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Norm equivalences and ideals of composition algebras

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Abstract. We show that linear bijections between quaternion algebras over a commutative ring preserving norms and identity elements are basically the same as isomorphisms or anti-isomorphisms. We also show that one-sided ideals of octonion algebras over a commutative ring are extended from the base.

1. Introduction

By a *composition algebra* over a commutative ring k we mean a non-associative k-algebra C with the following properties: C is finitely generated projective as a k-module, contains an identity element, and is equipped with a quadratic form $n_C : C \to k$ (the *norm*), uniquely determined by the condition that it is non-singular (see Subsection 2.2 below for the definition) and *permits composition*: $n_C(1_C) = 1$ and $n_C(xy) = n_C(x)n_C(y)$ for all $x, y \in C$. The rank of a composition algebra (provided it has one) is known to be one of the numbers $r = 1, 2, 4, 8$. More precisely, $C \cong k$ for $r = 1$, C is a commutative associative *quadratic étale* algebra for $r = 2$, an associative but not commutative *quaternion* algebra for $r = 4$, and an alternative but neither commutative nor associative *octonion* (or *Cayley*) algebra for $r = 8$. Octonion algebras, in particular, derive much of their importance from their intimate connection with the exceptional groups of type G_2; we refer to Gille–Neher [10] for details.

Over a field or, more generally, a semi-local ring, composition algebras are well understood. By [28, Cor. 22.16] (or [29, Thm. 1.6.2] in the field case), they can all be obtained from a (repeated) application of the Cayley–Dickson construction [23, § 6], and thanks to Witt cancellation of quadratic forms (Baeza [3, III, Cor. (4.3)]), they are classified by their norms: two composition algebras over a semi-local ring are isomorphic if and only if their norms are equivalent (Petersson–Racine [28, Thm. 26.7]); see [29, Thm. 1.7.1] for the field case. Unfortunately, however, if the base ring is arbitrary, these nice and useful properties are no longer valid. In fact, there exist octonion algebras that as quadratic spaces are either indecomposable themselves or, if the base ring contains $\frac{1}{2}$, their subspaces of trace-zero elements are. In any event, they cannot be realized by the Cayley–Dickson construction, the most interesting examples of this kind being provided by the Dickson–Coxeter octonions living on the E_8-lattice over the integers [6, 5, 26], and by the constructions of Knus–Parimala–Sridharan [18, Thm. (7.7)] over the polynomial ring in two variables with coefficients in an appropriate field of characteristic not two; for a simplification of this construction, see Thakur [30]. Moreover, Gille [9, Thm. 3.3] has exhibited examples of octonion algebras that have isometric norms but fail to be isomorphic; see also Alsaody–Gille [2, Cor. 6.7] for a characterization of octonion algebras with equivalent norms in terms of isotopes in the sense of McCrimmon [22]. On the positive side, there is a theorem of Knus–Paques [17, Thm. (3.10)] (with precursors due to Knus–Ojanguren [15, Proof of Prop. 2.1] and Knus–Ojanguren–Sridharan [16, Prop. 4.4] excluding low characteristics), which implies that quaternion algebras over any commutative ring are always classified by their norms. For a refinement of this result, see Knus [12, V, Cor. (4.3.2)].

Our aim in this paper is twofold. On the one hand, we will be concerned with an explicit version of the Knus–Paques theorem. More specifically, we show that unit preserving isometries of quaternion algebras over any commutative ring are basically the same as isomorphisms or anti-isomorphisms (Theorem 3.2). This result, which is well-known to hold over fields (Knus–Merkurjev–Rost–Tignol [13, VIII, Ex. 2]), yields the Knus–Paques theorem at once (Corollary 3.3). On the other hand, we will investigate the (one- or two-sided) ideal structure of composition algebras. Our main result (Theorem 4.1) extends earlier ones due to Mahler [21], Van der Blij–Springer [31], and Allcock [1] from the Dickson–Coxeter octonions over the integers to arbitrary composition algebras and implies among other things that one-sided ideals of any octonion algebra over any commutative ring are always extended from the base; this fact is all the more remarkable since the analogous result for quaternion or quadratic étale algebras breaks down even if the base ring is an algebraically closed field. What we have shown may also be expressed by saying that octonion algebras over a commutative ring are arithmetically simple in the sense of Legrand [19]. The paper concludes with an application of Theorem 4.1 to the nilradical of a composition algebra (Corollary 4.3).

The methods we employ to establish our results are quite elementary, relying only on properties of finitely generated projective modules and composition algebras that are basically standard.

2. Terminology and known results

2.1. **Notation.** Let k be a commutative ring remaining fixed throughout this paper. We denote by $\mathrm{Spec}(k)$ the prime spectrum of k, i.e., the totality of prime ideals in k equipped with the Zariski topology. Recall that a basis for this topology is furnished by the principal open sets $D(f)$, $f \in k$, which consist of all prime ideals in k not containing f. For $\mathfrak{p} \in \mathrm{Spec}(k)$, we denote by $k_\mathfrak{p}$ the local ring of k at $\mathfrak{p}$, with maximal ideal $\mathfrak{p}_\mathfrak{p}$ and residue field $\kappa(\mathfrak{p}) = k_\mathfrak{p}/\mathfrak{p}_\mathfrak{p} = \mathrm{Quot}(k/\mathfrak{p})$. The category of unital commutative associative k-algebras will be denoted by k-**alg**, its morphisms being k-algebra homomorphisms taking 1 into 1. If M is a (left) k-module, then its base change or scalar extension from k to $R \in k$-**alg** will be denoted by $M_R := M \otimes R$, unadorned tensor products always being taken over k, ditto for (non-associative) algebras instead of modules. For $\mathfrak{p} \in \mathrm{Spec}(k)$, we abbreviate $M_\mathfrak{p} := M_{k_\mathfrak{p}}$, $M(\mathfrak{p}) := M_{\kappa(\mathfrak{p})}$.

2.2. **Quadratic forms.** Let M be a k-module and $q : M \to k$ a quadratic form, so q is homogeneous of degree 2 and the map $M \times M \to k$, $(x, y) \mapsto q(x, y) := q(x + y) - q(x) - q(y)$ is (symmetric) bilinear, called the *bilinearization* of q. We say that q is *non-singular* if M is finitely generated projective and the linear map from M to its dual canonically induced by the bilinearization of q is an isomorphism. The property of a quadratic form to be non-singular is stable under base change, so if $q : M \to k$ is a non-singular quadratic form over k, its scalar extension $q_R : M_R \to R$ is a non-singular quadratic form over R for any $R \in k$-**alg**.

2.3. **Basic properties of composition algebras.** Recall that a composition algebra over k is a (non-associative) k-algebra C that is unital (i.e., contains an identity element), finitely generated projective as a k-module, and equipped with its norm $n_C : C \to k$, a non-singular quadratic form permitting composition:

$$n_C(1_C) = 1, \quad n_C(xy) = n_C(x)n_C(y) \quad (x, y \in C).$$

Note that the non-singularity condition prevents the base ring from being counted as a composition algebra unless it contains $\frac{1}{2}$. In order to avoid this awkward phenomenon, the notion of a composition algebra has to be modified; rather than insisting that the norm be non-singular, one should merely require that it be separable in the sense of Loos [20, 3.2]. This approach has been worked out in [28, Chap. 4], with a summary given in [27, § 4]. It should also be noted that what we call non-singular (resp. separable) quadratic forms are called regular (resp. non-singular) ones in [2].

Now let C be a composition algebra over k. Referring to [12, V, (7.1)], [23, 24] for details, we collect a few properties of C that will be used frequently later on.

(a) C is alternative, so its associator $[x, y, z] := (xy)z - x(yz)$ is alternating in $x, y, z \in C$. Hence the expression xyx is unambiguous and the Moufang identities hold.

$$x(y(xz)) = (xyx)z, \quad (xy)(zx) = x(yz)x, \quad ((zx)y)x = z(xyx). \tag{1}$$

Writing

$$t_C : C \to k, \quad x \mapsto t_C(x) := n_C(1_C, x)$$

for the *trace* of C (a linear form) and

$$\iota_C : C \to C, \quad x \mapsto \bar{x} := t_C(x)1_C - x$$

for its *conjugation* (an algebra involution), the following identities hold for all $x, y, z \in C$:

$$x^2 = t_C(x)x - n_C(x)1_C, \tag{2}$$

$$x\bar{x} = n_C(x)1_C = \bar{x}x, \tag{3}$$

$$n_C(x, \bar{y}) = t_C(x)t_C(y) - n_C(x, y) = t_C(xy). \tag{4}$$

In particular, we conclude from (3) that x is invertible in C if and only if $n_C(x)$ is invertible in k, in which case $x^{-1} = n_C(x)^{-1}\bar{x}$. The set of invertible elements in C (resp. k) will be denoted by $C^\times$ (resp. $k^\times$).

(b) Composition algebras are stable under base change; C_R is a composition algebra over R for any $R \in k$-**alg**, the norm of C_R being the R-quadratic extension of the norm of C.

(c) The trace t_C is an *associative* linear form in the sense that $t_C([x, y]) = t_C([x, y, z]) = 0$ for all $x, y, z \in C$, where $[x, y] := xy - yx$ is the commutator of x and y.

(d) $1_C \in C$ is *unimodular* in the sense that some linear form $\lambda : C \to k$ has $\lambda(1_C) = 1$, making C a faithful k-module and $k1_C$ a free k-module of rank 1.

(e) If C has rank $r > 2$, then C is *central* in the sense that its *center*, i.e.,

$$\mathrm{Cent}(C) := \{x \in C \mid [x, C] = [x, C, C] = \{0\}\},$$

satisfies $\mathrm{Cent}(C) = k1_C$. Similarly, if C is an octonion algebra, then C is *nuclear* in the sense that its *nucleus*, i.e.,

$$\mathrm{Nuc}(C) := \{x \in C \mid [x, C, C] = \{0\}\},$$

satisfies $\mathrm{Nuc}(C) = k1_C$ (see [2, Lem. 2.8]).

2.4. **The Cayley–Dickson construction.** Let B be an associative composition algebra over k and $\mu \in k$ an invertible element. Then the k-algebra C defined on the direct sum $B \oplus Bj$ of two copies of B as a k-module by the multiplication

$$(u_1 \oplus v_1 j)(u_2 \oplus v_2 j) = (u_1 u_2 + \mu \bar{v}_2 v_1) \oplus (v_2 u_1 + v_1 \bar{u}_2)j \tag{5}$$

for $u_i, v_i \in B$, $i = 1, 2$, is a composition algebra whose unit element, norm, and conjugation for $u, v \in B$ are respectively given by

$$1_C = 1_B \oplus 0 \cdot j, \quad n_C(u \oplus vj) = n_B(u) - \mu n_B(v), \quad \overline{u \oplus vj} = \bar{u} \oplus (-vj) \tag{6}$$

in terms of the corresponding data for B. We write $C =: \mathrm{Cay}(B, \mu)$ and call it the composition algebra arising from B, μ by the *Cayley–Dickson construction*. The assignment $u \mapsto u \oplus 0 \cdot j$ gives an embedding, i.e., an injective homomorphism of unital k-algebras, allowing us to identify $B \subseteq C$ as a unital subalgebra. We then have $u + vj = u \oplus vj$ for all $u, v \in B$, and $B^\perp = Bj$ is the orthogonal complement of B in C relative to the bilinearized norm. By (6) and Subsection 2.3 (a), therefore, $j \in B^\perp$ is invertible in C.

Conversely, let C be a composition algebra over k and $B \subset C$ a proper composition subalgebra. Then B is associative, and for any $l \in B^{\perp} \cap C^{\times}$ (which always exists if k is a semi-local ring), the inclusion $B \hookrightarrow C$ has a unique extension to an embedding from the Cayley–Dickson construction $\mathrm{Cay}(B, \mu)$, $\mu := -n_C(l) \in k^{\times}$, into C that sends j to l. This embedding is an isomorphism provided C, B have rank $r, \frac{r}{2}$, respectively.

3. Unital norm equivalences

3.1. **The Knus–Paques theorem.** Let B, B' be quaternion algebras over k. Using the theory of Clifford algebras, Knus–Paques [17, Thm. (3.10)] have shown that B and B' are isomorphic if and only if, for some $\beta \in k^{\times}$, the quadratic forms βn_B and $n_{B'}$ are isometric. In the present paper, we give this equivalence an explicit form. More specifically, calling an isometry from n_B to $n_{B'}$ that preserves identity elements a *unital norm equivalence*, and guided by [13, VIII, Ex. 2], we will establish the following result.

Theorem 3.2. *Let B, B' be quaternion algebras over k. A map $f : B \to B'$ is a unital norm equivalence if and only if there exists a decomposition $k = k_+ \oplus k_-$ of k as a direct sum of ideals such that the induced decompositions*

$$(7)\qquad \begin{aligned} &B = B_+ \oplus B_-, \quad B_{\pm} := B_{k_{\pm}}, \quad B' = B'_+ \oplus B'_-, \quad B'_{\pm} := B'_{k_{\pm}}, \\ &f = f_+ \oplus f_-, \qquad f_{\pm} := f_{k_{\pm}} \end{aligned}$$

make $f_+ : B_+ \to B'_+$ an isomorphism of quaternion algebras over k_+ and $f_- : B_- \to B'_-$ an anti-isomorphism of quaternion algebras over k_-.

Proof. If such a decomposition exists, f is clearly a unital norm equivalence. Conversely, let this be so. We put $X := \mathrm{Spec}(k)$ and conclude that the subsets

$$X_+ := \{\mathfrak{p} \in X \mid f_{\mathfrak{p}} : B_{\mathfrak{p}} \to B'_{\mathfrak{p}} \text{ is an isomorphism}\},$$
$$X_- := \{\mathfrak{p} \in X \mid f_{\mathfrak{p}} : B_{\mathfrak{p}} \to B'_{\mathfrak{p}} \text{ is an anti-isomorphism}\}$$

of X are Zariski-open (since B is finitely generated as a k-module, so the conditions imposed on $X_{\pm}$ may be characterized by finitely many equations) and disjoint (since quaternion algebras are not commutative). Suppose we can show that they cover X. Then Bourbaki [4, II, § 4, Prop. 15] yields a complete orthogonal system $(\varepsilon_+, \varepsilon_-)$ of idempotents in k satisfying $X_{\pm} = D(\varepsilon_{\pm})$. Now it suffices to put $k_{\pm} = k\varepsilon_{\pm}$, which implies for any $\mathfrak{p}_{\pm} \in \mathrm{Spec}(k_{\pm})$ that $\mathfrak{p} := \mathfrak{p}_{\pm} \oplus k_{\mp} \in D(\varepsilon_{\pm})$ makes $f_{\pm\mathfrak{p}_{\pm}} = (f_{\mathfrak{p}})_{k_{\pm\mathfrak{p}_{\pm}}}$ an isomorphism in case of the plus-sign and an anti-isomorphism in case of the minus-sign. Hence f_+ is an isomorphism and f_- is an anti-isomorphism.

We are thus reduced to showing $X = X_+ \cup X_-$. Equivalently, we may assume that k is a local ring and must show that f is either an isomorphism or an anti-isomorphism. Writing $\mathfrak{m}$ for the maximal ideal of k, the quaternion algebra $B(\mathfrak{m})$ over the field $\kappa(\mathfrak{m})$ contains a quadratic étale subalgebra generated by an element of trace 1, which in turn lifts to a quadratic étale subalgebra $D = k[u] \subseteq B$ for some trace-one element $u \in B$. Since units, norms, and traces are preserved by f, so are squares (by (2)), and we conclude that $D' := f(D) = k[u']$, $u' := f(u)$, is a quadratic étale subalgebra of B', making $f|_D : D \to D'$ an isomorphism. Now apply Subsection 2.4 to reach B from D by means of the Cayley–Dickson construction; there exists a unit $\mu \in k^\times$ such that the inclusion $D \hookrightarrow B$ extends to an identification $B = \mathrm{Cay}(D, \mu) = D \oplus Dj$, $j \in D^\perp$, $n_B(j) = -\mu$. Moreover, setting $j' := f(j) \in D'^\perp \subseteq B'$, we obtain $n_{B'}(j') = -\mu$ and conclude that $f|_D$ extends to an isomorphism $g : B \to B'$ satisfying $g(j) = j'$. Hence $f_1 := g^{-1} \circ f : B \to B$ is a unital norm equivalence inducing the identity on D and satisfying $f_1(j) = j$. Since f_1 stabilizes $D^\perp = Dj$, we find a k-linear bijection $\varphi : D \to D$ such that $f_1(vj) = \varphi(v)j$ for all $v \in D$. Then $\varphi(1_D) = 1_D$, and since n_B permits composition, φ leaves n_D invariant and is thus a unital norm equivalence of D, hence an automorphism. By [12, III, Lem. (4.1.1)], therefore, we are left with the following cases.

Case 1. $\varphi = \mathbf{1}_D$. Then $f_1 = \mathbf{1}_B$, and $f = g : B \to B'$ is an isomorphism.

Case 2. $\varphi = \iota_D$. One checks easily that the *reflection in* D (Jacobson [11, p. 12]), i.e., the map $\psi : B \to B$ defined by $\psi(v + wj) := v - wj$ for all $v, w \in D$, is an automorphism satisfying $f_1 = \psi \circ \mathrm{Int}(j) \circ \iota_B$, where $\mathrm{Int}(j)$ stands for the inner automorphism $x \mapsto jxj^{-1}$ of B affected by j. Hence $f = g \circ \psi \circ \mathrm{Int}(j) \circ \iota_B : B \to B'$ is an anti-isomorphism. □

Corollary 3.3 (Knus–Paques [17, Thm. (3.10)]). *For quaternion algebras B, B' over k, the following conditions are equivalent.*

(i) *B and B' are isomorphic.*
(ii) *B and B' are norm-equivalent, i.e., their norms are isometric.*
(iii) *B and B' are norm-similar, i.e., there exists a scalar $\beta \in k^\times$ such that βn_B and $n_{B'}$ are isometric.*

Proof. The implications (i) ⇒ (ii) ⇒ (iii) being obvious, it remains to establish the implication (iii) ⇒ (i), so let us assume for some $\beta \in k^\times$ that $g : B \to B'$ is an isometry from βn_B to $n_{B'}$. Then $u := g^{-1}(1_{B'}) \in B$ has $n_B(u) = \beta^{-1}$ and hence is invertible. Thus, since n_B permits composition, the assignment $x \mapsto g(ux)$ defines a unital norm equivalence $f : B \to B'$, and we obtain a decomposition $k = k_+ \oplus k_-$ as a direct sum of ideals such that the induced decompositions (7) enjoy the properties spelled out in Theorem 3.2. Hence $f_+ \oplus (f_- \circ \iota_B) : B \to B'$ is an isomorphism of quaternion algebras. □

3.4. **Remarks.** (a) Another proof of Theorem 6 using group schemes may be found in a forthcoming monograph by Gille–Neher [10]. A scheme-theoretic proof of Corollary 3.3 is due to Gille [9, Thm. 2.4].

(b) It is easy to see that a unital norm equivalence between quadratic étale algebras is an isomorphism. More precisely, given a unital norm equivalence $f : D \to D'$ of quadratic étale algebras, and arguing as in the proof of Theorem 3.2, one obtains decompositions as in the theorem such that $f_+ = \mathbf{1}_{D_+}$ is the identity of D_+ and $f_- = \iota_{D_-}$ is the conjugation of D_-. This result is well-known [12, III, Prop. (4.1.2)].

(c) At the other extreme, Theorem 3.2 breaks down for octonion algebras, even if the base ring is an algebraically closed field. In order to illuminate this well-known phenomenon, we refer to McCrimmon's notion of isotopy [22], a special case of which may be described as follows.

Let C be an octonion algebra over k, an arbitrary commutative ring. For $p \in C^\times$, the k-module C together with the multiplication $(xp^{-1})(py)$ is again an octonion algebra over k, denoted by C^p and called the *p-isotope* of C; it has the same identity element, norm, trace, and conjugation as C. For another invertible element $q \in C^\times$, the Moufang identities (1) imply $(C^p)^q = C^{pq}$, and using Subsection 2.3 (e), one checks that $C^p = C^q$ if and only if $q = \alpha p$ for some $\alpha \in k^\times$. Now suppose $B \subseteq C$ is a quaternion subalgebra, and let $p \in B^\times$. Then $C = B \oplus B^\perp$ as k-modules, and localizing if necessary, a straight-forward verification shows that $f : C \xrightarrow{\sim} C^p$, $f(u+v) := p^{-1}up + v$ for $u \in B$, $v \in B^\perp$ is an isomorphism. In particular, f is a unital norm equivalence but not an automorphism of C unless $p \in k1_C$. It follows from Gille [9, Thm. 3.3] and Alsaody–Gille [2, Cor. 6.7] that, if C is split and k is chosen judiciously, some $p \in C^\times$ has $C^p \not\cong C$. Such a p, therefore, cannot be embedded into a quaternion subalgebra of C, in stark contrast to the general situation over fields (Springer–Veldkamp [29, Prop. 1.6.4]).

The preceding examples are in some sense typical; it follows from the principle of triality [29, Thm. 3.2.1] combined with the formalism of [25] that, if k is a field, any rotation (resp. reflection) of the quadratic space (C, n_C) fixing the identity element is an isomorphism (resp. anti-isomorphism) from C to C^p for some $p \in C^\times$. Whether the same conclusion holds over any commutative ring seems to be an open problem.

4. One-sided ideals

In order to put our approach to ideals of octonion algebras in perspective, it seems appropriate to discuss them in the broader context of arbitrary composition algebras. We begin by distinguishing three types of ideals in a composition algebra C: the *general type*, i.e., one-sided ideals, the *middle ground*, i.e., arbitrary two-sided ideals, and the *special type*, i.e., ideals of (C, ι_C) as an algebra with involution, equivalently, ideals stable under conjugation.

The final result of this paper basically says that, if C has rank $r > 1$, all its ideals of the right type are extended from the base ring provided as r increases so does the generality of the type. More precisely, we can prove the following.

Theorem 4.1. *Let C be a composition algebra of rank $r > 1$ over k and identify $k = k1_C \subseteq C$ canonically via Subsection 2.3 (d). Then the assignment $\mathfrak{a} \mapsto \mathfrak{a}C$ defines an inclusion-preserving bijection from the ideals of k to*
(i) *the one-sided ideals of C for $r = 8$,*
(ii) *the two-sided ideals of C for $r = 4$,*
(iii) *the ideals of (C, ι_C) as an algebra with involution for $r = 2$.*
The inverse of this map is given by the assignment $I \mapsto I \cap k$.

Proof. We proceed in several steps by proving the following intermediate assertions.

1°. *For $r = 8$, i.e., if C is an octonion algebra, its one-sided ideals are two-sided ones.* In order to see this, let $I \subseteq C$ be a one-sided ideal. Replacing I by $\bar{I}$ if necessary, we may assume that I is a right ideal in C. Since the assertion is local on k, we may further assume that k is a local ring. From Subsection 2.4, we conclude $C = \mathrm{Cay}(B, \mu) = B \oplus Bj$ for some quaternion algebra B over k and some invertible scalar $\mu \in k$. For $l = 1, 2$, let $\pi_l : C \to B$ be the l-th projection defined by $x = \pi_1(x) + \pi_2(x)j$ for $x \in C$. Then $I_l := \pi_l(I)$ is a k-submodule of B satisfying $I \subseteq I_1 \oplus I_2 j$. From (5), we deduce

$$(u + vj)w = uw + (v\bar{w})j, \quad (u + vj)(wj) = \mu\bar{w}v + (wu)j$$

for all $u, v, w \in B$. Assuming $u + vj \in I$, the first equation shows that I_l is a right ideal in B, while the second one gives $BI_2 \subseteq I_1$, $BI_1 \subseteq I_2$. Hence $I_1 = 1_B I_1 \subseteq I_2 = 1_B I_2 \subseteq I_1$, so $I_0 := I_1 = I_2$ is a two-sided ideal in B such that $I \subseteq I_0 \oplus I_0 j$. Now let $u, v, w_1, w_2 \in B$, and assume again $u + vj \in I$. Then I contains the quantity

$$\begin{aligned} ((u + vj)w_1)w_2 - (u + vj)(w_2 w_1) &= uw_1 w_2 + (v\bar{w}_1\bar{w}_2)j \\ &\qquad - u(w_2 w_1) - (v\overline{w_2 w_1})j \\ &= u[w_1, w_2]. \end{aligned}$$

Let $\mathfrak{m}$ be the maximal ideal of k, and let $\kappa := k/\mathfrak{m}$ be its residue field. Since $[B(\mathfrak{m}), B(\mathfrak{m})]$, being the space of trace-zero elements in the quaternion algebra $B(\mathfrak{m})$ over the field κ, contains invertible elements, so does $[B, B]$, and we conclude $u, v \in I$. Thus $I = I_0 \oplus I_0 j$ is a two-sided ideal in C.

2°. *Let I be a two-sided ideal in C. Then $I \cap k = \{0\}$ implies $I = \{0\}$ provided $r > 2$ or I is stable under conjugation.* Suppose first I is stable under conjugation, and let $x \in I$. Then $t_C(x) = x + \bar{x} \in I \cap k = \{0\}$. But $I \subseteq C$ is an ideal, so $x\bar{z} \in I$ for all $z \in C$, and from (4), we conclude $n_C(x, z) = t_C(x\bar{z}) = 0$, hence $x = 0$ by non-singularity.

Next consider the case $r > 2$. Since $I \cap \bar{I} \subseteq C$ is a two-sided ideal stable under conjugation and satisfying $I \cap \bar{I} \cap k \subseteq I \cap k = \{0\}$, the previous case yields $I \cap \bar{I} = \{0\}$. Now let $x \in I$ be nonzero. Then $\bar{x} = t_C(x) - x$ does not belong to I, forcing $t_C(x) \neq 0$. Thus the linear map $t_C : I \to k$ is injective. From Subsection 2.3 (c), we deduce the relations

$$t_C([x, y]) = t_C([x, y, z]) = 0$$

for $y, z \in C$, which therefore imply $[x, y] = [x, y, z] = 0$, so x belongs to the center of C, which by Subsection 2.3 (e) is k. Hence $I \subseteq I \cap k = \{0\}$, which proves our claim.

3°. *There is a k-submodule $M \subseteq C$ such that*

$$C = k \oplus M \tag{8}$$

and $(\mathfrak{a}C) \cap k = \mathfrak{a}$ *for all ideals* $\mathfrak{a} \subseteq k$. Indeed, the first part follows immediately from 1_C being unimodular (cp. Subsection 2.3 (d)) and implies $(\mathfrak{a}C) \cap k = (\mathfrak{a} \oplus \mathfrak{a}M) \cap k = \mathfrak{a}$. Thus 3° holds.

It remains to establish the following assertion.

4°. *Let* I *be a two-sided ideal in* C, *and assume* $r > 2$ *or that* I *is stable under conjugation. Then* $I = (I \cap k)C$. The right-hand side being contained in the left, it suffices to show

$$I \subseteq (I \cap k)C. \tag{9}$$

Setting $\mathfrak{a} := I \cap k$, we pass to the base change $k' := k/\mathfrak{a} \in k$-**alg** and consider the composition algebra $C' := C \otimes k' = C/\mathfrak{a}C$ over k', denote by $I' = I/\mathfrak{a}C$ the image of I under the canonical map $x \mapsto x'$ from C to C', and claim that $\mathfrak{a}' := I' \cap k' = \{0\}$. To see this, we first note that (8) implies

$$C' = k' \oplus M', \quad M' := M \otimes k' = M/\mathfrak{a}M.$$

Given $\alpha_1 \in \mathfrak{a}' \subseteq I'$, we can therefore find elements $\xi \in k$, $y \in M$ satisfying $x := \xi + y \in I$ and $x' = \alpha_1$. This yields $\xi' = \alpha_1$, $y' = 0$, hence $y \in \mathfrak{a}M \subseteq \mathfrak{a}C \subseteq I$ and $\xi = x - y \in I \cap k = \mathfrak{a}$. But then $\alpha_1 = \xi' = 0$, and we have arrived at the desired conclusion $\mathfrak{a}' = \{0\}$. Now 2° implies $I' = \{0\}$, hence $I \subseteq \mathfrak{a}C$, as claimed in (9). Thus the proof of 4° is complete. □

Corollary 4.2. *Let* C *be a composition algebra over* k. *Then the assignments*

$$\mathfrak{a} \mapsto \mathfrak{a}C, \quad I \mapsto I \cap k$$

define inclusion-preserving inverse bijections between the ideals of k *and the ideals of* C *that are stable under conjugation.*

Proof. k splits into the direct sum of ideals k_i, $0 \le i \le 3$, such that $C_i := C_{k_i}$ is a composition algebra of rank 2^i over k_i for all i. Applying Theorem 4.1 to C_i for $i > 0$ (the theorem being trivial for $i = 0$), the assertion follows. □

Corollary 4.3. *Let* C *be a composition algebra over* k. *Then the nilradicals of* C *(as an alternative algebra) and of* k *are related by the formula*

$$\mathrm{Nil}(C) = \mathrm{Nil}(k) \cdot C.$$

Proof. Since $\mathrm{Nil}(C) \cap k$, being a nilideal in k, is contained in $\mathrm{Nil}(k)$ and $\mathrm{Nil}(C)$ itself is stable under conjugation, Corollary 4.2 implies $\mathrm{Nil}(C) \subseteq \mathrm{Nil}(k) \cdot C$. Conversely, since finitely generated nilideals of a commutative ring are nilpotent, one checks that $\mathrm{Nil}(k) \cdot C$ is a nilideal of C and hence contained in $\mathrm{Nil}(C)$. □

4.4. **Concluding remarks.** (a) Since, by [12, III, Thm. (5.1.1)], quaternion algebras are Azumaya algebras, part (ii) of Theorem 4.1 is not new [14, III, Corollaire 5.2]. Neither is part (iii) since, as Erhard Neher has pointed out to me, a quadratic étale D over k by [12, V, (4.1)] is a Galois extension with Galois group the constant group scheme $\mathbb{Z}/2\mathbb{Z}$, so the assertion follows from Ford [8, 12.2.2 (7)].

(b) Part (i) of Theorem 4.1 breaks down for quaternion algebras since even the split quaternions (of 2×2-matrices with entries in k) allow one-sided ideals not extended from the base ring. Similarly, part (ii) breaks down for quadratic étale algebras since even the split one (direct sum of two copies of k) allows (two-sided) ideals not extended from k.

(c) Step 1^{o} in our proof of Theorem 4.1 is devoted to showing that one-sided ideals of octonion algebras are, in fact, two-sided. After localizing, we do so by appealing to the Cayley–Dickson construction. This seems to be in keeping with the final remark 6.10 of [23] and also relates to an earlier result of Erdmann [7, Korollar 1 of Satz 7], who showed over fields of characteristic not 2 that algebras arising from the base field by a more than twofold iteration of the Cayley–Dickson construction fail to admit nontrivial one-sided ideals.

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