallest eigenvalue of the Laplacian, in *Problems in Analysis (Papers Dedicated to Salomon Bochner, 1969)*, 1970, pp. 195–199. MR 0402831.

[CL20] O. Chodosh and C. Li, Generalized soap bubbles and the topology of manifolds with positive scalar curvature, *arXiv preprint* (2020). arXiv 2008.11888.

[Cho69] G. Choquet, *Lectures on Analysis. Vol. I: Integration and Topological Vector Spaces*, W. A. Benjamin, New York-Amsterdam, 1969. MR 0250011.

[CC72] C. Constantinescu and A. Cornea, *Potential Theory on Harmonic Spaces*, Springer-Verlag, New York-Heidelberg, 1972. MR 0419799.

[DPP09] G. De Philippis and E. Paolini, A short proof of the minimality of Simons cone, *Rend. Semin. Mat. Univ. Padova* **121** (2009), 233–241. MR 2542144. DOI 10.4171/RSMUP/121-14.

[Del92] P. Delanoë, Generalized stereographic projections with prescribed scalar curvature, in *Geometry and Nonlinear Partial Differential Equations (Fayetteville, AR, 1990)*, *Contemp. Math.* **127**, Amer. Math. Soc., Providence, RI, 1992, pp. 17–25. MR 1155406. DOI 10.1090/conm/127/1155406.

[Dur83] P. L. Duren, *Univalent Functions*, *Grundlehren der Mathematischen Wissenschaften* **259**, Springer-Verlag, New York, 1983. MR 708494.

[DV14] B. Dyda and A. V. Vähäkangas, A framework for fractional Hardy inequalities, *Ann. Acad. Sci. Fenn. Math.* **39** (2014), 675–689. MR 3237044. DOI 10.5186/aasfm.2014.3943.

[EO93] D. E. Edmunds and B. Opic, Weighted Poincaré and Friedrichs inequalities, *J. London Math. Soc. (2)* **47** (1993), 79–96. MR 1200980. DOI 10.1112/jlms/s2-47.1.79.

[EG15] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, revised ed., *Textbooks in Mathematics*, CRC Press, Boca Raton, FL, 2015. MR 3409135.

[Fra14] J. M. Fraser, Assouad type dimensions and homogeneity of fractals, *Trans. Amer. Math. Soc.* **366** (2014), 6687–6733. MR 3267023. DOI 10.1090/S0002-9947-2014-06202-8.

[GO79] F. W. Gehring and B. G. Osgood, Uniform domains and the quasihyperbolic metric, *J. Analyse Math.* **36** (1979), 50–74. MR 581801.

[Gd90] É. Ghys and P. de la Harpe (eds.), *Sur les groupes hyperboliques d'après Mikhael Gromov*, *Progress in Mathematics* **83**, Birkhäuser, Boston, 1990. MR 1086648.

[GT98] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, *Classics in Mathematics*, Springer-Verlag, Berlin, 1998. MR 1814364.

[GW93] S. Giulini and W. Woess, The Martin compactification of the Cartesian product of two hyperbolic spaces, *J. Reine Angew. Math.* **444** (1993), 17–28. MR 1241792. DOI 10.1515/crll.1993.444.17. Available at `http://gdz.sub.uni-goettingen.de/dms/resolveppn/?PPN=GDZPPN002211076`.

[Giu84] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, *Monographs in Mathematics* **80**, Birkhäuser, Basel, 1984. MR 775682. DOI 10.1007/978-1-4684-9486-0.

[Gro87] M. Gromov, Hyperbolic groups, in *Essays in Group Theory*, *Math. Sci. Res. Inst. Publ.* **8**, Springer, New York, 1987, pp. 75–263. MR 919829.

[Gro96] M. Gromov, Positive curvature, macroscopic dimension, spectral gaps and higher signatures, in *Functional Analysis on the Eve of the 21st Century, Vol. II (New Brunswick, NJ, 1993)*, *Progr. Math.* **132**, Birkhäuser, Boston, 1996, pp. 1–213. MR 1389019. DOI 10.1007/s10107-010-0354-x.

[Gro14] M. Gromov, Plateau-Stein manifolds, *Cent. Eur. J. Math.* **12** (2014), 923–951. MR 3188456. DOI 10.2478/s11533-013-0387-5.

[Hei01] J. Heinonen, *Lectures on Analysis on Metric Spaces*, *Universitext*, Springer-Verlag, New York, 2001. MR 1800917. DOI 10.1007/978-1-4613-0131-8.

[Hel69] L. L. Helms, *Introduction to Potential Theory*, *Pure and Applied Mathematics, Vol. XXII*, Wiley-Interscience A Division of John Wiley & Sons, New York-London-Sydney, 1969. MR 0261018.

[Her11] G. Herglotz, über Potenzreihen mit positivem, reellen Teil im Einheitskreis, *Ber. Verh. Sächs. Akad. Wiss. Leipzig* (1911), 501–511.

[Her62] R.-M. Hervé, Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel, *Ann. Inst. Fourier (Grenoble)* **12** (1962), 415–571. MR 139756. DOI 10.5802/aif.125.

[HS74] D. Hoffman and J. Spruck, Sobolev and isoperimetric inequalities for Riemannian submanifolds, *Comm. Pure Appl. Math.* **27** (1974), 715–727. MR 365424. DOI 10.1002/cpa.3160270601.

[IP94] D. Ioffe and R. Pinsky, Positive harmonic functions vanishing on the boundary for the Laplacian in unbounded horn-shaped domains, *Trans. Amer. Math. Soc.* **342** (1994), 773–791. MR 1211410. DOI 10.2307/2154652.

[JK82] D. S. Jerison and C. E. Kenig, Boundary behavior of harmonic functions in non-tangentially accessible domains, *Adv. in Math.* **46** (1982), 80–147. MR 676988. DOI 10.1016/0001-8708(82)90055-X.

[KK66] J. Kadlec and A. Kufner, Characterization of functions with zero traces by integrals with weight functions. I, *Časopis Pěst. Mat.* **91** (1966), 463–471. MR 0212559.

[KLV13] A. Käenmäki, J. Lehrbäck, and M. Vuorinen, Dimensions, Whitney covers, and tubular neighborhoods, *Indiana Univ. Math. J.* **62** (2013), 1861–1889. MR 3205534. DOI 10.1512/iumj.2013.62.5155.

[Kem72] J. T. Kemper, A boundary Harnack principle for Lipschitz domains and the principle of positive singularities, *Comm. Pure Appl. Math.* **25** (1972), 247–255. MR 293114. DOI 10.1002/cpa.3160250303.

[KL18] M. Kemper and J. Lohkamp, Potential Theory on Gromov Hyperbolic Manifolds of Bounded Geometry, *arXiv preprint* (2018). arXiv 1805.02178.

[KS80] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, *Pure and Applied Mathematics* **88**, Academic Press, New York-London, 1980. MR 567696.

[Koi17] J. Koivisto, Non-amenability and visual Gromov hyperbolic spaces, *Groups Geom. Dyn.* **11** (2017), 685–704. MR 3668056. DOI 10.4171/GGD/412.

[Leh17] J. Lehrbäck, Hardy inequalities and Assouad dimensions, *J. Anal. Math.* **131** (2017), 367–398. MR 3631460. DOI 10.1007/s11854-017-0013-8.

[LV16] J. Lehrbäck and A. V. Vähäkangas, In between the inequalities of Sobolev and Hardy, *J. Funct. Anal.* **271** (2016), 330–364. MR 3501850. DOI 10.1016/j.jfa.2016.04.028.

[Lie85] G. M. Lieberman, Regularized distance and its applications, *Pacific J. Math.* **117** (1985), 329–352. MR 779924.

[Loh18] J. Lohkamp, Hyperbolic unfoldings of minimal hypersurfaces, *Ana. Geom. Metr. Spaces* **6** (2018), 96–128. MR 3849619. DOI 10.1515/agms-2018-0006.

[Loh20a] J. Lohkamp, Potential theory on minimal hypersurfaces I: Singularities as Martin boundaries, *Potential Ana.* **53** (2020), 1493–1528. MR 4159389. DOI 10.1007/s11118-019-09815-6.

[Loh20b] J. Lohkamp, Potential theory on minimal hypersurfaces II: Hardy structures and Schrödinger operators, *Potential Ana.* (2020). DOI 10.1007/s11118-020-09869-x.

[MM92] X. Ma and R. C. McOwen, Complete conformal metrics with zero scalar curvature, *Proc. Amer. Math. Soc.* **115** (1992), 69–77. MR 1101988. DOI 10.2307/2159566.

[Mag12] F. Maggi, *Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory*, *Cambridge Studies in Advanced Mathematics* **135**, Cambridge University Press, Cambridge, 2012. MR 2976521. DOI 10.1017/CBO9781139108133.

[MR18] A. Martínez-Pérez and J. M. Rodríguez, Cheeger isoperimetric constant of Gromov hyperbolic manifolds and graphs, *Commun. Contemp. Math.* **20** (2018), 1750050. MR 3833900. DOI 10.1142/S021919971750050X. arXiv 1605.04394.

[MR20] A. Martínez-Pérez and J. M. Rodríguez, A note on isoperimetric inequalities of Gromov hyperbolic manifolds and graphs, *arXiv preprint* (2020). arXiv 2004.01755.

[MS79] O. Martio and J. Sarvas, Injectivity theorems in plane and space, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **4** (1979), 383–401. MR 565886. DOI 10.5186/aasfm.1978-79.0413.

[Mas74] U. Massari, Esistenza e regolarità delle ipersuperfici di curvatura media assegnata in R^n, *Arch. Rational Mech. Anal.* **55** (1974), 357–382. MR 355766. DOI 10.1007/BF00250439.

[MM84] U. Massari and M. Miranda, *Minimal Surfaces of Codimension One*, *North-Holland Mathematics Studies* **91**, North-Holland Publishing Co., Amsterdam, 1984. MR 795963.

[Mir70] C. Miranda, *Partial Differential Equations of Elliptic Type*, second ed., *Ergebnisse der Mathematik und ihrer Grenzgebiete* **2**, Springer, Berlin, 1970. MR 0284700.

[Mir73] M. Miranda, Existence and regularity of hypersurfaces of R^n with prescribed mean curvature, in *Partial Differential Equations* (Proc. Sympos. Pure Math., Univ. California, Berkeley, 1971), *Proc. Sympos. Pure Math.* **XXIII**, American Mathematical Society, Providence, R.I., 1973, p. 1–9. MR 0336530.

[Mor09] F. Morgan, *Geometric Measure Theory*, fourth ed., Elsevier/Academic Press, Amsterdam, 2009. MR 2455580.

[Mun00] J. R. Munkres, *Topology*, second ed., Prentice Hall, Upper Saddle River, NJ, 2000. MR 3728284.

[Pet16] P. Petersen, *Riemannian Geometry*, third ed., *Graduate Texts in Mathematics* **171**, Springer, Cham, 2016. MR 3469435.

[Pet15] C. Petit, Boundary behavior of harmonic functions on Gromov hyperbolic manifolds, *Int. Math. Res. Not.* (2015), 212–239. MR 3340300. DOI 10.1093/imrn/rnt199.

[Pin95] R. G. Pinsky, *Positive Harmonic Functions and Diffusion*, *Cambridge Studies in Advanced Mathematics* **45**, Cambridge University Press, Cambridge, 1995. MR 1326606. DOI 10.1017/CBO9780511526244.

[SY79] R. Schoen and S. T. Yau, On the structure of manifolds with positive scalar curvature, *Manuscripta Math.* **28** (1979), 159–183. MR 535700. DOI 10.1007/BF01647970.

[Sim84] L. Simon, *Lectures on Geometric Measure Theory*, *Proceedings of the Centre for Mathematical Analysis, Australian National University* **3**, Australian National University, Centre for Mathematical Analysis, Canberra, 1984. MR 756417.

[Sma00] N. Smale, A construction of homologically area minimizing hypersurfaces with higher dimensional singular sets, *Trans. Amer. Math. Soc.* **352** (2000), 2319–2330. MR 1695037. DOI 10.1090/S0002-9947-99-02576-3.

[Ste70] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, *Princeton Mathematical Series* **30**, Princeton University Press, Princeton, N.J., 1970. MR 0290095.

[Tam82] I. Tamanini, Boundaries of Caccioppoli sets with Hölder-continuous normal vector, *J. Reine Angew. Math.* **334** (1982), 27–39. MR 667448. DOI 10.1515/crll.1982.334.27.

[Tes08] R. Tessera, Large scale Sobolev inequalities on metric measure spaces and applications, *Rev. Mat. Iberoam.* **24** (2008), 825–864. MR 2490163. DOI 10.4171/RMI/557.

[Väi88] J. Väisälä, Uniform domains, *Tohoku Math. J.* **40** (1988), 101–118. MR 927080. DOI 10.2748/tmj/1178228081.

[Whi09] B. White, Which ambient spaces admit isoperimetric inequalities for submanifolds?, *J. Differential Geom.* **83** (2009), 213–228. MR 2545035.