Classifying maps into uniform tracial sequence algebras

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Abstract. We classify *-homomorphisms from nuclear C^*-algebras into uniform tracial sequence algebras of nuclear $\mathcal{Z}$-stable C^*-algebras via tracial data.

INTRODUCTION

Over the last 10 years, the application of von Neumann techniques has been a major theme in the structure theory of simple nuclear C^*-algebras through the pioneering work of Matui and Sato [17, 18]. A starting point for a number of these applications is the following well-known consequence of Connes' revolutionary work on the characterisation of hyperfinite von Neumann algebras [9]: maps from a separable nuclear C^*-algebra A into $\mathcal{R}^\omega$ (the ultrapower of the hyperfinite II_1 factor) are classified up to unitary equivalence by the trace they induce on A (see for example [7, Prop. 2.1]). Most recently, this result played a key role in Schafhauser's breakthrough new approach to the classification of monotracial separable nuclear C^*-algebras which absorb the universal uniformly hyperfinite algebra tensorially [21].

For a nuclear C^*-algebra B with a unique trace, Connes' theorem allows us to view $\mathcal{R}^\omega$ as a tracial ultrapower of B. When B has multiple traces, a somewhat different reduced product construction is needed in order to be able to handle them all uniformly. This led to the uniform tracial ultrapower B^ω, formalised in [6]. This, and its precursor in terms of ultraproducts of W^*-bundles [3, 19], has been a crucial tool in recent developments. In particular, in our recent joint work with Winter [6], we introduced a new tool—complemented partitions of unity (CPoU)—for studying these ultraproducts and used this to show that Jiang–Su stability and finite nuclear dimension are equivalent in the Toms–Winter conjecture [6, Thm. A].

The principal purpose of this paper is to use the new techniques in [6] to classify maps from separable nuclear C^*-algebras into uniform tracial sequence algebras of $\mathcal{Z}$-stable nuclear C^*-algebras, analogous to the consequence of Connes' result for maps into $\mathcal{R}^\omega$. In the theorem which follows, the uniform tracial sequence algebra B^∞ associated to B is the C^*-algebra of bounded sequences in B, modulo those converging to zero uniformly over all trace norms. So classifying maps from A to B^∞ up to unitary equivalence is a way of encoding a classification of uniform trace norm approximately multiplicative maps from A into B up to approximate unitary equivalence in uniform trace norm.

Theorem A. *Let A be a separable nuclear C^*-algebra, and let B be a separable nuclear $\mathcal{Z}$-stable C^*-algebra with $T(B)$ compact and non-empty. Then, for any continuous affine function $\alpha : T(B^\infty) \to T(A)$, there exists a *-homomorphism $\phi : A \to B^\infty$ which induces α. Moreover, ϕ is unique up to unitary equivalence.*

Just as the classification of embeddings from separable nuclear C^*-algebras into $\mathcal{R}^\omega$ is vital in [21], Theorem A will form the starting point of the forthcoming joint work of Carrión, Gabe, Schafhauser, and the last two named authors, which will give an abstract approach to the classification of simple separable unital nuclear Jiang–Su stable C^*-algebras satisfying the UCT [8]. Our reason for setting up Theorem A with the uniform trace norm sequence algebra B^∞ as opposed to the uniform tracial ultrapower B^ω is so it can be applied exactly as written in [8].[1]

The role of nuclearity and $\mathcal{Z}$-stability of B in Theorem A is to obtain CPoU from [6, Thm. I]—the main technical result of that work. Although the defining property of CPoU is in the form of the existence of certain partitions of unity for trace spaces, its principal consequence is a local to global tracial type-satisfaction process for B^∞. In less fancy language, this means that if (suitable) properties hold approximately in trace in each tracial GNS-representation of B^∞, then they hold exactly in B^∞. This gives B^∞ a von Neumann algebra-like flavor (though it is certainly not a von Neumann algebra). Theorem A is obtained in this fashion; we glue together the classification of maps from separable nuclear C^*-algebras into finite von Neumann algebras from Connes' theorem over all traces using CPoU.

We also record in Proposition 2.1, another, somewhat easier, application of CPoU for use in [8], which showcases another von Neumann algebra-like property: every unitary in B^∞ is an exponential.

1. Preliminaries

Let B be a C^*-algebra. We let $T(B)$ denote the set of tracial states (which we abbreviate as "traces") on B. For $\tau \in T(B)$, we define the associated 2-semi-norm on B by

$$(1.1) \qquad \|b\|_{2,\tau} := \sqrt{\tau(|b|^2)}.$$

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[1]One can obtain an ultrapower version of Theorem A by working with an ultrapower B^ω in place of the sequence algebra B^∞ throughout the paper.

We let $\pi_\tau : B \to \mathcal{B}(\mathcal{H}_\tau)$ be the GNS representation associated to τ and continue to use $\|\cdot\|_{2,\tau}$ to denote the induced 2-norm on $\pi_\tau(B)''$.

Define the uniform tracial sequence algebra

$$(1.2) \qquad B^\infty := \ell^\infty(B)/\{(b_n)_{n=1}^\infty \mid \lim_{n\to\infty} \sup_{\tau \in T(B)} \|b_n\|_{2,\tau} = 0\}.$$

We will typically use representative sequences in $\ell^\infty(B)$ to denote elements of B^∞.

The ultraproduct versions of these sequence algebras are obtained using a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ in place of ∞, and many of their basic properties are the same. For example, when B is separable and $T(B)$ is non-empty and compact, B^∞ is unital, with the unit represented by an approximate unit $(e_n)_{n=1}^\infty$ for B in just the same way as the ultrapower version of this result [6, Prop. 1.11].

Given a sequence $(\tau_n)_{n=1}^\infty$ in $T(B)$ and a free ultrafilter ω, define the associated *limit trace* $\tau : B^\infty \to \mathbb{C}$ by

$$(1.3) \qquad \tau((b_n)_{n=1}^\infty) := \lim_{n\to\omega} \tau_n(b_n).$$

We let $T_\infty(B)$ denote the set of all limit traces on B^∞.

On B^∞, one has a uniform 2-norm[2] $\|\cdot\|_{2,T_\infty(B)}$ given by

$$(1.4) \qquad \|b\|_{2,T_\infty(B)} := \sup_{\tau \in T_\infty(B)} \|b\|_{2,\tau}, \quad b \in B^\infty.$$

More explicitly, for $b = (b_n)_{n=1}^\infty \in B^\infty$, one has

$$(1.5) \qquad \|b\|_{2,T_\infty(B)} = \limsup_n \sup_{\tau \in T(B)} \|b_n\|_{2,\tau}.$$

In this paper, we will frequently use Kirchberg's ϵ-test, which appears as [15, Lem. A.1]. However, we need a slightly different version, as we work with "sequence algebras" rather than ultrapowers; we state here the version we need and remark that the proof is nearly identical to that of [15, Lem. A.1].

Lemma 1.1 (Kirchberg's ϵ-test [15]). *Let $(X_n)_{n=1}^\infty$ be a sequence of non-empty sets, and for $k, n \in \mathbb{N}$, let $f_n^{(k)} : X_n \to [0,\infty]$ be a function. Define functions $f^{(k)} : X_1 \times X_2 \times \cdots \to [0,\infty]$ by*

$$(1.6) \qquad f^{(k)}(x_1, x_2, \dots) := \limsup_n f_n^{(k)}(x_n).$$

If, for every $\epsilon > 0$ and $k_0 \in \mathbb{N}$, there exists $x \in X_1 \times X_2 \times \cdots$ such that $f^{(k)}(x) < \epsilon$ for $k = 1, \dots, k_0$, then there exists $y \in X_1 \times X_2 \times \cdots$ such that $f^{(k)}(y) = 0$ for all $k \in \mathbb{N}$.

We next record a useful fact, which is a direct consequence of Cuntz and Pedersen's investigation of traces on C^*-algebras [10, Prop. 2.7] and the fact that the weak*-continuous functionals on the dual of a Banach space correspond precisely to elements of the Banach space.

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[2]To see it is a norm, use (1.5).

Proposition 1.2 (Cuntz–Pedersen)**.** *Let A be a C^*-algebra and $f : T(A) \to \mathbb{R}$ a continuous affine function. Then there is a self-adjoint element $a \in A$ such that*

$$(1.7) \qquad\qquad \tau(a) = f(\tau), \quad \tau \in T(A).$$

In this paper, we will be using CPoU as a property of the uniform tracial sequence algebra B^∞. In [6, Def. 3.1], CPoU was originally defined as a property of the uniform tracial ultrapower B^ω, but standard methods allow it to be rephrased as a local property of B instead; see [6, Prop. 3.2]. The same methods allow it to be rephrased as a property of B^∞, analogous to the original definition for B^ω, as recorded in the lemma below. For the purposes of this paper, one can take this as the definition of CPoU.

Lemma 1.3. *Let B be a separable C^*-algebra with $T(B)$ non-empty and compact. If B has CPoU, then, for any $\|\cdot\|_{2,T_\infty(B)}$-separable subset S of B^∞, any $\delta > 0$, and any $a_1, \ldots, a_k \in (B^\infty)_+$ satisfying*

$$(1.8) \qquad\qquad \min\{\tau(a_1), \ldots, \tau(a_k)\} < \delta, \quad \tau \in T_\infty(B),$$

there exist orthogonal projections $e_1, \ldots, e_k \in B^\infty \cap S'$ which sum to 1_{B^∞} such that

$$(1.9) \qquad\qquad \tau(a_i e_i) \leq \delta\tau(e_i), \quad i = 1, \ldots, k, \, \tau \in T_\infty(B).$$

We will access CPoU through one of the main technical results of [6], which we recall below. Note that, for unital C^*-algebras, the tracial state space $T(B)$ is automatically compact.

Theorem 1.4 ([6, Thm. I])**.** *Let B be a separable, nuclear, $\mathcal{Z}$-stable C^*-algebra with $T(B)$ compact and non-empty. Then B has CPoU.*

2. Results

We start by recording an application of CPoU regarding unitaries in uniform tracial sequence algebras of $\mathcal{Z}$-stable nuclear C^*-algebras for use in [8].

Proposition 2.1. *Let B be a separable C^*-algebra with CPoU and $T(B)$ compact and non-empty. Let S be a $\|\cdot\|_{2,T_\infty(B)}$-separable subset of B^∞ closed under taking adjoints. Then every unitary $u \in B^\infty \cap S'$ can be written as an exponential $u = e^{\pi i h}$ for some self-adjoint $h \in B^\infty \cap S'$ of norm at most 1. In particular, this holds whenever B is separable, unital, nuclear, and $\mathcal{Z}$-stable with $T(B) \neq \varnothing$.*

Proof. The final sentence of the proposition follows from the rest by Theorem 1.4. Fix $\epsilon > 0$. By Kirchberg's ϵ-test (Lemma 1.1),[3] it suffices to prove

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[3]Since S is $\|\cdot\|_{2,T_\infty(B)}$-separable, testing that a sequence $(b_n) \subset \ell^\infty(B)$ represents an element of $B^\infty \cap S'$ requires only countably many constraints. Indeed, $B^\infty \cap S' = B^\infty \cap S_0'$ for any countable $\|\cdot\|_{2,T_\infty(B)}$-dense subset $S_0 \subset S$, as multiplication is jointly $\|\cdot\|_{2,T_\infty(B)}$-continuous on $\|\cdot\|$-bounded sets.

that there exists a self-adjoint $h \in B^\infty \cap S'$ of norm at most 1 such that

$$\|u - e^{\pi i h}\|_{2, T_\infty(B)} \leq \epsilon. \tag{2.1}$$

For each $\tau \in T(B^\infty)$, using Borel functional calculus, there exists a self-adjoint $x \in \pi_\tau(B^\infty \cap S')''$ of norm at most 1 such that $\pi_\tau(u) = e^{\pi i x}$. By the Kaplansky density theorem, we may approximate x by a self-adjoint contraction in $\pi_\tau(B^\infty \cap S')$, which can then be lifted to a self-adjoint contraction $h_\tau \in B^\infty \cap S'$ such that

$$\|u - e^{\pi i h_\tau}\|_{2,\tau} < \epsilon. \tag{2.2}$$

Set $a_\tau := |u - e^{\pi i h_\tau}|^2 \in (B^\infty)_+$ so that $\tau(a_\tau) < \epsilon^2$. By continuity and compactness, there exist $\tau_1, \ldots, \tau_k \in T(B^\infty)$ such that, for every $\tau \in T(B^\infty)$,

$$\min\{\tau(a_{\tau_1}), \ldots, \tau(a_{\tau_k})\} < \epsilon^2. \tag{2.3}$$

Using CPoU as in Lemma 1.3, there exists a partition of unity consisting of projections $p_1, \ldots, p_k \in B^\infty \cap \{u, h_{\tau_1}, \ldots, h_{\tau_k}\}' \cap S'$ such that

$$\tau(p_j a_{\tau_j}) \leq \epsilon^2 \tau(p_j), \quad \tau \in T_\infty(B), \, j = 1, \ldots, k. \tag{2.4}$$

Set $h := \sum_{j=1}^k p_j h_{\tau_j} \in B^\infty \cap S'$. Since the p_i are orthogonal and commute with the self-adjoint contractions h_{τ_j}, this is a self-adjoint contraction. We note that, for $j = 1, \ldots, k$,

$$p_j e^{\pi i h} = p_j e^{\pi i h_{\tau_j}}. \tag{2.5}$$

Using this, for $\tau \in T_\infty(B)$, we compute

$$\tau(|u - e^{\pi i h}|^2) = \sum_{j=1}^k \tau(p_j|u - e^{\pi i h}|^2) = \sum_{j=1}^k \tau(p_j|u - e^{\pi i h_{\tau_j}}|^2) \tag{2.6}$$

$$= \sum_{j=1}^k \tau(p_j a_j) \leq \sum_{j=1}^k \epsilon^2 \tau(p_j) = \epsilon^2. \qquad \square$$

We next turn to the uniqueness aspect of Theorem A. Recall the by-now well-known consequence of Connes' characterisation of hyperfiniteness from [9] that if A is separable and nuclear and $\mathcal{M}$ is a finite von Neumann algebra, then $*$-homomorphisms $\phi, \psi : A \to \mathcal{M}$ are strong*-approximately unitary equivalent if and only if $\tau \circ \phi = \tau \circ \psi$ for all $\tau \in T(\mathcal{M})$ (see [7, Prop. 2.1], for example[4]).

Theorem 2.2. *Let A be a separable nuclear C^*-algebra, and let B be a separable C^*-algebra with CPoU and with $T(B)$ compact and non-empty. If $\phi, \psi : A \to B^\infty$ are $*$-homomorphisms such that $\tau \circ \phi = \tau \circ \psi$ for all $\tau \in T(B^\infty)$, then ϕ and ψ are unitarily equivalent.*

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[4]In the statement of [7, Prop. 2.1], the codomain $\mathcal{M}$ is required to be countably decomposable; however, this hypothesis is not needed in the proof. It might also be noted that, in our application, in the proof of Theorem 2.2, the codomain $\pi_\tau(B^\infty)''$ is countably decomposable as it has a faithful trace.

Proof. Fix $\epsilon > 0$ and a finite set $\mathcal{F} \subset A$. Since A is separable, by using Kirchberg's ϵ-test (Lemma 1.1), it suffices to prove that there is a unitary $u \in B^\infty$ such that

$$(2.7) \qquad \|\phi(x) - u^*\psi(x)u\|_{2,T_\infty(B)} \leq \epsilon, \quad x \in \mathcal{F}.$$

Set

$$(2.8) \qquad \eta := \frac{\epsilon}{\sqrt{|\mathcal{F}|}}.$$

Fix, for the moment, a trace $\tau \in T(B^\infty)$, and recall that $\pi_\tau : B^\infty \to \pi_\tau(B^\infty)''$ is the corresponding GNS representation. Then, since A is nuclear and $\pi_\tau \circ \phi$, $\pi_\tau \circ \psi : A \to \pi_\tau(B^\infty)''$ agree on the traces of $\pi_\tau(B^\infty)''$, it follows that these maps are strong*-approximately unitarily equivalent. By Kaplansky's density theorem, the unitaries implementing this can be taken from $\pi_\tau(B^\infty)$ (as done in the proof of Proposition 2.1). Since the strong*-topology is given by $\|\cdot\|_{2,\tau}$ on bounded sets, it follows that there exists a unitary $u_\tau \in B^\infty$ such that

$$(2.9) \qquad \|\phi(x) - u_\tau^*\psi(x)u_\tau\|_{2,\tau} < \eta, \quad x \in \mathcal{F}.$$

Set

$$(2.10) \qquad a_\tau := \sum_{x \in \mathcal{F}} |\phi(x) - u_\tau^*\psi(x)u_\tau|^2 \in (B^\infty)_+$$

so that $\tau(a_\tau) < |\mathcal{F}|\eta^2 = \epsilon^2$.

By continuity and compactness, there exist $\tau_1, \ldots, \tau_k \in T(B^\infty)$ such that, for every $\tau \in T(B^\infty)$,

$$(2.11) \qquad \min\{\tau(a_{\tau_1}), \ldots, \tau(a_{\tau_k})\} < \epsilon^2.$$

Using CPoU as in Lemma 1.3, there exist orthogonal projections $e_1, \ldots, e_k \in B^\infty \cap (\psi(\mathcal{F}) \cup \phi(\mathcal{F}) \cup \{u_{\tau_1}, \ldots, u_{\tau_k}\})'$ which sum to 1_{B^∞} such that

$$(2.12) \qquad \tau(a_{\tau_i}e_i) \leq \epsilon^2\tau(e_i), \quad \tau \in T_\infty(B).$$

Set

$$(2.13) \qquad u := \sum_{i=1}^k e_i u_{\tau_i}.$$

Since the e_i are orthogonal projections summing to 1_{B^∞} and using the fact that they commute with the unitaries u_{τ_j}, it follows that u is itself a unitary. Moreover, for $x \in \mathcal{F}$ and $\tau \in T_\infty(B)$, using the fact that the e_i are orthogonal projections which commute with the u_{τ_i}, and both $\phi(x)$ and $\psi(x)$, we have

$$(2.14) \qquad |\phi(x) - u^*\psi(x)u| = \sum_{i=1}^k e_i|\phi(x) - u_{\tau_i}^*\psi(x)u_{\tau_i}|.$$

Hence

$$(2.15) \qquad \|\phi(x) - u^*\psi(x)u\|_{2,\tau}^2 \overset{(2.14)}{=} \sum_{i=1}^{k} \tau(e_i|\phi(x) - u_{\tau_i}^*\psi(x)u_{\tau_i}|^2)$$

$$\overset{(2.10)}{\leq} \sum_{i=1}^{k} \tau(e_i a_{\tau_i}) \overset{(2.12)}{\leq} \sum_{i=1}^{k} \epsilon^2 \tau(e_i) = \epsilon^2.$$

Taking the supremum over all $\tau \in T_\infty(B)$, (2.7) follows. $\qquad\square$

In order to get our existence result into B^∞, we begin with two existence results into von Neumann algebras; these rely on the quasidiagonality of amenable traces on cones established in [4], which in turn builds on the earlier results of [22, 12]. Recall that a trace τ on A is *amenable* if, given a finite subset $\mathcal{F} \subset A$ and $\epsilon > 0$, there is a c.p.c. map $\phi : A \to M_n$ for some n (ϕ can be taken to be unital when A is unital) such that

$$(2.16) \qquad \|\phi(ab) - \phi(a)\phi(b)\|_{2,\mathrm{tr}_{M_n}} < \epsilon, \quad a,b \in \mathcal{F},$$

and

$$(2.17) \qquad |\mathrm{tr}_{M_n}(\phi(a)) - \tau(a)| < \epsilon, \quad a \in \mathcal{F}.$$

We write $T_{\mathrm{am}}(A)$ for the set of amenable traces on A. The trace τ is said to be *quasidiagonal* if (2.16) can be strengthened to the operator norm estimate $\|\phi(ab) - \phi(a)\phi(b)\| < \epsilon$ for $a \in \mathcal{F}$. We write $T_{\mathrm{qd}}(A)$ for the set of quasidiagonal traces on A. See [5] for details on these approximation properties.

Lemma 2.3. *Let A be a C^*-algebra, let $\mathcal{M}$ be a type II_1 von Neumann algebra, and let $\lambda \in T_{\mathrm{am}}(A)$. Then, given a finite set $\mathcal{F} \subset A$ and $\epsilon > 0$, there exist a finite-dimensional C^*-algebra F, a c.p.c. map $\theta : A \to F$, and a unital *-homomorphism $\eta : F \to \mathcal{M}$ such that*

$$(2.18) \qquad \|\theta(a)\theta(b)\| < \epsilon \quad \text{for } a,b \in \mathcal{F} \text{ satisfying } ab = 0,$$

$$(2.19) \qquad |\tau(\eta \circ \theta(a)) - \lambda(a)| < \epsilon \quad \text{for } a \in \mathcal{F} \text{ and } \tau \in T(\mathcal{M}).$$

Proof. Set $\mathcal{G} := \{\mathrm{id}_{(0,1]} \otimes a \in C_0((0,1]) \otimes A \mid a \in \mathcal{F}\}$. By [4, Prop. 3.2], the trace $\delta_1 \otimes \lambda$ is quasidiagonal on $C_0((0,1]) \otimes A$, where δ_1 is the functional of evaluation at 1 on $C_0((0,1]$. Thus, there exist a matrix algebra F and a c.p.c. map $\phi : C_0((0,1]) \otimes A \to F$ such that

$$\|\phi(x)\phi(y) - \phi(xy)\| < \epsilon, \quad x,y \in \mathcal{G},$$
$$(2.20) \qquad |\mathrm{tr}_F \circ \phi(x) - (\delta_1 \otimes \lambda)(x)| < \epsilon, \quad x \in \mathcal{G}.$$

Define $\theta : A \to F$ by $\theta(a) := \phi(\mathrm{id}_{(0,1]} \otimes a)$ so that it immediately follows that (2.18) is satisfied. As $\mathcal{M}$ is type II_1, F can be embedded unitally in $\mathcal{M}$. Let $\eta : F \hookrightarrow \mathcal{M}$ be any such embedding. By the uniqueness of the trace on F, (2.19) is also satisfied. $\qquad\square$

Lemma 2.4. *Let A be a C^*-algebra, let $\mathcal{M}$ be a type II_1 von Neumann algebra, and let $\alpha : T(\mathcal{M}) \to T_{\mathrm{am}}(A)$ be affine and continuous. Let $\mathcal{F} \subset A_{\mathrm{sa}}$ be a finite*

set. Then, given $\epsilon > 0$, there exist a finite-dimensional C^*-algebra F, a c.p.c. map $\theta : A \to F$, and a unital *-homomorphism $\eta : F \to \mathcal{M}$ such that

$$(2.21) \qquad \|\theta(a)\theta(b)\| < \epsilon \quad \text{for } a, b \in \mathcal{F} \text{ satisfying } ab = 0$$

and

$$(2.22) \qquad |\tau(\eta \circ \theta(a)) - \alpha(\tau)(a)| < \epsilon, \quad a \in \mathcal{F}, \tau \in T(\mathcal{M}).$$

Moreover, if, for each $a \in \mathcal{F}$, we are given an element $c_a \in \mathcal{M}_{\mathrm{sa}}$ satisfying

$$(2.23) \qquad \tau(c_a) = \alpha(\tau)(a), \quad \tau \in T(\mathcal{M}),$$

then, for each $a \in \mathcal{F}$, there exist $x_1^{(a)}, \ldots, x_{10}^{(a)}, y_1^{(a)}, \ldots, y_{10}^{(a)} \in \mathcal{M}$ such that

$$(2.24) \qquad \left\| \eta \circ \theta(a) - c_a - \sum_{i=1}^{10} [x_i^{(a)}, y_i^{(a)}] \right\| < \epsilon.$$

If $\alpha(T(\mathcal{M})) \subset T_{\mathrm{qd}}(A)$, then θ can be taken to satisfy

$$(2.25) \qquad \|\theta(a)\theta(b) - \theta(ab)\| < \epsilon \quad \text{for } a, b \in \mathcal{F}.$$

Proof. The idea is to glue maps from the previous lemma over the centre $Z(\mathcal{M})$ of $\mathcal{M}$, in a manner similar to the proof of [4, Lem. 2.5]. As, for $a \in \mathcal{F}$, the elements c_a satisfying (2.23) automatically exist by Proposition 1.2, we shall use them throughout the proof. Suppose $\mathcal{F} = \{a_1, \ldots, a_n\}$. By the structure of commutative von Neumann algebras [2, Thm. III.1.5.18], let (X, μ) be a locally finite measure space such that $Z(\mathcal{M}) \cong L^\infty(X, \mu)$. Let $E : \mathcal{M} \to L^\infty(X, \mu)$ be the centre-valued trace on $\mathcal{M}$ (see [2, Thm. III.2.5.7]). Choose natural numbers $C \geq 4$ and k such that $C > \sup_{a \in \mathcal{F}} \|c_a\|$ and $C/k < \epsilon$. Set $I := \{-Ck+1, \ldots, Ck\}^n$, and for $r = (r_1, \ldots, r_n) \in I$, let $p_r \in L^\infty(X, \mu)$ be the characteristic function of the set

$$(2.26) \qquad \left\{ x \in X \mid \frac{r_j - 1}{k} \leq E(c_{a_j})(x) < \frac{r_j}{k}, j = 1, \ldots, n \right\}.$$

By construction, $(p_r)_{r \in I}$ forms a partition of unity consisting of projections, and, as every trace on $\mathcal{M}$ factors though E (see [2, Thm. III.2.5.7 (iv)]),

$$(2.27) \qquad \tau(c_{a_j}) \approx_{1/k} \sum_{r \in I} \frac{r_j}{k} \tau(p_r), \quad \tau \in T(\mathcal{M}).^{[5]}$$

In particular, for any $r \in I$ and $j = 1, \ldots, n$, we have

$$(2.28) \qquad \alpha(\tau)(a_j) \overset{(2.23)}{=} \tau(c_{a_j}) \approx_{1/k} \frac{r_j}{k}, \quad \tau \in T(p_r \mathcal{M}).$$

(Note that we implicitly extend τ to $\mathcal{M}$, by setting it to be zero on $(1 - p_r)\mathcal{M}$, before we apply α in the previous equation.)

Let $I_0 := \{r \in I \mid p_r \neq 0\}$. For each $r \in I_0$, fix $\sigma_r \in T(p_r\mathcal{M})$, and set $\lambda_r := \alpha(\sigma_r)$. By Lemma 2.3, applied to $p_r\mathcal{M}$ and λ_r, there exist a finite-dimensional algebra F_r, a c.p.c. map $\theta_r : A \to F_r$, and a unital *-homomorphism $\eta_r : F_r \to p_r\mathcal{M}$ such that

$$(2.29) \qquad \|\theta_r(a)\theta_r(b)\| < \epsilon$$

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[5] To improve the readability of this proof, we write $z_1 \approx_\eta z_2$ as shorthand for $|z_1 - z_2| \leq \eta$.

for $a, b \in \mathcal{F}$ satisfying $ab = 0$, and

$$(2.30) \qquad |\tau(\eta_r \circ \theta_r(a)) - \lambda_r(a)| < \frac{\epsilon}{2}$$

for $a \in \mathcal{F}$ and $\tau \in T(p_r \mathcal{M})$. Set $F := \bigoplus_{r \in I_0} F_r$. Define $\theta : A \to F$ by $\theta(a) := \bigoplus_{r \in I_0} \theta_r(a)$ and $\eta : F \to \mathcal{M}$ by $\eta((x_r)_{r \in I_0}) := \sum_{r \in I_0} \eta_r(x_r)$. By construction, η is a unital *-homomorphism and θ satisfies (2.21).

Note that if each λ_r is quasidiagonal, this can be used directly in place of Lemma 2.3 in the previous paragraph enabling θ_r to be chosen $(\mathcal{F}, \epsilon)$-approximately multiplicative. Therefore, if $\alpha(T(\mathcal{M})) \subset T_{\mathrm{qd}}(A)$, then θ can be taken to satisfy (2.25).

Fix $\tau \in T(\mathcal{M})$ for the moment. For each $r \in I_0$, set $\tau_r := \frac{\tau(p_r \cdot)}{\tau(p_r)} \in T(p_r \mathcal{M})$,[6] so τ can expressed as the convex combination

$$(2.31) \qquad \tau = \sum_{r \in I_0} \tau(p_r) \tau_r.$$

Thus,

$$
\begin{aligned}
(2.32) \qquad \tau(\eta \circ \theta(a)) &= \sum_{r \in I_0} \tau(p_r) \tau_r (\eta_r \circ \theta_r(a)) \\
&\overset{(2.30)}{\approx}_{\epsilon/2} \sum_{r \in I_0} \tau(p_r) \lambda_r(a) \\
&\overset{(2.28)}{\approx}_{2/k} \sum_{r \in I_0} \tau(p_r) \alpha(\tau_r)(a) \\
&= \alpha\Big(\sum_{r \in I_0} \tau(p_r) \tau_r \Big)(a) \\
&\overset{(2.31)}{=} \alpha(\tau)(a)
\end{aligned}
$$

for all $a \in \mathcal{F}$. Since $\frac{2}{k} \le \frac{C}{2k} < \frac{\epsilon}{2}$ and $\tau \in T(\mathcal{M})$ was arbitrary, this establishes (2.22).

Now fix $a \in \mathcal{F}$ for the moment, and let us explain why $x_i^{(a)}, y_i^{(a)}$ can be found to satisfy (2.24). Set $h := E(\eta \circ \theta(a) - c_a) \in L^\infty(X, \mu)$, which by (2.22) satisfies $\|h\| \le \epsilon$. Observe that $E(\eta \circ \theta(a) - c_a - h) = 0$ so that, by [11, Thm. 3.2], there exist $x_1^{(a)}, \dots, x_{10}^{(a)}, y_1^{(a)}, \dots, y_{10}^{(a)} \in \mathcal{M}$ such that[7]

$$(2.33) \qquad \eta \circ \theta(a) - c_a - h = \sum_{i=1}^{10} [x_i^{(a)}, y_i^{(a)}].$$

Hence,

$$(2.34) \qquad \Big\| \eta \circ \theta(a) - c_a - \sum_{i=1}^{10} [x_i^{(a)}, y_i^{(a)}] \Big\| = \|h\| < \epsilon. \qquad \square$$

---

[6] We can choose τ_r arbitrarily in case $\tau(p_r) = 0$.

[7] We can also arrange that $\max_{1 \le i \le 10} \|x_i^{(a)}\| \|y_i^{(a)}\| \le 12 \cdot 12 \cdot \|\eta \circ \theta(a) - c_a - h\|$, but we do not need to control the norms of these elements on this occasion.

We now turn to the existence component of Theorem A. For this, we will need a so-called "no silly traces" result to show that the limit traces on B^∞ generate all traces on B^∞. For the purposes of Theorem A, we could use (a sequence algebra version of) the original result of this type, [19, Thm. 8] for $\mathcal{Z}$-stable exact C^*-algebras B. This gives a no silly traces result for the C^*-algebra ultraproduct, from which it follows that there are no silly traces on the uniform tracial ultraproduct (and this is easily modified to sequence algebras).

However, in Theorem 2.6 below, we prefer not to impose the hypothesis that B is $\mathcal{Z}$-stable, and instead simply ask that it has CPoU. Correspondingly, we first show how to obtain a no silly traces result for the uniform tracial sequence algebra just assuming CPoU. While CPoU is involved to handle possibly non-$\mathcal{Z}$-stable C^*-algebras, to some extent, the present result is easier than Ozawa's [19, Thm. 8] in that we can use the uniform bounds on the number of commutators of a self-adjoint operator in a finite von Neumann algebra which vanishes in all traces from [11], rather than the more delicate growth rate estimates used in [19] which are required to eliminate silly traces from the C^*-norm sequence algebra or ultrapower.

We note also that no silly traces for the tracial product B^∞ does not imply no silly traces for the norm product, as demonstrated by the unique trace example by Robert (based on earlier examples by Villadsen) in [20, Thm. 1.4].[8]

Proposition 2.5. *Let B be a separable C^*-algebra with $T(B)$ compact and non-empty and which has CPoU. Then the weak*-closed convex hull of $T_\infty(B)$ is $T(B^\infty)$.*

Proof. Fix a self-adjoint contraction $z \in B^\infty$. Let $\delta := \sup_{\tau \in T_\infty(B)} |\tau(z)|$. By [6, Lem. 4.4] (which is extracted from [19, Thm. 8]), it suffices to prove that $\sup_{\tau \in T(B^\infty)} |\tau(z)| = \delta$. This will be achieved by producing a self-adjoint $c \in B^\infty$ with $\|c\| \leq \delta$ and contractions $x^{(1)}, \ldots, x^{(10)}, y^{(1)}, \ldots, y^{(10)} \in B^\infty$ such that $z - c = K \sum_{i=1}^{10} [x^{(i)}, y^{(i)}]$, where $K := 12 \cdot 12(1 + \delta)$.

Choose a representative sequence $(z_n)_{n=1}^\infty$ of self-adjoint contractions for $z \in B^\infty$. Then $\limsup_{n \to \infty} \sup_{\tau \in T(B)} |\tau(z_n)| \leq \delta$, and so, by rescaling, we may assume $\sup_{\tau \in T(B)} |\tau(z_n)| \leq \delta$ for each n.

Fix $n \in \mathbb{N}$ for the moment. For each $\tau \in T(B)$, let π_τ be its GNS-representation and $\mathcal{M}_\tau := \pi_\tau(B)''$. Then $\sup_{\rho \in T(\mathcal{M}_\tau)} |\rho(\pi_\tau(z_n))| \leq \delta$. Letting $\tilde{c}_{n,\tau} \in \mathcal{M}_\tau$ be the result of applying the centre-valued trace in $\mathcal{M}_\tau$ to $\pi_\tau(z_n)$, we have $\|\tilde{c}_{n,\tau}\| \leq \delta$. By [11, Thm. 3.2], there exist contractions $\tilde{x}_{n,\tau}^{(1)}, \ldots, \tilde{x}_{n,\tau}^{(10)}$, $\tilde{y}_{n,\tau}^{(1)}, \ldots, \tilde{y}_{n,\tau}^{(10)} \in \mathcal{M}_\tau$ with $\pi_\tau(z_n) - \tilde{c}_n = K \sum_{i=1}^{10} [\tilde{x}_{n,\tau}^{(i)}, \tilde{y}_{n,\tau}^{(i)}]$. By Kaplansky's

---

[8]Let A be the C^*-algebra from [20, Thm. 1.4] so that condition (iii) (a) from [1, Thm. 3.24] fails for A. Then no norm ultraproduct of A can have unique trace (by the equivalence of condition (iii) (a) and the uniqueness of trace on an ultraproduct, which makes up the first part of the proof of (iii) $\Leftrightarrow$ (ii) in [1, Thm. 3.24]; see paragraph 3 of the proof, which notes this explicitly). Note that A has the Diximer property needed to apply this result by [13] as it is simple, unital, and has unique trace. Therefore, the norm product A_∞ of infinitely many copies of A has traces which are not in the closed convex hull of the limit traces.

density theorem, there exists a self-adjoint $c_{n,\tau} \in B$ with $\|c_{n,\tau}\| \leq \delta$ and contractions $x_{n,\tau}^{(1)}, \ldots, x_{n,\tau}^{(10)}, y_{n,\tau}^{(1)}, \ldots, y_{n,\tau}^{(10)} \in B$ with

$$(2.35) \qquad \left\| z_n - c_{n,\tau} - K \sum_{i=1}^{10} [x_{n,\tau}^{(i)}, y_{n,\tau}^{(i)}] \right\|_{2,\tau} < \gamma_n,$$

where $\gamma_n < \frac{1}{n}$.

Let $a_{n,\tau} := |z_n - c_{n,\tau} - K \sum_{i=1}^{10} [x_{n,\tau}^{(i)}, y_{n,\tau}^{(i)}]|^2$. By compactness, there exist $\tau_{n,1}, \ldots, \tau_{n,k_n}$ such that $\min_{\rho \in T(B)} \{\rho(a_{n,\tau_{n,1}}), \ldots, \rho(a_{n,\tau_{n,k_n}})\} < \gamma_n^2$. As every trace in $T_\infty(B)$ restricts to a trace on B, the same minimum holds over $\rho \in T_\infty(B)$. Let $S_n \subset B^\infty$ be the separable subalgebra generated by z_n together with $c_{n,\tau_{n,1}}, \ldots, c_{n,\tau_{n,k_n}}$ and the contractions $x_{n,\tau_{n,1}}^{(1)}, \ldots, x_{n,\tau_{n,k_n}}^{(10)}$ and $y_{n,\tau_{n,1}}^{(1)}, \ldots, y_{n,\tau_{n,k_n}}^{(10)}$.

By CPoU in the form of Lemma 1.3, there exist pairwise orthogonal projections $e_{n,1}, \ldots, e_{n,k_n}$ in $B^\infty \cap S_n'$ which sum to 1_{B^∞} and have $\rho(a_{n,\tau_{n,j}} e_{n,j}) \leq \gamma_n^2 \rho(e_{n,j})$ for $j = 1, \ldots, k_n$ and all $\rho \in T_\infty(B)$.

Define

$$\tilde{c}_n := \sum_{j=1}^{k_n} c_{n,\tau_{n,j}} e_{n,j} \in B^\infty,$$

$$\tilde{x}_n^{(i)} := \sum_{j=1}^{k_n} x_{n,\tau_{n,j}}^{(i)} e_{n,j} \in B^\infty,$$

$$\tilde{y}_n^{(i)} := \sum_{j=1}^{k_n} y_{n,\tau_{n,j}}^{(i)} e_{n,j} \in B^\infty.$$

Then $\|\tilde{c}_n\| \leq \delta$, and all the $\tilde{x}_n^{(i)}$ and $\tilde{y}_n^{(i)}$ are contractions. Let $\rho \in T_\infty(B)$. Using the properties of the $e_{n,j}$, we have

$$(2.36) \qquad \left\| z_n - \tilde{c}_n - K \sum_{i=1}^{10} [\tilde{x}_n^{(i)}, \tilde{y}_n^{(i)}] \right\|_{2,\rho}^2 = \rho\left(\sum_{j=1}^{k_n} a_{n,\tau_{n,j}} e_{n,j} \right) \leq \gamma_n^2.$$

Taking norm preserving lifts from B^∞ to $\ell^\infty(B)$ and then choosing elements $c_n, x_n^{(i)}, y_n^{(i)} \in B$ sufficiently far down the representative sequences for $\tilde{c}_n, \tilde{x}_n^{(i)}, \tilde{y}_n^{(i)} \in B^\infty$, we have

$$(2.37) \qquad \sup_{\tau \in T(B)} \left\| z_n - c_n - K \sum_{i=1}^{10} [x_n^{(i)}, y_n^{(i)}] \right\|_{2,\tau} \leq \gamma_n$$

and $\|c_n\| \leq \delta$. Assembling these into $c := (c_n)_{n=1}^\infty$, $x^{(i)} := (x_n^{(i)})_{n=1}^\infty$ and $y^{(i)} := (y_n^{(i)})_{n=1}^\infty$ in B^∞ provides the elements demanded in the first paragraph of the proof. $\qquad \square$

We can now give our more general version of the existence aspect of Theorem A. The condition that A be nuclear is weakened to the condition that the range of α is contained in the set $T_{\mathrm{am}}(A)$ of amenable traces on A. The

second part of the following theorem, regarding the form of a representative sequence for ϕ, is not needed for Theorem A, but we anticipate it playing a role in future nuclear dimension computations. Recall that a c.p. map $\phi : A \to B$ is said to be *order zero* if $\phi(a)\phi(b) = 0$ for all $a, b \in A_+$ with $ab = 0$; see [24] for the structure theory of these maps.

Theorem 2.6. *Let A be a separable C^*-algebra, and let B be a separable C^*-algebra with $T(B)$ compact and non-empty which has CPoU and no finite-dimensional representations. Given a continuous affine function $\alpha : T(B^\infty) \to T_{\mathrm{am}}(A)$, there exists a *-homomorphism $\phi : A \to B^\infty$ such that*

$$\tag{2.38} \tau \circ \phi = \alpha(\tau), \quad \tau \in T(B^\infty).$$

Moreover, ϕ can be represented by a sequence $(\phi_n)_{n=1}^\infty$ of c.p.c. maps $A \to B$ each of which factorises as $\phi_n = \psi_n \circ \theta_n$ for a c.p.c. map $\theta_n : A \to F_n$ with F_n finite-dimensional, and a c.p.c. order zero map $\psi_n : F_n \to B$. The maps θ_n can be taken to be approximately order zero, and if the range of α lies in the quasidiagonal traces on A, the θ_n can be taken to be approximately multiplicative.

Proof. We first note that any c.p.c. order zero map $\phi : A \to B^\infty$ satisfying (2.38) is automatically a *-homomorphism. Indeed, let $(e_n)_{n=1}^\infty$ be an increasing approximate unit for A. Then

$$\tag{2.39} \phi(a_1)\phi(a_2) = \lim_{n \to \infty} \phi(e_n)\phi(a_1 a_2), \quad a_1, a_2 \in A,$$

as a consequence of [24, Cor. 4.1]. It therefore suffices to prove that

$$\lim_{n \to \infty} \phi(e_n) = 1_{B^\infty}$$

in $\| \cdot \|_{2, T_\infty(B)}$. We compute that

$$\tag{2.40} \begin{aligned} \|1_{B^\infty} - \phi(e_n)\|_{2, T_\infty(B)}^2 &\leq \sup_{\tau \in T(B^\infty)} \tau(|1_{B^\infty} - \phi(e_n)|^2) \\ &\leq \sup_{\tau \in T(B^\infty)} \tau(1_{B^\infty} - \phi(e_n)) \\ &\leq \sup_{\tau \in T(B^\infty)} 1 - \tau(\phi(e_n)) \\ &= \sup_{\tau \in T(B^\infty)} (1 - \alpha(\tau)(e_n)) \to 0, \end{aligned}$$

as by Dini's theorem, $\alpha(\tau)(e_n)$ converges to 1 uniformly on $T(B^\infty)$.

Notice that, due to Proposition 2.5, it is enough to establish equation (2.38) for limit traces. Fix $\epsilon > 0$ and finite sets $\mathcal{F} \subset A$ and $\mathcal{G} \subset A_{\mathrm{sa}}$. We will prove that there is a finite-dimensional C^*-algebra F, a c.p.c. map $\theta : A \to F$, and a c.p.c. order zero map $\psi : F \to B^\infty$ such that, for $\phi = \psi \circ \theta$, we have

$$\tag{2.41} \|\theta(a)\theta(b)\| \leq \epsilon \quad \text{for } a, b \in \mathcal{F} \text{ satisfying } ab = 0,$$

$$\tag{2.42} |\tau(\phi(a)) - \alpha(\tau)(a)| \leq \epsilon \quad \text{for } a \in \mathcal{G} \text{ and } \tau \in T_\infty(B).$$

In the special case that $\alpha(T(B^\infty)) \subseteq T_{\mathrm{qd}}(A)$, we will show that we can additionally replace (2.41) by the stronger condition

$$(2.43) \qquad \|\theta(a)\theta(b) - \theta(ab)\| \leq \epsilon, \quad a, b \in \mathcal{F}.$$

Once this is achieved, an application of Kirchberg's ϵ-test (in the form of Lemma 1.1) can be used to obtain the required ϕ (and ψ_n, θ_n) in a very similar fashion to [3, Lem. 7.4]. We set this out for the passage from (2.41) and (2.42) to the required ϕ such that (θ_n) are approximately order zero in the next paragraph. The passage from (2.43) and (2.42) to obtaining a ϕ such that the maps $(\theta_n)_{n=1}^\infty$ are approximately multiplicative is similar (and slightly easier).

For each n, let X_n denote the set of triples (F_n, θ_n, ψ_n), where F_n is a finite-dimensional C^*-algebra F_n, $\theta_n : A \to F_n$ is c.p.c., and $\psi_n : F_n \to B$ is c.p.c. order zero.[9] Fix a countable dense subset $(x_k)_{k=1}^\infty$ of A_{sa}. Noting that the collection of pairs (a, b) in A_+ with $ab = 0$ is a subspace of the separable metric space $A \times A$, we may also fix a countable dense subset $(a_k, b_k)_{k=1}^\infty$ of these orthogonal pairs. Set $X := \prod_{n=1}^\infty X_n$, and define functions $f_n^{(k)} : X_n \to [0, \infty]$ by

$$(2.44) \qquad f_n^{(k)}(F_n, \theta_n, \psi_n) := \max_{j \leq k}\big(\|\theta_n(a_j b_j)\| $$
$$+ \sup_{\tau \in T(B)} |\tau(\psi_n(\theta_n(x_j))) - \alpha(\tau(x_j))|\big)$$

for $(F_n, \theta_n, \psi_n) \in X_n$. Given $\epsilon > 0$ and $k_0 \in \mathbb{N}$, let $\mathcal{F} := \{a_k, b_k : k \leq k_0\}$ and $\mathcal{G} := \{x_k : k \leq k_0\}$, and take F, θ, ψ satisfying (2.41) and (2.42). Then ψ lifts to a sequence $(\tilde{\psi}_n)_{n=1}^\infty$ of c.p.c. order zero maps $F \to B$ by projectivity of c.p.c. order zero maps with finite-dimensional domains ([23, Prop. 1.2.4], which rephrases Loring's work [16, Thm. 4.9] on projectivity of cones over finite-dimensional C^*-algebras to this setting). The sequence $(F, \theta, \tilde{\psi}_n)_{n=1}^\infty$ in $\prod_{n=1}^\infty X_n$ satisfies

$$(2.45) \qquad \limsup_n f_n^{(k)}(F, \theta, \tilde{\psi}_n) \leq 2\epsilon, \quad k = 1, \ldots, k_0.$$

Applying the ϵ-test gives a sequence (F_n, θ_n, ψ_n) in $\prod_{n=1}^\infty X_n$ with

$$(2.46) \qquad \limsup_n f_n^{(k)}(F_n, \theta_n, \psi_n) = 0, \quad k \in \mathbb{N}.$$

Defining $\phi_n := \psi_n \circ \theta_n$ and $\phi : A \to B^\infty$ to be the map induced by $(\phi_n)_{n=1}^\infty$ gives the required ϕ.

With the ϵ-test in place, we now commence the construction of maps satisfying (2.41) and (2.42). By Proposition 1.2, for each $a \in \mathcal{G}$, we may choose a self-adjoint element $c_a \in B^\infty$ such that

$$(2.47) \qquad \tau(c_a) = \alpha(\tau)(a), \quad \tau \in T(B^\infty).$$

Also, set

$$(2.48) \qquad \delta := \frac{\epsilon}{\sqrt{|\mathcal{G}|}}.$$

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[9]Although all the X_n represent the same set, we use the subscript n for direct comparison with the notation of the ϵ-test.

We work with the weak*-closure $\overline{T_\infty(B)}$ of $T_\infty(B)$ which is weak*-compact. Fix $\tau \in \overline{T_\infty(B)}$ for the moment. Consider the finite von Neumann algebra $\mathcal{M}_\tau := \pi_\tau(B^\infty)''$.

We claim that $\mathcal{M}_\tau$ is type II$_1$. As $T(B)$ is compact, any approximate unit $(e_n)_{n=1}^\infty$ for B satisfies $\inf_{\rho \in T(B)} \rho(e_n) \to 1$ by Dini's theorem. Then

$$(2.49) \qquad \inf_{\rho \in T_\infty(B)} \rho(e_n) = \inf_{\rho \in \overline{T_\infty(B)}} \rho(e_n) \to 1,$$

and hence $\tau(e_n) \to 1$. Thus, $\|1_{\mathcal{M}_\tau} - \pi_\tau(e_n)\|_{2,\tau} \to 0$, and $\pi_\tau(e_n)$ converges *-strongly to $1_{\mathcal{M}_\tau}$. Suppose that $\mathcal{M}_\tau$ has a non-zero type I$_k$ summand for some $k \in \mathbb{N}$, with corresponding central projection p, so that $p\mathcal{M}_\tau$ has a separating family of finite-dimensional representations. Since B has no non-zero finite-dimensional representations, we must have $p\pi_\tau(B) = 0$. On the other hand, $p\pi_\tau(e_n) \neq 0$ for sufficiently large n. This contradiction proves the claim.

Let $T(\pi_\tau) : T(\mathcal{M}_\tau) \to T(B^\infty)$ be the map induced by π_τ. By Lemma 2.4 (applied to the map $\alpha \circ T(\pi_\tau) : T(\mathcal{M}_\tau) \to T_{\mathrm{am}}(A)$), there exist a finite-dimensional C^*-algebra F_τ, a c.p.c. map $\theta_\tau : A \to F_\tau$, and a unital *-homomorphism $\eta_\tau : F_\tau \to \mathcal{M}_\tau$ as well as $x_1^{(a)}, \ldots, x_{10}^{(a)}, y_1^{(a)}, \ldots, y_{10}^{(a)} \in \mathcal{M}_\tau$ for $a \in \mathcal{G}$ such that

$$(2.50) \qquad \|\theta_\tau(a)\theta_\tau(b)\| < \epsilon \quad \text{for } a, b \in \mathcal{F} \text{ satisfying } ab = 0$$

and

$$(2.51) \qquad \left\| \eta_\tau \circ \theta_\tau(a) - \pi_\tau(c_a) - \sum_{i=1}^{10} [x_i^{(a)}, y_i^{(a)}] \right\| < \delta \quad \text{for } a \in \mathcal{G}.$$

By the Kaplansky density theorem, at the cost of replacing the norm estimate in (2.51) by a $\|\cdot\|_{2,\tau}$-estimate, we may assume that the elements $x_i^{(a)}, y_i^{(a)}$ belong to $\pi_\tau(B^\infty)$, and thus lift to elements $x_i^{(\tau,a)}, y_i^{(\tau,a)} \in B^\infty$. Using the order zero Kaplansky density theorem [14, Lem. 1.1], we may approximate η_τ in $\|\cdot\|_{2,\tau}$ by an order zero map $F_\tau \to \pi_\tau(B^\infty)$, and then this can be lifted to an order zero map $\psi_\tau : F_\tau \to B^\infty$ by [23, Prop. 1.2.4]. Starting from (2.51), we can perform these approximations and lifts so that

$$(2.52) \qquad \left\| \psi_\tau(\theta_\tau(a)) - c_a - \sum_{i=1}^{10} [x_i^{(a,\tau)}, y_i^{(a,\tau)}] \right\|_{2,\tau} < \delta, \quad a \in \mathcal{G}.$$

Let us now set

$$(2.53) \qquad s_\tau := \sum_{a \in \mathcal{G}} \left| \psi_\tau(\theta_\tau(a)) - c_a - \sum_{i=1}^{10} [x_i^{(a,\tau)}, y_i^{(a,\tau)}] \right|^2 \in (B^\infty)_+$$

so that, by (2.52), we get $\tau(s_\tau) < |\mathcal{G}|\delta^2 = \epsilon^2$.

By continuity and compactness of $\overline{T_\infty(B)}$, there exist $\tau_1, \ldots, \tau_k \in \overline{T_\infty(B)}$ such that, for every $\tau \in \overline{T_\infty(B)}$,

$$(2.54) \qquad \min\{\tau(s_{\tau_1}), \ldots, \tau(s_{\tau_k})\} < \epsilon^2.$$

Set

(2.55)
$$S := \psi_{\tau_1}(F_{\tau_1}) \cup \cdots \cup \psi_{\tau_k}(F_{\tau_k}) \cup \{c_a : a \in \mathcal{G}\}$$
$$\cup \{x_j^{(a,\tau_i)}, y_j^{(a,\tau_i)} \mid a \in \mathcal{G}, \ i = 1, \ldots, k, \ j = 1, \ldots, 10\},$$

a separable subset of B^∞. Using CPoU as in Lemma 1.3, there exist orthogonal projections $e_1, \ldots, e_k \in B^\infty \cap S'$ which sum to 1_{B^∞} such that

(2.56)
$$\tau(s_{\tau_i} e_i) \le \epsilon^2 \tau(e_i), \quad \tau \in T_\infty(B).$$

Set $F := \bigoplus_{i=1}^k F_{\tau_i}$, and define $\theta : A \to F$ and $\psi : F \to B^\infty$ by

(2.57)
$$\theta(a) := (\theta_{\tau_1}(a), \ldots, \theta_{\tau_k}(a)), \quad a \in A,$$

and

(2.58)
$$\psi(x_1, \ldots, x_k) := \sum_{i=1}^k e_i \psi_{\tau_i}(x_i), \quad (x_1, \ldots, x_k) \in F.$$

Then (2.41) is an immediate consequence of (2.50). Since the e_i are orthogonal positive elements commuting with the images of the c.p. order zero maps ψ_{τ_i}, it follows that ψ is c.p. and order zero. Moreover, ψ is contractive since $\psi(1_A) \le \sum_{i=1}^k e_i = 1_{B^\infty}$.

Finally, for $a \in \mathcal{G}$ and $\tau \in T_\infty(B)$, writing $\phi = \psi \circ \theta$, we compute

(2.59)
$$|\tau(\phi(a)) - \alpha(\tau)(a)|^2$$
$$\overset{(2.47)}{=} |\tau(\psi(\theta(a)) - c_a)|^2$$
$$= \left| \tau \left(\sum_{i=1}^k e_i \left(\psi_{\tau_i}(\theta_{\tau_i}(a)) - c_a - \sum_{j=1}^{10} [x_j^{(a,\tau_i)}, y_j^{(a,\tau_i)}] \right) \right) \right|^2$$
$$\le \tau \left(\sum_{i=1}^k e_i \left| \psi_{\tau_i}(\theta_{\tau_i}(a)) - c_a - \sum_{j=1}^{10} [x_j^{(a,\tau_i)}, y_j^{(a,\tau_i)}] \right|^2 \right)$$
$$\le \tau \left(\sum_{i=1}^k e_i s_{\tau_i} \right) \le \sum_{i=1}^k \epsilon^2 \tau(e_i) = \epsilon^2,$$

where on the third line, we use the fact that the e_i make up a pairwise orthogonal partition of unity of projections commuting with S. This proves (2.42) and completes the proof with θ_n approximately order zero.

When $\alpha(T_\infty(B))$ consists of quasidiagonal traces, (2.50) can be replaced by an $(\mathcal{F}, \epsilon)$-approximate multiplicativity condition using the last clause of Lemma 2.4. The map θ, as defined in (2.57), will then satisfy (2.43), and this completes the proof with θ_n approximately multiplicative. □

We end by recording how Theorem A follows as special cases of the existence and uniqueness results of this section.

Proof of Theorem A. This is a consequence of Theorems 2.2 and 2.6. The CPoU hypothesis on B needed in both these theorems is automatic for separable nuclear $\mathcal{Z}$-stable C^*-algebras by Theorem 1.4; moreover, $\mathcal{Z}$-stability is an obstruction to having finite-dimensional representations. The hypothesis in Theorem 2.6 that α takes values in the amenable traces on A is automatic as all traces on a nuclear C^*-algebra are amenable, essentially by Connes' theorem.[10] $\qquad\square$

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