Mathematik

Tangent spaces to the Teichmüller space from the energy-conscious perspective

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Abstract

Usually, the description of tangent spaces to the Teichmueller space $\mathcal{T}(\Sigma_g)$ of a (closed) Riemann surface Σ_g of genus $g \geq 2$ $\big($which we can identify with the quotient space $\mathbb{H}^2/\Gamma_g$ of the upper half plane $\mathbb{H}^2$ by a discrete cocompact subgroup Γ_g of $\mathrm{PSL}(2,\mathbb{R})\big)$ comes in two different flavours: the space of holomorphic quadratic differentials on Σ_g which are holomorphic sections of the tensor square of the canonical line bundle of Σ_g and the first cohomology group $H^1(\Gamma_g;\mathfrak{g})$ of the fundamental group Γ_g of Σ_g with coefficients in the vector space $\mathfrak{g}$ of Killing vector fields on $\mathbb{H}^2$ (or on $\mathbb{D}$), a.k.a the Lie algebra of $\mathrm{PSL}(2,\mathbb{R})$. This thesis is concerned with connecting the above-mentioned descriptions using the notion of a *harmonic vector field* on the upper half plane $\mathbb{H}^2$ (equivalently, on $\mathbb{D}$) that takes inspiration from the theory of harmonic maps between compact hyperbolic Riemann surfaces.

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Introduction

In Riemann surface theory, Teichmueller theory, and the theory of moduli spaces, on the one hand, we benefit a lot from cross-pollination of techniques coming from sometimes disparate fields like topology, complex analysis, algebraic geometry, and arithmetic geometry. However, on the other hand, passing from one structure/definition to another is quite often an arduous task, making the use of different techniques simultaneously rather tricky.

Incase of a closed Riemann surface Σ_g of genus $g \geq 2$, to make a smooth transition from *complex structures* (see Definition 1.2.1) to *hyperbolic structures* (see Definition 1.2.9), we need the Uniformization theorem; from the lens of the Korn-Lichtenstein theorem we watch metamorphosis of *almost complex structures* (see Definition 1.2.2) into *complex structures*. Usually, the problems even get worse when passing from a single Riemann surface to either the parametrization space $\mathcal{T}(\Sigma_g)$ - famously known as the Teichmueller space of Σ_g - parameterizing hyperbolic structures/complex structures/almost complex structures on Σ_g or the bundles of Riemann surfaces. As already mentioned, this problem is not only confined to structures but it is also valid when it comes to connecting different definitions and different descriptions of a mathematical object in Teichmueller theory. For instance, the description of the Teichmueller space $\mathcal{T}(\Sigma_g)$ of a closed oriented surface Σ_g of genus $g \geq 2$ enjoys a multifaceted viewpoint, i.e., we can view $\mathcal{T}(\Sigma_g)$ as

- the quotient space of the space $\mathcal{C}(\Sigma_g)$ of complex structures on Σ_g by the action of the group $\mathrm{Diff}_0^+(\Sigma_g)$ of orientation preserving diffeomorphisms on Σ_g that are isotopic to the identity (see [63]);

- the quotient space of $\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ by the action of the Lie group $\mathrm{PSL}(2,\mathbb{R})$, where Γ_g is the fundamental group of Σ_g and $\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ is the space of homomorphisms $\Gamma_g \longrightarrow \mathrm{PSL}(2,\mathbb{R})$ which describe a discrete and cocompact action of Γ_g on $\mathbb{H}^2$.

The above-mentioned viewpoints are brought together in one-to-one correspondence by the parametrized Uniformization Theorem. In the literature, $\mathcal{T}(\Sigma_g)$ is also defined as a connected component of the representation variety $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ (see [22], [23], [45]). In its own right, this description is a great motivation to study the Teichmueller space $\mathcal{T}(\Sigma_g)$ in detail, this thesis will not discuss it further.

Other than the Teichmueller space $\mathcal{T}(\Sigma_g)$, there are many examples of spaces in Teichmueller theory that enjoy a kaleidoscopic picture. One famous example is tangent spaces to the Teichmueller space $\mathcal{T}(\Sigma_g)$. Tangent spaces to the Teichmüller space $\mathcal{T}(\Sigma_g)$ are best described using the theory of *infinitesimal deformations*. The main slogan of the theory is to *deform* a point in the Teichmueller space $\mathcal{T}(\Sigma_g)$ be it

- a homomorphism ρ representing $[\rho] \in \mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$;
- or a complex structure on Σ_g

with respect to a (real) parameter t and then analyze the local structure of the corresponding spaces. Recall the Taylor expansion of a smooth function f (on a smooth manifold M) around a point $x \in M$. The first order derivative at x provides good information of f. In the same way, certain cohomology groups provide basic and satisfactory information on deformations of a homomorphism $\rho \in \mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$. Formally speaking, deformation of a homomorphism $\rho \in \mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ has the following meaning: we take a curve of maps ρ_t where $\rho_0 = \rho$ is a homomorphism, and ask for (infinitesimal) conditions which ensure that this curve ρ_t satisfies the homomorphism condition

$$\rho_t(\gamma_1\gamma_2) = \rho_t(\gamma_1)\rho_t(\gamma_2), \quad \forall \gamma_1, \gamma_2 \in \Gamma_g.$$

Solving $\frac{d\rho_t}{dt}\big|_{t=0}$ up to the first order determines a 1-cocycle with values in the vector space of Killing vector fields on $\mathbb{H}^2$, a.k.a the Lie algebra $\mathfrak{g}$ of $\mathrm{PSL}(2,\mathbb{R})$. As a result, $T_\rho\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ is nothing but the space of $\mathfrak{g}$-valued 1-cocycles $Z^1(\Gamma_g; \mathfrak{g}_{\mathrm{Ad}_\rho})$. Next, by considering "trivial" deformations ρ_t of ρ given by conjugation via elements of $\mathrm{PSL}(2,\mathbb{R})$ and solving the above-mentioned homomorphism condition up to the first-order determines a 1-coboundary $c \in B^1(\Gamma_g; \mathfrak{g}_{\mathrm{Ad}_\rho})$. Hence,

$$T_{[\rho]}\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R}) \cong H^1(\Gamma_g; \mathfrak{g}_{\mathrm{Ad}_\rho}).$$

Therefore, $H^1(\Gamma_g; \mathfrak{g}_{\mathrm{Ad}_\rho})$ serves as the cohomological description of tangent spaces to the Teichmueller space $\mathcal{T}(\Sigma_g)$. The space of infinitesimal deformations of a complex structure on Σ_g is parametrized by the space $\mathrm{HQD}(\Sigma_g)$ of *holomorphic quadratic differentials* on Σ_g (see [36] and [47]), where a holomorphic quadratic differential is a holomorphic section of Q_{Σ_g}, the tensor square of the canonical line bundle K_{Σ_g} of Σ_g. Hence, the analytic description of tangent spaces to the Teichmueller space $\mathcal{T}(\Sigma_g)$ is given by $\mathrm{HQD}(\Sigma_g)$. For precise descriptions, see **Section 1.3** in **Chapter 1**.

So, the main aim of this thesis is to construct explicit maps from $\mathrm{HQD}(\Sigma_g)$ to $H^1(\Gamma_g, \mathfrak{g}_{\mathrm{Ad}\rho})$ and vice-versa, i.e.,

$$\mathrm{HQD}(\Sigma_g) \underset{?}{\overset{?}{\longleftrightarrow}} H^1(\Gamma_g, \mathfrak{g}_{\mathrm{Ad}\rho}) \tag{1}$$

Now, we can ask ourselves the following question: what recipes are we going to use in the construction of maps from $\mathrm{HQD}(\Sigma_g)$ to $H^1(\Gamma_g, \mathfrak{g}_{\mathrm{Ad}\rho})$ and vice-versa?

Since the inception of Teichmueller's theorems, the use of *quasiconformal maps* in classical

Teichmueller theory is prevalent. However, in this thesis, we don't focus much on quasiconformal maps. We take an unconventional road that *minimizes energy* to connect the above-mentioned descriptions of tangent spaces to the Teichmueller space $\mathcal{T}(\Sigma_g)$. Our essential recipe will be the notion of a *harmonic vector field* on the upper half plane $\mathbb{H}^2$ or the Poincaré disk $\mathbb{D}$ in constructing maps from $\mathrm{HQD}(\Sigma_g)$ to $H^1(\Gamma_g, \mathfrak{g}_{\mathrm{Ad}\rho})$ and vice-versa.

The notion of a *harmonic vector field* on $\mathbb{H}^2$ (or on $\mathbb{D}$) takes inspiration from the definition (see Definition 2.1.4) of a *harmonic map* $\phi : \Sigma_1 \longrightarrow \Sigma_2$ between Riemann surfaces equipped with conformal metrics. Harmonic maps are critical points of the energy functional

$$E(\phi) = \int_{\Sigma_1} \|d\phi\|^2 d\mu,$$

where $\|\cdot\|$ is the Hilbert-Schmidt norm and $d\mu$ is the measure on Σ_1 determined by the Riemannian metric on Σ_1. The integrand is also known as the *energy density* (see (2.4)). Equivalently, harmonic maps satisfy the *Euler-Lagrange partial differential equations* associated with the energy functional (see (2.3)). These PDEs are non-linear and elliptic. Harmonic maps exist in the homotopy class of any diffeomorphism when the target surface is equipped with a strictly negatively curved metric, and are unique ([12], [27]). Harmonic maps are related to holomorphic quadratic differentials intimately, hence play an important role in Teichmueller theory. This relation arises from the fact that

> a diffeomorphism $\phi : (\Sigma_1, \sigma) \longrightarrow (\Sigma_2, \rho)$ between two Riemann surfaces equipped with conformal metrics is harmonic iff the quadratic differential $(\phi^*\rho)^{(2,0)}$ on the source surface Σ_1 is holomorphic (see Example 2.1.10 and [32, Lemma 1.1]). ★

The use of harmonic maps in Teichmueller theory goes all the way back to Gerstenhaber and Rauch's program (see [20] and [53]) to prove Teichmueller's Theorems using harmonic maps. In order to state our main results, we need to define a harmonic vector field on the upper half plane $\mathbb{H}^2$ or the Poincaré disk $\mathbb{D}$: let U be an open subset of M, where M is either the upper half plane $\mathbb{H}^2$ or the Poincaré disk $\mathbb{D}$. Let $\{\phi_t\}_{t\in[0,\epsilon)}$ be a smooth family of smooth maps

$$\phi_t : U \longrightarrow M$$

where ϕ_0 is the inclusion. Then $\xi = \frac{d\phi_t}{dt}|_{t=0}$ is a vector field on U.

Definition 0.0.1 (Definition 2.2.1)**.** The vector field ξ on U is harmonic if there exists a smooth family of smooth maps $\{\phi_t : U \longrightarrow M\}_{t\in[0,\epsilon)}$ which satisfies the following:

1. ϕ_0 is the inclusion map,
2. $\left.\dfrac{d\phi_t}{dt}\right|_{t=0} = \xi$,
3. $\forall x \in U : \left.\dfrac{d}{dt}\right|_{t=0} \tau(\phi_t)(x) = 0$, where τ is the *tension field* (see Definition 2.1.2).

An infinitesimal version of (★) is given by the following:

Proposition 0.0.2 (Proposition 2.2.4)**.** *A smooth vector field ξ on $\mathbb{H}^2$ or on $\mathbb{D}$ is harmonic iff $\left(\mathcal{L}_\xi \boldsymbol{g}_{\mathbb{H}^2}\right)^{(2,0)}$ or $\left(\mathcal{L}_\xi \boldsymbol{g}_{\mathbb{D}}\right)^{(2,0)}$ is holomorphic.*

Our first main theorem is based on the above Proposition and the fact that a holomorphic vector field on $U \subset \mathbb{H}^2$ is a harmonic vector field on $U \subset \mathbb{H}^2$.

Theorem 0.0.3 (Theorem 2.2.7)**.** *Let $\mathcal{HOL}$ denote the sheaf of holomorphic vector fields on $\mathbb{H}^2$, $\mathcal{HARM}$ denote the sheaf of harmonic vector fields on $\mathbb{H}^2$ and $\mathcal{HQD}$ denote the sheaf of holomorphic quadratic differentials on $\mathbb{H}^2$. Then the following sequence of sheaves*

$$\mathcal{HOL} \xrightarrow{\alpha} \mathcal{HARM} \xrightarrow{\beta} \mathcal{HQD} \tag{2}$$

is a short exact sequence of sheaves on $\mathbb{H}^2$. In (2), α is the inclusion map and β is given by the formula in Proposition 0.0.2.

Remark 0.0.4. Theorem 0.0.3 is also valid if we replace $\mathbb{H}^2$ with $\mathbb{D}$.

Our next main Theorem is about proving the global surjectivity of the map β in (2) in Theorem 0.0.3.

Theorem 0.0.5 (Theorem 2.2.13 + Theorem 2.2.19)**.** *Let $q = f(z)dz^2$ be a holomorphic quadratic differential on $\mathbb{H}^2$. Suppose that q satisfies the following boundedness conditions*

1. *q is bounded in the hyperbolic metric $\boldsymbol{g}_{\mathbb{H}^2}$, i.e.,*

$$\|q\|_{\boldsymbol{g}_{\mathbb{H}^2}} = |f(z)| \|dz^2\|_{\boldsymbol{g}_{\mathbb{H}^2}} \leq D,$$

where $\|dz^2\|_{\boldsymbol{g}_{\mathbb{H}^2}} = \Im(z)^2$ and D is a positive real number.

2. *The first and second covariant derivative of q w.r.t ∇, the linear connection on $T^*\mathbb{H}^2 \otimes_{\mathbb{C}} T^*\mathbb{H}^2$, are bounded in the hyperbolic metric $\boldsymbol{g}_{\mathbb{H}^2}$.*

Then there exists a harmonic vector field ξ^{reg} on $\mathbb{H}^2$ such that $\beta(\xi^{\text{reg}}) = q$, where β is introduced in Theorem 0.0.3. An explicit formula is

$$\xi^{\text{reg}}(z) = \lim_{c \to \infty} \left(\xi_c(z) - \left(\xi_c(\iota) + \left.\frac{\partial \xi_c}{\partial z}\right|_{z=\iota} \cdot (z - \iota) \right) \right),$$

where

$$\xi_c(z) = \left(\int_{y_*(z)}^{c} \iota \zeta^2 \overline{f(\bar{z} + 2\iota\zeta)} d\zeta \right) \eta(z)$$

and c is a positive real number. The harmonic vector field ξ^{reg} transformed from $\mathbb{H}^2$ to the open unit disc $\mathbb{D}$ by the Cayley transform C extends to a continuous vector field, say χ, on $\overline{\mathbb{D}}$ defined as follows:

$$\chi(C(z)) = \begin{cases} C_*(\xi^{\mathrm{reg}}(z)) & z \in \mathbb{H}^2 \\ C_*(\xi^{\mathrm{reg}}(z)) & z \in \partial\mathbb{H}^2 \setminus \{\infty\} \\ 0 & z = \{\infty\} \end{cases}$$

where $C_*(\xi^{\mathrm{reg}}(z))$ is the pushforward of $\xi^{\mathrm{reg}}(z)$ by the Cayley transform C.

Remark 0.0.6. We have introduced a simple terminology reg short for "regularisation" to characterise our required harmonic vector field.

Remark 0.0.7. The global surjectivity of the map β in Theorem 0.0.3 is proven independently by S. Wolpert in [70, Section 2]. See the beginning of Subsection 2.2.1 in **Chapter 2**.

Chapter 2 is dedicated to proving Proposition 0.0.2, Theorem 0.0.3 and Theorem 0.0.5. It also discusses the main advantages of the method which is used in **Chapter 2** in proving Theorem 0.0.5 over Scott Wolpert's method.

Theorem 0.0.5 implies that the coboundary $\delta\chi$

$$\chi \longmapsto \big(\gamma \longmapsto \chi(\gamma)\gamma^{-1} - \chi\big), \quad \forall \gamma \in \Gamma$$

where Γ is a discrete cocompact subgroup of $\mathrm{Isom}^+(\mathbb{D})$, defines a 1-cocycle with values in the vector space HOL of holomorphic vector fields on $\mathbb{D}$. Note that we view χ as a 0-cocycle with values in the vector space of harmonic vector fields on $\mathbb{D}$.

In **Chapter 3**, we ensure that we get an explicit map from the vector space of Γ-invariant holomorphic quadratic differentials $\mathrm{HQD}(\mathbb{D}, \Gamma)$ on $\mathbb{D}$ to $H^1(\Gamma; \mathfrak{g})$ by using the theory of L^2-vector fields on $\mathbb{S}^1$. One of the main actors in Chapter 3 is the notion of a *tangential L^2-vector field* on $\mathbb{S}^1$ (see Definition 3.1.2 and Example 3.1.3). The main upshots of **Chapter 3** are the following results:

Theorem 0.0.8 (Theorem 3.1.4 + Theorem 3.1.5)**.** *Given a holomorphic quadratic differential $q = f dz^2$ on the Poincaré disk $\mathbb{D}$ which satisfies the following boundedness conditions:*

1. *q is bounded in the hyperbolic metric on $\mathbb{D}$, i.e.,*

$$\|q\|_{g_{\mathbb{D}}} \leq D,$$

   where D is a positive real number.

2. *The first and the second covariant derivative of q w.r.t the linear connection on $T^*\mathbb{D} \otimes_{\mathbb{C}} T^*\mathbb{D}$ are bounded in $g_{\mathbb{D}}$.*

Then there exists a harmonic vector field χ on $\mathbb{D}$ which admits an L^2-extension to the closed unit disk $\overline{\mathbb{D}}$ such that $(\mathcal{L}_\chi \mathbf{g}_\mathbb{D})^{(2,0)} = q$. Moreover, the restriction of that extension to the boundary circle $\mathbb{S}^1$ is tangential and χ is unique upto the addition of holomorphic vector fields on $\mathbb{D}$ which extend tangentially to the boundary circle $\mathbb{S}^1$. Also, χ is unique upto the addition of the vector space $\mathfrak{g}$ of Killing vector fields on $\mathbb{D}$.

Corollary 0.0.9 (Corollary 3.1.6)**.** *Let Γ denote a subgroup of $\mathrm{Isom}^+(\mathbb{D})$, where $\mathrm{Isom}^+(\mathbb{D})$ is the group of orientation preserving isometries of $\mathbb{D}$. If $q = f dz^2$ and χ are related as in Theorem 0.0.8 and if in addition to (1) and (2) in Theorem 0.0.8, q is Γ-invariant, i.e.,*

$$f(\gamma(z))\gamma'(z)^2 = f(z), \quad \forall \gamma \in \Gamma, z \in \mathbb{D},$$

then $\delta\chi$ defined by

$$\gamma \longmapsto \chi(\gamma)\gamma'^{-1} - \chi, \quad \forall \gamma \in \Gamma$$

is a 1-cocycle c for the group Γ with coefficients in the Lie algebra $\mathfrak{g}$ of $\mathrm{Isom}^+(\mathbb{D})$ and its cohomology class $[c]$ depends only on q.

Corollary 0.0.10 (Corollary 3.1.8)**.** *Let Γ be a discrete cocompact subgroup of $\mathrm{Isom}^+(\mathbb{D})$. Then we have an injective mapping*

$$\Phi : \mathrm{HQD}(\mathbb{D}, \Gamma) \longrightarrow H^1(\Gamma; \mathfrak{g})$$

$$q \longmapsto [c],$$

where $\mathrm{HQD}(\mathbb{D}, \Gamma)$ denotes the vector space of Γ-invariant holomorphic quadratic differentials on $\mathbb{D}$ and $c = \delta\chi$.

Chapter 4 is dedicated to constructing a map in the other direction in (1), i.e., from the cohomological description of tangent spaces to the analytic description

$$H^1(\Gamma; \mathfrak{g}) \xrightarrow{?} \mathrm{HQD}(\mathbb{D}, \Gamma),$$

where Γ denotes a discrete cocompact subgroup of $\mathrm{PSU}(1,1)$ and $\mathfrak{g}$ denotes the Lie algebra of $\mathrm{PSU}(1,1)$. Given any 1-cocycle c representing $[c] \in H^1(\Gamma; \mathfrak{g})$, we first construct a smooth vector field ψ on $\mathbb{D}$ such that $\delta\psi = c$ and ψ admits an L^2-extension to the closed unit disk $\overline{\mathbb{D}}$ whose restriction to the boundary circle $\mathbb{S}^1$ is tangential. This construction relies on the existence of a Γ-invariant partition of unity on $\mathbb{D}$. See Section 4.1 in **Chapter 4**.

Lemma 0.0.11 (Lemma 4.1.1)**.** *There exists a smooth function φ on $\mathbb{D}$ such that*

1. $0 \le \varphi \le 1$.
2. *For each $z \in \mathbb{D}$, there is a neighborhood U of z and a finite subset S of Γ such that $\varphi = 0$ on $\gamma(U)$ for every $\gamma \in \Gamma - S$.*

3. $\sum_{\gamma\in\Gamma}\varphi(\gamma(z))=1$ *on* $\mathbb{D}$.

Remark 0.0.12. We suspect that Lemma 0.0.11 is a simpler version of results on *Kleinian groups* (see [37]).

Lemma 0.0.13 (Lemma 4.1.3 + Corollary 4.1.7)**.** *Given any* $[c]\in H^1(\Gamma;\mathfrak{g})$ *we set*

$$\psi(z)=-\sum_{\gamma\in\Gamma}\varphi(\gamma(z))c_\gamma(z),\quad z\in\mathbb{D},$$

where φ *is introduced in Lemma 0.0.11.* ψ *is a* C^∞*-vector field on* $\mathbb{D}$ *such that* $\delta\psi=c$*. Moreover,* ψ *admits a unique* L^2*-extension to the closed unit disk* $\overline{\mathbb{D}}$ *whose restriction* $\psi^\sharp$ *to the boundary circle* $\mathbb{S}^1$ *is tangential.*

Remark 0.0.14. The above-mentioned construction of a vector field on the boundary circle $\mathbb{S}^1$ from a cocycle c representing $[c]\in H^1(\Gamma;\mathfrak{g})$ is in the spirit of *universal Teichmueller theory*. See [17], [19], [40], [41], [44] for more details.

For the construction of ψ in Lemma 0.0.13 we can either use the Γ-invariant partition of unity method or the difficult theory of Chapter 2 and Chapter 3 which produces a harmonic solution. Lemma 0.0.13 is valid for all of these but the construction of an L^2-extension of ψ to $\overline{\mathbb{D}}$ relies on the existence of harmonic vector fields. Therefore, it is worth asking the following:

Problem 0.0.15 (Problem 4.4.1)**.** *Is there a more direct way of proving Lemma 0.0.13 which does not take harmonicity into account?*

The final results of this thesis are based on the reincarnation (see Subsection 4.2.1) and adaptation of the *Poisson integral formula* in the case of continuous tangential vector fields on $\mathbb{S}^1$. First, we construct a harmonic vector field on the open unit disk $\mathbb{D}$ from a continuous tangential vector field X on $\mathbb{S}^1$. Note that a continuous tangential vector field X on $\mathbb{S}^1$ can be written as $X=fY$ where f is a real-valued continuous function on $\mathbb{S}^1$ and Y is the norm 1 tangential vector field on $\mathbb{S}^1$ given by $z\longmapsto\iota z$.

Theorem 0.0.16 (Theorem 4.2.11 + Lemma 4.2.15)**.** *Let* $\mathcal{S}_{C^0}(T\mathbb{S}^1)$ *be the Banach space of (tangential) continuous vector fields on* $\mathbb{S}^1$ *and* $\mathcal{S}_{C^0}(T\mathbb{D})$ *be the space of continuous vector fields on the open disk* $\mathbb{D}$*. A linear map*

$$\mathcal{F}:\mathcal{S}_{C^0}(T\mathbb{S}^1)\longrightarrow\mathcal{S}_{C^0}(T\mathbb{D})$$

is given by the normalized convolution

$$\mathcal{F}(X)=f*\boldsymbol{K},$$

where $\boldsymbol{K}$ *is the Poisson Kernel vector field given by*

$$\boldsymbol{K}(z)=\frac{\iota(1-|z|^2)^3}{|1-z|^2\cdot(1-\bar{z})^2}.$$

Moreover, $\mathcal{F}(X)$ is a harmonic vector field on the open unit disk $\mathbb{D}$, and $\mathcal{F}(X)$ and X make up a continuous vector field on the the closed unit disk $\overline{\mathbb{D}}$.

We adapt Theorem 0.0.16 in the case of tangential L^2-vector fields on $\mathbb{S}^1$ as follows:

Corollary 0.0.17 (Corollary 4.2.16)**.** *For an L^2-tangential vector field X on $\mathbb{S}^1$, X is an L^2-boundary extension of the smooth vector field $\mathcal{F}(X)$ on the open unit disk $\mathbb{D}$.*

Remark 0.0.18. We suspect that Corollary 0.0.17 is an infinitesimal version of the problem of finding harmonic extensions of quasiconformal maps (from $\mathbb{S}^1$ to itself) to the open unit disk $\mathbb{D}$ or the upper half plane $\mathbb{H}^2$. See [26] for more details.

We have not shown that there exists a *unique* harmonic extension of a tangential L^2-vector field X on $\mathbb{S}^1$ to the closed unit disk $\overline{\mathbb{D}}$. And this brings us to our second open problem:

Problem 0.0.19 (Problem 4.4.2)**.** *Given a tangential L^2-vector field X on the boundary circle $\mathbb{S}^1$, does there exist a unique harmonic extension to the closed unit disk $\overline{\mathbb{D}}$?*

From Theorem 0.0.16 and Corollary 0.0.17, we get the following result:

Theorem 0.0.20 (Theorem 4.3.1)**.** *Let Γ be a discrete cocompact subgroup of $\mathrm{PSU}(1,1)$. For every cocycle c representing a cohomology class $[c] \in H^1(\Gamma;\mathfrak{g})$, there exists a smooth vector field ψ on the open unit disk $\mathbb{D}$ such that $c = \delta\psi$. Moreover, any such ψ admits an L^2-extension to $\overline{\mathbb{D}}$ whose restriction $\psi^\sharp$ to the boundary circle $\mathbb{S}^1$ is tangential. There exists a homomorphism*

$$\begin{aligned} \Psi : H^1(\Gamma;\mathfrak{g}) &\longrightarrow \mathrm{HQD}(\mathbb{D},\Gamma) \\ [c] &\longmapsto \left(\mathcal{L}_{\mathcal{F}(\psi^\sharp)}\boldsymbol{g}_{\mathbb{D}}\right)^{(2,0)}, \end{aligned}$$

where the map $\mathcal{F}$ is introduced in Theorem 0.0.16 and $\mathcal{F}(\psi^\sharp)$ is a harmonic vector field on the open disk $\mathbb{D}$.

Corollary 0.0.21 (Corollary 4.3.2)**.**

$$\Phi \circ \Psi = \mathrm{Id},$$

where Φ is defined in Corollary 0.0.10 and Ψ is defined in Theorem 0.0.20.

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Chapter 1

Preliminaries

Throughout this chapter we denote a closed oriented surface of genus $g \geq 2$ by Σ_g. The main goal of this chapter is to gather some necessary results, prove that the Teichmueller space $\mathcal{T}(\Sigma_g)$ is a $6g-6$ dimensional manifold using techniques from differential topology, and discuss briefly about tangent spaces to the Teichmueller space. We have attempted to follow a coherent narrative.

1.1 Some facts from hyperbolic geometry

The upper half plane $\mathbb{H}^2$ with the metric $\mathbf{g}_{\mathbb{H}^2} = \frac{dx^2+dy^2}{y^2}$ and the Poincaré disk $\mathbb{D}$ with the metric $\mathbf{g}_{\mathbb{D}} = \frac{4dx^2+dy^2}{(1-(x^2+y^2))^2}$ are the common models for the hyperbolic plane. Semicircles and half lines orthogonal to $\mathbb{R}$ are the geodesics in the upper half plane model $\mathbb{H}^2$. In the Poincaré disk model $\mathbb{D}$, if two points z_1 and z_2 are on the same diameter then the geodesic from z_1 to z_2 is the Euclidean line segment joining them, otherwise the geodesic is the arc of circle, orthogonal to $\mathbb{S}^1$. Both $\mathbb{H}^2$ and $\mathbb{D}$ have curvature -1 w.r.t $\mathbf{g}_{\mathbb{H}^2}$ and $\mathbf{g}_{\mathbb{D}}$. Both $\mathbf{g}_{\mathbb{H}^2}$ and $\mathbf{g}_{\mathbb{D}}$ are invariant under

$$\mathrm{Aut}(\mathbb{H}^2) = \{f \in \mathrm{Aut}(\overline{\mathbb{C}}) | f(\mathbb{H}^2) = \mathbb{H}^2\},$$

where $\mathrm{Aut}(\overline{\mathbb{C}})$ is the automorphism group of the Riemann sphere $\overline{\mathbb{C}}$, and

$$\mathrm{Aut}(\mathbb{D}) = \{f \in \mathrm{Aut}(\overline{\mathbb{C}}) | f(\mathbb{D}) = \mathbb{D}\}.$$

Note that

$$\mathrm{Aut}(\mathbb{H}^2) \cong \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2),$$

where $\mathrm{Isom}^+(\mathbb{H}^2)$ is the group of orientation preserving isometries of $\mathbb{H}^2$. Every element of $\mathrm{Isom}^+(\mathbb{H}^2)$ has a form $\gamma(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$. We classify elements of $\mathrm{PSL}(2,\mathbb{R})$ based on an extremal problem on hyperbolic translation length as follows: for every $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ except the identity element, set

$$\alpha(\gamma) = \inf_{z \in \mathbb{H}^2} d_{\mathbb{H}^2}(z, \gamma(z)),$$

where $d_{\mathbb{H}^2}(-,-)$ denotes the hyperbolic distance, then

1. γ is *elliptic* if $\alpha(\gamma) = 0$ and there exists a point $z \in \mathbb{H}^2$ with $\alpha(\gamma) = d_{\mathbb{H}^2}(z, \gamma(z))$. In other words, z is a fixed point of γ;
2. γ is *parabolic* if $\alpha(\gamma) = 0$ but there exists no point $z \in \mathbb{H}^2$ with $\alpha(\gamma) = d_{\mathbb{H}^2}(z, \gamma(z))$;
3. γ is *hyperbolic* if $\alpha(\gamma) > 0$ and there exists a point $z \in \mathbb{H}^2$ with $\alpha(\gamma) = d_{\mathbb{H}^2}(z, \gamma(z))$.

Since $\mathbb{H}^2$ is isometric to $\mathbb{D}$, normal forms of above elements are given as follows:

1. Any elliptic element is conjugate to a rotation $z \longmapsto \lambda z$ in $\mathrm{Aut}(\mathbb{D})$, for some λ with $|\lambda| = 1$;
2. Any parabolic element is conjugate to either $z \longmapsto z+1$ or to $z \longmapsto z-1$ in $\mathrm{Aut}(\mathbb{H}^2)$, and these maps are not conjugate to each other;
3. Any hyperbolic element is conjugate to $z \longmapsto \lambda z$ in $\mathrm{Aut}(\mathbb{H}^2)$, where $\lambda > 1$.

Since elements of $\mathrm{PSL}(2, \mathbb{R})$ have matrix representations, they are also classified by `trace`, i.e., for a non-identity $\gamma \in \mathrm{PSL}(2, \mathbb{R})$ the following holds:

1. γ is parabolic iff $\mathtt{trace}^2(\gamma) = 4$;
2. γ is elliptic iff $0 \leqq \mathtt{trace}^2(\gamma) < 4$;
3. γ is hyperbolic iff $\mathtt{trace}^2(\gamma) > 4$.

[6], [9], [16], [24], [31], [38], [50], [52], [54], [58], [60], [62], [64], and [71] as well articles of [5], [28], and [59] are great references to absorb different flavours of hyperbolic geometry.

1.2 The Teichmüller space, a kaleidoscopic view

Understanding and generalizing a 'mathematical structure' on a 'mathematical object' is an important concept in every discipline of pure mathematics. In (Riemann) surface theory the study of *conformal structure*, *complex structure*, and *almost complex structure* on Σ_g has received much attention. We begin by giving an overview of the above-mentioned structures on Σ_g and also emphasize the interplay between them.

Definition 1.2.1 (Complex structure)**.** A *complex structure* J on Σ_g is an equivalence class of complex atlases, where two atlases, say, $\{U_i, f_i\}$ and $\{V_i, g_i\}$ are equivalent iff their union forms a new complex atlas.

Definition 1.2.2 (Almost complex structure)**.** *An almost complex structure* on Σ_g is a smooth bundle endomorphism $J : T\Sigma_g \longrightarrow T\Sigma_g$ such that

1. $\forall x \in \Sigma_g : J_x^2 = -I_x$,
2. $\forall$ nonzero $v \in T_x\Sigma_g : (v, J_x(v))$ is an oriented basis for $T_x\Sigma_g$.

Equivalently, an almost complex structure is a smooth section of the fiber bundle

$$\mathrm{GL}(\Sigma_g) \times_{\mathrm{GL}^+(2,\mathbb{R}^2)} \mathrm{GL}^+(2,\mathbb{R}^2)/\mathrm{GL}(1,\mathbb{C}^1) \longrightarrow \Sigma_g.$$

$\mathrm{GL}(1,\mathbb{C})$ is the multiplicative group of non-zero complex numbers embedded in the group $\mathrm{GL}^+(2,\mathbb{R}^2)$ of the real 2×2 matrices with positive determinant.

Definition 1.2.3 (Conformal structure)**.** A *conformal structure* on Σ_g is an equivalence class of Riemannian metrics on Σ_g where two Riemannian metrics h_1 and h_2 are equivalent if the following holds

$$h_1 = e^{2u} h_2,$$

where u is a real valued C^∞-function on Σ_g.

We denote the set of almost complex structures on Σ_g by $\mathcal{A}(\Sigma_g)$ and the set of complex structures on Σ_g by $\mathcal{C}(\Sigma_g)$. $\mathcal{A}(\Sigma_g)$ is endowed with the C^∞-topology and is clearly contractible because the homogeneous space $\mathrm{GL}^+(2,\mathbb{R}^2)/\mathrm{GL}(1,\mathbb{C}^1)$ is contractible. Getting an almost complex structure on Σ_g from a complex structure on Σ_g is obvious but the question of whether Σ_g admits a complex structure whose underlying almost complex structure is the given one is answered by the *Newlander-Nirenberg theorem*. Here is the precise formulation:

Theorem 1.2.4 (Korn-Lichtenstein Theorem [8], [49])**.** *There is an obvious (forgetful) map*

$$\begin{aligned} \Xi : \mathcal{C}(\Sigma_g) &\longrightarrow \mathcal{A}(\Sigma_g) \\ c \ni (U \subset \Sigma_g, \phi) &\longmapsto \left(J_\phi(x) := d\phi_x^{-1} \hat{J} d\phi_x, x \in U, \hat{J} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \end{aligned}$$

which is a bijection.

Remark 1.2.5. J_ϕ is independent of the choice of ϕ in the description of the map Ξ above.

Let $\mathrm{Diff}^+(\Sigma_g)$ be the topological group of all orientation preserving diffeomorphisms of Σ_g and let $\mathrm{Diff}_0^+(\Sigma_g)$ be the open subgroup of those orientation preserving diffeomorphisms which are homotopic to the identity. The group $\mathrm{Diff}^+(\Sigma_g)$ and $\mathrm{Diff}_0^+(\Sigma_g)$ acts on $\mathcal{A}(\Sigma_g)$ by

$$(f^*J)_x := (df_x)^{-1} J_{f(x)} df_x; \quad f \in \mathrm{Diff}^+(\Sigma_g).$$

The above action makes the bijective map $\mathcal{C}(\Sigma_g) \longrightarrow \mathcal{A}(\Sigma_g)$ in Theorem 1.2.4 $\mathrm{Diff}^+(\Sigma_g)$-equivariant. Futhermore, we call a Riemannian metric h on Σ_g with an almost complex structure J conformal if J is orthogonal w.r.t h. From the Uniformization theorem, Σ_g is biholomorphically equivalent to the quotient space $\mathbb{H}^2/\Gamma$, where Γ is a group of holomorphic automorphisms of $\mathbb{H}^2$ acting freely and properly discontinuously and is identified with a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$, i.e., a *Fuchsian group*. In other words, in any conformal class of Riemannian metrics on Σ_g, there exists a unique Riemannian metric of constant curvature -1. In summary, almost complex structures, complex structures, conformal structures, and Riemannian metrics of constant curvature -1 are equivalent notions for Σ_g.

1.2.1 Classical definition

We choose a basepoint $x_0 \in \Sigma_g$. The fundamental group $\pi_1(\Sigma_g, x_0)$ is generated by the homotopy classes $[a_1], [b_1], \ldots, [a_g], [b_g]$ induced from simple closed curves $a_1, b_1, \ldots, a_g, b_g$ with base point x_0 satisfying the following relation: $[[a_1], [b_1]] \cdots [[a_g], [b_g]] = 1$, where 1 is the unit element. We denote the fundamental group $\pi_1(\Sigma_g, x_0)$ by Γ_g. By abuse of notation, we denote the generators of Γ_g by $a_1, b_1, \ldots, a_g, b_g$ satisfying the fundamental relation $[a_1, b_1] \cdots [a_g, b_g] = 1$. From the Uniformization theorem, Γ_g is isomorphic to a discrete cocompact subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Before giving the classical definition of the Teichmueller space, we describe elements of Γ_g.

Proposition 1.2.6 ([35])**.** *Every non-identity element of Γ_g is hyperbolic.*

Proof We prove the proposition by contradiction. Assume that $\gamma \in \Gamma_g - \{1\}$ is either parabolic or elliptic. Note that Γ_g acts freely on $\mathbb{H}^2$ and hence cannot have elliptic elements. Now, assume that $\gamma \in \Gamma_g - \{ 1\}$ is a parabolic element. Since every parabolic element of $\mathrm{PSL}(2, \mathbb{R})$ is conjugate in $\mathrm{PSL}(2, \mathbb{R})$ to either $z \longmapsto z + 1$ or $z \longmapsto z - 1$ (see Subsection 1.1), we work with $\gamma(z) = z + 1$ for the rest of the proof. Let a be a positive real number. Let us denote the image of the segment joining ιa to $\gamma(\iota a)$ by the projection map $p : \mathbb{H}^2 \longrightarrow \mathbb{H}^2/\Gamma_g$ by C_a. We note that C_a is a closed curve. See Figure 1.1 below.

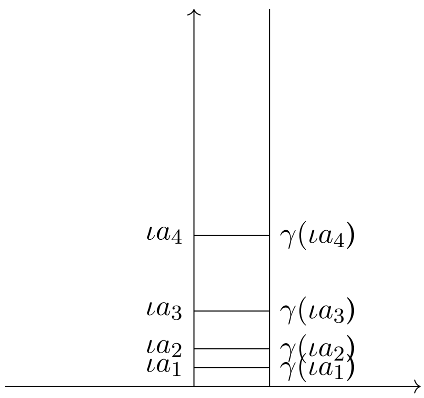

Figure 1.1: Line segments joining ιa_i to $\gamma(\iota a_i)$

Recall the Poincaré metric on $\mathbb{H}^2$ induces a hyperbolic metric on the compact surface $\mathbb{H}^2/\Gamma_g$. Let $l(C_a)$ be the hyperbolic length of C_a w.r.t to a hyperbolic metric on $\mathbb{H}^2/\Gamma_g$. We have one-to-one correspondence between the free homotopy classes of closed curves on the compact surface $\mathbb{H}^2/\Gamma_g$ and the set of conjugacy classes in the fundamental group $\pi_1(\mathbb{H}^2/\Gamma_g)$. So we could view C_a as an element of the fundamental group $\pi_1(\mathbb{H}^2/\Gamma_g)$. C_a is null-homotopic because for a sequence of positive numbers $\{a_i\}_{i=1}^{\infty}$, $l(C_{a_i}) \longrightarrow 0$ as $i \longrightarrow \infty$. In order to get a contradiction we have to show that C_a is not null-homotopic as an element of the fundamental group $\pi_1(\mathbb{H}^2/\Gamma_g) \simeq \Gamma_g$. It's obvious because we started with a non-identity element $\gamma \in \Gamma_g$. □

Lemma 1.2.7 ([29], [30]). *Let $\gamma_1, \gamma_2 \in \mathrm{PSL}(2,\mathbb{R}) - \{1\}$ be hyperbolic, where 1 denotes the identity element of $\mathrm{PSL}(2,\mathbb{R})$. Let $\mathrm{Fix}(\gamma_1)$ and $\mathrm{Fix}(\gamma_2)$ be the set of fixed points of γ_1 and γ_2, where the set of fixed points of an element $\gamma \in \mathrm{PSL}(2,\mathbb{R}) - \{1\}$ is the set of all $z \in \mathbb{R} \cup \{\infty\}$ satisfying $\gamma(z) = z$. Then γ_1 and γ_2 commute iff they have atleast one common fixed point, i.e.,*

$$z \in \mathrm{Fix}(\gamma_1) \cap \mathrm{Fix}(\gamma_2) \neq \emptyset.$$

Remark 1.2.8. We denote the centralizer of Γ_g in $\mathrm{PSL}(2,\mathbb{R})$ by $C_{\Gamma_g}\mathrm{PSL}(2,\mathbb{R})$. From Lemma 1.2.7, it is easy to see that $C_{\Gamma_g}\mathrm{PSL}(2,\mathbb{R})$ is trivial. Here is an argument: from Lemma 1.2.7, γ_1 and γ_2 are noncommuting iff $\mathrm{Fix}(\gamma_1) \cap \mathrm{Fix}(\gamma_2) = \emptyset$. Now, let's assume that $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ commutes with γ_1 and γ_2. Then γ fixes the axis of γ_1 and γ_2, since $\gamma(\mathrm{Ax}\gamma_i) = \mathrm{Ax}\gamma\gamma_i\gamma^{-1} = \mathrm{Ax}\gamma_i$, for $i = 1, 2$. Thus γ maps $\mathrm{Fix}(\gamma_i), i = 1, 2$ to itself. However we cannot conclude that $\gamma(z) = z$, $z \in \mathrm{Fix}(\gamma_i), i = 1, 2$. We have two possibilities:

1. γ is hyperbolic with the same axis as of γ_1 and γ_2.
2. γ is elliptic of order 2, i.e., γ interchanges the fixed points of γ_1 and γ_2. And $\gamma\gamma_i\gamma^{-1} = \gamma_i^{-1}$.

We can exclude both the possibilities because according to (1), γ has 4 fixed points, hence a contradiction. And from (2), $\gamma \notin C_{\Gamma_g}\mathrm{PSL}(2,\mathbb{R})$. Hence, $C_{\Gamma_g}\mathrm{PSL}(2,\mathbb{R})$ is trivial.

Definition 1.2.9. The Teichmüller space of Σ_g is defined as the space of equivalence classes of *marked hyperbolic surfaces*. By a *marked hyperbolic surface* we mean a pair (S, ϕ) where S is a hyperbolic surface, i.e., a closed oriented surface of genus $g \geq 2$ endowed with a fixed hyperbolic metric (a Riemannian metric of constant sectional curvature -1) and $\phi : \Sigma_g \longrightarrow S$ is an orientation preserving diffeomorphism. Equivalence relation is defined as follows:

$$(S, \phi) \sim (S', \psi),$$

if there exists an isometry $h : S \longrightarrow S'$ such that ψ is isotopic to $h \circ \phi$. We denote the Teichmüller space of Σ_g by $\mathcal{T}(\Sigma_g)$.

Remark 1.2.10. Note that there is a glitch in the above definition as we have not introduced a topology on the Teichmüller space $\mathcal{T}(\Sigma_g)$. There is a notion of the *Teichmueller metric* which gives a topology on $\mathcal{T}(\Sigma_g)$. See [14] and [30] for a complete understanding.

1.2.2 $\mathcal{T}(\Sigma_g)$ as a representation variety

Let Γ be a finitely generated group and G be a connected Lie group. The most interesting case for us is when $\Gamma = \Gamma_g$ and $G = \mathrm{Isom}^+(\mathbb{H}^2) \cong \mathrm{PSL}(2,\mathbb{R})$. Let $\mathrm{Hom}(\Gamma, G)$ denote the space of all homomorphisms $\Gamma \longrightarrow G$ with the compact-open topology. The space $\mathrm{Hom}(\Gamma, G)$ has the structure of an algebraic variety which can be described as follows: Let $a_1, \ldots, a_n$ denote the generators and $R_1, \ldots, R_r, \ldots$ the corresponding relators in a presentation of Γ. Note that G can be described as a closed subgroup of $\mathrm{GL}(k,\mathbb{R})$ for some large k. Therefore, we can think of G as a real algebraic subgroup of $\mathrm{GL}(k,\mathbb{R})$. The representation variety $\mathrm{Hom}(\Gamma, G)$ is isomorphic to the algebraic subvariety (in G^n)

$$\{(g_1,\dots,g_n)\in G^n | R_1(g_1,\dots,g_n)=e,\dots,R_r(g_1,\dots,g_n)=e,\dots\},$$

where e is the identity element in G. The isomorphism type of the variety $\mathrm{Hom}(\Gamma, G)$ does not depend on the choice of the presentation of Γ (see [34], [43]). Note that the spaces $\mathrm{Hom}(\Gamma, G)$ are not generally manifolds. The natural symmetries of the space $\mathrm{Hom}(\Gamma, G)$ come from the action of $\mathrm{Aut}(\Gamma)\times\mathrm{Aut}(G)$ where the action is described as: if $\gamma\in\mathrm{Aut}(\Gamma)$ and $\alpha\in\mathrm{Aut}(G)$, then $\rho^{(\gamma,\alpha)}\in\mathrm{Hom}(\Gamma, G)$ is defined as:

$$\rho^{(\gamma,\alpha)}(x)=(\alpha\circ\rho\circ\gamma^{-1})(x).$$

We will be mainly concerned with the quotient space of $\mathrm{Hom}(\Gamma, G)$ by $\mathrm{Inn}(G)$ which will be denoted by $\mathrm{Hom}(\Gamma, G)/G$. Note that $\mathrm{Inn}(G)$ does not act freely on $\mathrm{Hom}(\Gamma, G)$ in some cases. The isotropy group of a point $\rho\in\mathrm{Hom}(\Gamma, G)$ is the centralizer $C_G(\rho)$ in $\mathrm{Inn}(G)$ and $\mathrm{Inn}(G)$ acts freely on $\mathrm{Hom}(\Gamma, G)$ if $C_G(\rho)$ is trivial for all $\rho\in\mathrm{Hom}(\Gamma, G)$. In the case of our interest, i.e., when $\Gamma=\Gamma_g$ and $G=\mathrm{PSL}(2,\mathbb{R})$, we overcome this pathology (see Remark 1.2.8). The quotient space $\mathrm{Hom}(\Gamma, G)/G$ is not generally a Hausdorff space unless G is a compact Lie group.

Definition 1.2.11.

$$\mathrm{Hom}_{\mathrm{DF}}(\Gamma, G) := \{\rho\in\mathrm{Hom}(\Gamma, G)|\rho \text{ is injective with discrete image}\},$$
$$\mathrm{Hom}_0(\Gamma, G) := \{\rho\in\mathrm{Hom}_{\mathrm{DF}}(\Gamma, G)|G/\rho(\Gamma) \text{ is compact}\}.$$

Remark 1.2.12. It is clear that $\mathrm{Hom}_0(\Gamma, G)\subset\mathrm{Hom}_{\mathrm{DF}}(\Gamma, G)\subset\mathrm{Hom}(\Gamma, G)$. $\mathrm{Hom}_0(\Gamma, G)$ is an open subset of $\mathrm{Hom}(\Gamma, G)$ [66], [67].

Definition 1.2.13. The Teichmueller space $\mathcal{T}(\Sigma_g)$ of Σ_g is (also) defined as the quotient space $\mathrm{Hom}_0(\Gamma_g,\mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$, where $\mathrm{Hom}_0(\Gamma_g,\mathrm{PSL}(2,\mathbb{R}))$ is defined in Definition 1.2.11.

The above definition will be the main definition of the Teichmueller space in this thesis. Now, we prove the following general fact using techniques from differential topology:

Proposition 1.2.14. $\mathrm{Hom}_0(\Gamma_g,\mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ *has a preferred structure of smooth manifold of dimension* $6g-6$.

Proof: We prove the statement in the following steps:

Step I: Here we prove that $\mathrm{Hom}_0(\Gamma_g,\mathrm{PSL}(2,\mathbb{R}))$ is a smooth manifold of dimension $6g-3$. Since a homomorphism $\rho:\Gamma_g\longrightarrow\mathrm{PSL}(2,\mathbb{R})$ is determined by choosing the $2g$ images $\rho(a_i),\rho(b_i), 1\le i\le g$, there is a natural inclusion of $\mathrm{Hom}(\Gamma_g,\mathrm{PSL}(2,\mathbb{R}))$ into the direct product $\mathrm{PSL}(2,\mathbb{R})^{2g}$ of $2g$ copies of $\mathrm{PSL}(2,\mathbb{R})$. Consider the following map

$$R:\mathrm{PSL}(2,\mathbb{R})^{2g}\longrightarrow\mathrm{PSL}(2,\mathbb{R})$$

given by

$$R(A_1,B_1,\dots,A_g,B_g)=A_1B_1A_1^{-1}B_1^{-1}\cdots A_gB_gA_g^{-1}B_g^{-1}. \tag{1.1}$$

Claim: We assume that A_1 and B_1 are noncommuting hyperbolic elements. Then the differential of R at $(A_1, B_1, \dots, A_g, B_g) \in \mathrm{PSL}(2,\mathbb{R})^{2g}$ is surjective.

Proof of the Claim: By precomposing the map R given in (1.1) with the map

$$\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R}) \longrightarrow \mathrm{PSL}(2,\mathbb{R})^{2g}$$

$(A, B) \longmapsto (A, B, 1, \dots, 1)$, we get a map

$$\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R}) \longrightarrow \mathrm{PSL}(2,\mathbb{R})$$

given by

$$(A, B) \longmapsto ABA^{-1}B^{-1}. \tag{1.2}$$

We denote this composite map by R as well. Therefore, proving the above-mentioned claim amounts to proving the following statement: Let $\mathfrak{g}$ denote the Lie algebra of $\mathrm{PSL}(2,\mathbb{R})$. If A and B are noncommuting hyperbolic elements, then the differential of the map R given in (1.2)

$$dR(A,B) : T_{(A,B)}(\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})) \longrightarrow T_{R(A,B)}\mathrm{PSL}(2,\mathbb{R}) \tag{1.3}$$

is surjective. For the calculation of the differential $dR(A, B)$ we can replace $\mathrm{PSL}(2,\mathbb{R})$ with $\mathrm{SL}(2,\mathbb{R})$. A simple calculation shows that

$$T_A\mathrm{SL}(2,\mathbb{R}) = A \cdot \mathfrak{gl}(2,\mathbb{R}),$$

where $\mathfrak{gl}(2,\mathbb{R})$ is the Lie algebra of $\mathrm{SL}(2,\mathbb{R})$, equivalently, the tangent space at the identity. From this discussion on tangent spaces, we can write (1.3) as

$$dR(A,B) : A\mathfrak{gl}(2,\mathbb{R}) \times B\mathfrak{gl}(2,\mathbb{R}) \longrightarrow R(A,B)\mathfrak{gl}(2,\mathbb{R}). \tag{1.4}$$

Now, we prove the surjectivity of the map given by (1.4). First, we calculate the differential of R at (A, B). Let $u, v \in \mathfrak{gl}(2,\mathbb{R})$. For $t \to 0$, we have

$$\begin{aligned}
R(A\exp tu, B\exp tv) - R(A,B) &\approx A(I+tu)B(I+tv)(I-tu)A^{-1}(I-tv)B^{-1} - ABA^{-1}B^{-1} \\
&\approx (A+Atu)(B+Btv)(A^{-1}-tuA^{-1})(B^{-1}-tvB^{-1}) - ABA^{-1}B^{-1} \\
&\approx (AB+ABtv+AtuB)(A^{-1}B^{-1} - A^{-1}tvB^{-1} - tuA^{-1}B^{-1}) \\
&\quad - ABA^{-1}B^{-1} \\
&\approx ABA^{-1}B^{-1} - ABA^{-1}tvB^{-1} - ABtuA^{-1}B^{-1} + ABtvA^{-1}B^{-1} \\
&\quad + AtuBA^{-1}B^{-1} - ABA^{-1}B^{-1} \\
&\approx -ABA^{-1}tvB^{-1} - ABtuA^{-1}B^{-1} + ABtvA^{-1}B^{-1} + AtuBA^{-1}B^{-1} \\
&\approx AB\big(-A^{-1}tvA - tu + tv + B^{-1}tuB\big)A^{-1}B^{-1}.
\end{aligned} \tag{1.5}$$

Recall that the adjoint representation Ad of $\mathrm{SL}(2,\mathbb{R})$ on $\mathfrak{gl}(2,\mathbb{R})$ is defined by

$$(\mathrm{Ad}A)w := A^{-1}wA, \quad w \in \mathfrak{gl}(2,\mathbb{R}).$$

Therefore, the differential $dR(A, B) : A\mathfrak{gl}(2, \mathbb{R}) \times B\mathfrak{gl}(2, \mathbb{R}) \longrightarrow R(A, B)\mathfrak{gl}(2, \mathbb{R})$ is given by the following:

$$(Au, Bv) \longmapsto AB\big((\mathrm{Ad}B)u - u + v - (\mathrm{Ad}A)v\big)A^{-1}B^{-1}, \quad u, v \in \mathfrak{gl}(2, \mathbb{R}). \tag{1.6}$$

It is enough to show that the map $\mathfrak{gl}(2, \mathbb{R}) \times \mathfrak{gl}(2, \mathbb{R}) \longrightarrow \mathfrak{gl}(2, \mathbb{R})$ given by

$$(u, v) \longmapsto (\mathrm{Ad}B)u - u + v - (\mathrm{Ad}A)v, \quad u, v \in \mathfrak{gl}(2, \mathbb{R}) \tag{1.7}$$

is surjective.

Proof of surjectivity of the map given in (1.7) : Note that $\mathrm{SL}(2, \mathbb{R})$ preserves a non-degenerate bilinear form on its Lie algebra $\mathfrak{gl}(2, \mathbb{R})$. Moreover, $\mathrm{PSL}(2, \mathbb{R})$ embeds into the isometry group of the Killing form on $\mathfrak{gl}(2, \mathbb{R})$. So, we think of B as an element of one parameter subgroup generated by $b \in \mathfrak{gl}(2, \mathbb{R})$ of the isometry group of the Killing form on $\mathfrak{gl}(2, \mathbb{R})$. The image of the linear map $u \longmapsto (\mathrm{Ad}B)u - u$ from $\mathfrak{gl}(2, \mathbb{R})$ to itself is precisely the 2-dimensional subspace of $\mathfrak{gl}(2, \mathbb{R})$ which is perpendicular (in the sense of the Killing form) to b. Similarly, the image of the linear map $v \longmapsto v - (\mathrm{Ad}A)v$ from $\mathfrak{gl}(2, \mathbb{R})$ to itself is precisely the 2-dimensional subspace of $\mathfrak{gl}(2, \mathbb{R})$ which is perpendicular (in the sense of Killing form) to $a \in \mathfrak{gl}(2, \mathbb{R})$. Since we have chosen A and B such that they are noncommuting hyperbolic elements, a and b are linearly independent in $\mathfrak{gl}(2, \mathbb{R})$. The reader can also verify these two statements in coordinates, i.e., by making choices for B (and A respectively), u (and v respectively) and plugging these into $u \longmapsto (\mathrm{Ad}B)u - u$ and $v \longmapsto v - (\mathrm{Ad}A)v$. Therefore, the map $\mathfrak{gl}(2, \mathbb{R}) \times \mathfrak{gl}(2, \mathbb{R}) \longrightarrow \mathfrak{gl}(2, \mathbb{R})$ given in (1.7) is surjective. □

We denote the subset of $\mathrm{PSL}(2, \mathbb{R})^{2g}$ consisting of elements $A_1, B_1, \ldots, A_g, B_g$ such that A_1, B_1 are noncommuting hyperbolic elements by W. Since W is open in $\mathrm{PSL}(2, \mathbb{R})^{2g}$, hence W is a manifold of dimension $6g$. From the above-mentioned claim, 1 is a regular value of the restriction map $R|_W : W \longrightarrow \mathrm{PSL}(2, \mathbb{R})$. In fact, every value of the map $R|_W$ is a regular value. Hence, $R|_W^{-1}(1)$ is a submanifold of W of dimension $6g - 3$. Note that $R|_W^{-1}(1)$ is nothing but $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2, \mathbb{R})) \cap W$. From Remark 1.2.12, we know that $\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$ is an open subset of $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$, therefore, $\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$ is a $6g - 3$ dimensional smooth manifold.

Step II: In this step, we study the action of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$. Given $g \in \mathrm{PSL}(2, \mathbb{R})$ and $\rho \in \mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$, we define $\rho^g : \Gamma_g \longrightarrow \mathrm{PSL}(2, \mathbb{R})$ by setting

$$\rho^g(\gamma) = g\rho(\gamma)g^{-1}, \quad \forall \gamma \in \Gamma_g. \tag{1.8}$$

The map $(g, \rho) \longmapsto \rho^g$ is a continuous action of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$. We want to show that the action is free and that the orbit space of this action is again a smooth manifold. Consider the following map

$$\psi_1 : \mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2, \mathbb{R})) \longrightarrow \mathrm{Conf}_3(\partial\mathbb{H}^2),$$

$$\rho \longmapsto (z_1, z_2, z_3)$$

where $\mathrm{Conf}_3(\partial\mathbb{H}^2)$ is the space of ordered configurations of distinct 3 points in the boundary $\partial\mathbb{H}^2$. In the above map, z_1, z_2 are *attractive* and *repelling* fixed points of A_1, i.e.,

$$\lim_{n\to\infty} A_1^n(z) = z_1, \forall z \in \mathbb{H}^2, \quad \lim_{n\to-\infty} A_1^n(z) = z_2, \forall z \in \mathbb{H}^2,$$

and z_3 is the attractive fixed point of B_1. Moreover, the group $\mathrm{PSL}(2,\mathbb{R})$ acts sharply transitively on ordered triples in $\partial\mathbb{H}^2$, we can also think of ψ_1 as a map

$$\psi_1 : \mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R})) \longrightarrow \mathrm{PSL}(2,\mathbb{R}).$$

Note that we have identified $\mathrm{PSL}(2,\mathbb{R})$ with $\mathrm{Conf}_3(\partial\mathbb{H}^2)$ by the map $g \longmapsto g\cdot(0,1,\infty)$. Observe that ψ_1 is a $\mathrm{PSL}(2,\mathbb{R})$-equivariant map, i.e.,

$$\psi_1(g\cdot\rho) = g\cdot\psi_1(\rho), \quad \forall g \in \mathrm{PSL}(2,\mathbb{R}),$$

where the action on the LHS is by conjugation and the action on the RHS is by left-multiplication. In other words, if we change ρ by conjugating it by an element $g \in \mathrm{PSL}(2,\mathbb{R})$, the three distinct points z_1, z_2, z_3 in $\partial\mathbb{H}^2$ are also transformed by the same element $g \in \mathrm{PSL}(2,\mathbb{R})$. The only thing we have to show now is that ψ_1 is differentiable. Here is an argument: $\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ is also a closed subset of $\mathrm{PSL}(2,\mathbb{R})^{2g}$. Now, ψ_1 extends to a small open neighborhood $\mathscr{U}$ of $\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ in $\mathrm{PSL}(2,\mathbb{R})^{2g}$. We know that an element $\rho \in \mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ is determined by hyperbolic elements $(A_1, B_1, \ldots, A_g, B_g) \in \mathrm{PSL}(2,\mathbb{R})^{2g}$ satisfying the relation $[A_1, B_1]\cdots[A_g, B_g] = 1$. Since the set of hyperbolic elements form an open subset of $\mathrm{PSL}(2,\mathbb{R})$ (see Subsection 1.1), then an open neighborhood $\mathscr{U} \subseteq \mathrm{PSL}(2,\mathbb{R})^{2g}$ of $\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ also contains hyperbolic elements $A_1', B_1', \ldots, A_g', B_g'$ which may not satisfy $[A_1', B_1']\cdots[A_g', B_g'] = 1$. The upshot is ψ_1 is smooth because it is the restriction of a map defined on an open neighborhood $\mathscr{U}$ of $\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ which is obviously smooth. $\mathrm{PSL}(2,\mathbb{R})$-equivariance of ψ_1 makes immediately clear that ψ_1 is everywhere regular. Therefore, $\psi_1^{-1}(1)$ is a submanifold of codimension 3 of $\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$. We denote $\psi_1^{-1}(1)$ by Z. Tying it all together, the action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ admits a transversal, i.e., there exists a submanifold Z of $\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ of codimension 3 such that the action of $\mathrm{PSL}(2,\mathbb{R})$ gives us a diffeomorphism

$$\psi_2 : \mathrm{PSL}(2,\mathbb{R}) \times Z \longrightarrow \mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$$

$$\psi_2(g,z) = gzg^{-1}.$$

Therefore, the orbit space $\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ is diffeomorphic to Z. □

Remark 1.2.15. Note that a different choice of generators for Γ_g will give the same structure of smooth manifold on $\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$.

Remark 1.2.16. We were only made aware of Earle and Eells' paper [11], where they only give a sketch proof of Step 1 at the end of this thesis research. Many thanks to Johannes Ebert.

1.3 Tangent spaces to the Teichmüller space

1.3.1 Cohomological description

Let Γ be a finitely generated group and G be a connected Lie group with Lie algebra $\mathfrak{g}$. We can obtain a (linear) action of Γ on $\mathfrak{g}$ by fixing a homomorphism $\rho_0 : \Gamma \longrightarrow G$ and composing ρ_0 with the adjoint representation of G and hence make $\mathfrak{g}$ a $k\Gamma$-module where $k = \mathbb{R}$ or $\mathbb{C}$. We denote $\mathfrak{g}$ with the above-mentioned Γ-module structure by $\mathfrak{g}_{\mathrm{Ad}\rho_0}$. A map $c : \Gamma \longrightarrow \mathfrak{g}$ is called a *1-cocycle* if

$$c(\gamma_1\gamma_2) = c(\gamma_1) + \mathrm{Ad}(\rho_0(\gamma_1))c(\gamma_2), \quad \forall \gamma_1, \gamma_2 \in \Gamma. \tag{1.9}$$

c is a *1-coboundary* if it has the following form

$$c(\gamma) = u - \mathrm{Ad}(\rho_0(\gamma))u \tag{1.10}$$

for some $u \in \mathfrak{g}$. The (real vector) space of 1-cocycles is denoted by $Z^1(\Gamma; \mathfrak{g}_{\mathrm{Ad}\rho_0})$ and the (real vector) space of 1-coboundaries is denoted by $B^1(\Gamma; \mathfrak{g}_{\mathrm{Ad}\rho_0})$. Their quotient is the group cohomology

$$H^1(\Gamma; \mathfrak{g}_{\mathrm{Ad}\rho_0}) = Z^1(\Gamma; \mathfrak{g}_{\mathrm{Ad}\rho_0})/B^1(\Gamma; \mathfrak{g}_{\mathrm{Ad}\rho_0}).$$

When $\Gamma = \pi_1(M)$ for a topological space M, $H^1(\Gamma; \mathfrak{g}_{\mathrm{Ad}\rho_0})$ can be identified with $H^1(M; \mathfrak{g}_{\mathrm{Ad}\rho_0})$, the first cohomology of M with coefficients in the local system given by $\mathfrak{g}_{\mathrm{Ad}\rho_0}$. For more details on group cohomology, the reader is referred to [7]. We are interested in the case when $\Gamma = \Gamma_g$, $G = \mathrm{PSL}(2,\mathbb{R})$, and $\mathfrak{g}$ is the Lie algebra of $\mathrm{PSL}(2,\mathbb{R})$.

Proposition 1.3.1 ([43, Theorem 2.6], [51, Chapter VI])**.** $T_{[\rho_0]}\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R}) \cong H^1(\Gamma_g; \mathfrak{g}_{\mathrm{Ad}\rho_0})$.

Proof We construct a linear map

$$\Psi : T_{[\rho_0]}\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R}) \longrightarrow H^1(\Gamma_g; \mathfrak{g}_{\mathrm{Ad}\rho_0})$$

as follows: to the first order, a curve of maps $(\rho_t)_{t\in[0,\epsilon)}$ in $\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ through the point ρ_0 depending smoothly on the real parameter t is described as:

$$\rho_t(\gamma) = \exp\big(tc(\gamma) + O(t^2)\big)\rho_0(\gamma), \quad \forall \gamma \in \Gamma_g.$$

The infinitesimal condition for ρ_t to be a homomorphism is given as:

$$\begin{aligned}
\rho_t(\gamma_1\gamma_2) &= (e + tc_{\gamma_1\gamma_2} + O(t^2))\rho_0(\gamma_1\gamma_2)\\
&= \rho_0(\gamma_1\gamma_2) + tc_{\gamma_1\gamma_2}\rho_0(\gamma_1\gamma_2) + O(t^2)\\
&= \big(\rho_0(\gamma_1) + tc_{\gamma_1}\rho_0(\gamma_1)\big)\big(\rho_0(\gamma_2) + tc_{\gamma_2}\rho_0(\gamma_2)\big) + O(t^2)\\
&= \rho_0(\gamma_1\gamma_2) + t\big(\rho_0(\gamma_1)c_{\gamma_2}\rho_0(\gamma_2) + c_{\gamma_1}\rho_0(\gamma_1)\rho_0(\gamma_2)\big) + O(t^2)\\
&= \rho_0(\gamma_1\gamma_2) + t\big(\rho_0(\gamma_1)c_{\gamma_2} + c_{\gamma_1}\rho_0(\gamma_1)\big)\rho_0(\gamma_2) + O(t^2)\\
&= \rho_0(\gamma_1\gamma_2) + t\big(\rho_0(\gamma_1)c_{\gamma_2}\rho_0(\gamma_1)^{-1}\rho_0(\gamma_1) + c_{\gamma_1}\rho_0(\gamma_1)\big)\rho_0(\gamma_2) + O(t^2)\\
&= \rho_0(\gamma_1\gamma_2) + t\big(\rho_0(\gamma_1)c_{\gamma_2}\rho_0(\gamma_1)^{-1} + c_{\gamma_1}\big)\rho_0(\gamma_1)\rho_0(\gamma_2) + O(t^2)\\
&= \rho_0(\gamma_1\gamma_2) + t\big(\mathrm{Ad}(\rho_0(\gamma_1))c_{\gamma_2} + c_{\gamma_1}\big)\rho_0(\gamma_1\gamma_2) + O(t^2).
\end{aligned}$$

From the above equation, notice that

$$c_{\gamma_1\gamma_2} = \mathrm{Ad}(\rho_0(\gamma_1))c_{\gamma_2} + c_{\gamma_1}.$$

Therefore, we define $\Psi\big(\frac{d}{dt}\rho_t|_{t=0}\big)$ to be the cocycle $c \in Z^1(\Gamma_g; \mathfrak{g}_{\mathrm{Ad}\rho_0})$. We show next that Ψ is injective. Suppose that the cocycle c determined by ρ_t is a coboundary, i.e., $c(\gamma) = u - \mathrm{Ad}(\rho_0(\gamma))u$ for some $u \in \mathfrak{g}$ (see (1.10)). Then the curve $\rho_t(\gamma) = g_t\rho_0(\gamma)g_t^{-1}$ induced by a path $g_t = e + tu + O(t^2), u \in \mathfrak{g}$ is tangent at $t = 0$ to the orbit $\mathrm{PSL}(2,\mathbb{R})\rho_0$ for all $\gamma \in \Gamma_g$. Moreover, Ψ is surjective because of the fact that $\mathrm{dim}H^1(\Gamma_g; \mathfrak{g}_{\mathrm{Ad}\rho_0}) = 6g - 6$. The fact follows from a non-trivial result (see [21]) that given a connected Lie group G and $\rho_0 \in \mathrm{Hom}(\Gamma_g, G)$,

$$\mathrm{dim}Z^1(\Gamma_g; \mathfrak{g}_{\mathrm{Ad}\rho_0}) = (2g-1)\mathrm{dim}G + \mathrm{dim}C_G(\rho(\Gamma_g)),$$

where $C_G(\rho(\Gamma_g))$ denotes the the centralizer of $\rho(\Gamma_g)$ in G and

$$\mathrm{dim}B^1(\Gamma_g; \mathfrak{g}_{\mathrm{Ad}\rho_0}) = \mathrm{dim}G - \mathrm{dim}C_G(\rho(\Gamma_g)).$$

For the case of our interest, i.e., when $G = \mathrm{PSL}(2,\mathbb{R})$, $C_G(\rho(\Gamma_g))$ is trivial (see Remark 1.2.8). Therefore,

$$\mathrm{dim}Z^1(\Gamma_g; \mathfrak{g}_{\mathrm{Ad}\rho_0}) = (2g-1)\mathrm{dim}G = 6g-3, \quad \mathrm{dim}B^1(\Gamma_g; \mathfrak{g}_{\mathrm{Ad}\rho_0}) = \mathrm{dim}G = 3.$$

□

Remark 1.3.2. Note that $\rho_0 \in \mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ can be lifted to a homomorphism $\widetilde{\rho_0} : \Gamma_g \longrightarrow \mathrm{SL}(2,\mathbb{R})$ because the Euler class $e(\rho_0)$ of the oriented $\mathbb{S}^1$-bundle associated to ρ_0 equals twice the Euler number of $\mathbb{R}^2$-bundle associated to $\widetilde{\rho_0}$, i.e., $e(\rho_0) = 2g - 2$. See [46, Appendix C] for more details. As a result, in the above proof, the expressions of $\rho_t(\gamma_1\gamma_2)$ and g_t are justified.

1.3.2 Analytic description: Holomorphic quadratic differentials

Let K_{Σ_g} be the canonical line bundle, that is, the line bundle over Σ_g such that the fiber K_x over any point $x \in \Sigma_g$ is the complex cotangent space $T_x^*\Sigma_g$ to Σ_g at x. Let Q_{Σ_g} be the tensor square of the canonical line bundle K_{Σ_g}. The bundle Q_{Σ_g} and its sections provide a glimpse into one of the important aspects of the Teichmueller theory.

Definition 1.3.3. A holomorphic quadratic differential on Σ_g is a holomorphic section of Q_{Σ_g}.

We will denote a holomorphic quadratic differential on Σ_g by q. Locally, q on Σ_g as specified in any atlas $\{(U_i, z_i)\}$ can be described as $f_i(z_i)dz_i^2$, where each f_i is a holomorphic function on U_i of Σ_g and $dz_i^2 := dz_i \otimes dz_i$ is a section of Q_{Σ_g}. Let's denote the space of holomorphic quadratic differentials on Σ_g by $\mathrm{HQD}(\Sigma_g)$. Since K_{Σ_g} has degree $2g-2$, the Riemann-Roch formula (see [15]) implies that

$$\mathrm{dim}(\mathrm{HQD}(\Sigma_g)) = \mathrm{deg}(Q_{\Sigma_g}) - g + 1 = 3g - 3.$$

Note that the bundle Q_{Σ_g} appears in a splitting of the bundle $S^2(T\Sigma_g)$ of (real) symmetric bilinear forms on $T\Sigma_g$. This splitting is described as follows: one summand is the 1-dimensional real vector subbundle spanned by the everywhere nonzero section of the hyperbolic metric **g** on Σ_g. The other summand is the image of the bundle of quadratic differentials under the following embedding:

$$\psi : \hom_{\mathbb{C}}(T\Sigma_g \otimes_{\mathbb{C}} T\Sigma_g, \mathbb{C}) \longrightarrow S^2(T\Sigma_g) \tag{1.11}$$

where $\psi(q)$ is the real part of q, viewed as a (family) of symmetric $\mathbb{R}$-bilinear forms. This subbundle is the *trace-free* summand by definition. It is a 2-dimensional subbundle of a 3-dimensional (real) vector bundle which comes with a structure of 1-dimensional complex vector bundle. We illustrate the above splitting as follows:

Example 1.3.4. Let U be an open subset of $\mathbb{C}$ with the complex structure induced from $\mathbb{C}$. Then TU is identified with a trivial bundle $\mathbb{C} \times U \longrightarrow U$ and therefore, $\hom_{\mathbb{C}}(TU \otimes_{\mathbb{C}} TU)$ is also identified with a trivial bundle $\mathbb{C} \times U \longrightarrow U$. Therefore, quadratic differentials on U whether holomorphic or not, are identified with complex valued functions on U. For such a function f, we get

$$\psi(f)(z) = \begin{bmatrix} \Re(f(z)) & -\Im(f(z)) \\ -\Im(f(z)) & -\Re(f(z)) \end{bmatrix},$$

where ψ is the map given in (1.11). This is very easy to check. The preferred ordered basis of $T_zU \cong \mathbb{C}$ as a real vector space is $\{1, \iota\}$. If $f(z) = x + y\iota$ then $\Re(1 \cdot f(z) \cdot 1) = x$, $\Re(\iota \cdot f(z) \cdot \iota) = -x$, $\Re(1 \cdot f(z) \cdot \iota) = -y$.

From the above discussion, it follows automatically that a holomorphic quadratic differential q on Σ_g gives a one parameter family $\{g(t)\}_{t\in[0,\epsilon)}$ of deformations of **g** on Σ_g such that $g(0) = \mathbf{g}$ and $\frac{dg(t)}{dt}\big|_{t=0} = \psi(q)$. In other words, for t close to 0, $g(t) = \mathbf{g} + t\psi(q)$. We view $g(t)$ as a curve in the space $\mathcal{M}$ of Riemannian metrics on Σ_g. Recall that a Riemannian metric on Σ_g determines an almost complex structure J on Σ_g which further determines a complex structure on Σ_g. This is due to the Korn-Lichtenstein theorem (Theorem 1.2.4). Consequently, we get a one parameter family of complex curves $\{\Sigma_g^t\}_{t\in[0,\epsilon)}$. From the "Uniformization theorem", each of these complex curves in the family $\{\Sigma_g^t\}_{t\in[0,\epsilon)}$ has a preferred hyperbolic metric. Hence, we view $\{\Sigma_g^t\}_{t\in[0,\epsilon)}$ as a smooth curve in the Teichmueller space $\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ such that

$$\left.\frac{d\Sigma_g^t}{dt}\right|_{t=0} \in T_{[\rho_0]}\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R}).$$

In summary, we have a linear map from $\mathrm{HQD}(\Sigma_g)$ to $T_{[\rho_0]}\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$. The injectivity of this linear map follows from [55], [69]. Furthermore, this map is a bijective linear map because the dimension of $\mathrm{HQD}(\Sigma_g)$ and $T_{[\rho_0]}\mathrm{Hom}_0(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ (as real vector spaces) is same.

Chapter 2

Explicit expressions of harmonic vector fields on $\mathbb{H}^2$

2.1 Harmonic maps

Conventions: All manifolds are finite dimensional, connected, and Riemannian of class C^∞, unless otherwise stated. All vector bundles and their sections are smooth, unless otherwise specified. Now we review some basic notions from the theory of harmonic maps. We make an effort to do our computations coordinate free first and then in coordinates. The reader is referred to the textbook [32] for proofs and much more details on harmonic maps. Other references on harmonic maps include [13], [42], [56], and [68].

Let (M,g) and (N,h) be m and n dimensional manifolds with the Levi-Civita connections ∇^g and ∇^h, respectively. Let $\phi : M \longrightarrow N$ be a smooth map. The differential

$$d\phi \in \Gamma(M, T^*M \otimes \phi^{-1}TN)$$

can be viewed as a $\phi^{-1}(TN)$-valued 1-form on M, i.e., $d\phi \in \mathscr{A}^1(\phi^{-1}(TN))$. Before we define the notion of a *harmonic map*, observe the following:

1. There exists a unique connection, $\phi^{-1}\nabla^h$, induced by ϕ on $\phi^{-1}(TN)$. Note that $\phi^{-1}(TN)$ is a vector bundle on M defined by ϕ.
2. The bundle $T^*M \otimes \phi^{-1}TN$ has a connection ∇, naturally induced by ∇^g and $\phi^{-1}\nabla^h$.

Definition 2.1.1. $\nabla d\phi \in \Gamma(M, \otimes^2 T^*M \otimes \phi^{-1}TN)$ is called the *second fundamental form of ϕ*.

Definition 2.1.2. $\mathrm{Trace}(\nabla d\phi) \in \Gamma(M, \phi^{-1}TN)$ is called the *tension field* of ϕ. It is usually denoted by $\tau(\phi)$.

Definition 2.1.3. ϕ is said to be *totally geodesic* if $\nabla d\phi = 0$.

Definition 2.1.4. ϕ is said to be *harmonic* if

$$\tau(\phi) = 0. \tag{2.1}$$

We call τ the *Eells-Sampson Laplacian*.

In co-ordinate form: By taking coordinate charts, the second fundamental form of ϕ at $x = (x^1, \dots, x^m) \in U \subset M$ can be represented as:

$$(\nabla d\phi)^{\alpha}_{ij}(x) = \frac{\partial^2 \phi^{\alpha}}{\partial x^i \partial x^j}(x) - \Gamma^k_{ij} \frac{\partial \phi^{\alpha}}{\partial x^k}(x) + \Upsilon^{\alpha}_{\beta\gamma}(\phi(x)) \frac{\partial \phi^{\beta}}{\partial x^i}(x) \frac{\partial \phi^{\gamma}}{\partial x^j}(x), \tag{2.2}$$

where Γ^k_{ij}, $\Upsilon^{\alpha}_{\beta\gamma}$ denote the Christoffel symbols of ∇^g and ∇^h. Note that we have used the Einstein-Summation convention in (2.2). In coordinate charts,

$$\tau^{\alpha}(\phi)(x) = g^{ij}\big(\nabla(d\phi)^{\alpha}(x)\big),$$

where g^{ij} denotes the inverse of the metric tensor g_{ij}. (2.1) in co-ordinate form can be expressed as:

$$\sum_{i,j=1}^{m} g^{ij} \left(\frac{\partial^2 \phi^{\alpha}}{\partial x^i \partial x^j} - \sum_{k=1}^{m} \Gamma^k_{ij}(x) \frac{\partial \phi^{\alpha}}{\partial x^k} + \sum_{\beta,\gamma=1}^{n} \Upsilon^{\alpha}_{\beta\gamma}(\phi(x)) \frac{\partial \phi^{\beta}}{\partial x^i} \frac{\partial \phi^{\gamma}}{\partial x^j} \right) = 0, \quad 1 \leq \alpha \leq n. \tag{2.3}$$

Note that in (2.3), the term

$$\sum_{i,j=1}^{m} g^{ij} \left(\frac{\partial^2 \phi^{\alpha}}{\partial x^i \partial x^j} - \sum_{k=1}^{m} \Gamma^k_{ij} \frac{\partial \phi^{\alpha}}{\partial x^k} \right)$$

is the Laplace-Beltrami operator on M, a contribution of ∇^g in T^*M and the other term

$$\sum_{i,j=1}^{m} g^{ij} \left(\sum_{\beta,\gamma=1}^{n} \Upsilon^{\alpha}_{\beta\gamma}(\phi(x)) \frac{\partial \phi^{\beta}}{\partial x^i} \frac{\partial \phi^{\gamma}}{\partial x^j} \right)$$

which is a non-linear term containing the Christoffel symbols of ∇^h is a contribution of $\phi^{-1}\nabla^h$ in $\phi^{-1}TN$. (2.3) is the *Euler-Lagrange equation* for the *energy* E of ϕ which can be defined under some conditions, for example when M is compact, as:

$$E(\phi) = \int_M e(\phi) d\mu_g,$$

where $d\mu_g$ denotes the measure on M induced by g and $e(\phi)$ is the *energy density* of ϕ. The energy density $e(\phi)$ of ϕ is defined by

$$e(\phi)(x) = \frac{1}{2}\|d\phi(x)\|^2 = \frac{1}{2}\mathrm{trace}(\phi^* h)(x),$$

where $\|d\phi(x)\|$ is the Hilbert-Schmidt norm of the differential map

$$d\phi(x) : T_x M \longrightarrow T_{\phi(x)}N.$$

The energy density $e(\phi)$ of ϕ has the following expression in local coordinates

$$e(\phi) = \frac{1}{2} g^{ij}(x) h_{\beta\gamma}(\phi) \frac{d\phi^{\beta}}{dx^i} \frac{d\phi^{\gamma}}{dx^j}, \quad x \in M. \tag{2.4}$$

When M is compact, we can define ϕ to be a harmonic map if it's a critical point of E.

Remark 2.1.5. Harmonic maps are critical points of the energy functional and hence should not be seen as energy minimizers. Below we give the formulation of the energy extremal problem in the case of harmonic maps:
Given a smooth map $\phi : (M, g) \longrightarrow (N, h)$, let

$$E^*[\phi] = \inf\{E(\phi') : \phi' = \text{smooth}, \phi' \text{ is homotopic to } \phi\}$$

A smooth map ϕ such that $E(\phi) = E^*[\phi]$ is called an energy minimizer. For the existence and the uniqueness of energy minimizers when the target manifold is equipped with a strictly negatively curved metric, see [12], [27].

Now, if we have two complex manifolds Σ_1 and Σ_2 for M and N, and on these manifolds, we have conformal metrics,

$$\sigma(z)^2 dz d\bar{z} = \sigma(z)^2(dx^2 + dy^2) \quad (z = x + \iota y)$$

and

$$\rho(u)^2 du d\bar{u} = \rho(u)^2(du_1^2 + du_2^2) \quad (u = u_1 + \iota u_2)$$

then the Laplace-Beltrami operator on Σ_1 is given by $\frac{1}{\sigma(z)^2}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}$. According to J. Jost (see [32, Chapter 1]), (2.3) in these coordinates takes the form

$$\frac{1}{\sigma(z)^2}\phi_{z\bar{z}} + \frac{1}{\sigma(z)^2}\frac{2\rho_\phi}{\rho}\phi_z\phi_{\bar{z}} = 0, \tag{2.5}$$

where a subscript denotes a partial derivative and ρ_ϕ denotes the Wirtinger derivative of ρ at the point $\phi(z)$. Therefore, a conformal map between Riemann surfaces with conformal metrics is harmonic. From (2.5), we can see that in the case of a smooth map $\phi : (\Sigma_1, \sigma(z)^2 dz d\bar{z}) \longrightarrow (\Sigma_2, \rho(u)^2 du d\bar{u})$ between Riemann surfaces with conformal Riemannian metrics, the Riemannian metric on Σ_1 is not needed to decide whether ϕ is harmonic but the Riemannian metric on Σ_2 matters. More generally, it is also true for a smooth map from a Riemann surface to a Riemannian manifold. In summary, we see harmonic maps as a very efficient tool to compare the Riemannian metric structure of Σ_2 to the conformal structure of Σ_1. Next we discuss some basic examples of harmonic maps.

Example 2.1.6. Totally geodesic maps are harmonic. Clear from Definition (2.1.3).

Example 2.1.7. The identity map $(M, g) \longrightarrow (M, g)$ is harmonic.

Example 2.1.8. Let $M = \mathbb{S}^1$ and N is compact without boundary, then every homotopy class of maps of M into N contains a closed geodesic, hence a harmonic map.

To discuss the next two examples we will make a small investment in algebra which will lead us to consider natural quantities: Recall Definition 1.2.2. Extending an almost complex structure $J : T\Sigma_g \longrightarrow T\Sigma_g$ on Σ_g to the complexified tangent bundle $(T\Sigma_g)^c := T\Sigma_g \otimes_{\mathbb{R}} \mathbb{C}$ amounts to having a decomposition of the complexified tangent space $(T_x\Sigma_g)^c$ at each $x \in \Sigma_g$ into $(T_x\Sigma_g)^{(1,0)}$ and $(T_x\Sigma_g)^{(0,1)}$ corresponding to eigenvalues ι and $-\iota$. That is,

$$(T_x\Sigma_g)^{(1,0)} = \{v \in (T_x\Sigma_g)^c | Jv = \iota v\}, \quad (T_x\Sigma_g)^{(0,1)} = \{v \in (T_x\Sigma_g)^c | Jv = -\iota v\}.$$

$(T_x\Sigma_g)^{(1,0)}$ and $(T_x\Sigma_g)^{(0,1)}$ are called holomorphic and antiholomorphic tangent spaces, spanned by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \iota\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right),$$

where $z = x + \iota y$. In a similar fashion, we can complexify the dual tangent bundle $T^*\Sigma_g$ and again, for every $x \in \Sigma_g$, we can decompose $(T_x^*\Sigma_g)^c$ into its $\pm\iota$ eigenspaces - $(T_x^*\Sigma_g)^{(1,0)}$ and $(T_x^*\Sigma_g)^{(0,1)}$. $(T_x^*\Sigma_g)^{(1,0)}$ and $(T_x^*\Sigma_g)^{(0,1)}$ are spanned by

$$dz = dx + \iota dy, \quad d\bar{z} = dx - \iota dy$$

respectively. Using the above decompositions, we can then decompose complexified tensor bundles and hence sections of tensor bundles. Most importantly, we will consider a symmetric tensor s in the complexification of the bundle $T^*\Sigma_g \otimes T^*\Sigma_g$. Note that s can be written in terms of $dz^2 := dz \otimes dz$, $d\bar{z}^2 := d\bar{z} \otimes d\bar{z}$, and $|dz^2| := \frac{1}{2}(dz \otimes d\bar{z} + d\bar{z} \otimes dz)$. Tensors that have only $(2,0)$ part can be written locally as $f dz^2$ for some locally defined complex valued function f and are famously known as quadratic differentials (see Subsection 1.3.2).

Example 2.1.9. When $M = \Omega \subset \mathbb{R}^n$ and $N = \mathbb{R}$, then the harmonic map equations are the harmonic function equations, i.e.,

$$\Delta\phi = 0.$$

If M is a surface with a complex structure and $N = \mathbb{R}$, then in the complex language the Laplace equation can be written as:

$$4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}\phi = 0.$$

Let's try to observe something really important by rewriting the above equation as follows:

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial}{\partial z}\phi\right) = 0. \tag{2.6}$$

We can also write (2.6) in more fancy way as follows:

$$\bar{\partial}\big((d\phi)^{(1,0)}\big) = 0,$$

where the object in the parentheses is a "holomorphic object" (if and only if the equation holds). In other words, tied to the harmonicity of a map ϕ on a surface (with a complex structure) is a "holomorphic object" which is a holomorphic 1-form in the present case.

Example 2.1.10. A diffeomorphism $\phi : (\Sigma_1, \sigma(z)^2 dz d\bar{z}) \longrightarrow (\Sigma_2, \rho(u)^2 du d\bar{u})$ is harmonic iff $(2,0)$-part of the pullback metric $\phi^*(\rho(u)^2 du d\bar{u})$ is holomorphic. This can be seen as follows: we denote the conformal metric $\rho(u)^2 du d\bar{u}$ on Σ_2 by h. The pullback of h by ϕ has the following local expression:

$$\begin{aligned}\phi^*(h) &= (\phi^*(h))^{(2,0)} + (\phi^*(h))^{(1,1)} + (\phi^*(h))^{(0,2)} \\ &= \langle \phi_* \partial_z, \phi_* \partial_z \rangle_h dz^2 + \left(\|\phi_* \partial_z\|_h^2 + \|\phi_* \partial_{\bar{z}}\|_h^2\right)\sigma^2(z) dz d\bar{z} + \langle \phi_* \partial_{\bar{z}}, \phi_* \partial_{\bar{z}} \rangle_h d\bar{z}^2.\end{aligned} \tag{2.7}$$

Note that in the first equality we used the complex eigenspace decomposition

$$\phi^*(h) = (\phi^*(h))^{(2,0)} + (\phi^*(h))^{(1,1)} + (\phi^*(h))^{(0,2)}$$

under the action of J on $T\Sigma_g$. Also,

$$\begin{aligned}\langle \phi_* \partial_z, \phi_* \partial_z \rangle_h dz^2 &= h\Big(\frac{\partial \phi}{\partial z}, \frac{\partial \phi}{\partial z}\Big) dz^2 \\ &= \left(h\Big(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial x}\Big) - h\Big(\frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial y}\Big) - 2\iota h\Big(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\Big)\right) dz^2 \\ &= \left(|\phi_x|^2 - |\phi_y|^2 - 2\iota h(\phi_x, \phi_y)\right) dz^2 \\ &= 4\rho^2 \phi_z \bar{\phi}_z dz^2\end{aligned} \tag{2.8}$$

and

$$\left(\|\phi_* \partial_z\|_h^2 + \|\phi_* \partial_{\bar{z}}\|_h^2\right)\sigma^2(z) = e(\phi), \tag{2.9}$$

the energy density of ϕ, expressed locally in (2.4). Now, (2.7) has the following form using the simplified expressions in (2.8) and (2.9)

$$\phi^*(h) = 4\rho^2 \phi_z \bar{\phi}_z dz^2 + e(\phi) dz d\bar{z} + \overline{4\rho^2 \phi_z \bar{\phi}_z dz^2}$$

Now, from [32, Lemma 1.1], it is easy to see that

$$\begin{aligned}\partial_{\bar{z}}((\phi^*(h))^{(2,0)}) &= \partial_{\bar{z}}(4\rho^2 \phi_z \bar{\phi}_z dz^2) \\ &= \rho^2\left(\bar{\phi}_z \tilde{\tau}(\phi) + \phi_z \overline{\tilde{\tau}(\phi)}\right),\end{aligned}$$

where $\tilde{\tau}(\phi) = \phi_{z\bar{z}} + \frac{2\rho_\phi}{\rho}\phi_z \phi_{\bar{z}}$. Therefore, $\partial_{\bar{z}}((\phi^*(h))^{(2,0)}) = 0$ when ϕ is harmonic, i.e., when $\tau(\phi) = 0$ (see (2.5)) and hence $\tilde{\tau}(\phi) = 0$. We denote $(\phi^*(h))^{(2,0)}$ by q. Conversely, if q is holomorphic, i.e.,

$$\bar{\phi}_z \tau(\phi) + \phi_z \overline{\tau(\phi)} = 0$$

and if $\tau(\phi) \neq 0$ at a point $p \in \Sigma_1$, this would imply $|\phi_z| = |\bar{\phi}_z| = |\phi_{\bar{z}}|$ and hence the Jacobian at p is zero which contradicts the fact that the Jacobian is non zero everywhere since ϕ is a diffeomorphism. Furthermore, $q \equiv 0$ iff ϕ is conformal.

Example 2.1.11. The only holomorphic quadratic differential q on $\mathbb{S}^2$ is identically zero. This follows from the following: first we write the holomorphic quadratic differential q as $f dz^2$ using a complex coordinate $z \in \mathbb{C}$, obtained from the streographic projection of the sphere minus the south pole. We also get another complex coordinate $w = 1/z$ using the streographic projection of the sphere minus the north pole. In these coordinates, $q = f/w^4 dw^2$. We can easily deduce that f is a bounded holomorphic function on $\mathbb{C}$ and hence constant by the Liouville theorem. Since $f \longrightarrow 0$ as $z \longrightarrow \infty$, that constant must be zero. Combining the above result with Example 2.1.10, we have: every non-constant harmonic map from $\mathbb{S}^2$ to an arbitrary Riemannian manifold N is conformal.

2.2 The notion of a harmonic vector field

We introduce the notion of a *harmonic vector field* on a Riemannian manifold M which is regarded as the infinitesimal generator of local harmonic diffeomorphisms. Note that some sources use the term *harmonic vector field* to mean vector fields which have harmonic associated 1-form [72] and vector fields as sections of the tangent bundle with *lift metrics* [48]. Let U be an open subset of a Riemannian manifold M. Let $\{\phi_t\}_{t\in[0,\epsilon)}$ be a smooth family of smooth maps

$$\phi_t : U \longrightarrow M$$

where ϕ_0 is the inclusion. Then $\xi = \frac{d\phi_t}{dt}|_{t=0}$ is a vector field on U.

Definition 2.2.1. The vector field ξ on U is harmonic if there exists a smooth family of smooth maps $\{\phi_t : U \longrightarrow M\}_{t\in[0,\epsilon)}$ which satisfies the following:

1. ϕ_0 is the inclusion map,
2. $\left.\dfrac{d\phi_t}{dt}\right|_{t=0} = \xi$,
3. $\forall x \in U : \left.\dfrac{d}{dt}\right|_{t=0} \tau(\phi_t)(x) = 0\,.$

Remark 2.2.2. Given ξ we can always find the family $\{\phi_t\}_{t\in[0,\epsilon]}$ satisfying (1) and (2) in Definition 2.2.1.

Remark 2.2.3. The choice of $\{\phi_t\}_{t\in[0,\epsilon)}$ is unimportant in (3) in Definition 2.2.1.

Here τ is the *Eells-Sampson Laplacian* which has been introduced in (2.1). Condition 3 in Definition 2.2.1 can be expressed in co-ordinate form as:

$$\left.\frac{d}{dt}\right|_{t=0}\left(\sum_{i,j=1}^{m} g^{ij}(x)\left(\frac{\partial^2\phi_t^\alpha}{\partial x^i\partial x^j}-\sum_{k=1}^{m}\Gamma_{ij}^k(x)\frac{\partial\phi_t^\alpha}{\partial x^k}+\sum_{\beta,\gamma=1}^{m}\Gamma_{\beta\gamma}^\alpha(\phi_t(x))\frac{\partial\phi_t^\beta}{\partial x^i}\frac{\partial\phi_t^\gamma}{\partial x^j}\right)\right)=0,\qquad 1\le\alpha\le m. \tag{2.10}$$

Now, for each $1 \le i \le m$, $\nabla_t \frac{\partial\phi_t}{\partial x^i} = \nabla_i \frac{\partial\phi_t}{\partial t}$. Therefore, (2.10) becomes

$$\sum_{i,j=1}^{m} g^{ij}(x)\left(\frac{\partial^2}{\partial x^i\partial x^j}\left(\left.\frac{d\phi_t^\alpha}{dt}\right|_{t=0}\right)-\sum_{k=1}^{m}\Gamma_{ij}^k(x)\frac{\partial}{\partial x^k}\left(\left.\frac{d\phi_t^\alpha}{dt}\right|_{t=0}\right)+\sum_{\beta,\gamma=1}^{m}\left.\frac{d}{dt}\right|_{t=0}\left(\Gamma_{\beta\gamma}^\alpha(\phi_t(x))\frac{\partial\phi_t^\beta}{\partial x^i}\frac{\partial\phi_t^\gamma}{\partial x^j}\right)\right)=0,$$

where $1 \leq \alpha \leq m$. Since $\xi^\alpha = \frac{d\phi_t^\alpha}{dt}\big|_{t=0}$, we have

$$\sum_{i,j=1}^{m} g^{ij}(x)\Bigg(\frac{\partial^2\xi^\alpha}{\partial x^i\partial x^j} - \sum_{k=1}^{m}\Gamma_{ij}^k(x)\frac{\partial\xi^\alpha}{\partial x^k} + \sum_{\beta,\gamma=1}^{m}\Big(\big(\Gamma_{\beta\gamma}^\alpha\big)'(\phi_0(x))\cdot\xi\Big)\frac{\partial\phi_0^\beta}{\partial x^i}\frac{\partial\phi_0^\gamma}{\partial x^j} + \Gamma_{\beta\gamma}^\alpha(\phi_0(x))\left(\frac{\partial\xi^\beta}{\partial x^i}\frac{\partial\phi_0^\gamma}{\partial x^j} + \frac{\partial\phi_0^\beta}{\partial x^i}\frac{\partial\xi^\gamma}{\partial x^j}\right)\Bigg) = 0, \tag{2.11}$$

where $1 \leq \alpha \leq m$ and $\big(\Gamma_{\beta\gamma}^\alpha\big)'$ denotes the derivative of $\Gamma_{\beta\gamma}^\alpha$. Since $\phi_0 : U \longrightarrow M$ is the inclusion map, we rewrite (2.11):

$$\sum_{i,j=1}^{m} g^{ij}(x)\Bigg(\frac{\partial^2\xi^\alpha}{\partial x^i\partial x^j} - \sum_{k=1}^{m}\Gamma_{ij}^k(x)\frac{\partial\xi^\alpha}{\partial x^k} + \sum_{\beta,\gamma=1}^{m}\Big(\big(\Gamma_{\beta\gamma}^\alpha\big)'(x)\cdot\xi\Big)\delta_{\beta i}\delta_{\gamma j} + \Gamma_{\beta\gamma}^\alpha(x)\left(\frac{\partial\xi^\beta}{\partial x^i}\delta_{\gamma j} + \delta_{\beta i}\frac{\partial\xi^\gamma}{\partial x^j}\right)\Bigg) = 0, \tag{2.12}$$

where $1 \leq \alpha \leq m$, $\frac{\partial\phi_0^\gamma}{\partial x^j} = \delta_{\gamma j}$ and $\frac{\partial\phi_0^\beta}{\partial x^i} = \delta_{\beta i}$.

We now assume that M is $\mathbb{H}^2$ with the standard hyperbolic metric $\mathbf{g}_{\mathbb{H}^2}$, coordinatized as an open subset of $\mathbb{C}$. Rewriting (2.12), we get

$$\sum_{i,j=1}^{2} \mathbf{g}_{\mathbb{H}^2}^{ij}(x)\Bigg(\frac{\partial^2\xi^\alpha}{\partial x^i\partial x^j} - \sum_{k=1}^{2}\Gamma_{ij}^k(x)\frac{\partial\xi^\alpha}{\partial x^k} + \sum_{\beta,\gamma=1}^{2}\Big(\big(\Gamma_{\beta\gamma}^\alpha\big)'(x)\cdot\xi\Big)\delta_{\beta i}\delta_{\gamma j} + \sum_{\beta,\gamma=1}^{2}\Gamma_{\beta\gamma}^\alpha(x)\left(\frac{\partial\xi^\beta}{\partial x^i}\delta_{\gamma j} + \delta_{\beta i}\frac{\partial\xi^\gamma}{\partial x^j}\right)\Bigg) = 0, \tag{2.13}$$

where $1 \leq \alpha \leq 2$. The Christoffel symbols Γ_{11}^1, Γ_{22}^1, Γ_{12}^2 and Γ_{21}^2 for $\mathbf{g}_{\mathbb{H}^2}$ vanish. Also $g_{\mathbb{H}^2}^{11} = g_{\mathbb{H}^2}^{22} = y^2$ and $g_{\mathbb{H}^2}^{12} = g_{\mathbb{H}^2}^{21} = 0$. Hence (2.13) simplifies to:

$$\begin{aligned} y^2\frac{\partial^2\xi^\alpha}{\partial^2 x} + y^2\frac{\partial^2\xi^\alpha}{\partial^2 y} - \left(y^2\Gamma_{11}^2\frac{\partial\xi^\alpha}{\partial y} + y^2\Gamma_{22}^2\frac{\partial\xi^\alpha}{\partial y}\right) + y^2\big(\big(\Gamma_{11}^\alpha\big)'(x)\cdot\xi\big) + y^2\big(\big(\Gamma_{22}^\alpha\big)'(x)\cdot\xi\big) \\ + y^2\left(\Gamma_{11}^\alpha\left(\frac{\partial\xi^1}{\partial x} + \frac{\partial\xi^1}{\partial x}\right) + \Gamma_{12}^\alpha\left(0 + \frac{\partial\xi^2}{\partial x}\right) + \Gamma_{21}^\alpha\left(\frac{\partial\xi^2}{\partial x} + 0\right)\right) \\ + y^2\left(\Gamma_{12}^\alpha\left(\frac{\partial\xi^1}{\partial y} + 0\right) + \Gamma_{21}^\alpha\left(0 + \frac{\partial\xi^1}{\partial y}\right) + \Gamma_{22}^\alpha\left(\frac{\partial\xi^2}{\partial y} + \frac{\partial\xi^2}{\partial y}\right)\right) = 0; \quad 1 \leq \alpha \leq 2. \end{aligned} \tag{2.14}$$

The other Christoffel symbols for $\mathbf{g}_{\mathbb{H}^2}$ are given as follows:

$$\Gamma_{12}^1 = \Gamma_{21}^1 = \Gamma_{22}^2 = -\frac{1}{y}, \quad \Gamma_{11}^2 = \frac{1}{y}.$$

Substituting these values into (2.14), we obtain the following two equations which describe the conditions for ξ to be a *harmonic vector field* on U:

$$\xi_{xx}^1 + \xi_{yy}^1 - \frac{2}{y}(\xi_x^2 + \xi_y^1) = 0 \tag{2.15}$$

$$\xi^2_{xx} + \xi^2_{yy} + \frac{2}{y}(\xi^1_x - \xi^2_y) = 0 \tag{2.16}$$

If the flow (ϕ_t) and the vector field ξ are related as above, then we can describe ϕ_t up to the first order in terms of ξ using the standard coordinates in $\mathbb{H}^2 \subset \mathbb{C}$:

$$\phi_t(p) \approx p + t\xi(p)$$

(for $p \in U$ and sufficiently small t). We define a family of Riemannian metrics on U as follows:

$$t \longmapsto \rho_t = \phi_t^* \mathbf{g}_{\mathbb{H}^2} \tag{2.17}$$

More precisely the map in (2.17) can be viewed as

$$t \longmapsto (D\phi_t : T_pU \longrightarrow T_{\phi_t(p)}\mathbb{H}^2)^* \mathbf{g}_{\mathbb{H}^2} \tag{2.18}$$

(2.18) to the *first order* can be expressed as follows:

$$t \longmapsto (\mathrm{Id} + t \cdot d\xi : T_pU \longrightarrow T_{\phi_t(p)}\mathbb{H}^2)^* \mathbf{g}_{\mathbb{H}^2},$$

where $d\xi$ is the derivative of ξ (where ξ is viewed as a smooth map $\mathbb{C} \longrightarrow \mathbb{C}$) at p, and it is an $\mathbb{R}$-linear map. Using the *first order* approximation, ρ_t is given as:

$$\begin{aligned}
\rho_t &\approx (\mathrm{Id} + t \cdot d\xi)^T (\mathbf{g}_{\mathbb{H}^2} + t\mathbf{g}'_{\mathbb{H}^2}(\xi)(\mathrm{Id} + t \cdot d\xi) \\
&\approx \mathbf{g}_{\mathbb{H}^2} + t \cdot d\xi^T \mathbf{g}_{\mathbb{H}^2} + t\mathbf{g}'_{\mathbb{H}^2}(\xi) + t \cdot d\xi \mathbf{g}_{\mathbb{H}^2} \\
&\approx \mathbf{g}_{\mathbb{H}^2} + (t \cdot d\xi^T + t \cdot d\xi)\mathbf{g}_{\mathbb{H}^2} + t\mathbf{g}'_{\mathbb{H}^2}(\xi)
\end{aligned}$$

In the above expression, $d\xi^T$ denotes the transpose of $d\xi$ when written in the local coordinates. Calculating

$$\frac{d\rho_t}{dt}\Big|_{t=0}$$

gives us a section of $S^2(TU)$, the vector bundle of (real) symmetric bilinear forms on TU and this is denoted by $\mathcal{L}_\xi \mathbf{g}_{\mathbb{H}^2}$, the Lie derivative of $\mathbf{g}_{\mathbb{H}^2}$ w.r.t ξ. Therefore,

$$\mathcal{L}_\xi \mathbf{g}_{\mathbb{H}^2} = (d\xi^T + d\xi)\mathbf{g}_{\mathbb{H}^2} + \mathbf{g}'_{\mathbb{H}^2}(\xi) \tag{2.19}$$

in our preferred coordinates. Now, to obtain a local expression for $\mathcal{L}_\xi \mathbf{g}_{\mathbb{H}^2} \in \Gamma(S^2(TU))$, we represent $d\xi$ by the following matrix

$$d\xi = \begin{bmatrix} \xi^1_x & \xi^1_y \\ \xi^2_x & \xi^2_y \end{bmatrix}$$

Using the above expression for $d\xi$, the right-hand side of (2.19) can be represented as

$$\begin{aligned}
\mathcal{L}_\xi \mathbf{g}_{\mathbb{H}^2} &= \frac{1}{y^2}\begin{bmatrix} 2\xi^1_x & \xi^1_y + \xi^2_x \\ \xi^2_x + \xi^1_y & 2\xi^2_y \end{bmatrix} + \begin{bmatrix} \frac{-2}{(\xi^2)^3} & 0 \\ 0 & \frac{-2}{(\xi^2)^3} \end{bmatrix} \\
&= \underbrace{\frac{1}{y^2}\begin{bmatrix} \xi^1_x - \xi^2_y & \xi^1_y + \xi^2_x \\ \xi^1_y + \xi^2_x & \xi^2_y - \xi^1_x \end{bmatrix}}_{\text{TF}} + \frac{1}{y^2}\begin{bmatrix} \xi^1_x + \xi^2_y & 0 \\ 0 & \xi^1_x + \xi^2_y \end{bmatrix} + \begin{bmatrix} \frac{-2}{(\xi^2)^3} & 0 \\ 0 & \frac{-2}{(\xi^2)^3} \end{bmatrix}
\end{aligned}$$

Recall from Subsection 1.3.2 that the bundle $S^2(TU)$ of (real) symmetric bilinear forms on TU splits into 1-dimensional real vector subbundle spanned by the everywhere nonzero section $\mathbf{g}_{\mathbb{H}^2}$ and the image of the embedding (recall (1.11))

$$\psi : \hom_{\mathbb{C}}(TU \otimes_{\mathbb{C}} TU, \mathbb{C}) \longrightarrow S^2(TU),$$

where $\psi(q)$ is the real part of $q = f dz^2$. In particular, $\psi\big((\mathcal{L}_\xi \mathbf{g}_{\mathbb{H}^2})^{(2,0)}\big)$ is the real part of $(\mathcal{L}_\xi \mathbf{g}_{\mathbb{H}^2})^{(2,0)}$. Using the above splitting it is straightforward to check that the trace-free component of $\mathcal{L}_\xi \mathbf{g}_{\mathbb{H}^2}$ is $\psi\big((\mathcal{L}_\xi \mathbf{g}_{\mathbb{H}^2})^{(2,0)}\big) = \psi(f dz^2)$ where $f(z) = \mathrm{TF}_{11} - \iota \mathrm{TF}_{12}$. Notice that

$$\overline{f(z)} = \mathrm{TF}_{11} + \iota \mathrm{TF}_{12} = \frac{2}{y^2}\frac{\partial \xi}{\partial \bar{z}} = \frac{-8}{(z-\bar{z})^2}\frac{\partial \xi}{\partial \bar{z}}. \tag{2.20}$$

Furthermore, (2.20) is equivalent (upto to a constant factor) to the following *potential equation* (see Appendix A) described by S. Wolpert in his paper [70]

$$\overline{f(z)} = \frac{1}{(z-\bar{z})^2}\frac{\partial \xi}{\partial \bar{z}}. \tag{2.21}$$

Moreover, (2.15) and (2.16) are precisely the conditions that the corresponding quadratic differential $(\mathcal{L}_\xi \mathbf{g}_{\mathbb{H}^2})^{(2,0)}$ is holomorphic, i.e., f is holomorphic. Therefore, we can summarize our discussion as follows:

Proposition 2.2.4. *ξ is a harmonic vector field on U iff the quadratic differential $(\mathcal{L}_\xi \mathbf{g}_{\mathbb{H}^2})^{(2,0)}$ associated with it is holomorphic. In the standard coordinates, $(\mathcal{L}_\xi \mathbf{g}_{\mathbb{H}^2})^{(2,0)} = f dz^2$ where $\overline{f(z)} = \frac{-8}{(z-\bar{z})^2}\frac{\partial \xi}{\partial \bar{z}}$.*

Remark 2.2.5. Proposition 2.2.4 is an infinitesimal version of Lemma 1.1 in [32] and Example 2.1.10. In fact the statement in [32] is more general since it applies to harmonic maps between oriented 2-dimensional Riemannian manifolds.

Corollary 2.2.6. *Every holomorphic vector field on U is harmonic.*

Proof Let ξ be a holomorphic vector field on U. Then $\mathcal{L}_\xi \mathbf{g}_{\mathbb{H}^2}$ in (2.19) has diagonal form and therefore $(\mathcal{L}_\xi \mathbf{g}_{\mathbb{H}^2})^{(2,0)}$, the trace-free part of $\mathcal{L}_\xi \mathbf{g}_{\mathbb{H}^2}$, is zero. □

2.2.1 Constructing harmonic vector fields on $U \subseteq \mathbb{H}^2$

Theorem 2.2.7. *Let $\mathcal{HOL}$ denote the sheaf of holomorphic vector fields on $\mathbb{H}^2$, $\mathcal{HARM}$ denote the sheaf of harmonic vector fields on $\mathbb{H}^2$ and $\mathcal{HQD}$ denote the sheaf of holomorphic quadratic differentials on $\mathbb{H}^2$. Then the following sequence of sheaves*

$$\mathcal{HOL} \xrightarrow{\alpha} \mathcal{HARM} \xrightarrow{\beta} \mathcal{HQD} \tag{2.22}$$

is a short exact sequence of sheaves on $\mathbb{H}^2$. In (2.22), α is the inclusion map and β is given by the formula in Proposition 2.2.4.

Before we prove Theorem 2.2.7, we discuss the following result by S. Wolpert [70, Section 2]: let η be the vector field on $\mathbb{H}^2$ given by $\eta(z) = (1,0)$ everywhere. Given a holomorphic quadratic differential $q = f(z)dz^2$ on $\mathbb{H}^2$, there exists a global solution ξ of the potential equation $\frac{\partial \xi}{\partial \bar{z}} = (z-\bar{z})^2\overline{f(z)}$ (see (2.21)) and an explicit formula for ξ is given as:

$$\xi(z) = \left(\overline{\int_w^z (\bar{z}-\zeta)^2 f(\zeta) d\zeta}\right)\eta(z), \tag{2.23}$$

where $w \in \mathbb{H}^2$ is fixed and $\zeta, z \in \mathbb{H}^2$. The formula for ξ in (2.23) is path independent since the integrand is holomorphic.

Proof of Theorem 2.2.7: Exactness at the term $\mathcal{HARM}$ in (2.22) follows from Theorem 2.2.4 and Corollary 2.2.6. Now, Let $q = f(z)dz^2$ be defined in a neighborhood V of $w \in \mathbb{H}^2$, where $w \in \mathbb{H}^2$ is fixed. To prove the local surjectivity of β we have to get a solution for a harmonic vector field ξ whose associated holomorphic quadratic differential is q in a possibly smaller neighborhood $U \subset V$ of w. It is clear that (2.23) gives the required solution for ξ upto a constant factor. □

Corollary 2.2.8. *If a sequence of harmonic vector fields defined on an open set U in $\mathbb{H}^2$ converges uniformly on compact subsets of U, and if all of them determine the same holomorphic quadratic differential q on U, then the limit vector field is again harmonic and still determines the same holomorphic quadratic differential q on U.*

We will now describe a more pedestrian approach to finding harmonic vector fields with prescribed holomorphic quadratic differential. This has certain advantages over Wolpert's formula, as we will see in Subsection 2.2.2. First, we give an explicit expression for a harmonic vector field on $U \subset \mathbb{H}^2$ whose associated holomorphic quadratic differential q is given.

Lemma 2.2.9. *Let U be an open subset of $\mathbb{H}^2$ with the usual hyperbolic metric. Let η be the vector field on U given by $\eta(z) = (1,0)$ everywhere: vectors parallel to the real axis, pointing left to right, of euclidean length 1. Let f be a holomorphic function on U. The quadratic differential q associated to the vector field $\xi = y^n f\eta$ is represented as:*

$$q = -n\iota y^{n-3}\overline{f}dz^2, \quad n \geq 3. \tag{2.24}$$

Proof We use the recipe in Proposition 2.2.4 to prove the Lemma. And it suffices to prove for $n = 3$. From (2.20), we have

$$\frac{\partial \xi}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}(y^3 f) = \frac{\partial}{\partial \bar{z}}\left(\frac{(z-\bar{z})^3}{-8\iota}f\right) = \frac{3(z-\bar{z})^2}{8\iota}f = \frac{3\iota(z-\bar{z})^2}{-8}f = \frac{(z-\bar{z})^2}{-8}\overline{(-3\iota\bar{f})},$$

so that $q = -3\iota\bar{f}dz^2$.

□

Using Lemma 2.2.9, we can find an explicit expression for a harmonic vector field ξ on $\mathbb{H}^2$ whose associated holomorphic quadratic differential is

$$q = z^n dz^2 (n \geq 0)$$

using the obvious expression $z^n = (\overline{z} + 2\iota y)^n = \sum_{k=0}^{n} \binom{n}{k} (2\iota y)^{n-k} \overline{z}^k$.

Lemma 2.2.10. *An explicit expression for a harmonic vector field ξ on $\mathbb{H}^2$ whose associated holomorphic quadratic differential is $q = f(z)dz^2$, where $f(z) = z^n$, for some $n \geq 0$ (a holomorphic function on $\mathbb{H}^2$), is given as:*

$$\begin{aligned}
\xi(z) &= \left(\sum_{k=0}^{n} \binom{n}{k} \frac{(-2)(-2\iota)^{n-k-1}}{n-k+3} y^{n-k+3} z^k \right) \eta(z) \\
&= \left(\sum_{k=0}^{n} \binom{n}{k} (-2)(-2\iota)^{n-k-1} \Big(\int_0^y \zeta^{n-k+2} d\zeta \Big) z^k \right) \eta(z) \\
&= \left(\int_0^y \left(\sum_{k=0}^{n} \binom{n}{k} (-2)(-2\iota)^{n-k-1} \zeta^{n-k+2} z^k \right) d\zeta \right) \eta(z) \\
&= \left(\int_0^{\Im(z)} -\iota\zeta^2 (z - 2\iota\zeta)^n d\zeta \right) \eta(z)
\end{aligned}$$

Lemma 2.2.11. *An explicit expression for a harmonic vector field ξ on $U \subset \mathbb{H}^2$ whose associated holomorphic quadratic differential is $q = f(z)dz^2$, where $f(z) = (z-a)^n (n \geq 0)$ is a holomorphic function on $U \subset \mathbb{H}^2$ and $a \in \mathbb{H}^2$ fixed, is given as:*

$$\begin{aligned}
\xi(z) &= \left(\sum_{k=0}^{n} \binom{n}{k} (-\bar{a})^{n-k} \Big(\int_0^{\Im(z)} -\iota\zeta^2 (z - 2\iota\zeta)^k d\zeta \Big) \right) \eta(z) \\
&= \left(\int_0^{\Im(z)} -\iota\zeta^2 \Big(\sum_{k=0}^{n} \binom{n}{k} (-\bar{a})^{n-k} (z - 2\iota\zeta)^k \Big) d\zeta \right) \eta(z) \\
&= \left(\int_0^{\Im(z)} -\iota\zeta^2 \big(z - \bar{a} - 2\iota\zeta \big)^n d\zeta \right) \eta(z) \\
&= \left(\int_0^{\Im(z)} -\iota\zeta^2 \overline{f(\overline{z} + 2\iota\zeta)} d\zeta \right) \eta(z).
\end{aligned} \tag{2.25}$$

Another Proof of Theorem 2.2.7: Exactness at the term $\mathcal{HARM}$ in (2.22) follows from Theorem 2.2.4 and Corollary 2.2.6. Let $q = f(z)dz^2$ be defined in a neighborhood V of $a \in \mathbb{H}^2$, where $a \in \mathbb{H}^2$ is fixed. To prove the local surjectivity of β we have to get a solution for a harmonic vector field whose associated holomophic quadratic differential is q in a possibly smaller neighborhood $U \subset V$ of a. Note that we can't use the expression in (2.25). As ζ runs from 0 to $\Im(z)$, $f(\bar{z}+2\iota\zeta)$ does not even make sense when $\zeta = 0$. We try the following

$$\xi_c(z) = \left(\int_c^{\Im(z)} -\iota\zeta^2 \overline{f(\bar{z}+2\iota\zeta)} d\zeta \right) \eta(z), \tag{2.26}$$

where c is any positive real number.

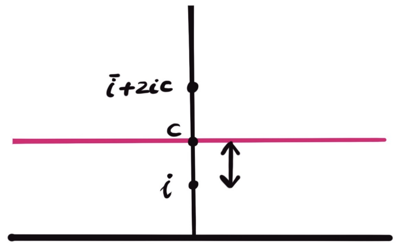

Figure 2.1: The expression for $\xi_c(\iota)$ defined along the hyperbolic line joining ι and c

But there is a caveat: as ζ runs from c to $\Im(z)$, $f(\bar{z}+2\iota\zeta)$ may not be defined on the upper half plane since we are assuming that f is defined only on $V \subset \mathbb{H}^2$. By making the best possible choice of c which is $\Im(a)$ in this case, we get the required solution as follows

$$\xi_{\Im(a)}(z) = \left(\int_{\Im(z)}^{\Im(a)} \iota\zeta^2 \overline{f(\bar{z}+2\iota\zeta)} d\zeta \right) \eta(z), \tag{2.27}$$

defined on

$$U = \{z \in V | \bar{z} + 2\iota t \in V \text{ for all } t \in [\Im(z), \Im(a)]\}.$$

Evaluating the expression in (2.27) at a, we get

$$\begin{aligned} \xi_{\Im(a)}(a) &= \left(\int_{\Im(a)}^{\Im(a)} \iota\zeta^2 \overline{f(\bar{a}+2\iota\zeta)} d\zeta \right) \eta(a) \\ &= 0. \end{aligned}$$

□

Remark 2.2.12. Let q be a quadratic differential which is defined everywhere on $\mathbb{H}^2$ and is bounded in the hyperbolic metric $\mathbf{g}_{\mathbb{H}^2}$, i.e.,

$$\|q\|_{\mathbf{g}_{\mathbb{H}^2}} = |f(z)| \|dz^2\|_{\mathbf{g}_{\mathbb{H}^2}} \leq D,$$

where $\|dz^2\|_{\mathbf{g}_{\mathbb{H}^2}} = \Im(z)^2$ and D is a positive real number. Note that ξ_c in (2.26) has a continuous extension on $\mathbb{R}$. In other words, for z such that $\Im(z) = 0$, we define

$$\xi_c(z) = \lim_{\epsilon \to 0} \left(\int_\epsilon^c \iota\zeta^2 \overline{f(z+2\iota\zeta)} d\zeta \right) \eta(z). \tag{2.28}$$

To prove that the above limit exists, we use the Cauchy criterion of convergence of improper integrals,

$$\begin{aligned}\left|\int_{\epsilon_1}^{\epsilon_2} \iota\zeta^2\overline{f(\bar{z}+2\iota\zeta)}d\zeta\right| &\leq \int_{\epsilon_1}^{\epsilon_2} \zeta^2\frac{D}{4\zeta^2}d\zeta \\ &= \frac{D}{4}(\epsilon_2-\epsilon_1).\end{aligned}$$

From the above estimate, it is clear that the limit in (2.28) exists.

Theorem 2.2.13. *Let $q = f(z)dz^2$ be a holomorphic quadratic differential on $\mathbb{H}^2$. Suppose that q satisfies the following boundedness conditions*

1. *q is bounded in the hyperbolic metric $g_{\mathbb{H}^2}$, i.e.*

$$\|q\|_{g_{\mathbb{H}^2}} = |f(z)|\|dz^2\|_{g_{\mathbb{H}^2}} \leq D, \tag{2.29}$$

where $\|dz^2\|_{g_{\mathbb{H}^2}} = \Im(z)^2$ and D is a positive real number.

2. *The first and second covariant derivative of q w.r.t the linear connection ∇ on $T^*\mathbb{H}^2 \otimes_{\mathbb{C}} T^*\mathbb{H}^2$, are bounded in the hyperbolic metric $g_{\mathbb{H}^2}$.*

Then there exists a harmonic vector field ξ^{reg} on $\mathbb{H}^2$ such that $\beta(\xi^{\mathrm{reg}}) = q$, where β is introduced in Theorem 2.2.7. An explicit formula is

$$\xi^{\mathrm{reg}}(z) = \lim_{c\to\infty}\left(\xi_c(z) - \left(\xi_c(\iota) + \left.\frac{\partial\xi_c}{\partial z}\right|_{z=\iota}\cdot(z-\iota)\right)\right), \tag{2.30}$$

where

$$\xi_c(z) = \left(\int_{\Im(z)}^{c} \iota\zeta^2\overline{f(\bar{z}+2\iota\zeta)}d\zeta\right)\eta(z)$$

and c is a positive real number.

Remark 2.2.14. We have introduced a simple terminology reg short for "regularisation" to characterise our required harmonic vector field.

Remark 2.2.15. The boundedness conditions on q in the above theorem are satisfied if q is invariant under the action of a discrete cocompact subgroup Γ of $\mathrm{PSL}(2,\mathbb{R})$, i.e.,

$$f(\gamma(z))\gamma'(z)^2 = f(z), \quad z\in\mathbb{H}^2, \quad \forall\gamma\in\Gamma.$$

Remark 2.2.16. In Theorem 2.2.13, ∇ is a first order linear differential operator

$$\mathscr{A}^0(\mathbb{H}^2, T^*\mathbb{H}^2\otimes_{\mathbb{C}} T^*\mathbb{H}^2) \longrightarrow \mathscr{A}^1(\mathbb{H}^2, T^*\mathbb{H}^2\otimes_{\mathbb{C}} T^*\mathbb{H}^2),$$

where on the L.H.S. we have sections of the vector bundle $T^*\mathbb{H}^2\otimes_{\mathbb{C}} T^*\mathbb{H}^2 \longrightarrow \mathbb{H}^2$ and on the R.H.S we have the space of $T^*\mathbb{H}^2\otimes_{\mathbb{C}} T^*\mathbb{H}^2$-valued 1-forms, i.e., sections of the vector bundle $\mathbf{hom}(T\mathbb{H}^2, T^*\mathbb{H}^2 \otimes_{\mathbb{C}} T^*\mathbb{H}^2)$. Recall that the Levi-Civita connection ∇ of the hyperbolic plane can be extended complex linearly to the complexification of the tangent and cotangent bundles - $(T\mathbb{H}^2)^c$ and $(T^*\mathbb{H}^2)^c$ - of the plane and their tensor products, and then decomposed as

$$\nabla = \nabla_{\frac{\partial}{\partial z}} \oplus \nabla_{\frac{\partial}{\partial \bar{z}}}.$$

Recall the discussion just before Example 2.1.9. We view $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ as sections of the complexified tangent bundle $(T\mathbb{H}^2)^c$, and dz and $d\bar{z}$ as sections of the complexified cotangent bundle $(T^*\mathbb{H}^2)^c$. Furthermore, $dz(\frac{\partial}{\partial z}) = 1$ and $dz(\frac{\partial}{\partial \bar{z}}) = 0$. For example, applied to functions $f : \mathbb{H}^2 \longrightarrow \mathbb{C}$, we have $\nabla_{\frac{\partial}{\partial z}} f = f_z dz$ and $\nabla_{\frac{\partial}{\partial \bar{z}}} f = f_{\bar{z}} d\bar{z}$. Now, for the hyperbolic plane with the hyperbolic metric $\mathbf{g}_{\mathbb{H}^2} = \rho^2 dz d\bar{z}$, where $\rho(z) = 1/\Im(z)$, we get the following:

$$\begin{aligned} \nabla \frac{\partial}{\partial z} = \frac{2\rho_z}{\rho} dz \otimes \frac{\partial}{\partial z}, \quad \nabla dz = dz \otimes \nabla_{\frac{\partial}{\partial z}} dz = -\frac{2\rho_z}{\rho} dz \otimes dz \\ \nabla_{\frac{\partial}{\partial \bar{z}}} \frac{\partial}{\partial z} = 0, \quad \nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z} = \frac{2\rho_z}{\rho} \frac{\partial}{\partial z}. \end{aligned} \tag{2.31}$$

Equations in (2.31) are taken from [39]. To get boundedness conditions on f_z and f_{zz} from boundedness conditions on q and on the first and second covariant derivative of $q = f dz^2$, i.e.,

$$\begin{aligned} \|q\|_{\mathbf{g}_{\mathbb{H}^2}} &\leq D \\ \|\nabla q\|_{\mathbf{g}_{\mathbb{H}^2}} &\leq D_1 \\ \|\nabla^2 q\|_{\mathbf{g}_{\mathbb{H}^2}} &\leq D_2, \end{aligned} \tag{2.32}$$

D_1 and D_2 are positive real numbers, we need to compute ∇q and $\nabla^2 q$. Consider the first covariant derivative of q w.r.t ∇:

$$\begin{aligned} \nabla f dz^2 &= \nabla(f) dz^2 + f \nabla(dz \otimes dz) \\ &= f_z dz^3 + f_{\bar{z}} d\bar{z} \otimes dz^2 + f(\nabla dz \otimes dz + dz \otimes \nabla dz) \\ &= f_z dz^3 + 0 + f \cdot -\frac{4\rho_z}{\rho} dz^3, \end{aligned} \tag{2.33}$$

where the last equality follows from (2.31) and the fact that f is a holomorphic function. From (2.33), we have

$$\|f_z dz^3 + f \cdot -\frac{4\rho_z}{\rho} dz^3\|_{\mathbf{g}_{\mathbb{H}^2}} \leq D_1$$

which implies

$$|f_z| \leq \frac{K_1}{\Im(z)^3}, \tag{2.34}$$

where K_1 is a positive constant that depends upon the bounds for f. Now, using (2.31) and (2.33) consider the second covariant derivative of q w.r.t ∇:

$$\begin{aligned}\nabla\big(f_z dz^3 + f\cdot -\frac{4\rho_z}{\rho}dz^3\big) &= \nabla f_z dz^3 + f_z\nabla(dz\otimes dz\otimes dz) + \nabla f\cdot -\frac{4\rho_z}{\rho}dz^3\\ &\quad - f\cdot\nabla\left(\frac{4\rho_z}{\rho}\right)dz^3 + f\cdot -\frac{4\rho_z}{\rho}\nabla(dz\otimes dz\otimes dz)\\ &= f_{zz}dz^4 + f_{z\bar z}d\bar z\otimes dz^3 + f_z\cdot -\frac{6\rho_z}{\rho}dz^4 + f_z\cdot -\frac{4\rho_z}{\rho}dz^4\\ &\quad + f_{\bar z}\cdot -\frac{4\rho_z}{\rho}d\bar z\otimes dz^3 - f\cdot\rho^2 dz^4 + f\cdot\frac{4\rho_z}{\rho}\cdot\frac{6\rho_z}{\rho}dz^4\\ &= f_{zz}dz^4 + 0 + f_z\cdot -\frac{6\rho_z}{\rho}dz^4 + f_z\cdot -\frac{4\rho_z}{\rho}dz^4 + 0\\ &\quad - f\cdot\rho^2 dz^4 + f\cdot\frac{24\rho_z^2}{\rho^2}dz^4\\ &= f_{zz}dz^4 + f_z\cdot -\frac{10\rho_z}{\rho}dz^4 - f\cdot\rho^2 dz^4 + f\cdot\frac{24\rho_z^2}{\rho^2}dz^4.\end{aligned} \tag{2.35}$$

From (2.35), the second covariant derivative of q being bounded in the hyperbolic metric implies the following:

$$|f_{zz}| \le \frac{K_2}{\Im(z)^4}, \tag{2.36}$$

where K_2 is a positive constant that depends upon the bounds for f and f_z.

Before we begin with the proof of Theorem 2.2.13 which establishes the global surjectivity of β in Theorem 2.2.7, we discuss the following abortive attempts to get a (global) harmonic vector field on the whole upper half plane $\mathbb{H}^2$.

Remark 2.2.17. Assume that q is bounded in the hyperbolic metric, i.e.

$$\|q\|_{\mathbf{g}_{\mathbb{H}^2}} = |f(z)|\|dz^2\|_{\mathbf{g}_{\mathbb{H}^2}} \le D,$$

where D is a positive real number. We try to define

$$\xi(z) = \left(\int_{\Im(z)}^{\infty} \iota\zeta^2\overline{f(\bar z + 2\iota\zeta)}d\zeta\right)\eta(z) = \lim_{c\to\infty}\left(\int_{\Im(z)}^{c} \iota\zeta^2\overline{f(\bar z + 2\iota\zeta)}d\zeta\right)\eta(z), \tag{2.37}$$

hoping that the above limit exists. In this case, we say that the improper integral in (2.37) *converges* and its value is that of the limit. From the above mentioned boundedness condition on q we get the following

$$|f(z)| \le \frac{D}{\Im(z)^2}, \forall z\in\mathbb{H}^2. \tag{2.38}$$

From the Cauchy criterion of convergence of improper integrals, the improper integral

$$\int_{\Im(z)}^{\infty} \iota\zeta^2\overline{f(\bar z + 2\iota\zeta)}d\zeta$$

in (2.37) converges iff for every $\epsilon > 0$ there is a $K \geq \Im(z)$ so that for all $A, B \geq K$ we have

$$\Big| \int_A^B \iota\zeta^2 \overline{f(\bar{z} + 2\iota\zeta)} d\zeta \Big| < \epsilon.$$

Using (2.38), we have

$$\begin{aligned} \Big| \int_A^B \iota\zeta^2 \overline{f(\bar{z} + 2\iota\zeta)} d\zeta \Big| &\leq \int_A^B \zeta^2 \,\big|\, \overline{f(\bar{z} + 2\iota\zeta)} \,\big|\, d\zeta \\ &\leq \int_A^B \frac{D\zeta^2}{(2\zeta - \Im(z))^2} d\zeta \end{aligned} \tag{2.39}$$

Now, we assume that $A \geq \Im(z)$. Then the denomiator $(2\zeta - \Im(z))^2$ in the second inequality in (2.39) is atleast as big as ζ^2. Rewriting (2.39), we get

$$\begin{aligned} \Big| \int_A^B \iota\zeta^2 \overline{f(\bar{z} + 2\iota\zeta)} d\zeta \Big| &\leq \int_A^B \frac{D\zeta^2}{\zeta^2} d\zeta \\ &= \int_A^B D d\zeta \\ &= D(B - A). \end{aligned}$$

From the above estimate, there is no conclusion that limit in (2.37) exists.

Remark 2.2.18. Assume that both q and its first covariant derivative w.r.t ∇ are bounded in the hyperbolic metric $\mathbf{g}_{\mathbb{H}^2}$. From Remark 2.2.16 and (2.34), the covariant derivative of q (w.r.t ∇) being bounded in the hyperbolic metric $\mathbf{g}_{\mathbb{H}^2}$ implies the following:

$$|f_z| \leq \frac{K_1}{\Im(z)^3}, \tag{2.40}$$

where f_z denotes the first complex derivative of f, f being a holomorphic function on $\mathbb{H}^2$. We try to define

$$\xi(z) = \lim_{c \to \infty} (\xi_c(z) - \xi_c(\iota)) \tag{2.41}$$

hoping that the above limit exists. We view $\xi_c(\iota)$ as the zeroth order Taylor approximation of $\xi_c(z)$ at $z = \iota$. Moreover, $\xi_c(\iota)$ is a constant vector field, hence a holomorphic vector field, depending on c. Note that the expression in (2.41) resembles the idea of Weierstrass in constructing the Weierstrass $\mathcal{P}$-function. Naively speaking, we want to compare the integral along a vertical hyperbolic line $\mathcal{L}_1$ joining some point z to $\bar{z} + 2\iota c$ with the integral along a vertical hyperbolic line $\mathcal{L}_2$ joining ι to $(2c - 1)\iota$. Infact, $\mathcal{L}_1$ and $\mathcal{L}_2$ are asymptotic lines in the hyperbolic plane $\mathbb{H}^2$. Let's first spell out the expression $\xi_c(z) - \xi_c(\iota)$ on the R.H.S of (2.41).

Case I: $2c \geq 1 \geq \Im(z)$

$$\begin{aligned}\xi_c(z)-\xi_c(\iota) &= \left(\int_{\Im(z)}^{c} \iota\zeta^2\overline{f(\bar z+2\iota\zeta)}d\zeta - \int_1^c \iota\zeta^2\overline{f(\bar\iota+2\iota\zeta)}d\zeta\right)\eta(z)\\ &= \left(\int_{\Im(z)}^{1} \iota\zeta^2\overline{f(\bar z+2\iota\zeta)}d\zeta + \int_1^c \iota\zeta^2\overline{f(\bar z+2\iota\zeta)}d\zeta - \int_1^c \iota\zeta^2\overline{f(\bar\iota+2\iota\zeta)}d\zeta\right)\eta(z)\\ &= \Bigg(\underbrace{\int_1^c \iota\zeta^2\left(\overline{f(\bar z+2\iota\zeta)-f(\bar\iota+2\iota\zeta)}\right)d\zeta}_{I_c} - \int_1^{\Im(z)} \iota\zeta^2\overline{f(\bar z+2\iota\zeta)}d\zeta\Bigg)\eta(z)\end{aligned} \tag{2.42}$$

Case II: $2c \geq \Im(z) \geq 1$

$$\begin{aligned}\xi_c(z)-\xi_c(\iota) &= \left(\int_{\Im(z)}^{c} \iota\zeta^2\overline{f(\bar z+2\iota\zeta)}d\zeta - \int_1^c \iota\zeta^2\overline{f(\bar\iota+2\iota\zeta)}d\zeta\right)\eta(z)\\ &= \left(\int_{\Im(z)}^{c} \iota\zeta^2\overline{f(\bar z+2\iota\zeta)}d\zeta - \int_1^{\Im(z)} \iota\zeta^2\overline{f(\bar\iota+2\iota\zeta)}d\zeta - \int_{\Im(z)}^c \iota\zeta^2\overline{f(\bar\iota+2\iota\zeta)}d\zeta\right)\eta(z)\\ &= \Bigg(\underbrace{\int_{\Im(z)}^c \iota\zeta^2\left(\overline{f(\bar z+2\iota\zeta)-f(\bar\iota+2\iota\zeta)}\right)d\zeta}_{II_c} - \int_1^{\Im(z)} \iota\zeta^2\overline{f(\bar\iota+2\iota\zeta)}d\zeta\Bigg)\eta(z)\end{aligned} \tag{2.43}$$

Since $\int_1^{\Im(z)} \iota\zeta^2\overline{f(\bar z+2\iota\zeta)}d\zeta$ and $\int_1^{\Im(z)} \iota\zeta^2\overline{f(\bar\iota+2\iota\zeta)}d\zeta$ on the R.H.S of (2.42) and (2.43) are independent of c we only work with I_c and II_c to determine whether the limit in (2.41) exists or not. Now if $A, B \geq c$, we have

$$I_B - I_A = \int_A^B \iota\zeta^2\left(\overline{f(\bar z+2\iota\zeta)-f(\bar\iota+2\iota\zeta)}\right)d\zeta, \tag{2.44}$$

and

$$II_B - II_A = \int_A^B \iota\zeta^2\left(\overline{f(\bar z+2\iota\zeta)-f(\bar\iota+2\iota\zeta)}\right)d\zeta. \tag{2.45}$$

Using (2.40), we have the following estimate for (2.44) and (2.45)

$$\left|\int_A^B \iota\zeta^2\left(\overline{f(\bar z+2\iota\zeta)-f(\bar\iota+2\iota\zeta)}\right)d\zeta\right| \leq \int_A^B \zeta^2\cdot\frac{K_1}{(2\zeta-\Im(z))^3}\cdot|\bar z-\bar\iota|d\zeta, \tag{2.46}$$

where the inequality in (2.46) follows from (2.40). Now, we assume that $A \geq \Im(z)$. Then the denomiator $(2\zeta-\Im(z))^3$ in the inequality in (2.46) is atleast as big as ζ^3. Rewriting (2.46), we get

$$\begin{aligned}\left|\int_A^B \iota\zeta^2\left(\overline{f(\bar z+2\iota\zeta)-f(\bar\iota+2\iota\zeta)}\right)d\zeta\right| &\leq |\bar z-\bar\iota|\int_A^B \zeta^2\cdot\frac{K_1}{\zeta^3}d\zeta\\ &= |\bar z-\bar\iota|\cdot K_1\log\left(\frac{B}{A}\right).\end{aligned}$$

Observe that the attempt in (2.41) is much better than the attempt in (2.37). But it does not serve our purpose.

Proof of Theorem 2.2.13: Recall (2.36). To begin with we note that the second boundedness condition on q can be translated as follows:

$$|f_{zz}| \leq \frac{K_2}{\Im(z)^4}, \quad \forall z \in \mathbb{H}^2, \tag{2.47}$$

where f_{zz} denote the second complex derivative of f, f being a holomorphic function on $\mathbb{H}^2$. To prove that $\xi^{\text{reg}}(z)$ converges we use the Cauchy criterion of convergence of improper integrals which has been stated in Remark 2.2.17. We notice that

$$\xi_c(\iota) + \frac{\partial \xi_c}{\partial z}\bigg|_{z=\iota} \cdot (z-\iota)$$

in (2.30) is the *holomorphic part* of the first order Taylor approximation of $\xi_c(z)$ at $z = \iota$. Let's denote it by $T^{\text{hol}}_{1,\iota}(\xi_c(z))$. Also, $\frac{\partial \xi_c}{\partial z}\big|_{z=\iota}$ is nothing complicated but a complex number because $\xi_c'(z)|_{z=\iota}$ as an $\mathbb{R}$-linear map from $\mathbb{C}$ to $\mathbb{C}$ can be written uniquely as a sum of a $\mathbb{C}$-linear map and a $\mathbb{C}$-conjugate linear map. Let's denote the integrand $\iota\zeta^2\overline{f(\bar{z}+2\iota\zeta)}$ in the expression of $\xi_c(z)$ by $F(\zeta, z)$. As both $F(\zeta, z)$ and its partial derivatives are continuous in ζ and z, we can express $\xi_c'(z)|_{z=\iota}$ using the Leibniz rule as follows:

$$\begin{aligned}
\xi_c'(z)|_{z=\iota} &= \left(-\iota\Im(z)^2\overline{f(\bar{z}+2\iota\Im(z))}\cdot\Im'(z) + \int_{\Im(z)}^{c} \iota\zeta^2 \left(\frac{\partial}{\partial z}\overline{f(\bar{z}+2\iota\zeta)}dz + \frac{\partial}{\partial \bar{z}}\overline{f(\bar{z}+2\iota\zeta)}d\bar{z} \right) d\zeta \right)\bigg|_{z=\iota} \\
&= \left(-\iota\Im(z)^2\overline{f(z)}\cdot\Im'(z) + \int_{\Im(z)}^{c} \iota\zeta^2 \frac{\partial}{\partial z}\overline{f(\bar{z}+2\iota\zeta)}d\zeta \right)\bigg|_{z=\iota} \\
&= \underbrace{-\iota\overline{f(\iota)}\cdot\Im'(z)|_{z=\iota}}_{K} + \left(\int_{\Im(z)}^{c} \iota\zeta^2 \frac{\partial}{\partial z}\overline{f(\bar{z}+2\iota\zeta)}d\zeta \right)\bigg|_{z=\iota}
\end{aligned} \tag{2.48}$$

where the second equality in (2.48) follows from the fact that f is a holomorphic function, hence we get

$$\frac{\partial}{\partial \bar{z}}\overline{f(\bar{z}+2\iota\zeta)} = \overline{\left(\frac{\partial}{\partial z} f(\bar{z}+2\iota\zeta)\right)} = 0.$$

Note that we have omitted dz in $\frac{\partial}{\partial z}\overline{f(\bar{z}+2\iota\zeta)}dz$ because dz as a linear map can be viewed as the 2×2 identity matrix. Since the summand $\iota\overline{f(\iota)}\cdot\Im'(z)|_{z=\iota}$ in (2.48) does not depend on c, therefore it does not hurt to drop it in the expression of $T^{\text{hol}}_{1,\iota}(\xi_c(z))$ for convergence investigation. We will denote the corrected term by $\Psi_c(z)$. Using (2.48), $\Psi_c(z)$ can be written as:

$$\Psi_c(z) = \left(\int_1^c \iota\zeta^2 \left(\overline{f(\bar{\iota}+2\iota\zeta)} + \left(\frac{\partial}{\partial z}\overline{f(\bar{z}+2\iota\zeta)} \right)\bigg|_{z=\iota} \cdot (z-\iota) \right) d\zeta \right) \eta(z). \tag{2.49}$$

Then

$$\xi^{\mathrm{reg}}(z) = \lim_{c\to\infty} \big(\xi_c(z) - \Psi_c(z) - K\big). \tag{2.50}$$

Let's first spell out the expression $\xi_c(z) - \Psi_c(z)$.

Case I: $2c \geq 1 \geq \Im(z)$

$$\begin{aligned}
\xi_c(z) - \Psi_c(z) = \Bigg(&\int_{\Im(z)}^{c} \iota\zeta^2 \overline{f(\bar z + 2\iota\zeta)} d\zeta \\
&- \int_1^c \iota\zeta^2 \bigg(\overline{f(\bar\iota + 2\iota\zeta)} + \Big(\frac{\partial}{\partial z}\overline{f(\bar z + 2\iota\zeta)}\Big)\bigg|_{z=\iota} \cdot (z-\iota)\bigg) d\zeta \Bigg) \eta(z) \\
= \Bigg(&\int_{\Im(z)}^{1} \iota\zeta^2 \overline{f(\bar z + 2\iota\zeta)} d\zeta + \int_1^c \iota\zeta^2 \overline{f(\bar z + 2\iota\zeta)} d\zeta \\
&- \int_1^c \iota\zeta^2 \bigg(\overline{f(\bar\iota + 2\iota\zeta)} + \Big(\frac{\partial}{\partial z}\overline{f(\bar z + 2\iota\zeta)}\Big)\bigg|_{z=\iota} \cdot (z-\iota)\bigg) d\zeta \Bigg) \eta(z) \\
= \Bigg(&\underbrace{\int_1^c \iota\zeta^2 \bigg(\overline{f(\bar z + 2\iota\zeta) - f(\bar\iota + 2\iota\zeta)} - \Big(\frac{\partial}{\partial z}\overline{f(\bar z + 2\iota\zeta)}\Big)\bigg|_{z=\iota} \cdot (z-\iota)\bigg) d\zeta}_{I_c} \\
&- \int_1^{\Im(z)} \iota\zeta^2 \overline{f(\bar z + 2\iota\zeta)} d\zeta \Bigg) \eta(z).
\end{aligned} \tag{2.51}$$

Case II: $2c \geq \Im(z) \geq 1$

$$\begin{aligned}
\xi_c(z) - \Psi_c(z) = \Bigg(&\int_{\Im(z)}^{c} \iota\zeta^2 \overline{f(\bar z + 2\iota\zeta)} d\zeta \\
&- \int_1^{\Im(z)} \iota\zeta^2 \bigg(\overline{f(\bar\iota + 2\iota\zeta)} + \Big(\frac{\partial}{\partial z}\overline{f(\bar z + 2\iota\zeta)}\Big)\bigg|_{z=\iota} \cdot (z-\iota)\bigg) d\zeta \\
&- \int_{\Im(z)}^c \iota\zeta^2 \bigg(\overline{f(\bar\iota + 2\iota\zeta)} + \Big(\frac{\partial}{\partial z}\overline{f(\bar z + 2\iota\zeta)}\Big)\bigg|_{z=\iota} \cdot (z-\iota)\bigg) d\zeta \Bigg) \eta(z) \\
= \Bigg(&\underbrace{\int_{\Im(z)}^c \iota\zeta^2 \bigg(\overline{f(\bar z + 2\iota\zeta) - f(\bar\iota + 2\iota\zeta)} - \Big(\frac{\partial}{\partial z}\overline{f(\bar z + 2\iota\zeta)}\Big)\bigg|_{z=\iota} \cdot (z-\iota)\bigg) d\zeta}_{II_c} \\
&- \int_1^{\Im(z)} \iota\zeta^2 \bigg(\overline{f(\bar\iota + 2\iota\zeta)} + \Big(\frac{\partial}{\partial z}\overline{f(\bar z + 2\iota\zeta)}\Big)\bigg|_{z=\iota} \cdot (z-\iota)\bigg) d\zeta \Bigg) \eta(z).
\end{aligned} \tag{2.52}$$

Since the integrals

$$\int_1^{\Im(z)} \iota\zeta^2 \overline{f(\bar{z}+2\iota\zeta)} d\zeta$$

and

$$\int_1^{\Im(z)} \iota\zeta^2 \Big(\overline{f(\bar{\iota}+2\iota\zeta)} + \Big(\frac{\partial}{\partial z}\overline{f(\bar{z}+2\iota\zeta)}\Big)\Big|_{z=\iota} \cdot (z-\iota)\Big) d\zeta$$

in R.H.S of (2.51) and (2.52) are independent of c, we work with I_c and II_c in (2.51) and (2.52) to prove the convergence of ξ^{reg}. Now if $A, B \geq c$, we have

$$I_B - I_A = \int_A^B \iota\zeta^2 \Big(\overline{f(\bar{z}+2\iota\zeta) - f(\bar{\iota}+2\iota\zeta)} - \Big(\frac{\partial}{\partial z}\overline{f(\bar{z}+2\iota\zeta)}\Big)\Big|_{z=\iota} \cdot (z-\iota)\Big) d\zeta,$$

and

$$II_B - II_A = \int_A^B \iota\zeta^2 \Big(\overline{f(\bar{z}+2\iota\zeta) - f(\bar{\iota}+2\iota\zeta)} - \Big(\frac{\partial}{\partial z}\overline{f(\bar{z}+2\iota\zeta)}\Big)\Big|_{z=\iota} \cdot (z-\iota)\Big) d\zeta.$$

Using the Remainder Estimation Theorem for f, we have

$$|I_B - I_A| = |II_B - II_A| \leq \int_A^B \zeta^2 \cdot \max_w |f^{(2)}(w)| \cdot |(\bar{z}+2\iota\zeta) - (\bar{\iota}+2\iota\zeta)|^2 d\zeta, \tag{2.53}$$

where w is varying on the line segment connecting $\bar{z}+2\iota\zeta$ and $\bar{\iota}+2\iota\zeta$. We assume $A, B > \Im(z)$. Using (2.47), we rewrite (2.53) as follows:

$$\begin{aligned} |I_B - I_A| = |II_B - II_A| &\leq \int_A^B \zeta^2 \cdot \max_w \frac{K_2}{(\Im(w))^4} \cdot |\bar{z}-\bar{\iota}|^2 d\zeta \\ &\leq |\bar{z}-\bar{\iota}|^2 \int_A^B \zeta^2 \cdot \frac{K_2}{(2\zeta - \Im(z))^4} d\zeta \end{aligned} \tag{2.54}$$

Also, the denomiator $(2\zeta - \Im(z))^4$ is atleast as big as ζ^4. As a result (2.54) has the following form:

$$\begin{aligned} |I_B - I_A| = |II_B - II_A| &\leq |\bar{z}-\bar{\iota}|^2 \int_A^B \zeta^2 \cdot \frac{K_2}{\zeta^4} d\zeta \\ &= |\bar{z}-\bar{\iota}|^2 \cdot \frac{K_2}{4}\Big(-\frac{1}{B} + \frac{1}{A}\Big). \end{aligned} \tag{2.55}$$

These estimates show that ξ^{reg} is a well defined vector field. But they also show that ξ^{reg} is locally a uniform limit of harmonic vector fields which determine the same holomorphic quadratic differential. Therefore, ξ^{reg} is a harmonic vector field by Corollary 2.2.8. □

2.2.2 Extending harmonic vector fields on $\mathbb{H}^2$ to the boundary circle $\mathbb{S}^1$

We refer to the extended real axis $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ as the boundary at infinity of $\mathbb{H}^2$. We are using the unit disc model so that we have a well defined notion of the tangent space at the point $\{\infty\} \in \partial\mathbb{H}^2$ as there is a natural 1-1 correspondence between $\partial\mathbb{D}$ and $\partial\mathbb{H}^2$. The starting point is to compare the length of a vector $v \in T_z\mathbb{H}^2$ for some $z \in \mathbb{H}^2$ (measured in the Euclidean metric) with the length of the pushforward of v (measured in the Euclidean metric) by a conformal map between $\mathbb{H}^2$ and $\mathbb{D}$. Consider the Cayley transformation

$$C(z) = \frac{z - \iota}{z + \iota} \tag{2.56}$$

mapping the upper half plane model of $\mathbb{H}^2$ to the unit disc model $\mathbb{D}$ of $\mathbb{H}^2$. We have

$$|dC_z(v)| = \frac{|v|}{|z|^2}, \quad \forall v \in T_z\mathbb{H}^2. \tag{2.57}$$

Theorem 2.2.19. *The harmonic vector field ξ^{reg} in Theorem 2.2.13, transformed from $\mathbb{H}^2$ to the open unit disc $\mathbb{D} \subset \mathbb{C}$ by the Cayley transform C given by (2.56) extends to a continuous vector field, say χ, on $\overline{\mathbb{D}}$ defined as follows:*

$$\chi(C(z)) = \begin{cases} C_*(\xi^{\text{reg}}(z)) & z \in \mathbb{H}^2 \\ C_*(\xi^{\text{reg}}(z)) & z \in \partial\mathbb{H}^2 \setminus \{\infty\} \\ 0 & z = \{\infty\} \end{cases} \tag{2.58}$$

where $C_*(\xi^{\text{reg}}(z))$ is the pushforward of $\xi^{\text{reg}}(z)$ by the Cayley transform C.

Before we prove Theorem 2.2.19, we discuss the one and only disadvantage of Wolpert's formula (2.23) in the following remark:

Remark 2.2.20. Recall Wolpert's global solution ξ (see (2.23)) for the potential equation (2.21). Given that $q = f dz^2$ is bounded in the hyperbolic metric $\mathbf{g}_{\mathbb{H}^2}$, i.e., $|f(z)| \leq \frac{D}{\Im(z)^2}$ where D is a positive constant, ξ extends to the real line $\mathbb{R}$. This can be seen as follows: f is not defined for z such that $\Im(z) = 0$. So the integral in (2.23) is an improper integral, so for z such that $\Im(z) = 0$, we define

$$\xi(z) = \lim_{\epsilon \to 0} \left(\left(\overline{\int_w^{z+\iota\epsilon} (\overline{z + \iota\epsilon} - \zeta)^2 f(\zeta) d\zeta} \right) \eta(z) \right).$$

The above limit exists, as can be seen by taking w to be ι and using $|f(z)| \leq \frac{D}{\Im(z)^2}$. We have no reason to believe that ξ extends to the point $\{\infty\}$ in the boundary $\mathbb{R} \cup \{\infty\}$. Here is an argument: for the sake of convenience, we choose the line segment from $w = \iota$ to $z = c\iota$ as the path of integration in the expression of ξ, where $c > 1$ is a positive real number. Then,

$$\xi(c\iota) = \overline{\int_\iota^{c\iota} (\overline{c\iota} - \zeta)^2 f(\zeta) d\zeta}.$$

Therefore,

$$\begin{aligned}|\xi(c\iota)| &\leq D\int_{\iota}^{c\iota}\frac{|\overline{c\iota}-\zeta|^2}{\Im(\zeta)^2}d\zeta\\ &= D\int_{\iota}^{c\iota}\frac{|c\iota-\zeta|^2}{\Im(\zeta)^2}d\zeta\\ &\leq D\cdot|c\iota-\iota|\cdot\max_{\zeta}\frac{|c\iota-\zeta|^2}{\Im(\zeta)^2},\end{aligned}$$

where ζ is varying on the line segment from ι to $c\iota$. From the above estimate, it is clear that ξ is $O(|z|^3)$ at the point $\{\infty\}$ in the boundary $\mathbb{R}\cup\{\infty\}$.

Proof of Theorem 2.2.19: Recall Remark 2.2.12. For z such that $\Im(z)=0$, the definition of ξ^{reg} makes perfectly good sense because the convergence of the improper integral in the expression of ξ^{reg} for z such that $\Im(z)=0$ follows from the conditions given in (2.38), (2.40), and (2.47). Now, we claim that for a sequence $\{z_n\}$ of points in $\mathbb{H}^2$ such that $|z_n|\to\infty$, where $|\cdot|$ denotes the absolute value

$$\lim_{|z_n|\to\infty}|C_*(\xi^{\text{reg}}(z_n))|=0, \tag{2.59}$$

where $|C_*(\xi^{\text{reg}}(z))|$ denotes the length of the pushforward of $\xi^{\text{reg}}(z)$ measured in the Euclidean metric. Using (2.57), we rewrite (2.59) as follows

$$\lim_{|z_n|\to\infty}\frac{|\xi^{\text{reg}}(z_n)|}{|z_n|^2}=0. \tag{2.60}$$

The main idea is to split the integral $\int_0^c \iota\zeta^2\overline{f(z_n+2\iota\zeta)}d\zeta$ at height h such that $h=|z_n|$ and estimate the resulting integrals in different ways. Using (2.48), (2.49), and (2.50), our expression for $\xi^{\text{reg}}(z_n)$ takes the following form:

$$\xi^{\text{reg}}(z_n)=\underbrace{\xi_h(z_n)-\left(\xi_h(\iota)+\frac{\partial\xi_h}{\partial z}\bigg|_{z=\iota}\cdot(z_n-\iota)\right)}_{\xi_1^{\text{reg}}(z_n)}+\underbrace{\lim_{c\to\infty}\left(\xi_{h,c}(z_n)-\Psi_{h,c}(z_n)-K\right)}_{\xi_2^{\text{reg}}(z_n)},$$

where

$$\xi_h(z_n)=\left(\int_0^h\iota\zeta^2\overline{f(z_n+2\iota\zeta)}d\zeta\right)\eta(z_n),\quad \xi_{h,c}(z_n)=\left(\int_h^c\iota\zeta^2\overline{f(z_n+2\iota\zeta)}d\zeta\right)\eta(z_n),$$

$$\xi_h(\iota)=\left(\int_1^h\iota\zeta^2\overline{f(\bar{\iota}+2\iota\zeta)}d\zeta\right)\eta(\iota),\quad \frac{\partial\xi_h}{\partial z}\bigg|_{z=\iota}=\left(\int_1^h\iota\zeta^2\left(\frac{\partial}{\partial z}\overline{f(z_n+2\iota\zeta)}\right)\bigg|_{z=\iota}d\zeta\right)\eta(\iota),$$

$$\Psi_{h,c}(z_n)=\left(\int_h^c\iota\zeta^2\left(\overline{f(\bar{\iota}+2\iota\zeta)}+\left(\frac{\partial}{\partial z}\overline{f(z_n+2\iota\zeta)}\right)\bigg|_{z=\iota}\cdot(z_n-\iota)\right)d\zeta\right)\eta(z_n).$$

Note that we have treated $h = |z_n|$ as a constant independent of z_n. Using (2.34), (2.36), and (2.38), each individual term - $\xi_h(z_n)$, $\xi_h(\iota)$ and $\frac{\partial \xi_h}{\partial z}|_{z=\iota} \cdot (z_n - \iota)$ - in the expression of $\xi_1^{\mathrm{reg}}(z_n)$ satisfies the following inequalities when estimated in the Poincare metric $\mathbf{g}_{\mathbb{H}^2}$:

$$\begin{aligned}
|\xi_h(z_n)| &\leq \frac{D}{4}|z_n|, \\
|\xi_h(\iota)| &\leq \frac{D}{4}|z_n|, \\
\left| \frac{\partial \xi_h}{\partial z}\Big|_{z=\iota} \cdot (z_n - \iota) \right| &\leq \frac{K_1}{8}|z_n|.
\end{aligned}$$

At this point (2.57) comes in handy and show us immediately that $C_*(\xi_1^{\mathrm{reg}}(z_n)) \to 0$ as $|z_n| \to \infty$. From the estimate given in (2.55) in the proof of Theorem 2.2.13 we have $C_*(\xi_2^{\mathrm{reg}}(z_n)) \to 0$ as $|z_n| \to \infty$. □

Chapter 3

Going from the analytic description to the cohomological description

3.1 Vector fields on $\mathbb{D}$ and $\mathbb{S}^1$

We will denote the Hilbert space of measurable functions f on $\mathbb{S}^1$ such that

$$\int_{\mathbb{S}^1} |f(x)|^2 dx < +\infty$$

modulo the equivalence relation of almost-everywhere equality by $L^2(\mathbb{S}^1)$. We are not going to prove the completeness of $L^2(\mathbb{S}^1)$. The main idea to prove completeness of $L^2(\mathbb{S}^1)$ is that a Cauchy sequence of L^2-functions has a subsequence that converges pointwise off a set of measure 0. There is a different definition of $L^2(\mathbb{S}^1)$, namely the completion of $C^0(\mathbb{S}^1)$, the space of continuous $\mathbb{C}$-valued functions on $\mathbb{S}^1$, with respect to the norm

$$\|f\| := \frac{1}{\sqrt{2\pi}}\left(\int_{\mathbb{S}^1} |f(z)|^2 dz\right)^{1/2} \tag{3.1}$$

The Fourier basis elements are the exponential functions $\psi_k(z) := z^k$ for $z \in \mathbb{S}^1$. The exponential functions $\{\psi_k | k \in \mathbb{Z}\}$ form an orthonormal set in $L^2(\mathbb{S}^1)$. But it's not clear immediately that they form an orthonormal Hilbert basis (see [4]). From orthonormality, the *Fourier coefficients* $a_k \in \mathbb{C}$ of f are the inner products

$$a_k = \langle f, \psi_k \rangle = \frac{1}{2\pi}\int_{\mathbb{S}^1} f(z)\overline{\psi_k(z)}dz.$$

The Fourier expansion of $f \in L^2(\mathbb{S}^1)$ is

$$f(z) = \sum_{k\in\mathbb{Z}} a_k \psi_k(z)$$

where the equality means convergence of the partial sums to f in the L^2-norm, or

$$\lim_{N\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{S}^1} \left| \sum_{k=-N}^{N} a_k\psi_k(z) - f(z) \right|^2 dz = 0.$$

The convenient algebraic property of ψ_k is that the basis is multiplicative. And multiplication of functions corresponds to the *convolution of Fourier series;* this is actually obvious in our context since

$$\psi_k \cdot \psi_l = \psi_{k+l}. \tag{3.2}$$

From now on we will denote $L^2(\mathbb{S}^1)$ by $\mathcal{H}$. There is an orthogonal sum splitting $\mathcal{H} = \mathcal{H}^1 \oplus \mathcal{H}^2$ where $\mathcal{H}^1$ is the closure of the span of $\{\psi_k | k < 0\}$ and consequently, $\mathcal{H}^2$ is the closure of the span of $\{\psi_k | k \geq 0\}$. An element of $\mathcal{H}^2$, say

$$f := \sum_{k\geq 0} a_k f_k$$

has a canonical extension to a function (in the L^2-sense) defined on the unit disk $\mathbb{D}$ in $\mathbb{C}$ by the formula

$$z \longmapsto \sum_{k\geq 0} a_k z^k.$$

This is in fact a convergent power series in the open unit disk $\mathbb{D}$, so defines a holomorphic function on the open unit disk $\mathbb{D}$ in $\mathbb{C}$. So we should see $\mathcal{H}^2$ as the linear subspace of $\mathcal{H}$ consisting of those L^2-functions on $\mathbb{S}^1$ which extend holomorphically to the open unit disk $\mathbb{D}$ in $\mathbb{C}$. Equivalently, think of $\mathcal{H}^2$ as the complex vector space of L^2-vector fields on $\mathbb{S}^1$ which extend holomorphically to the open unit disk, i.e.

$$\mathcal{H}^2 = \{X : \mathbb{S}^1 \longrightarrow \mathbb{R}^2 | X \,\mathrm{is}\, L^2, X(z) \in T_z\mathbb{R}^2 \cong \mathbb{R}^2 \cong \mathbb{C}, \forall z \in \mathbb{S}^1\},$$

where the norm on X is taken in the sense of (3.1).

Remark 3.1.1. A smooth or continuous vector field X on the open unit disk $\mathbb{D}$ has an L^2-*extension* to the closed disk $\overline{\mathbb{D}}$ if the following holds: for every $\epsilon > 0$, we get a continuous vector field X_ϵ on $\mathbb{S}^1_{1-\epsilon}$, a circle of radius $1-\epsilon$ (which can be identified canonically with $\mathbb{S}^1$ by streching), by restricting X to $\mathbb{S}^1_{1-\epsilon}$. Now, letting $\epsilon \to 0$, we get a sequence $\{X_\epsilon\}$ in the Hilbert space of L^2-vector fields on the boundary circle $\mathbb{S}^1$. And if $\{X_\epsilon\}$ converges to a L^2-vector field on the boundary circle $\mathbb{S}^1$, then X has an L^2- extension to the closed disk $\overline{\mathbb{D}}$.

Definition 3.1.2. A vector field on $\mathbb{S}^1$ with values in $\mathbb{R}^2$ or $\mathbb{C}$ is called *tangential* if it makes $\mathbb{S}^1$ flows into itself.

We denote the space of tangential vector fields on $\mathbb{S}^1$ by $\mathfrak{X}_{\text{tangential}}(\mathbb{S}^1)$. It is a real vector space. To get more insight, consider the following example:

Example 3.1.3. Consider the following complex-valued vector field on $\mathbb{S}^1$:

$$X(x,y) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

In complex coordinates, we express X as $X(z) = \iota z$. It is clear that X is a tangential vector field on $\mathbb{S}^1$ since

$$\sigma(t,(x,y)) = (x\cos t - y\sin t, x\sin t + y\cos t)$$

is a flow generated by X and the flow through (x,y) is a circle whose centre is at origin. Clearly, $\sigma(t,(x,y)) = (x,y)$ if $t = 2n\pi, n \in \mathbb{Z}$. See L.H.S of Figure 3.1.

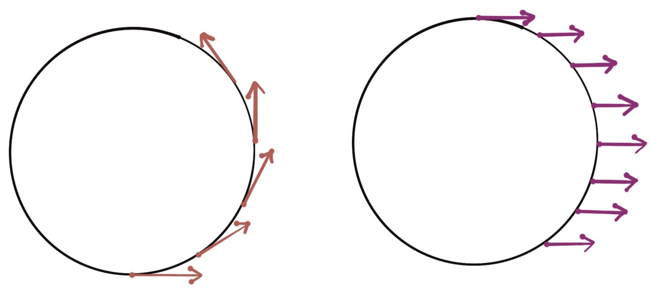

Figure 3.1: An example of a tangential vector field on $\mathbb{S}^1$

Note that the above example is only one solution of tangential vector fields on $\mathbb{S}^1$. But we get all other solutions by multiplying X in Example 3.1.3 with any real valued function on $\mathbb{S}^1$. Note that vector fields can be multiplied with functions. For simplicity, we think of multiplication of L^2-vector fields on $\mathbb{S}^1$ with real valued functions on $\mathbb{S}^1$ as multiplication of functions with functions.

Recall that we have expressed an L^2-function f on $\mathbb{S}^1$ with values in $\mathbb{C}$ as $\sum_{k\in\mathbb{Z}} a_k\psi_k$. It's a routine exercise in Fourier analysis to show that f is real valued iff $a_k = \overline{a_{-k}}$ for all k. Therefore the corresponding (real) Fourier expansion of f is

$$f(x) = \frac{1}{2}a'_0 + \sum_{k=1}^{\infty} a'_k\cos(kx) + b'_k\sin(kx),$$

where $a'_k = a_k + a_{-k}$ and $b'_k = \iota(a_k - a_{-k})$. So, the real-valued functions

$$\{1, \cos(kx), \sin(kx) | k = 1,2,3,\ldots\}$$

also form an orthogonal basis of the space $\mathcal{H}$, since

$$\cos(kx) = \frac{\exp(\iota kx) + \exp(-\iota kx)}{2} = \frac{z^k + z^{-k}}{2},$$

$$\sin(kx) = \frac{\exp(\iota kx) - \exp(-\iota kx)}{2\iota} = \frac{z^k - z^{-k}}{2\iota}.$$

Using (3.2), i.e., the fact that the Fourier transform of the product of functions is the convolution of the Fourier transforms, we have the following real Hilbert basis of $\mathfrak{X}_{\text{tangential}}(\mathbb{S}^1)$:

$$\left\{ \iota z, \frac{\iota z^{1+k} + \iota z^{1-k}}{2}, \frac{z^{1+k} - z^{1-k}}{2} \middle| k = 1, 2, 3, \dots \right\}. \quad (3.3)$$

Also, Killing vector fields on $\mathbb{D}$ are the infinitesimal generators of isometries of $\mathbb{D}$, hence Killing vector fields on $\mathbb{D}$ are tangential vector fields on $\mathbb{S}^1$. We will denote the three dimensional real vector space whose elements are Killing vector fields on $\mathbb{S}^1$ by $\mathfrak{X}_{\text{Killing}}(\mathbb{S}^1)$.

Theorem 3.1.4. *We have*

1. $\mathfrak{X}_{tangential}(\mathbb{S}^1) \cap \mathcal{H}^2 = \mathfrak{X}_{Killing}(\mathbb{S}^1)$.
2. $\mathfrak{X}_{tangential}(\mathbb{S}^1) + \mathcal{H}^2$ *is the vector space of all L^2-vector fields on $\mathbb{S}^1$.*

Proof (1). As any complex vector space has an underlying real vector space so the real Hilbert basis of the space $\mathcal{H}^2$ is given as

$$\{z^k, \iota z^k | k \geq 0\}.$$

The basis for $\mathfrak{X}_{\text{tangential}}(\mathbb{S}^1)$ is given by (3.3). Assume $X \in \mathfrak{X}_{\text{tangential}}(\mathbb{S}^1) \cap \mathcal{H}^2$. Then $X = \sum_{k\geq 0} a_k z^k + b_k i z^k$ and $X = a'_0 \iota z + \sum_{k\geq 1} a'_k \frac{\iota z^{1+k} + \iota z^{1-k}}{2} + b'_k \frac{z^{1+k} - z^{1-k}}{2}$. Since

$$\sum_{k\geq 0} a_k z^k + b_k i z^k = a'_0 \iota z + \sum_{k\geq 1} a'_k \frac{\iota z^{1+k} + \iota z^{1-k}}{2} + b'_k \frac{z^{1+k} - z^{1-k}}{2},$$

comparing the coefficients of z^k and ιz^k in each expression, we obtain $a'_0 = b_1$, $b_2 = b_0 = \frac{a'_1}{2}, a_2 = \frac{b'_1}{2} = -a_0$, and all other coefficients are zero. Therefore, X is a linear combination with real coefficients of ιz, $\frac{z^2-1}{2}$, and $\frac{\iota z^2 + \iota}{2}$ Note that ιz, $\frac{z^2-1}{2}$, and $\frac{\iota z^2+\iota}{2}$ are linearly independent. Hence the vector space $\mathfrak{X}_{\text{tangential}}(\mathbb{S}^1) \cap \mathcal{H}^2$ is a 3-dimensional space which is nothing but $\mathfrak{X}_{\text{Killing}}(\mathbb{S}^1)$.

(2) A real Hilbert basis of the space of L^2-vector fields on $\mathbb{S}^1$ is given by $\{z^k, \iota z^k | k \in \mathbb{Z}\}$. Then it is very easy to see that

$$X(z) - \left(\left(\sum_{k\in\{2,3,\dots\}} b_{1-k} \left(\iota z^{1+k} + \iota z^{1-k} \right) \right) - \sum_{k\in\{2,3,\dots\}} a_{1-k} \left(z^{1+k} - z^{1-k} \right) \right)$$
$$= a_0 + b_0 \iota + a_1 z + b_1 \iota z + a_2 z^2 + b_2 \iota z^2 + (a_3 + a_{-1}) z^3 + (b_3 - b_{-1}) \iota z^3 + \cdots,$$

where $X(z) = \sum_{k\in\mathbb{Z}} a_k z^k + b_k \iota z^k$, $z \in \mathbb{S}^1$. Therefore, $X = X_1 + X_2$, where $X_1 \in \mathfrak{X}_{\text{tangential}}(\mathbb{S}^1)$ and $X_2 \in \mathcal{H}^2$. $\square$

Before we state conclusions of this chapter we introduce some notions and conventions:

1. Let $\mathfrak{M}$ be a Γ-module, where Γ is a subgroup of $\mathrm{PSU}(1,1)$. A map $c : \Gamma \longrightarrow \mathfrak{M}$ is called a cocycle if
$$c_{\gamma_1 \circ \gamma_2} = \gamma_2^* c_{\gamma_1} + c_{\gamma_2}, \gamma_1, \gamma_2 \in \Gamma,$$
c_γ stands for $c(\gamma)$, $*$ denotes the action of Γ on $\mathfrak{M}$. If $m \in \mathfrak{M}$, its *coboundary* δm is the cocycle
$$\gamma \longmapsto \gamma^* m - m, \gamma \in \Gamma. \tag{3.4}$$
The first *cohomology group* $H^1(\Gamma; \mathfrak{M})$ is the quotient $Z^1(\Gamma; \mathfrak{M})/B^1(\Gamma; \mathfrak{M})$.

2. The most important cases of $\mathfrak{M}$ from the viewpoint of this thesis are
   (a) $\mathcal{S}^\infty(T\mathbb{D})$, the vector space of smooth vector fields on $\mathbb{D}$. Γ acts on $\mathcal{S}^\infty(T\mathbb{D})$ in the following manner
$$\gamma^* F = F(\gamma)\gamma'^{-1}, \quad \gamma \in \Gamma, F \in \mathcal{S}^\infty(T\mathbb{D}). \tag{3.5}$$
   (b) HOL, the vector space of holomorphic vector fields on $\mathbb{D}$. Γ acts on HOL in the same manner as in (3.5).
   (c) $\mathfrak{g}$, the vector space of Killing vector fields on $\mathbb{D}$. Note that we have already dealt with this case in Section 1.3.1 in Chapter 1.3.

3. Note that
$$\mathfrak{g} \subset \mathrm{HOL} \subset \mathcal{S}^\infty(T\mathbb{D}).$$

Recall Subsection 2.2 in Chapter 2. Given a holomorphic quadratic differential q on $\mathbb{D}$ which satisfies boundedness conditions, namely, q is bounded in the hyperbolic metric $\mathbf{g}_\mathbb{D}$ of $\mathbb{D}$, and the first and the second covariant derivative of q w.r.t the linear connection on $T^*\mathbb{D} \otimes_\mathbb{C} T^*\mathbb{D}$ are bounded in $\mathbf{g}_\mathbb{D}$, we obtain a harmonic vector field χ on $\mathbb{D}$ that extends continuously on the boundary circle $\mathbb{S}^1$ such that $(\mathcal{L}_\chi \mathbf{g}_\mathbb{D})^{(2,0)} = q$. Note that χ is not necessarily tangential to the boundary circle $\mathbb{S}^1$. We will denote the restriction of χ to $\mathbb{S}^1$ by $\chi|_{\mathbb{S}^1}$. Using Theorem 3.1.4 (2), we can write $\chi|_{\mathbb{S}^1}$ as $\chi_1 + \chi_2$, where $\chi_1 \in \mathfrak{X}_{\text{tangential}}(\mathbb{S}^1)$ and $\chi_2 \in \mathcal{H}^2$. Since χ is a harmonic vector field on $\mathbb{D}$ whose associated holomorphic quadratic differential is q, then the holomorphic quadratic differential associated with the vector field χ_1 is the same q. Because the holomorphic quadratic differential associated with χ_2 is zero. Notice that in the expression of $\chi_1 = \chi - \chi_2$ we are working with the holomorphic extension of χ_2 to the open unit disk $\mathbb{D}$. Now, the coboundary of χ, i.e.,
$$\delta\chi(\gamma) = \chi(\gamma)\gamma'^{-1} - \chi, \quad \forall \gamma \in \Gamma$$
is a cocycle with values in HOL because of the Γ-invariance of q. But our goal is to get a cocycle with values in $\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of $\mathrm{Isom}^+(\mathbb{D})$. Using Theorem 3.1.4 (1), we can easily see that for every $\gamma \in \Gamma$, $\delta(\chi_1)(\gamma) \in \mathfrak{X}_{\text{tangential}}(\mathbb{S}^1) \cap \mathcal{H}^2$ and therefore we get a cocycle in $\mathfrak{X}_{\text{Killing}}(\mathbb{S}^1) \cong \mathfrak{g}$. We summarize our discussion as

Theorem 3.1.5. *Given a holomorphic quadratic differential $q = f dz^2$ on the Poincaré disk $\mathbb{D}$ which satisfies the following boundedness conditions:*

1. *q is bounded in the hyperbolic metric on $\mathbb{D}$, i.e.,*

$$\|q\|_{g_{\mathbb{D}}} \leq D,$$

where D is a positive real number.

2. *The first and the second covariant derivative of q w.r.t the linear connection on $T^*\mathbb{D} \otimes_{\mathbb{C}} T^*\mathbb{D}$ are bounded in $\mathbf{g}_{\mathbb{D}}$.*

Then there exists a harmonic vector field χ on $\mathbb{D}$ which L^2-extends to the closed disk $\overline{\mathbb{D}}$ such that $(\mathcal{L}_\chi \mathbf{g}_{\mathbb{D}})^{(2,0)} = q$. Moreover, the restriction of that extension to the boundary circle $\mathbb{S}^1$ is tangential and χ is unique upto the addition of holomorphic vector fields on $\mathbb{D}$ which extend tangentially to the boundary circle $\mathbb{S}^1$. From Theorem 3.1.4 (1), χ is unique upto the addition of the vector space $\mathfrak{g}$ of Killing vector fields on $\mathbb{D}$.

Corollary 3.1.6. *Let Γ denote a subgroup of $\mathrm{Isom}^+(\mathbb{D})$ where $\mathrm{Isom}^+(\mathbb{D})$ is the group of orientation preserving isometries of $\mathbb{D}$. If $q = f dz^2$ and χ are related as in Theorem 3.1.5 and if in addition to (1) and (2) in Theorem 3.1.5, q is Γ-invariant, i.e.,*

$$f(\gamma(z))\gamma'(z)^2 = f(z), \quad \forall \gamma \in \Gamma, z \in \mathbb{D},$$

then $\delta\chi$ defined by

$$\gamma \longmapsto \chi(\gamma)\gamma'^{-1} - \chi, \quad \forall \gamma \in \Gamma$$

is a 1-cocycle c with coefficients in the Γ-module $\mathfrak{g}$ - the Lie algebra of $\mathrm{Isom}^+(\mathbb{D})$ and its cohomology class $[c]$ depends only on q.

Proof From Theorem 3.1.5, we know that χ is unique upto the addition of Killing vector fields on $\mathbb{D}$, hence for every $\gamma \in \Gamma$, $\delta\chi(\gamma)$ is a holomorphic vector field which extends tangentially to the boundary circle $\mathbb{S}^1$. Therefore, for every $\gamma \in \Gamma$, $\delta\chi(\gamma) \in \mathfrak{g}$. Recall that we have for every $\gamma \in \Gamma$, $c(\gamma) = \frac{\chi(\gamma)}{\gamma'} - \chi$. Since χ is well-defined upto addition of a Killing vector field X on $\mathbb{D}$, it follows that c is well defined upto addition of δX. Hence, the cohomology class $[c]$ of c is well defined. □

Remark 3.1.7. In Corollary 3.1.6, we view χ as a 0-cochain with values in the vector space of harmonic vector fields on $\mathbb{D}$.

Corollary 3.1.8. *Let Γ in Corollary 3.1.6 be a discrete cocompact subgroup of $\mathrm{Isom}^+(\mathbb{D})$. Then we have an injective mapping*

$$\begin{aligned} \Phi : \mathrm{HQD}(\mathbb{D}, \Gamma) &\longrightarrow H^1(\Gamma; \mathfrak{g}) \\ q &\longmapsto [c], \end{aligned} \tag{3.6}$$

where $\mathrm{HQD}(\mathbb{D}, \Gamma)$ denotes the vector space of Γ-invariant holomorphic quadratic differentials on $\mathbb{D}$ and $c = \delta\chi$.

Proof We assume that $\Phi(q) = [c] = 0$. Then, there exists an element $X \in \mathfrak{g}$ such that $c = \delta(X)$. By setting $Y = \chi - X$ we notice that the holomorphic quadratic differential associated to Y is q, and $\delta Y = 0$, i.e., Y is invariant under the action of Γ. Therefore, Y can be viewed as a harmonic vector field on the the surface $\mathbb{D}/\Gamma$. From [10, Proposition 4.2], on a two dimensional compact orientable Riemannian manifold without boundary, a harmonic vector field is a conformal vector field. Therefore, $q \equiv 0$. □

Chapter 4

Going from the cohomological description to the analytic description

4.1 Γ-invariant partition of unity on $\mathbb{D}$

Recall that a partition of unity subordinate to an open covering $\{U_i\}$ of a manifold M is a collection $\{\varphi_i\}$ of non-negative smooth functions such that

1. $\mathrm{supp}(\varphi_i) \subset U_i$.
2. Each $p \in M$ has a neighborhood that intersects with only finitely many $\mathrm{supp}(\varphi_i)$.
3. $\sum \varphi_i = 1$.

Let Γ be a discrete cocompact subgroup of $\mathrm{PSU}(1,1)$ where $\mathrm{PSU}(1,1)$ denotes the group of orientation preserving isometries of $\mathbb{D}$. Below we give the existence of a Γ-invariant partition of unity on $\mathbb{D}$.

Lemma 4.1.1. *There exists a smooth function φ on $\mathbb{D}$ such that*

1. $0 \leq \varphi \leq 1$.
2. *For each $z \in \mathbb{D}$, there is a neighborhood U of z and a finite subset S of Γ such that $\varphi = 0$ on $\gamma(U)$ for every $\gamma \in \Gamma - S$.*
3. $\sum_{\gamma\in\Gamma} \varphi(\gamma(z)) = 1$ *on $\mathbb{D}$.*

Proof We choose an open covering $\{U_i\}_{i\in I}$ of the closed surface $\mathbb{D}/\Gamma$ where each U_i is simply connected and a smooth partition of unity $\{\alpha_i\}$ subordinate to the covering

$\{U_i\}_{i\in I}$. For each U_i, we choose a single component V_i of $\pi^{-1}(U_i)$ where $\pi : \mathbb{D} \longrightarrow \mathbb{D}/\Gamma$ is the projection map, and set

$$\phi_i(z) = \begin{cases} \alpha_i(\pi(z)), & z \in V_i \\ 0, & z \in \mathbb{D} - V_i. \end{cases}$$

Note that the mapping π restricted to each component of $\pi^{-1}(U_i)$ is a one-to-one covering. It's clear that $\phi_i \in C^\infty(\mathbb{D})$, and that $\phi = \sum_i \phi_i(z), z \in \mathbb{D}$ has the required properties. □

Remark 4.1.2. We suspect that Lemma 4.1.1 is a simpler version of results on *Kleinian groups* (see [37]).

To go from the cohomological description of tangent spaces (to the Teichmueller space) to the analytic description which is given by the space of holomorphic quadratic differentials on Σ_g, we first construct a tangential vector field on the circle $\mathbb{S}^1$ (recall Section 3.1 from Chapter 3) from a cocycle c representing a cohomology class $[c] \in H^1(\Gamma; \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of the group of orientation preserving isometries of $\mathbb{D}$. We use Lemma 4.1.1 to get the following: given any $[c] \in H^1(\Gamma; \mathfrak{g})$ we set

$$\psi(z) = -\sum_{\gamma\in\Gamma} \varphi(\gamma(z))c_\gamma(z), \quad z \in \mathbb{D}.$$

Lemma 4.1.3 ([37])**.** *ψ is a C^∞-vector field on $\mathbb{D}$ such that for $A \in \Gamma$, $z \in \mathbb{D}$,*

$$(A^*\psi)(z) - \psi(z) = c_A(z). \tag{4.1}$$

Proof Recall (3.5). Consider the L.H.S of (4.1) in the Lemma, we have

$$\begin{aligned}(A^*\psi)(z) - \psi(z) &= -\sum_{\gamma\in\Gamma}\Big(\varphi(\gamma(Az))c_\gamma(Az)A'(z)^{-1} - \varphi(\gamma(z))c_\gamma(z)\Big)\\ &= -\sum_{\gamma\in\Gamma}\Big(\varphi(\gamma(Az))\Big(c_{\gamma\circ A}(z) - c_A(z)\Big) - \varphi(\gamma(z))c_\gamma(z)\Big)\\ &= \sum_{\gamma\in\Gamma}\varphi(\gamma(Az))c_A(z) = c_A(z).\end{aligned}$$

The second equality in the above equation follows from the fact that c is a cocycle. Therefore,

$$\delta\psi = c.$$

□

Remark 4.1.4. Let $\mathcal{S}^\infty(T\mathbb{D})$ denote the vector space of C^∞-vector fields on $\mathbb{D}$. From Lemma 4.1.3, we have $H^1(\Gamma; \mathcal{S}^\infty(T\mathbb{D})) = \{0\}$.

Corollary 4.1.5. *If* HOL *is the vector space of holomorphic vector fields on $\mathbb{D}$, then for every cocycle c representing a cohomology class $[c] \in H^1(\Gamma; \mathrm{HOL})$, there is a $\psi \in \mathcal{S}^\infty(T\mathbb{D})$ such that*

$$c = \delta\psi.$$

Proof The injection of HOL into $\mathcal{S}^\infty(T\mathbb{D})$ induces a mapping

$$H^1(\Gamma;\mathrm{HOL}) \longrightarrow H^1(\Gamma;\mathcal{S}^\infty(T\mathbb{D})).$$

□

Remark 4.1.6. Corollary 4.1.5 is true if we replace HOL by the vector space of Killing vector fields $\mathfrak{g}$ on $\mathbb{D}$ because of $\mathfrak{g} \subset \mathrm{HOL} \subset \mathcal{S}^\infty(T\mathbb{H}^2)$.

Let c be a 1-cocycle with values in the vector space $\mathfrak{g}$ of Killing vector fields on $\mathbb{D}$. From Chapter 2 and Chapter 3 we know that there exists a harmonic vector field χ with a tangential L^2-extension on the boundary circle $\mathbb{S}^1$ such that $\delta\chi = c$. From Lemma 4.1.3, Corollary 4.1.5, and Remark 4.1.6, we get another 0-cochain ψ in $\mathcal{S}^\infty(T\mathbb{D})$ such that $\delta\psi = c$. Therefore, $\chi - \psi$ is a 0-cocycle in $\mathcal{S}^\infty(T\mathbb{D})$ and $\chi - \psi$ is invariant under the action of Γ, i.e.,

$$\begin{aligned}(\chi - \psi) &= \gamma^*(\chi - \psi)\\ &= \big((\chi - \psi)(\gamma)\big)\gamma'^{-1}, \quad \forall \gamma \in \Gamma.\end{aligned} \tag{4.2}$$

Hence, $\chi - \psi$ is bounded in the hyperbolic metric on $\mathbb{D}$.

Corollary 4.1.7. *ψ admits an L^2-extension to the closed unit disk $\overline{\mathbb{D}}$ whose restriction $\psi^\sharp$ to the boundary circle $\mathbb{S}^1$ is tangential.*

Remark 4.1.8. Note that in Corollary 4.1.7 such an extension is unique and it depends only on c, not on the choice of φ in Lemma 4.1.1.

4.2 The Poisson map adapted to vector fields

To get a vector field which is harmonic on the interior of $\mathbb{D}$ from a tangential vector field on $\mathbb{S}^1$, we first give the reincarnation of *the Poisson integral formula* and then adapt it to the case of vector fields. Recall that the *Dirichlet problem* asks for finding a harmonic function F on the disk $\mathbb{D}$ given a continous function f on the boundary circle $\mathbb{S}^1$ such that they together make a continous function on the closed disk $\overline{\mathbb{D}}$. The Poisson integral map is an important tool to solve the Dirichlet problem:

$$F(re^{\iota\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{\iota\phi})\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}d\phi. \tag{4.3}$$

The term $\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}$ is called the *Poisson Kernel* and denoted by K. When $z = re^{\iota\theta}$ and $w = e^{\iota\phi}$, we have

$$K(w,z) = \frac{|w|^2-|z|^2}{|w-z|^2} = \Re\left(\frac{w+z}{w-z}\right). \tag{4.4}$$

Note that $K(w, z)$ is defined for $0 \leq |z| < |w| \leq 1$. We assume that $|w| = 1$, then

$$K(w, z) = \frac{1 - |z|^2}{|1 - z\bar{w}|^2},$$

since $|w - z| = |w\bar{w} - z\bar{w}| = |1 - z\bar{w}|$. Therefore,

$$\frac{1 - |z|^2}{|1 - z\bar{w}|^2} = \frac{1 - z\bar{z}}{(1 - z\bar{w})(1 - \bar{z}w)} = \sum_{n=0}^{\infty} \bar{z}^n w^n + \sum_{n=1}^{\infty} z^n \bar{w}^n.$$

So,

$$K(e^{\iota\phi}, re^{\iota\theta}) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{\iota n(\phi - \theta)} = K_r(\phi - \theta).$$

It is obvious that K is a positive function of w and z. So, (4.3) can also be written as

$$F(re^{\iota\theta}) = \frac{1}{2\pi} \int_0^{2\pi} K_r(\theta - \phi) f(e^{\iota\phi}),$$

where $K_r(\theta - \phi) = K_r(\phi - \theta)$. Before we give the reincarnation of (4.3) we set some conventions and state some necessary facts. The group $\mathrm{SU}(1, 1)$ is the set of matrices

$$\mathrm{SU}(1, 1) = \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \in \mathrm{GL}(2, \mathbb{C}) \big| |a|^2 - |b|^2 = 1 \right\},$$

with group multiplication given by matrix multiplication. Note that the group $\mathrm{SU}(1, 1)$ is isomorphic to the group $\mathrm{SL}(2, \mathbb{R})$ of 2×2 real matrices with determinant 1. We identify the circle group $\mathrm{SO}(2)$ with the subgroup of $\mathrm{SU}(1, 1)$ given by

$$\mathrm{SO}(2) = \left\{ \begin{bmatrix} \exp(\iota\theta) & 0 \\ 0 & \exp(-\iota\theta) \end{bmatrix} \Bigg| \quad \theta \in [0, 2\pi) \right\}.$$

Recall that $\mathrm{Aut}(\mathbb{D})$, the orientation preserving isometries of the Poincaré disk $\mathbb{D}$ with the hyperbolic metric $\mathbf{g}_{\mathbb{D}}$, is identified with

$$\mathrm{PSU}(1, 1) = \mathrm{SU}(1, 1)/\{\pm \mathrm{Id}\}$$

because every $\gamma \in \mathrm{PSU}(1, 1)$ acts on $\mathbb{D}$ by the following formula

$$\gamma(z) = \frac{az + b}{\bar{b}z + \bar{a}}, \gamma = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}, \quad |a|^2 - |b|^2 = 1, \quad \forall z \in \mathbb{D}.$$

4.2.1 Reincarnation of the Poisson integral formula

We denote the space of continuous functions on the circle $\mathbb{S}^1$ by $C^0(\mathbb{S}^1)$ and the space of continuous functions on the open unit disk $\mathbb{D}$ by $C^0(\mathbb{D})$. To construct and characterise the Poisson map

$$P : C^0(\mathbb{S}^1) \longrightarrow C^0(\mathbb{D})$$

given in (4.3) which is continuous w.r.t to the topology of uniform convergence on both the source and the target space, we first observe that $P(f)(0)$ is nothing but the *normalised Haar integral*[1]

$$\frac{1}{2\pi}\int_{\mathbb{S}^1} f.$$

By convention, integral of the constant function 1 over $\mathbb{S}^1$ is 2π. Therefore, $P(f)(0)$ is linear, positive, continous, and invariant under the circle group. To obtain the expression for $P(f)(z)$, $z \in \mathbb{D}$, we use the transitivity of the action of $\mathrm{PSU}(1,1)$ on the open unit disk $\mathbb{D}$, i.e., $P(f)(z) = P(f)(\gamma(0))$ for some $\gamma \in \mathrm{PSU}(1,1)$ such that $\gamma(0) = z$. Moreover,

$$P(f)(z) = P(f)(\gamma(0)) = P(f \cdot \gamma)(0) = \frac{1}{2\pi}\int_{\mathbb{S}^1} f \cdot \gamma, \tag{4.5}$$

where the second equality follows from the fact that the Poisson map P is $\mathrm{PSU}(1,1)$-*equivariant*, i.e., $P(f \cdot \gamma) = P(f) \cdot \gamma$, for all $\gamma \in \mathrm{PSU}(1,1)$ and all $f \in C^0(\mathbb{S}^1)$, where $\cdot$ denotes the action of $\mathrm{PSU}(1,1)$ on $C^0(\mathbb{S}^1)$ and $C^0(\mathbb{D})$ by pre-composition. The condition can also be understood as the following commutative diagram:

$$\begin{array}{ccc} C^0(\mathbb{S}^1) & \xrightarrow{P} & C^0(\mathbb{D}) \\ \downarrow{\scriptstyle \gamma\cdot} & & \downarrow{\scriptstyle \gamma\cdot} \\ C^0(\mathbb{S}^1) & \xrightarrow{P} & C^0(\mathbb{D}) \end{array}$$

The $\mathrm{PSU}(1,1)$-equivariance of the Poisson map follows from the uniqueness of solutions to the Dirichlet problem for Laplace's equation, i.e., for a given $f \in C^0(\mathbb{S}^1)$, the Dirichlet problem for Laplace's equation

$$\begin{aligned} \Delta F &= 0 \text{ on } \mathbb{D} \\ F &= f \text{ on } \mathbb{S}^1 \end{aligned}$$

has atmost one solution $F \in C^2(\mathbb{D}) \cap C^1(\overline{\mathbb{D}})$. Transforming $f \in C^0(\mathbb{S}^1)$ by an element $\gamma \in \mathrm{PSU}(1,1)$ gives us a new harmonic extension F_1 of $f \cdot \gamma$ on $\mathbb{D}$. From the weak maximum principle applied to the harmonic function $F \circ \gamma - F_1$, we have $F \circ \gamma - F_1 \leq \max_{\mathbb{S}^1}(F \circ \gamma - F_1) = 0$. Thus, $F \circ \gamma \leq F_1$ on $\mathbb{D}$. Similarly, we get $F_1 \leq F \circ \gamma$. Therefore, $F \circ \gamma$ and F_1 coincide. Note that the last equality in (4.5) follows from the fact that $P(f)(0)$ is the Haar integral. $P(f)(z)$ is well-defined, i.e., it does not depend on $\gamma \in \mathrm{PSU}(1,1)$

[1]Let G denote a locally compact group. The real vector space of the real valued continuous functions on G with compact support is denoted by $C_c(G)$. The set of nonnegative functions in $C_c(G)$ is denoted by $C_c^+(G)$. A continuous linear functional $I : C_c(G) \longrightarrow \mathbb{R}$ is called a *Haar integral* if the following hold: 1) if $f \in C_c^+(G)$, then $I(f) \geq 0$, 2) if $g \in G$ and $f \in C_c(G)$, then $I(gf) = I(f)$, 3) there exists a function $f \in C_c^+(G)$ with $I(f) > 0$. Note that for $r > 0$, rI is again a Haar integral. For more information, see [33], [61].

and is unique upto a positive scaling factor because if we take z to be the origin again, then the stabilizer subgroup of $\mathrm{PSU}(1,1)$ w.r.t to the origin is the circle group $SO(2)$ and the Haar integral is invariant under rotations. We list the following properties which are satisfied by P:

1. P is linear,
2. P is continous,
3. P is $\mathrm{PSU}(1,1)$-equivariant.

Proposition 4.2.1. *Given a point $z \in \mathbb{D}$, the map $w \longmapsto \frac{w+z}{w\bar{z}+1}$ is a hyperbolic isometry that sends the origin to the point z.*

Proof We check that indeed $\gamma(0) = z$. Let $\gamma(z) = \frac{w+z}{w\bar{z}+1}$, and let $\gamma(w) = f(w) + \iota g(w)$. By differentiating, we get $\frac{d\gamma(w)}{dw} = \frac{1-|z|^2}{(w\bar{z}+1)^2}$. Observe that

$$df(w)^2 + dg(w)^2 = (df(w) + \iota dg(w))(df(w) - \iota dg(w)) = d\gamma(w)\overline{d\gamma(w)}.$$

Therefore,

$$\frac{2\sqrt{df(w)^2+dg(w)^2}}{1-f(w)^2-g(w)^2} = \frac{2\sqrt{d\gamma(w)\overline{d\gamma(w)}}}{1-|\gamma(w)|^2} = \frac{2\sqrt{\frac{d\gamma(w)}{dw}dw\,\overline{\frac{d\gamma(w)}{dw}}\,\overline{dw}}}{1-|\gamma(w)|^2} = \frac{2\sqrt{\frac{(1-|z|^2)^2}{(w\bar{z}+1)^2(\bar{w}z+1)^2}}}{1-|\gamma(w)|^2}\sqrt{dx^2+dy^2}.$$

Simplifying $1 - |\gamma(w)|^2$ further, we get

$$\frac{2\sqrt{\frac{(1-|z|^2)^2}{(w\bar{z}+1)^2(\bar{w}z+1)^2}}}{1-|\gamma(w)|^2} = \frac{2}{1-|w|^2}.$$

Therefore, γ is a hyperbolic isometry. The final and remaining thing is to check that γ maps $\mathbb{D}$ to itself. Suppose that $|w| < 1$. We want to show that $|\frac{w+z}{w\bar{z}+1}| < 1$. This is equivalent to showing that $|w+z| < |w\bar{z}+1|$. Furthermore, it is enough to show that $(w+z)(\bar{w}+\bar{z}) < (w\bar{z}+1)(\bar{w}z+1)$, or equivalently, $w\bar{w} + z\bar{z} < w\bar{w}z\bar{z} + 1$, or $(1-w\bar{w})(1-z\bar{z}) > 0$, which is true since $1 - w\bar{w}$ and $1 - z\bar{z}$ are both positive. □

We summarize our discussion as follows:

Proposition 4.2.2. *Every continuous linear map $F : C^0(\mathbb{S}^1) \longrightarrow C^0(\mathbb{D})$ which is $\mathrm{PSU}(1,1)$-equivariant is a scalar multiple of the continous linear map $P : C^0(\mathbb{S}^1) \longrightarrow C^0(\mathbb{D})$ given by the following formula*

$$P(f)(z) = \frac{1}{2\pi}\int_{\mathbb{S}^1} f \cdot \gamma,$$

where $\gamma \in \mathrm{PSU}(1,1)$ is given in Proposition 4.2.1 such that $\gamma(0) = z$ and $f \in C^0(\mathbb{S}^1)$.

Remark 4.2.3. Alternatively, we can construct such a linear map $F : C^0(\mathbb{S}^1) \longrightarrow C^0(\mathbb{D})$ in Proposition 4.2.2 by plugging the Dirac distribution δ at the point $1 \in \mathbb{S}^1$ into the formula for P instead of a continous function f on the circle $\mathbb{S}^1$. We adopt the view that δ is the limit of step functions $\{\epsilon^{-1} g_\epsilon\}$ where g_ϵ is the characteristic function of an arc of length ϵ centered at $1 \in \mathbb{S}^1$. Therefore, we define $\delta \cdot \gamma = \gamma^* \delta$ to be the Dirac distribution at the point $\gamma^{-1}(1)$ times $\left|\left(\gamma'(\gamma^{-1}(1))\right)^{-1}\right|$. This suggests

$$F(\delta)(z) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \delta \cdot \gamma,$$

where $\gamma(0) = z$ and the explicit form of γ is given by Proposition 4.2.1. Using $\gamma(w) = \frac{w+z}{w\bar{z}+1}$, we see that

$$2\pi \cdot \big(F(\delta)(z)\big) = \left|\left(\gamma'(\gamma^{-1}(1))\right)^{-1}\right| = \frac{1-|z|^2}{|1-\bar{z}|^2} = \frac{1-|z|^2}{|1-z|^2}.$$

We denote the real valued (positive) function $z \longmapsto \frac{1-|z|^2}{|1-z|^2}$ defined on $\mathbb{D}$ for $0 \leq |z| < 1$ by K. The intuition is $F(\delta) = \frac{K}{2\pi}$ and therefore, we define

$$F(f) = f * K, \tag{4.6}$$

where $f \in C^0(\mathbb{S}^1)$ and $K(z) = \frac{1-|z|^2}{|1-z|^2}$, and $*$ denotes the convolution[2] of K and f. To show the $\mathrm{PSU}(1,1)$-equivariance, we first note that every element $A \in \mathrm{PSU}(1,1)$ has a unique expression $A = BC$ where $B \in \mathrm{SO}(2)$ and C is in the two-dimensional subgroup $\mathrm{Stab}_{\mathrm{PSU}(1,1)}(1)$ of $\mathrm{PSU}(1,1)$ consisting of all elements which fix the element 1 in the boundary circle $\mathbb{S}^1$. Also, $\mathrm{Stab}_{\mathrm{PSU}(1,1)}(1)$ acts transitively on $\mathbb{D}$. The general form of elements γ of the group $\mathrm{Stab}_{\mathrm{PSU}(1,1)}(1)$ is given by the following:

$$\gamma(z) = \frac{az+b}{\bar{b}z+\bar{a}}, |a|^2 - |b|^2 = 1, a+b = \overline{a+b}. \tag{4.7}$$

Hence, showing the $\mathrm{PSU}(1,1)$-equivariance of F is equivalent to showing the $\mathrm{SO}(2)$-equivariance and $\mathrm{Stab}_{\mathrm{PSU}(1,1)}(1)$-equivariance of F. It is easy to see that F in (4.6) is $\mathrm{SO}(2)$-equivariant. To show the $\mathrm{Stab}_{\mathrm{PSU}(1,1)}(1)$-equivariance of F in (4.6), we claim that

[2] The convolution of K and f is defined as: $(f * K)(z) := \frac{1}{2\pi} \int_{\mathbb{S}^1} f(w) K(zw^{-1}) dw$.

$\gamma^* K = cK$, where c is a positive constant and $\gamma \in \mathrm{Stab}_{\mathrm{PSU}(1,1)}(1)$. We have

$$\begin{aligned}
K(\gamma(z)) &= \frac{1-|\gamma(z)|^2}{|1-\gamma(z)|^2} = \frac{1-\gamma(z)\overline{\gamma(z)}}{\big(1-\gamma(z)\big)\big(1-\overline{\gamma(z)}\big)} \\
&= \frac{1-\frac{az+b}{\bar{b}z+\bar{a}}\cdot\frac{\bar{a}\bar{z}+\bar{b}}{b\bar{z}+a}}{\left(1-\frac{az+b}{\bar{b}z+\bar{a}}\right)\cdot\left(1-\frac{\bar{a}\bar{z}+\bar{b}}{b\bar{z}+a}\right)} = \frac{1-\frac{|a|^2|z|^2+az\bar{b}+b\bar{a}\bar{z}+|b|^2}{|b|^2|z|^2+az\bar{b}+b\bar{a}\bar{z}+|a|^2}}{\left(\frac{(b-\bar{a})\bar{z}-(\bar{b}-a)}{b\bar{z}+a}\right)\left(\frac{(\bar{b}-a)z-(b-\bar{a})}{\bar{b}z+\bar{a}}\right)} \\
&= \frac{1-|z|^2}{(b\bar{z}+a)(\bar{b}z+\bar{a})}\cdot\left(\frac{\big(b-\bar{a}\big)^2\cdot\big|1-\bar{z}\big|^2}{(b\bar{z}+a)(\bar{b}z+\bar{a})}\right)^{-1} = \frac{1-|z|^2}{\big(b-\bar{a}\big)^2\cdot\big|1-\bar{z}\big|^2} \\
&= \frac{\gamma'(1)^{-1}\big(1-|z|^2\big)}{\big|1-\bar{z}\big|^2} = \gamma'(1)^{-1}K(z),
\end{aligned}$$

where $\gamma'(1) = \big(\bar{b}+\bar{a}\big)^{-2}$. Note that K is the real part of a holomorphic function, hence harmonic. Therefore, $F(f)$ is also harmonic.

Corollary 4.2.4. *The map F in (4.6) is the Poisson map given in (4.3). Hence, the map P in Proposition 4.2.2 lands in the vector space of harmonic functions on the open unit disk $\mathbb{D}$.*

Let $\mathcal{S}_{C^0}(T\mathbb{S}^1)$ be the Banach space of (tangential) continuous vector fields on $\mathbb{S}^1$ and $\mathcal{S}_{C^0}(T\mathbb{D})$ be the space of continuous vector fields on the open disk $\mathbb{D}$. We want to mimick the reincarnation of the Poisson map in the case of vector fields.

Proposition 4.2.5. *Every continuous and $\mathrm{SO}(2)$-equivariant linear functional Λ from the real Banach space of continous tangential vector fields on $\mathbb{S}^1$ to $\mathbb{C}$ has the following form:*

$$\Lambda(X) = \left(\int_{\mathbb{S}^1} X\right)\cdot v,$$

where X is a tangential vector field on $\mathbb{S}^1$ and $v \in \mathbb{C}$.

Proposition 4.2.6. *Every continous linear map*

$$\mathcal{F} : \mathcal{S}_{C^0}(T\mathbb{S}^1) \longrightarrow \mathcal{S}_{C^0}(T\mathbb{D})$$

which is equivariant under the action of $\mathrm{PSU}(1,1)$ is a scalar multiple of the continous linear map

$$\mathcal{P} : \mathcal{S}_{C^0}(T\mathbb{S}^1) \longrightarrow \mathcal{S}_{C^0}(T\mathbb{D})$$

given by the following formula

$$\mathcal{P}(X)(z) = \mathcal{P}(X)(\gamma(0)) = \gamma'(0)\cdot\Big(\mathcal{P}(\gamma^*X)(0)\Big) = \gamma'(0)\cdot\left(\frac{1}{2\pi}\int_{\mathbb{S}^1}\gamma^*X\right), \tag{4.8}$$

for some $\gamma \in \mathrm{PSU}(1,1)$ such that $\gamma(0) = z$.

Remark 4.2.7. The third equality in the expression of $\mathcal{P}(X)(z)$ in (4.8) follows from Proposition 4.2.5.

Remark 4.2.8. The scalar in Proposition 4.2.6 can be any complex number. Also, note that $\mathcal{P}(X)(0) \in T_0\mathbb{D}$ and the second equality in (4.8) follows from the PSU$(1,1)$-equivariance of $\mathcal{P}$, i.e.,

$$\mathcal{P}(\gamma^* X) = \gamma^*(\mathcal{P}(X)), \quad \forall \gamma \in \mathrm{PSU}(1,1),$$

where $\gamma^* X = X(\gamma)\gamma'^{-1}, \gamma \in \mathrm{PSU}(1,1), X \in \mathcal{S}_{C^0}(T\mathbb{S}^1)$.

Remark 4.2.9. We can also construct such a linear map

$$\mathcal{F} : \mathcal{S}_{C^0}(T\mathbb{S}^1) \longrightarrow \mathcal{S}_{C^0}(T\mathbb{D})$$

in Proposition 4.2.6 by plugging the *Dirac vector field* $\boldsymbol{\delta}$ in the formula for $\mathcal{P}$ instead of a tangential vector field X on the circle $\mathbb{S}^1$. We adopt the view that $\boldsymbol{\delta}$ is the limit of vector fields $\{\epsilon^{-1} g_\epsilon\}$ where g_ϵ is the norm 1 (positively oriented) tangential vector field supported on an arc of length ϵ centered at $1 \in \mathbb{S}^1$. Therefore, we have

$$\mathcal{F}(\boldsymbol{\delta})(z) = \mathcal{F}(\boldsymbol{\delta})(\gamma(0)) = \gamma'(0) \cdot \Big(\mathcal{F}\big(\gamma^*\boldsymbol{\delta}\big)(0)\Big) = \gamma'(0) \cdot \left(\frac{1}{2\pi}\int_{\mathbb{S}^1} \gamma^*\boldsymbol{\delta}\right), \tag{4.9}$$

where $\gamma(0) = z$ and the explicit form of γ is given by Proposition 4.2.1. (4.9) is further simplified to

$$2\pi \cdot \big(\mathcal{F}(\boldsymbol{\delta})(z)\big) = \gamma'(0) \cdot \Big(\iota \cdot \big|\big(\gamma'(\gamma^{-1}(1))\big)^{-1}\big| \cdot \big(\gamma'(\gamma^{-1}(1))\big)^{-1}\Big). \tag{4.10}$$

Observe that the factor $\big|\big(\gamma'(\gamma^{-1}(1))\big)^{-1}\big|$ accounts for the streching of the arc length when we pull back the Dirac vector field under γ and the factor $\big(\gamma'(\gamma^{-1}(1))\big)^{-1}$ accounts for the streching of vectors. Using $\gamma(w) = \frac{w+z}{w\bar{z}+1}$, the expression

$$\gamma'(0) \cdot \Big(\iota \cdot \big|\big(\gamma'(\gamma^{-1}(1))\big)^{-1}\big| \cdot \big(\gamma'(\gamma^{-1}(1))\big)^{-1}\Big)$$

in (4.10) simplifies to

$$\frac{\iota(1-|z|^2)^3}{|1-\bar{z}|^2 \cdot (1-\bar{z})^2}. \tag{4.11}$$

We call the vector field given by (4.11) the *Poisson kernel vector field* and denote it by $\mathbf{K}$. By definition $\mathcal{F}(\boldsymbol{\delta}) = \frac{1}{2\pi}\mathbf{K}$. Let X be a tangential vector field on the boundary circle $\mathbb{S}^1$ of the form fY where f is a real-valued continuous function on the boundary and Y is the norm 1 tangential vector field on $\mathbb{S}^1$ given by $z \longmapsto \iota z$. From the above discussion, a vector field $\mathcal{F}(X)$ on $\mathbb{D}$ is given by the convolution of the Poisson Kernel vector field $\mathbf{K}$ with a given function f on $\mathbb{S}^1$, i.e.,

$$\mathcal{F}(X) = f * \mathbf{K}. \tag{4.12}$$

Proposition 4.2.10. *The map $\mathcal{F}$ in (4.12) satisfies the conditions of the map $\mathcal{F}$ in Proposition 4.2.6.*

Before we prove Proposition 4.2.10, we state and prove the following:

Theorem 4.2.11. *The Poisson Kernel vector field $\mathbf{K}$ given by (4.11) in Remark 4.10 is harmonic at every point $z \in \mathbb{D}$.*

Proof Recall Theorem 2.2.4 in Section 2.2 in Chapter 2 where we show that a vector field ξ on $\mathbb{D}$ is harmonic iff the quadratic differential $(L_\xi \mathbf{g}_{\mathbb{D}})^{(2,0)}$ associated with it is holomorphic. We first prove that $\mathbf{K}$ is harmonic at the origin in $\mathbb{D}$. We write the Taylor approximation of $\mathbf{K}$ up to the second order at the origin as follows:

$$
\begin{aligned}
\mathbf{K}(z) &= \frac{\iota(1-|z|^2)^3}{|1-\bar{z}|^2 \cdot (1-\bar{z})^2} \\
&= \frac{\iota(1-|z|^2)^3}{(1-\bar{z})\overline{(1-\bar{z})}(1-\bar{z})^2} \\
&= \iota(1-|z|^2)^3(1-\bar{z})^{-3}(1-z)^{-1} \\
&\approx \iota(1-3|z|^2)(1+\bar{z}+\bar{z}^2)^3(1+z+z^2) \\
&\approx \iota(1-3|z|^2)(1+3\bar{z}+3\bar{z}^2+3\bar{z}^2)(1+z+z^2) \\
&\approx \iota(1+3\bar{z}+6\bar{z}^2-3|z|^2)(1+z+z^2) \\
&\approx \iota(1+3\bar{z}+6\bar{z}^2-3|z|^2+z+3|z|^2+z^2) \\
&= \iota(1+z+3\bar{z}+z^2+6\bar{z}^2) \\
&= \iota\big(1+(x+\iota y)+3(x-\iota y)+x^2-y^2+2\iota xy+6(x^2-y^2)-12\iota xy\big) \\
&= \iota(1+4x-2\iota y+7x^2-7y^2-10\iota xy) \\
&= (2y+10xy, 1+4x+7x^2-7y^2).
\end{aligned} \tag{4.13}
$$

Note that the metric $\mathbf{g}_{\mathbb{D}}$ at the origin does not change. Following the criteria for harmonicity of a vector field from Section 2.2 in Chapter 2, we notice that the quadratic differential q associated to $\mathbf{K}$ is given as $(6\iota - 24\iota z)dz^2$. The function $f(z) = 6\iota - 24\iota z$ is holomorphic. Hence, $\mathbf{K}$ is harmonic at the origin in $\mathbb{D}$. Now, we claim that the vector field $\mathbf{K}$ when transformed using elements $\gamma \in \mathrm{PSU}(1,1)$ which fix the element 1 in the boundary circle $\mathbb{S}^1$, changes only by multiplying it by a non-zero real constant.

Proof of the claim: Recall the general form of elements γ of the group $\mathrm{PSU}(1,1)$ which fix the element 1 in the boundary circle $\mathbb{S}^1$ given by (4.7) in Remark 4.2.3. Now, γ acts on $\mathbf{K}$ in the usual way:

$$
\begin{aligned}
\gamma^*\mathbf{K} &= \mathbf{K}(\gamma(z))\gamma'(z)^{-1} \\
&= \frac{\iota\big(1-|\gamma(z)|^2\big)^3}{\big|1-\overline{\gamma(z)}\big|^2 \cdot \big(1-\overline{\gamma(z)}\big)^2} \cdot \gamma'(z)^{-1}.
\end{aligned} \tag{4.14}
$$

Using (4.7), the numerator and the denominator of the term $\frac{\iota\left(1-|\gamma(z)|^2\right)^3}{\left|1-\overline{\gamma(z)}\right|^2\cdot\left(1-\overline{\gamma(z)}\right)^2}$ in the RHS of (4.14) are explicitly written as:

$$\begin{aligned}
\iota\left(1-|\gamma(z)|^2\right)^3 &= \iota\left(1-\gamma(z)\overline{\gamma(z)}\right)^3 \\
&= \iota\left(1-\frac{az+b}{\bar{b}z+\bar{a}}\cdot\frac{\bar{a}\bar{z}+\bar{b}}{b\bar{z}+a}\right)^3 \\
&= \iota\left(1-\frac{|a|^2|z|^2+az\bar{b}+b\bar{a}\bar{z}+|b|^2}{|b|^2|z|^2+az\bar{b}+b\bar{a}\bar{z}+|a|^2}\right)^3 \\
&= \iota\left(\frac{1-|z|^2}{|b|^2|z|^2+az\bar{b}+b\bar{a}\bar{z}+|a|^2}\right)^3 \\
&= \frac{\iota\left(1-|z|^2\right)^3}{\left((\bar{b}z+\bar{a})(b\bar{z}+a)\right)^3}
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
\left|1-\overline{\gamma(z)}\right|^2\cdot\left(1-\overline{\gamma(z)}\right)^2 &= \left(1-\overline{\gamma(z)}\right)\overline{\left(1-\overline{\gamma(z)}\right)}\cdot\left(1-\overline{\gamma(z)}\right)^2 \\
&= \left(1-\overline{\gamma(z)}\right)\left(1-\gamma(z)\right)\cdot\left(1-\overline{\gamma(z)}\right)^2 \\
&= \left(\frac{(b-\bar{a})\bar{z}-(\bar{b}-a)}{b\bar{z}+a}\right)\left(\frac{(\bar{b}-a)z-(b-\bar{a})}{\bar{b}z+\bar{a}}\right)\left(\frac{(b-\bar{a})\bar{z}-(\bar{b}-a)}{b\bar{z}+a}\right)^2 \\
&= \frac{\left(b-\bar{a}\right)^2\cdot\left|1-\bar{z}\right|^2}{(b\bar{z}+a)(\bar{b}z+\bar{a})}\cdot\frac{\left(b-\bar{a}\right)^2\left(1-\bar{z}\right)^2}{\left(b\bar{z}+a\right)^2} \\
&= \frac{\left(b-\bar{a}\right)^4\left|1-\bar{z}\right|^2\left(1-\bar{z}\right)^2}{(b\bar{z}+a)^3(\bar{b}z+\bar{a})},
\end{aligned} \tag{4.16}$$

where in the last two equalities in (4.16) we have used the fact that $b-\bar{a}$ is real, i.e., $b-\bar{a}=\bar{b}-a$. Also, $\gamma'(z)^{-1}=\left(\bar{b}z+\bar{a}\right)^2$. Using (4.15) and (4.16), the explicit form of the RHS of (4.14) is

$$\begin{aligned}
\frac{\iota\left(1-|\gamma(z)|^2\right)^3}{\left|1-\overline{\gamma(z)}\right|^2\cdot\left(1-\overline{\gamma(z)}\right)^2}\cdot\gamma'(z)^{-1} &= \frac{\iota\left(1-|z|^2\right)^3(b\bar{z}+a)^3(\bar{b}z+\bar{a})}{\left((\bar{b}z+\bar{a})(b\bar{z}+a)\right)^3\left(b-\bar{a}\right)^4\left|1-\bar{z}\right|^2\left(1-\bar{z}\right)^2}\cdot\left(\bar{b}z+\bar{a}\right)^2 \\
&= \frac{1}{\left(b-\bar{a}\right)^4}\cdot\frac{\iota(1-|z|^2)^3}{|1-\bar{z}|^2\cdot(1-\bar{z})^2} \\
&= \frac{1}{\left(b-\bar{a}\right)^4}\mathbf{K}(z)=(\gamma'(1))^{-2}\mathbf{K}(z).
\end{aligned}$$

As mentioned in Remark 4.2.3, every element $A\in\mathrm{PSU}(1,1)$ has a unique expression $A=BC$ where $B\in\mathrm{SO}(2)$ and C is in the two-dimensional subgroup $\mathrm{Stab}_{\mathrm{PSU}(1,1)}(1)$

of $\mathrm{PSU}(1,1)$ consisting of all elements which fix the element 1 in the boundary circle $\mathbb{S}^1$. Therefore, **K** is $\mathrm{Stab}_{\mathrm{PSU}(1,1)}(1)$-invariant up to multiplication by real scalars. Note that the harmonicity of a vector field on the open unit disk $\mathbb{D}$ is preserved by conformal automorphisms of $\mathbb{D}$. Hence, **K** is harmonic everywhere on the open unit disk $\mathbb{D}$. □

Remark 4.2.12. $\mathrm{Stab}_{\mathrm{PSU}(1,1)}(1)$-invariance of **K** up to multiplication by real scalars suffices to ensure that **K** is harmonic on the open unit disk $\mathbb{D}$ because $\mathrm{Stab}_{\mathrm{PSU}(1,1)}(1)$ acts transitively on the open unit disk $\mathbb{D}$.

Remark 4.2.13. Since the Poisson Kernel vector field **K** is harmonic, $\mathcal{F}(X)$ given by (4.12) is also harmonic on $\mathbb{D}$, where X is a tangential vector field on $\mathbb{S}^1$.

Proof of Proposition 4.2.10 : The map $\mathcal{F}$, given by (4.12), is clearly $\mathrm{PSU}(1,1)$-equivariant. It follows from $\mathrm{Stab}_{\mathrm{PSU}(1,1)}(1)$-invariance of **K** up to multiplication by real scalars (see Proof of Proposition 4.2.11). Hence, it immediately follows that $\mathcal{F}$ satisfies all the conditions stated in Proposition 4.2.6. □

Corollary 4.2.14. *The map $\mathcal{F}$, given by (4.12), is same as the map $\mathcal{P}$ in Proposition 4.2.6. Hence, the map $\mathcal{P}$ in Proposition 4.2.6 lands in the vector space of harmonic vector fields on the open unit disk $\mathbb{D}$.*

Lemma 4.2.15. *For a continuous tangential vector field X on $\mathbb{S}^1$, $\mathcal{F}(X)$ and X together make up a continuous vector field on the closed unit disk $\overline{\mathbb{D}}$.*

Proof [Sketch] For every $\epsilon > 0$, we get a continuous vector field $\mathbf{K}_{1-\epsilon}$ on $\mathbb{S}^1$ by composing **K** with the map $z \longmapsto (1-\epsilon)z$. We first notice that

$$\mathbf{K}_{1-\epsilon}(z) = \frac{\iota\big(1 - |(1-\epsilon)z|^2\big)^3}{\big(1-(1-\epsilon)\bar{z}\big)^3 \cdot \big(1-(1-\epsilon)z\big)}, \tag{4.17}$$

where $|z| = 1$. Simplifying (4.17), we get

$$\mathbf{K}_{1-\epsilon}(z) \approx \frac{\iota 8\epsilon^3 z^3}{\big(1-(1-\epsilon)z\big) \cdot \big(z-(1-\epsilon)\big)^3}, \tag{4.18}$$

where we used the fact that $\bar{z} = z^{-1}$. We put $1 - \epsilon = s$ in (4.18) and get

$$\mathbf{K}_{1-\epsilon}(z) \approx \frac{\iota 8\epsilon^3 z^3}{(1-sz)\cdot(z-s)^3}.$$

Let $\lambda_z = |z - (1-\epsilon)|$. Notice that

$$|\mathbf{K}_{1-\epsilon}(z)| \leq \frac{8}{\epsilon}. \tag{4.19}$$

The estimate in (4.19) is independent of z. And now, let us try to make an upper bound for $|\mathbf{K}_{1-\epsilon}(z)|$ which is dependent on z. The following estimate for $|\mathbf{K}_{1-\epsilon}(z)|$ ensures that $\mathbf{K}_{1-\epsilon}$ is 'very small' outside the arc of length $\sqrt{2\epsilon}$ centered at 1.

$$|\mathbf{K}_{1-\epsilon}(z)| \leq \frac{8\epsilon^3}{\lambda_z^4}. \tag{4.20}$$

(4.19) and (4.20) have following two consequences:

1. if $X = fY$, where f is a real-valued continuous function on $\mathbb{S}^1$ and Y is the norm 1 continous tangential vector field on $\mathbb{S}^1$, we will have

$$(f * K_{1-\epsilon})(z) \approx f(z) \cdot \left(\frac{1}{2\pi}\int_{\mathbb{S}^1} K_{1-\epsilon}\right), \quad z \in \mathbb{S}^1. \tag{4.21}$$

   Therefore, it is enough to show that

$$\lim_{\epsilon\to 0}\left(\frac{1}{2\pi}\int_{\mathbb{S}^1} K_{1-\epsilon}\right) = \iota.$$

2. we may replace the ordinary Haar integral by the complex path integral at the price of dividing by ι.

Therefore,

$$\begin{aligned}\lim_{\epsilon\to 0}\left(\frac{1}{2\pi\iota}\int_{\mathbb{S}^1}\frac{\iota 8\epsilon^3 z^3}{(1-sz)\cdot(z-s)^3}dz\right) &= \lim_{\epsilon\to 0}\left(\frac{\iota 8\epsilon^3}{2\pi\iota}\int_{\mathbb{S}^1}\frac{z^3}{(1-sz)\cdot(z-s)^3}dz\right)\\ &= \lim_{\epsilon\to 0}\left(\frac{\iota 8\epsilon^3}{2\pi\iota}\big(2\pi\iota\cdot \mathrm{Res}(f,s)\big)\right),\end{aligned} \tag{4.22}$$

where $f(z) = \frac{z^3}{(1-sz)\cdot(z-s)^3}$, and

$$\mathrm{Res}(f,s) = \frac{6s - 12s^3 + 8s^5 - 2s^7}{2(1-s^2)^4} = \frac{4\epsilon + 2\epsilon^2 + 2\epsilon^3 - 30\epsilon^4 + 34\epsilon^5 - 14\epsilon^6 + 2\epsilon^7}{2\big(16\epsilon^4 - 32\epsilon^5 + 20\epsilon^6 - 8\epsilon^7 + \epsilon^8\big)}.$$

Rewriting (4.22), we get

$$\begin{aligned}\lim_{\epsilon\to 0}\left(\frac{\iota 8\epsilon^3}{2\pi\iota}\big(2\pi\iota\cdot \mathrm{Res}(f,s)\big)\right) &= \lim_{\epsilon\to 0}\left(8\iota\cdot\frac{4\epsilon + 2\epsilon^2 + 2\epsilon^3 - 30\epsilon^4 + 34\epsilon^5 - 14\epsilon^6 + 2\epsilon^7}{2\big(16\epsilon - 32\epsilon^2 + 20\epsilon^3 - 8\epsilon^4 + \epsilon^5\big)}\right)\\ &= 4\iota\left(\lim_{\epsilon\to 0}\frac{4\epsilon + 2\epsilon^2 + 2\epsilon^3 - 30\epsilon^4 + 34\epsilon^5 - 14\epsilon^6 + 2\epsilon^7}{16\epsilon - 32\epsilon^2 + 20\epsilon^3 - 8\epsilon^4 + \epsilon^5}\right)\\ &= \iota.\end{aligned}$$

□

Corollary 4.2.16. *For an L^2-tangential vector field X on $\mathbb{S}^1$, X is an L^2-boundary extension of the smooth vector field $\mathcal{F}(X)$ on the open unit disk $\mathbb{D}$.*

Proof Notice that in the proof of Lemma 4.2.15, we showed that

$$\lim_{\epsilon \to 0} \mathbf{K}_{1-\epsilon} = 2\pi \boldsymbol{\delta}.$$

Hence, Corollary 4.2.16 follows from Lemma 4.2.15 and [57, Proposition 5.4]. □

Remark 4.2.17. We suspect that Corollary 4.2.16 is an infinitesimal version of the problem of finding harmonic extensions of quasiconformal maps (from $\mathbb{S}^1$ to itself) to the open unit disk $\mathbb{D}$ or the upper half plane $\mathbb{H}^2$. See [26] for more details.

4.3 A detailed map from $H^1(\Gamma; \mathfrak{g})$ to $\mathrm{HQD}(\mathbb{D}, \Gamma)$

In this section, we summarize the main consequences of Section 4.1 and Section 4.2.

Theorem 4.3.1. *Let Γ be a discrete cocompact subgroup of $\mathrm{PSU}(1,1)$. For every cocycle c representing a cohomology class $[c] \in H^1(\Gamma; \mathfrak{g})$, there exists a smooth vector field ψ on the open unit disk $\mathbb{D}$ such that $c = \delta\psi$. Moreover, any such ψ admits an L^2-extension to $\overline{\mathbb{D}}$ whose restriction $\psi^\sharp$ to the boundary circle $\mathbb{S}^1$ is tangential. There exists a homomorphism*

$$\begin{aligned} \Psi : H^1(\Gamma; \mathfrak{g}) &\longrightarrow \mathrm{HQD}(\mathbb{D}, \Gamma) \\ [c] &\longmapsto \left(\mathcal{L}_{\mathcal{F}(\psi^\sharp)} g_{\mathbb{D}}\right)^{(2,0)}, \end{aligned} \tag{4.23}$$

where the map $\mathcal{F}$ is introduced in (4.12) and $\mathcal{F}(\psi^\sharp)$ is a harmonic vector field on the open disk $\mathbb{D}$.

Corollary 4.3.2.

$$\Phi \circ \Psi = \mathrm{Id},$$

where the map Φ is constructed in (3.6) (see Corollary 3.1.8) and the map Ψ in (4.23) (see Theorem 4.3.1).

Proof [Sketch] Recall from Corollary 4.1.7 (and Section 4.1) that given a cocycle c representing a cohomology class $[c] \in H^1(\Gamma; \mathfrak{g})$, there exists a smooth vector field ψ on the open unit disk $\mathbb{D}$ such that $c = \delta\psi$ and ψ admits a unique L^2-extension to $\overline{\mathbb{D}}$. We denote the restriction of that extension to the boundary circle $\mathbb{S}^1$ by $\psi^\sharp$, and $\psi^\sharp$ is tangential. Note that $\delta\psi^\sharp = c^\sharp$, where $c^\sharp$ is a 1-cocycle (determined by c) with values in the vector space of Killing vector fields on $\mathbb{S}^1$. The map $\mathcal{F}$ maps Killing vector fields on $\mathbb{S}^1$ to Killing vector fields on the open unit disk $\mathbb{D}$. Therefore, it is clear that $\delta\mathcal{F}(\psi^\sharp) = c$ and $\mathcal{F}(\delta\psi^\sharp(\gamma)) = c^\sharp(\gamma)$, for every $\gamma \in \Gamma$.

□

4.4 Open Problems

In this section, we state the following non-exhaustive list of open problems based on this chapter:

From Corollary 4.1.7 we know that given a cocycle c representing a cohomology class $[c] \in H^1(\Gamma; \mathfrak{g})$, there exists a smooth vector field ψ on the open unit disk $\mathbb{D}$ such that $c = \delta\psi$ and ψ admits a unique L^2-extension to the closed unit disk $\overline{\mathbb{D}}$ whose restriction $\psi^\sharp$ to the boundary circle $\mathbb{S}^1$ is tangential. For the construction of ψ we can either use the Γ-invariant partition of unity method or the difficult theory of Chapter 2 and Chapter 3 which produces a harmonic solution. Corollary 4.1.7 is valid for all of these but the construction of an L^2-extension of ψ to $\overline{\mathbb{D}}$ relies on the existence of harmonic vector fields.

Problem 4.4.1. *Is there a more direct way of proving Corollary 4.1.7 which does not take harmonicity into account?*

In Subsection 4.2.1, we have not shown that there exists a *unique* harmonic extension of a tangential L^2-vector field X on $\mathbb{S}^1$ to the closed unit disk $\overline{\mathbb{D}}$.

Problem 4.4.2. *Given a tangential L^2-vector field X on the boundary circle $\mathbb{S}^1$, does there exist a unique harmonic extension to the closed unit disk $\overline{\mathbb{D}}$?*

Appendix A

The genesis of the potential equation $F_{\bar{z}} = (z - \bar{z})^2\overline{\phi(z)}$

A.1 A swift introduction to Beltrami differentials

Let $(V, J_V), (W, J_W)$ be complex vector spaces which we treat as real vector spaces with linear operators J_V and J_W such that $J_V^2 = J_W^2 = -\mathrm{Id}$. A $\mathbb{R}$-linear map

$$f : (V, J_V) \longrightarrow (W, J_W)$$

can be written uniquely as a sum of $\mathbb{C}$-linear map f_1 and $\mathbb{C}$-antilinear map f_2, i.e.,

$$f_1 \circ J_V = J_W \circ f_1, \quad f_2 \circ J_V = -J_W \circ f_2.$$

Definition A.1.1. Given an invertible $\mathbb{R}$-linear map which is orientation preserving

$$f : (V, J_V) \longrightarrow (W, J_W)$$

of complex vector spaces, the *Beltrami form* of f is the map

$$\mu(f) := f_1^{-1} \circ f_2 \in \mathrm{End}_{\mathbb{R}}((V, J_V)). \tag{A.1}$$

Remark A.1.2. $\mu(f)$ anticommutes with J_V.

Now, we will restrict our discussion to one dimensional complex vector spaces. Any $\mathbb{R}$-linear map $f : (\mathbb{C}, \iota) \longrightarrow (\mathbb{C}, \iota)$ can be written as $f(z) = az + b\bar{z}$, $a, b, z \in \mathbb{C}$. Here $f_1(z) = az$ and $f_2(z) = b\bar{z}$. From [29, Exercise 4.8.5, Chapter 4], we have

$$\frac{\|f\|^2}{\det f} = \frac{|f_1| + |f_2|}{|f_1| - |f_2|},$$

where $\|\cdot\|$ denotes the operator norm on the vector space of $\mathbb{R}$-linear maps $(\mathbb{C}, \iota) \longrightarrow (\mathbb{C}, \iota)$, and $|\cdot|$ denotes the operator norm on the vector space of $\mathbb{C}$-linear maps $(\mathbb{C}, \iota) \longrightarrow (\mathbb{C}, \iota)$ and $\mathbb{C}$-antilinear maps $(\mathbb{C}, \iota) \longrightarrow (\mathbb{C}, \iota)$. Moreover, if $|a| > |b|$, then the map f is orientation-preserving. Hence, it immediately follows that $\|\mu(f)\| < 1$. The space of all Beltrami forms on $(\mathbb{C}, \iota)$ is defined as follows:

$$\mathrm{Bel}(\mathbb{C}) := \{\mu \in \mathrm{End}_{\mathbb{R}}(\mathbb{C}) | \exists c \in \mathbb{C}, |c| < 1, \mu(z) = c\bar{z}\}.$$

Now, we ask the following question: given $\mu \in \mathrm{Bel}(\mathbb{C})$, how do we find an orientation preserving $f : \mathbb{C} \longrightarrow \mathbb{C}$ with $\mu(f) = \mu$? The equation $\mu(f) = \mu$ is famously known as the *Beltrami equation*. The most sophisticated answer to the above question is that f solves $\mu(f) = \mu$ iff f maps an ellipse in $\mathbb{C}$ whose ratio of the major to the minor axis is $\frac{1+\|\mu\|}{1-\|\mu\|}$ to a circle in $\mathbb{C}$. Let's discuss how the above discussion translates to the case of Riemann surfaces X and Y and an orientation preserving C^1 map $f : X \longrightarrow Y$ between them. Note that $df(x) : T_xX \longrightarrow T_{f(x)}Y$ can be written as a sum of a $\mathbb{C}$-linear map and a $\mathbb{C}$-antilinear map. For example, when $f : U \subset \mathbb{C} \longrightarrow \mathbb{C}$, we have $df = df^{(1,0)} + df^{(0,1)}$ (see the discussion just before Example 2.1.9), where $df^{(1,0)} = \frac{\partial f}{\partial z}dz$ and $df^{(0,1)} = \frac{\partial f}{\partial \bar{z}}d\bar{z}$. For a function $f : X \longrightarrow \mathbb{C}$,

$$\mu(f) = (df^{(1,0)})^{-1} \circ df^{(0,1)}. \tag{A.2}$$

Compare (A.2) it with (A.1) given in Definition A.1.1. $\mu(f)$ is an antilinear bundle map $TX \longrightarrow TX$.

Definition A.1.3. A smooth *Beltrami differential* on X is a smooth antilinear bundle map $\mu : TX \longrightarrow TX$.

Remark A.1.4. We can think of a Beltrami differential μ as a smooth section of the bundle $\overline{T^*X} \otimes_{\mathbb{C}} TX$.

A.2 Filling in the gap

Let Σ_g be given as $\mathbb{H}^2/\Gamma$ where Γ is a discrete-cocompact subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Given a Γ-invariant Beltrami differential μ on $\mathbb{H}^2$ with $\|\mu\| < 1$, there exists a smooth map $f : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ such that $f_{\bar{z}} = \mu f_z$ (see [1], [2], [3], [29], [30]). For t real and small, $\{f^{t\mu}\}$ denotes the family of smooth maps determined by the Beltrami differential $t\mu$. Then the *deformation vector field*

$$F := \frac{d}{dt}f^{t\mu}|_{t=0}$$

on $\mathbb{H}^2$ satisfies the famous potential equation

$$F_{\bar{z}}\frac{d\bar{z}}{dz} = \mu.$$

Let $T\Sigma_g$ denote the holomorphic tangent bundle of Σ_g. Recall Definition A.1.3 and Remark A.1.4. A Beltrami differential μ can be thought of as a smooth differential form on Σ_g of type $(0,1)$ with values in the bundle $T\Sigma_g$. In classical Teichmueller theory, we have the following short exact sequence of sheaves:

$$0 \longrightarrow \mathcal{S}_{\text{Hol}}\big(T\Sigma_g\big) \xrightarrow{\ i\ } \mathcal{S}(T\Sigma_g) \xrightarrow{\ \frac{\partial}{\partial \bar{z}}\ } \mathcal{BEL} \longrightarrow 0 \tag{A.3}$$

where

$$\mathcal{S}_{\text{Hol}}\big(T\Sigma_g\big) \text{ is the sheaf of holomorphic sections of } T\Sigma_g \text{ on } \Sigma_g,$$
$$\mathcal{S}(T\Sigma_g) \text{is the sheaf of smooth sections of} T\Sigma_g,$$
$$\mathcal{BEL} \text{is the sheaf of (smooth) Beltrami differentials on} \Sigma_g.$$

Clearly, i is the inclusion map. Locally, a smooth section of $T\Sigma_g$ can be written as $f_i \frac{\partial}{\partial z_i}$, where f_i is a smooth function. Applying $\frac{\partial}{\partial \bar{z}}$ on $f_i \frac{\partial}{\partial z_i}$ gives us a Beltrami differential $\frac{\partial f_i}{\partial \bar{z}} d\bar{z}_i \otimes \frac{\partial}{\partial z_i}$. This definition of $\frac{\partial}{\partial \bar{z}}$ is independent of the choice of coordinates. Note that (A.3) is a special case of a more general construction $\big($called the *Dolbeault resolution* of the sheaf $\mathcal{S}_{\text{Hol}}\big(T\Sigma_g\big)\big)$ by Dolbeault. See [65, Chapter 4] for more details. In particular, the map $\frac{\partial}{\partial \bar{z}}$ in (A.3) contributes to a long exact sequence in sheaf cohomology. Therefore, we have

$$H^1\big(\Sigma_g; \mathcal{S}_{\text{Hol}}\big(T\Sigma_g\big)\big) \cong \frac{H^0(\Sigma_g; \mathcal{BEL})}{\bar{\partial} H^0\big(\Sigma_g; \mathcal{S}\big(T\Sigma_g\big)\big)}.$$

It is a well known fact that for Σ_g with a hyperbolic metric, a cohomology class in

$$\frac{H^0(\Sigma_g; \mathcal{BEL})}{\bar{\partial} H^0\big(\Sigma_g; \mathcal{S}\big(T\Sigma_g\big)\big)}$$

has a unique representative known as a *harmonic Beltrami differential*, i.e., a Beltrami differential which is annihilated by the appropriate Laplacian (see [65, Chapter 5]). In the famous potential equation

$$F_{\bar{z}} = (z - \bar{z})^2 \overline{f(z)},$$
$$\mu = (z - \bar{z})^2 \overline{f(z)} \frac{d\bar{z}}{dz} \tag{A.4}$$

is a harmonic Beltrami differential if f is holomorphic.

For $\mathbb{H}^2$, we have the following sequence:

$$0 \longrightarrow \mathcal{S}_{\text{Hol}}\big(T\mathbb{H}^2\big) \xrightarrow{\ i\ } \mathcal{S}\big(T\mathbb{H}^2\big) \xrightarrow{\ \frac{\partial}{\partial \bar{z}}\ } \mathcal{BEL} \longrightarrow 0 \tag{A.5}$$

In (A.5), $\mathcal{BEL}$ is the sheaf of smooth Beltrami differentials on $\mathbb{H}^2$. Note that the short exact sequence given by (2.22) in Thereom 2.2.7, i.e.,

$$0 \longrightarrow \mathcal{HOL} \xrightarrow{\ \alpha\ } \mathcal{HARM} \xrightarrow{\ \beta\ } \mathcal{HQD} \longrightarrow 0 \tag{A.6}$$

is similar to the one given by (A.5). In (A.6), $\mathcal{HOL}$ is the sheaf of holomorphic vector fields on $\mathbb{H}^2$, $\mathcal{HARM}$ is the sheaf of harmonic vector fields on $\mathbb{H}^2$ and $\mathcal{HQD}$ is the sheaf of holomorphic quadratic differentials on $\mathbb{H}^2$. So, the gap is following:

Question A.2.1. How do (A.5) and (A.6) relate?

The following diagram fills the gap:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{HOL} & \xrightarrow{\alpha} & \mathcal{HARM} & \xrightarrow{\beta} & \mathcal{HQD} & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow{\scriptstyle \varrho_1} & & \Big\downarrow{\scriptstyle \varrho_2} & & \\ 0 & \longrightarrow & \mathcal{S}_{\mathrm{Hol}}\big(T\mathbb{H}^2\big) & \xrightarrow{i} & \mathcal{S}\big(T\mathbb{H}^2\big) & \xrightarrow{\frac{\partial}{\partial \bar{z}}} & \mathcal{BEL} & \longrightarrow & 0 \end{array} \tag{A.7}$$

In (A.7), ϱ_1 is clearly the inclusion map because a harmonic vector field on $\mathbb{H}^2$ is a smooth vector field. And, ϱ_2 is defined as:

$$\varrho_2(q) = \frac{g_{\mathbb{H}^2}^{-1}}{2}\bar{q},$$

where q is a holomorphic quadratic differential on $\mathbb{H}^2$. Note that $\varrho_2(q)$ is a harmonic Beltrami differential (see (A.4)) on $\mathbb{H}^2$. Moreover, ϱ_2 has a coordinate independent meaning. Here is an argument: recall that a Riemannian metric on an almost complex manifold M is the real part of a unique Hermitian metric on M. Therefore, a Riemannian metric on M determines an isomorphism

$$TM \longrightarrow \overline{T^*M}.$$

That in turn determines an isomorphism

$$\mathrm{Hom}(\overline{TM}, TM) \longrightarrow \mathrm{Hom}(\overline{TM}, \overline{T^*M}).$$

Note that $\mathrm{Hom}(\overline{TM}, \overline{T^*M}) = \mathrm{Hom}(TM, T^*M)$. Therefore, the bundle of Beltrami differetials on $\mathbb{H}^2$ is isomorphic to the bundle of quadratic differentials on $\mathbb{H}^2$.

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