

On R -triviality of F_4 , II

Maneesh Thakur

(Communicated by Linus Kramer)

Abstract. Simple algebraic groups of type F_4 defined over a field k are the full automorphism groups of Albert algebras over k . Let A be an Albert algebra over a field k of arbitrary characteristic whose isotopes are all isomorphic. We prove that $\mathbf{Aut}(A)$ is R -trivial, in the sense of Manin. If k contains cube roots of unity and A is any Albert algebra over k , we prove that there is an isotope $A^{(v)}$ of A such that $\mathbf{Aut}(A^{(v)})$ is R -trivial.

1. INTRODUCTION

This work builds on and subsumes the results from [24]. We recall at this stage that simple algebraic groups of type F_4 defined over a field k are precisely the full groups of automorphisms of Albert algebras defined over k . In papers [24, 1], the authors proved (independently) that the automorphism group of an Albert algebra arising from the first Tits construction is R -trivial, the proof in [24] being characteristic free.

In this paper, we prove that, for an Albert algebra A over a field k of arbitrary characteristic whose isotopes are all isomorphic, the algebraic group $\mathbf{Aut}(A)$ is R -trivial. Since, for first Tits construction Albert algebras, all isotopes are isomorphic (see [15]), results in this paper improve the result proved in [1, 24].

We remark here that the class of Albert algebras whose isotopes are all isomorphic strictly contains the class of Albert algebras that are first Tits constructions (see [15]). The case of reduced Albert algebras which have no nonzero nilpotents and Albert division algebras in general is work in progress.

The knowledge that a group is R -trivial helps in studying its rationality properties. Recently, in the papers [25, 2], R -triviality of the structure group of Albert division algebras was proved, thereby proving the long-standing Tits–Weiss conjecture for Albert division algebras as well as the Kneser–Tits conjecture for groups of type $E_{8,2}^{78}$ and $E_{7,1}^{78}$. These groups have the structure group

This work was partially funded by the Deutsche Forschungsgemeinschaft under Germany’s Excellence Strategy EXC 2044-390685587, Mathematics Münster: Dynamics-Geometry-Structure.

of an Albert division algebra as their anisotropic kernel. In [22], we had proved R -triviality for $\mathbf{Aut}(A)$ for pure first Tits construction Albert algebras, while in [23], R -triviality of the structure group was proved to settle the Tits–Weiss conjecture for first Tits construction Albert division algebras, as well as reduced Albert algebras. In the paper [11], a proof of Kneser–Tits conjecture for groups of type $E_{8,2}^{66}$ is given by proving R -triviality of such groups. The fundamental result of P. Gille [4] connects the group of R -equivalent points of an isotropic, simple, simply connected algebraic group with its Whitehead group, thereby reducing the Kneser–Tits problem to proving R -triviality of the group or otherwise.

In this paper, we prove that, for an Albert algebra A over a field k of arbitrary characteristic whose isotopes are all isomorphic, the group $\mathbf{Aut}(A)$ is R -trivial. We get the R -triviality of the group of automorphisms of first Tits construction Albert algebras as a consequence. If k contains cube roots of unity and A is any Albert algebra over k , we prove that there exists $v \in A$ such that $\mathbf{Aut}(A^{(v)})$ is R -trivial.

2. PRELIMINARIES

We refer the reader to [13, 20, 7] for basic material on Albert algebras. In this section, we quickly introduce some notions related to Albert algebras that are indispensable. All base fields will be assumed to be infinite of arbitrary characteristic unless specified otherwise.

For what follows, we refer to [18, 17]. Let J be a finite-dimensional vector space over a field k . A *cubic norm structure* on J is a triple $(N, \#, c)$, $N : J \rightarrow k$ is a cubic form called the *norm*, $\# : J \rightarrow J$ is a quadratic map called the *adjoint* and $c \in J$ is a base point, called the *identity element* of the norm structure. There is an associated trace bilinear form $T : J \times J \rightarrow k$, and these satisfy some identities (see [18, 17]).

Let $x \in J$. Define

$$U_x(y) := T(x, y)x - x^\# \times y, \quad y \in J,$$

where $a \times b := (a + b)^\# - a^\# - b^\#$, $a, b \in J$. Then, with $1_J := c$ and the endomorphisms U_x as U -operators, J is a unital *quadratic Jordan algebra* (see [9]), denoted by $J(N, c)$. An element $x \in J$ is defined to be *invertible* if $N(x) \neq 0$ and $x^{-1} := N(x)^{-1}x^\#$. The structure $J(N, c)$ is a *division algebra* if U_x is surjective for all $x \neq 0$, or equivalently, $N(x) \neq 0$ for all $x \neq 0$. Special Jordan algebras of degree 3 provide important class of examples; we list them below for our purpose.

Example. Let D be a separable associative algebra over k of degree 3. Let N_D denote its norm and T_D the trace. Let $\# : D \rightarrow D$ be the adjoint map. Then $(N_D, \#, 1_D)$ is a cubic norm structure, where 1_D is the unit element of D . This yields a quadratic Jordan algebra structure on D , which we will denote by D_+ .

Let (B, σ) be a separable associative algebra over k with an involution σ of the second kind (over its center). With the unit element 1 of B and the restriction of the norm N_B of B to $(B, \sigma)_+ := \{b \in B \mid \sigma(b) = b\}$, we obtain a cubic norm structure and hence a Jordan algebra structure on $(B, \sigma)_+$ which is a substructure of B_+ .

Tits processes and Tits constructions. Let D be a finite-dimensional associative k -algebra of degree 3 with norm N_D and trace T_D . Let $\lambda \in k^\times$. On the k -vector space $D \oplus D \oplus D$, we define a cubic norm structure as below:

$$\begin{aligned} 1 &:= (1, 0, 0), \quad N((x, y, z)) := N_D(x) + \lambda N_D(y) + \lambda^{-1} N_D(z) - T_D(xyz), \\ (x, y, z)^\# &:= (x^\# - yz, \lambda^{-1} z^\# - xy, \lambda y^\# - zx). \end{aligned}$$

The Jordan algebra associated to this norm structure is denoted by $J(D, \lambda)$. The algebra D_+ is a subalgebra of $J(D, \lambda)$, corresponding to the first summand. The algebra $J(D, \lambda)$ is a division algebra if and only if $\lambda \notin N_D(D)$ (see [18, § 5.2]). This construction is called the *first Tits process* arising from the parameters D and λ .

Let K be a quadratic étale extension of k and B a separable associative algebra of degree 3 over K with an involution σ over K/k . Let $x \mapsto \bar{x}$ denote the nontrivial k -automorphism of K . For an *admissible pair* (u, μ) , i.e., $u \in (B, \sigma)_+$ such that $N_B(u) = \mu \bar{\mu}$ for some $\mu \in K^\times$, define a cubic norm structure on the k -vector space $(B, \sigma)_+ \oplus B$ as follows:

$$\begin{aligned} N((b, x)) &:= N_B(b) + T_K(\mu N_B(x)) - T_B(bxu\sigma(x)), \\ (b, x)^\# &:= (b^\# - xu\sigma(x), \bar{\mu}\sigma(x)^\#u^{-1} - bx), \quad 1 := (1_B, 0). \end{aligned}$$

The Jordan algebra obtained from this cubic norm structure is denoted by $J(B, \sigma, u, \mu)$. Note that $(B, \sigma)_+$ is a subalgebra of $J(B, \sigma, u, \mu)$ through the first summand. The algebra $J(B, \sigma, u, \mu)$ is a division algebra if and only if μ is not a norm from B (see [18, § 5.2]). This construction is called the *second Tits process* arising from the parameters (B, σ) , u and μ .

The Tits process starting with a central simple algebra D and $\lambda \in k^\times$ yields the *first Tits construction* Albert algebra $A = J(D, \lambda)$ over k . Similarly, in the Tits process, if we start with a central simple algebra (B, σ) with center a quadratic étale algebra K over k and an involution σ of the second kind, u, μ as described above, we get the *second Tits construction* Albert algebra $A = J(B, \sigma, u, \mu)$ over k . One knows that all Albert algebras can be obtained via Tits constructions.

An Albert algebra is a division algebra if and only if its (cubic) norm N is anisotropic over k (see [7, § 39]). If $A = J(B, \sigma, u, \mu)$ as above, then $A \otimes_k K \cong J(B, \mu)$ as K -algebras, where K is the center of B (see [7, § 39.4]).

Let A be an Albert algebra over k . If A arises from the first construction, but does not arise from the second construction, then we call A a *pure first construction* Albert algebra. Similarly, *pure second construction* Albert algebras are defined as those which do not arise from the first Tits construction.

For an Albert division algebra A , any subalgebra is either k or a cubic subfield of A or of the form $(B, \sigma)_+$ for a degree 3 central simple algebra B with an involution σ of the second kind over its center K , a quadratic étale extension of k (see [5, Chap. IX, § 12, Lem. 2], [17]).

Norm similarities, isotopes. Let J be a quadratic Jordan algebra of degree 3 over k and N its norm map. By a *norm similarity* of J , we mean a bijective k -linear map $f : J \rightarrow J$ such that $N(f(x)) = \nu(f)N(x)$ for all $x \in J$ and some $\nu(f) \in k^\times$. When k is infinite, the notions of norm similarity and isotopy for degree 3 Jordan algebras coincide (see [5, Chap. VI, Thm. 6, Thm. 7]). Let $v \in J$ be invertible. The k -vector space J with the new identity element $1^{(v)} := v^{-1}$ and the U -operator defined by $U_x^{(v)} := U_x U_p$ is called the v -isotope of J and is denoted by $J^{(v)}$. For a second Tits process $J(B, \sigma, u, \mu)$ as above and $v \in (B, \sigma)_+$ invertible, we have $J(B, \sigma, u, \mu)^{(v)} \cong J(B, \sigma_v, uv^\#, N(v)\mu)$ (see [13, § 3.11]).

Let J be a degree 3 Jordan algebra over k with norm map N . For $a \in J$, the U -operator U_a is given by $U_a(y) := T(a, y)a - a^\# \times y$, $y \in J$. When $a \in J$ is invertible, one knows that U_a is a norm similarity of J ; in fact, for any $x \in A$, $N(U_a(x)) = N(a)^2 N(x)$. It also follows that $N(x^\#) = N(x)^2$.

Albert algebras and algebraic groups. For a k -algebra X and a field extension L of k , X_L will denote the L -algebra $X \otimes_k L$. Let A be an Albert algebra over k with norm N , and let \bar{k} be an algebraic closure of k . It is well-known that the full group of automorphisms $\mathbf{Aut}(A) := \text{Aut}(A_{\bar{k}})$ is a simple algebraic group of type F_4 defined over k , and all simple groups of type F_4 defined over k arise this way. We will denote the group of k -rational points of $\mathbf{Aut}(A)$ by $\text{Aut}(A)$, i.e., $\text{Aut}(A) = \mathbf{Aut}(A)(k)$. It is known that A is a division algebra if and only if the norm form N of A is anisotropic (see [21, Thm. 17.6.5]). Albert algebras whose norm form is isotropic over k , i.e., has a nontrivial zero over k , are called *reduced*.

The *structure group* of A is the full group $\mathbf{Str}(A)$ of norm similarities of N , is a connected reductive group over k , of type E_6 . We denote by $\text{Str}(A)$ the group of k -rational points $\mathbf{Str}(A)(k)$. The automorphism group $\mathbf{Aut}(A)$ is the stabilizer of $1 \in A$ in $\mathbf{Str}(A)$.

R -triviality of algebraic groups. Let X be an irreducible variety over a field k with the set of k -rational points $X(k) \neq \emptyset$. We say that $x, y \in X(k)$ are R -equivalent if there exists a sequence $x_0 = x, x_1, \dots, x_n = y$ of points in $X(k)$ and rational maps $f_i : \mathbb{A}_k^1 \rightarrow X$, $1 \leq i \leq n$, defined over k and regular at 0 and 1, such that $f_i(0) = x_{i-1}$, $f_i(1) = x_i$ (see [8]).

Let G be a connected algebraic group defined over k . The set of points in $G(k)$ that are R -equivalent to $1 \in G(k)$ is a normal subgroup of $G(k)$, denoted by $RG(k)$. The set $G(k)/R$ of R -equivalence classes in $G(k)$ is in a canonical bijection with $G(k)/RG(k)$ and thus has a natural group structure. We identify $G(k)/R$ with the group $G(k)/RG(k)$. This group is useful in studying rationality properties of G .

Call G R -trivial if $G(L)/R = \{1\}$ for all field extensions L of k . A variety X defined over k is defined to be k -rational if X is birationally isomorphic over k to an affine space. One knows that if G is k -rational, then G is R -trivial (see [27, Chap. 6, Prop. 2]). Also, owing to the fact that tori of rank at most 2 are rational, one knows that algebraic groups of rank at most 2 are rational.

3. R -TRIVIALITY RESULTS

In this section, we prove results on R -triviality of various groups and also prove our main theorems. We need a few results from [23] which we recall below. We remark here that if A is a *reduced* Albert algebra, then there is an isotope $A^{(v)}$ of A such that $A^{(v)}$ contains nonzero nilpotents. Hence $\mathbf{Aut}(A^{(v)})$ is a simple abstract group [6, Thm. 14]. Since $R\mathbf{Aut}(A^{(v)})(k)$ is a normal subgroup of $\mathbf{Aut}(A^{(v)})(k)$ and $\mathbf{Aut}(A^{(v)})$ contains k -rational subgroups, it follows that $\mathbf{Aut}(A^{(v)})$ is R -trivial. This therefore proves that, for any reduced Albert algebra A , whose isotopes are all isomorphic, the group $\mathbf{Aut}(A)$ is R -trivial. We record this as the following result.

Proposition 3.1. *Let A be a reduced Albert algebra over a field k of arbitrary characteristic. Then there exists an invertible element $v \in A$ such that the algebraic group $\mathbf{Aut}(A^{(v)})$ is R -trivial.*

Corollary 3.2. *Let A be a reduced Albert algebra over a field k whose isotopes are all isomorphic. Then $\mathbf{Aut}(A)$ is R -trivial.*

Proof. As we discussed above, there exists an isotope $A^{(v)}$ of A which contains a nonzero nilpotent. For such an isotope, for any field extension M of k , the algebra $A^{(v)} \otimes M = (A \otimes M)^{(v \otimes 1)}$ also contains nonzero nilpotents. Since all isotopes of A are isomorphic, it follows that A contains nonzero nilpotents, and hence $A \otimes M$ contains nonzero nilpotents. Since the group of automorphisms of a reduced Albert algebra which contain nonzero nilpotents is an abstract simple group, it follows by the above discussion that $\mathbf{Aut}(A \otimes M) = R\mathbf{Aut}(A)(M)$. Hence $\mathbf{Aut}(A)$ is R -trivial. \square

Hence we may assume for the rest of the paper that A is a division Albert algebra. We need the following result [2, Prop. 3.1].

Proposition 3.3. *Let A be an Albert division algebra. Let $S \subset A$ be a 9-dimensional subalgebra. Then, with the notations as above, $\mathbf{Aut}(A, S)$ is rational over k , hence is R -trivial.*

We need [25, Lem. 6.1]; we include it here for convenience.

Lemma 3.4. *Let L/k be a cyclic cubic extension, K/k a quadratic étale extension and $\nu \in K^\times$ an element with $N_K(\nu) = 1$, and let $*$: $LK \rightarrow LK$ be the nontrivial L -automorphism of LK . Let J denote the Tits process $J = J(LK, *, 1, \nu)$. Let ρ be a generator of $\mathrm{Gal}(L/k)$. Then ρ extends to an automorphism of J .*

We now prove the following theorem.

Theorem 3.5. *Let A be a cyclic Albert division algebra and $L \subset A$ a cyclic cubic subfield. Then there exists $v \in L^\times$ such that $\text{Aut}(A^{(v)}/L) \subset R\mathbf{Aut}(A^{(v)})(k)$.*

Proof. Let $S \subset A$ be a 9-dimensional subalgebra such that $L \subset S$. By [12, § 6.4], we may assume $S = J(LK, *, v, \nu)$ for $v \in L^\times$ and $\nu \in K^\times$ with $N_L(v) = N_K(\nu) = 1$. We have

$$S^{(v)} = J(LK, *, v\nu^\#, N(v)\nu) = J(LK, *, 1, \nu).$$

Let ρ be a generator of $\text{Gal}(L/k)$.

By Lemma 3.4, ρ extends to an automorphism $\tilde{\rho}$ of $S^{(v)}$. By [23, Prop. 3.2], $\tilde{\rho}$ extends to an automorphism of $A^{(v)}$, which we denote, abusing notation, also by $\tilde{\rho}$. By Proposition 3.3, $\tilde{\rho} \in R\mathbf{Aut}(A^{(v)}, S^{(v)}) \subset R\mathbf{Aut}(A^{(v)})(k)$. Now let $\phi \in \text{Aut}(A^{(v)}/L)$. Then $\tilde{\rho}^{-1}\phi \notin \text{Aut}(A^{(v)}/L)$. By [23, Thm. 4.1], $\tilde{\rho}^{-1}\phi$ fixes a cubic subfield $M \subset A^{(v)}$ pointwise and $M \neq L$. The subalgebra $S' = \langle L, M \rangle$ of $A^{(v)}$ generated by L and M is 9-dimensional and is invariant under $\psi := \tilde{\rho}^{-1}\phi$. Hence, by Proposition 3.3, we have

$$\psi \in \text{Aut}(A^{(v)}, S') = R\mathbf{Aut}(A^{(v)}, S')(k) \subset R\mathbf{Aut}(A^{(v)})(k).$$

Since $\tilde{\rho} \in R\mathbf{Aut}(A^{(v)})(k)$, it follows that $\phi \in R\mathbf{Aut}(A^{(v)})(k)$. This proves that $\text{Aut}(A^{(v)}/L) \subset R\mathbf{Aut}(A^{(v)})(k)$. \square

Corollary 3.6. *Let A be a cyclic division algebra over a field k and $L \subset A$ a cyclic cubic subfield. Then there exists $v \in L^\times$ such that $\text{Aut}(A^{(v)}, L) \subset R\mathbf{Aut}(A^{(v)})$.*

Proof. Let $v \in L^\times$ be such that

$$\text{Aut}(A^{(v)}/L) \subset R\mathbf{Aut}(A^{(v)})(k).$$

Let $\phi \in \text{Aut}(A^{(v)}, L)$. We may assume that $\phi|_L = \rho \neq 1$. Then, with $\tilde{\rho}$ as above, it follows that $\tilde{\rho}^{-1}\phi$ fixes L pointwise, hence belongs to $\text{Aut}(A^{(v)}/L)$ which is contained in $R\mathbf{Aut}(A^{(v)})(k)$. Also, $\tilde{\rho} \in R\mathbf{Aut}(A^{(v)})$ by Proposition 3.3. Hence it follows that $\phi \in R\mathbf{Aut}(A^{(v)})(k)$. \square

We need the following result from [25] for our purpose here; we reproduce the proof for the sake of completeness.

Proposition 3.7 ([25, Thm. 6.4]). *Let A be an Albert division algebra over a field k of arbitrary characteristic. Let $M \subset A$ be a cubic subfield, and let \mathcal{S}_M denote the set of all 9-dimensional subalgebras of A that contain M . Then the group $\text{Str}(A, M)$ acts on \mathcal{S}_M .*

Proof. Let $\psi \in \text{Str}(A, M)$ and $S \in \mathcal{S}_M$ be arbitrary. Let $\phi \in \text{Aut}(A/S)$, $\phi \neq 1$. Then $A^\phi = S$. We have

$$\psi\phi\psi^{-1}(1) = \psi(\psi^{-1}(1)) = 1$$

since $\psi^{-1}(1) \in M \subset S$ and ϕ fixes S pointwise. Hence $\psi\phi\psi^{-1} \in \text{Aut}(A)$. It follows also that the fixed point subspace of $\psi\phi\psi^{-1}$ is precisely $\psi(S)$, and hence $\psi(S)$ is a subalgebra of A of dimension 9 and $M = \psi(M) \subset \psi(S)$; therefore $\psi(S) \in \mathcal{S}_M$. This completes the proof. \square

Proposition 3.8. *Let A be an Albert division algebra over a field k and $L \subset A$ be a cubic subfield. Let $\phi \in \text{Aut}(A)$. Then there exists $S \in \mathcal{S}_L$ and $\theta \in \text{Str}(A, S)$ such that $\theta\phi \in \text{Str}(A, L)$.*

Proof. Consider the cubic subfield $M := \phi(L)$ of A . If $M = L$, then we have $\phi \in \text{Aut}(A, L) \subset \text{Str}(A, L)$, and we are done by taking $\theta = 1$ and S as any 9-dimensional subalgebra with $S \in \mathcal{S}_L$. So assume that $M \neq L$. Then $S := \langle L, M \rangle$, the subalgebra of A generated by L and M is necessarily 9-dimensional, and applying [3, Thm. 5.2.7] to the embeddings $\phi : L \rightarrow S$ and $\iota : L \rightarrow S$, there exists $\theta \in \text{Str}(A, S)$ and $w \in L^\times$ such that $\theta\phi(x) = wx$ for all $x \in L^\times$. In particular, $\theta\phi(L) = L$. Thus $\theta\phi \in \text{Str}(A, L)$. \square

Proposition 3.9. *Let A be a cyclic Albert division algebra over a field k , and let $L \subset A$ be a cyclic cubic subfield. Let $S \in \mathcal{S}_L$ and $\psi \in \text{Str}(A, L)$. Assume that $S \cong \psi(S)$. Then there exists $\phi \in \text{Aut}(A, L)$ such that $\phi^{-1}\psi \in \text{Str}(A, S)$.*

Proof. We have $S = (B, \sigma)_+$ for a degree 3 central division algebra B over a quadratic étale extension K/k and σ a unitary involution on B . Since $\psi \in \text{Str}(A, L)$ and $S \cong \psi(S)$, it follows that $\psi(S)$ is isomorphic to an isotope of S by an element of L^\times . We may assume, by [18, Prop. 3.8], $\psi((B, \sigma)_+) \cong (B, \sigma_v)_+$ for some $v \in L$ with $N_L(v) = 1$ since $\psi(L) = L$. Hence we have $(B, \sigma)_+ \cong (B, \sigma_v)_+$, and in particular, the involutions σ and σ_v on B are conjugate. By [7, Cor. 19.31], there exist $w \in (LK)^\times$ and $\lambda \in k^\times$ such that $v = \lambda w w^*$, where $*$ is the nontrivial k -automorphism of K/k , extended L -linearly to an automorphism of LK .

The inclusion $L \subset (B, \sigma)_+$ extends to an isomorphism $\delta : J(LK, *, u, \mu) \rightarrow (B, \sigma)_+$ for a suitable admissible pair $(u, \mu) \in L^\times \times K^\times$ (see [19, §1.6]). Moreover, $L \subset (B, \sigma_v)_+$; hence there is an isomorphism

$$\eta : J(LK, *, u', \mu') \rightarrow \psi(S) = (B, \sigma_v)_+$$

for suitable (u', μ') extending the inclusion $L \hookrightarrow (B, \sigma_v)_+$. Since δ and η are identity on L , it follows that $J(LK, *, u', \mu')$ is the v -isotope of $J(LK, *, u, \mu)$. Hence, by [18, Prop. 3.9], we may assume $(u', \mu') = (uv^\#, N_L(v)\mu)$. We have therefore

$$u' = uv^\# = u(\lambda w w^*)^\# = u\lambda^2 w^\# (w^\#)^* = u w_0 w_0^*, \quad \mu' = N_L(v)\mu,$$

where $w_0 = \lambda w^\#$. We have an isomorphism (see [18, Prop. 3.7])

$$\phi_1 : J(LK, *, uv^\#, N_L(v)\mu) = J(LK, *, u w_0 w_0^*, \mu) \rightarrow J(LK, *, u, N_{LK}(w_0)^{-1}\mu).$$

We have

$$\begin{aligned} N_{LK}(w_0) \overline{N_{LK}(w_0)} &= N_{LK}(\lambda w^\#) \overline{N_{LK}(\lambda w^\#)} = (\lambda^3 N_{LK}(w)^2) \lambda^3 \overline{N_{LK}(w)^2} \\ &= (\lambda^3 N_{LK}(w) \overline{N_{LK}(w)})^2 = N_L(v)^2 = 1. \end{aligned}$$

By [19, Lem. 4.5], we have $w_1 \in LK$ with $w_1 w_1^* = 1$ and $N_{LK}(w_1) = N_{LK}(w_0)$. The map

$$\phi_2 : J(LK, *, u, N_{LK}(w_0)^{-1}\mu) \rightarrow J(LK, *, u, \mu)$$

given by $\phi_2((l, x)) = (l, x w_1^{-1})$ is an isomorphism by [18, Prop. 3.7]. Hence the composite $\phi_1^{-1} \phi_2^{-1} : J(LK, *, u, \mu) \rightarrow J(LK, *, u', \mu')$ is an isomorphism

which maps L to L . We have therefore the isomorphism $\phi := \eta(\phi_1^{-1}\phi_2^{-1})\delta^{-1} : S \rightarrow \psi(S)$, which satisfies $\phi(L) = L$ and $\phi^{-1}\psi(S) = S$, where ϕ denotes any extension of ϕ as above, to an automorphism of A , which is possible by [10, 12]. This completes the proof. \square

We need the following intermediate result.

Proposition 3.10. *Let A be a cyclic Albert division algebra over a field k of arbitrary characteristic with $f_5(A) = 0$. Let $L \subset A$ be a cubic cyclic subfield, and assume A contains a second Tits process $J(LK, *, 1, \nu)$ as a subalgebra. Then $\mathbf{Aut}(A)$ is R -trivial.*

Proof. Let ρ be a generator of $\text{Gal}(L/k)$, and let $\tilde{\rho}$ denote the extension of ρ to an automorphism of $J(LK, *, 1, \nu)$ and its further extension to an automorphism of A as well. By Proposition 3.3, $\tilde{\rho} \in R\mathbf{Aut}(A)(k)$.

Now let $\phi \in \mathbf{Aut}(A)$. First assume ϕ fixes L pointwise. Consider the automorphism $\psi := \tilde{\rho}\phi \in \mathbf{Aut}(A)$. Clearly, we have $\psi(L) = L$ and $\psi \notin \mathbf{Aut}(A/L)$. By [23, Thm. 4.1], there is a cubic subfield $M \subset A$ such that ψ fixes M pointwise. Then $M \neq L$, and the subalgebra $S := \langle L, M \rangle$ of A generated by L and M is 9-dimensional. Clearly, $\psi(S) = S$. Hence, by Proposition 3.3, $\psi = \tilde{\rho}\phi \in R\mathbf{Aut}(A)(k)$. Since $\tilde{\rho} \in R\mathbf{Aut}(A)(k)$, we have $\phi \in R\mathbf{Aut}(A)(k)$.

Now assume $\phi \in \mathbf{Aut}(A, L)$ and $\phi \notin \mathbf{Aut}(A/L)$. Without loss of generality, we assume $\phi|_L = \rho$. Then $\tilde{\rho}^{-1}\phi \in \mathbf{Aut}(A/L) \subset R\mathbf{Aut}(A)(k)$. It follows that $\phi \in R\mathbf{Aut}(A)(k)$. Finally, assume $M := \phi(L) \neq L$. By Proposition 3.8, there exist $S \in \mathcal{S}_L$ and $\theta \in \text{Str}(A, S)$ such that $\theta\phi \in \text{Str}(A, L)$. Hence $\phi^{-1}\theta^{-1}(L) = L$. Let $\gamma := \phi^{-1}\theta^{-1}$. Then $\gamma \in \text{Str}(A, L)$, and since $\theta(S) = S$, we have $\gamma(S) = \phi^{-1}(S) \cong S$. By Proposition 3.9, there exists $\eta \in \mathbf{Aut}(A, L)$ such that $\eta^{-1}\gamma \in \text{Str}(A, S)$. We therefore have

$$S = \eta^{-1}\gamma(S) = \eta^{-1}\phi^{-1}\theta^{-1}(S) = \eta^{-1}\phi^{-1}(S).$$

Hence $\eta^{-1}\phi^{-1} \in \mathbf{Aut}(A, S)$. Since, by Proposition 3.3,

$$\mathbf{Aut}(A, S) \subset R\mathbf{Aut}(A)(k) \quad \text{and} \quad \eta \in \mathbf{Aut}(A, L) \subset R\mathbf{Aut}(A)(k),$$

it follows that $\phi \in \mathbf{Aut}(A)(k)$.

Finally, if N is any field extension of k , we have $\mathbf{Aut}(A)(N) = \mathbf{Aut}(A \otimes N)$. If $A \otimes N$ is a division algebra, by considering the cyclic subfield $L \otimes N \subset A \otimes N$ and the Tits process subalgebra $J(L_N K, *, 1, \nu) \subset A \otimes N$, by exactly the same arguments as above, we get $\mathbf{Aut}(A_N) \subset R\mathbf{Aut}(A)(N)$. On the other hand, when A_N is reduced, the hypothesis $f_5(A) = 0$ implies A_N has nonzero nilpotents, and hence the abstract group $\mathbf{Aut}(A_N)$ is simple [6, Thm. 14]. Since $\mathbf{Aut}(A)$ has rank 2-subgroups defined over k , it follows $R\mathbf{Aut}(A)(N) \neq \{1\}$. Hence simplicity implies $\mathbf{Aut}(A_N) = R\mathbf{Aut}(A)(N)$. This completes the proof that $\mathbf{Aut}(A)$ is R -trivial. \square

We now proceed to prove our main results.

Theorem 3.11. *Let A be an Albert division algebra over a field k of arbitrary characteristic whose isotopes are all isomorphic. Then the algebraic group $\mathbf{Aut}(A)$ is R -trivial.*

Proof. By [26], since all isotopes of A are isomorphic to A , it follows that A contains a cyclic cubic subfield L . Also, if A_{red} denotes the reduced model of A over k (see [16]), then the hypothesis on A implies A_{red} contains nonzero nilpotents and $f_5(A) = 0$. By Corollary 3.6, there is $v \in L^\times$ such that $\text{Aut}(A^{(v)}, L) \subset R\mathbf{Aut}(A^{(v)})$. Hence $\text{Aut}(A, L) \subset R\mathbf{Aut}(A)(k)$. Let $S \subset A$ be a 9-dimensional subalgebra containing L . We may write $S = (B, \sigma)_+$ for a suitable degree 3 central division algebra B with a unitary involution over $K := Z(B)$. By [12, § 6.4], we may assume $S = J(LK, *, w, \nu)$ for a suitable admissible pair (w, ν) with $N_L(w) = N_K(\nu) = 1$. By hypothesis, $A^{(w)} \cong A$ and $A^{(w)} \supset S^{(w)}$. Also, $S^{(w)} = J(LK, *, w, \nu)^{(w)} \cong J(LK, *, 1, \nu)$. Hence we may assume that A contains $J(LK, *, 1, \nu)$ as a subalgebra. Thus A satisfies the hypothesis of Proposition 3.10, and the result now follows, i.e., $\mathbf{Aut}(A)$ is R -trivial. \square

Corollary 3.12. *Let A be an Albert algebra arising from the first Tits construction. Then $\mathbf{Aut}(A)$ is R -trivial.*

Proof. A is either split, in which case $\mathbf{Aut}(A)$ is split, hence R -trivial, or A is division and all isotopes of A are isomorphic, by [15]. Hence the theorem applies to A and $\mathbf{Aut}(A)$ is R -trivial. \square

When the base field contains cube roots of unity, we can prove a better result.

Theorem 3.13. *Let k be a field of arbitrary characteristic containing cube roots of unity. Let A be an Albert algebra over k . Then there exists $v \in A$ such that the group $\mathbf{Aut}(A^{(v)})$ is R -trivial.*

Proof. Let k and A be as in the hypothesis. When A is reduced, the assertion is proved in Proposition 3.1, without the assumption on k . So assume that A is a division algebra. Then there exists $w \in A$ such that the isotope $A^{(w)}$ is cyclic by [26] or [14]. Let $L \subset A^{(w)}$ be a cubic cyclic subfield. Then there exists a 9-dimensional subalgebra $S \subset A^{(w)}$ with $L \subset S$. We may assume that $S = (B, \sigma)_+$ for a central simple algebra (B, σ) with a unitary involution over its center K , which is a quadratic étale extension of k . By [19, § 1.6] and [12, § 6.4], we can extend the inclusion $L \subset (B, \sigma)_+$ to an isomorphism of the Tits process $J(LK, *, v_1, \nu)$ with $(B, \sigma)_+$ for some $v_1 \in L$ with $N_L(v_1) = 1 = N_K(\nu)$. We may therefore assume that the Tits process $J(LK, *, v_1, \nu) \subset A^{(w)}$. Passing further to the v_1 -isotope of $A_1 := A^{(w)}$, we have $J(LK, *, v_1, \nu)^{(v_1)} \subset A_1^{(v_1)}$, as $v_1 \in L \subset J(LK, *, v_1, \nu)$. We have

$$J(LK, *, v_1, \nu)^{(v_1)} \cong J(LK, *, v_1 v_1^\#, N_L(v_1)\nu) = J(LK, *, 1, \nu),$$

using $N_L(v_1) = v_1 v_1^\# = 1$.

Therefore, $J(LK, *, 1, \nu) \subset A_1^{(v_1)}$. By transitivity of isotopes, $A_1^{(v_1)} = A^{(v)}$ for some $v \in A$. Hence the Tits process $J(LK, *, 1, \nu)$ is a subalgebra of $A^{(v)}$. We next prove that $f_5(A^{(v)}) = 0$.

We may write $J(LK, *, 1, \nu) = (C, \tau)_+$ for a suitable degree 3 central division algebra C with a unitary involution τ over its center. First assume that the characteristic of k is not 3. Then the hypothesis on k allows a Kummer element $a \in L^\times$ with $T_L(a) = T_L(a^2) = 0$. Then $a^3 =: b \in k^\times$. The involution $\tau_a = \text{int}(a)\tau$ on C then is distinguished. We have

$$(C, \tau_a)_+ \cong (C, \tau)_+^{(a)} = J(LK, *, 1, \nu)^{(a)} \cong J(LK, *, a^2, b\nu) \cong J(LK, *, 1, \nu).$$

It follows from this that $f_5(A^{(v)}) = 0$ (see [7, § 40.7]). Now assume the characteristic of k is 3. Then, by [12, §§ 2.9, 2.10], it follows that τ is a distinguished involution on C , and since $(C, \tau)_+ \subset A^{(v)}$, again, we have $f_5(A^{(v)}) = 0$.

We have therefore shown that, under the hypothesis on k , there is $v \in A$ such that $J(LK, *, 1, \nu) \subset A^{(v)}$ and $f_5(A^{(v)}) = 0$. The result now follows from Proposition 3.10. \square

Acknowledgments. Part of this work was done while the author was visiting Linus Kramer of Mathematics Institute, Universität Münster in the summer of 2019. We thank Linus Kramer for his support and hospitality during this visit. The author also thanks Holger Petersson for his interest in this work and some fruitful discussions at Hagen during the above period. We are grateful to the referee for his/her efforts in reading the manuscript carefully and suggesting changes which have resulted in better readability of the paper. We thank V. Nandagopal of the School of Mathematics, TIFR for providing us technical support during the preparation of this manuscript.

REFERENCES

- [1] S. Alsaody, V. Chernousov, and A. Pianzola, R -triviality of groups of type F_4 arising from the first Tits construction, *Bull. Lond. Math. Soc.* **53** (2021), no. 2, 535–547. MR4239194
- [2] S. Alsaody, V. Chernousov, and A. Pianzola (appendix by Richard Weiss), On the Tits–Weiss conjecture and the Kneser–Tits conjecture for $E_{7,1}^{78}$ and $E_{8,2}^{78}$, arXiv:1911.12908v1 [math.RA] (2019).
- [3] S. Garibaldi and H. P. Petersson, Outer automorphisms of algebraic groups and a Skolem–Noether theorem for Albert algebras, *Doc. Math.* **21** (2016), 917–954. MR3548137
- [4] P. Gille, Le problème de Kneser–Tits, *Astérisque* No. 326 (2009), Exp. No. 983, vii, 39–81 (2010). MR2605318
- [5] N. Jacobson, *Structure and representations of Jordan algebras*, Amer. Math. Soc. Colloq. Publ., 39, American Mathematical Society, Providence, RI, 1968. MR0251099
- [6] N. Jacobson, Some groups of transformations defined by Jordan algebras. II. Groups of type F_4 , *J. Reine Angew. Math.* **204** (1960), 74–98. MR0159849
- [7] M. A. Knus, A. Merkurjev, M. Rost, and J. P. Tignol, *The book of involutions*, Amer. Math. Soc. Colloq. Publ., 44, American Mathematical Society, Providence, RI, 1998. MR1632779
- [8] Yu. I. Manin, *Cubic forms*, translated from the Russian by M. Hazewinkel, North-Holland, Amsterdam, 1974.

- [9] K. McCrimmon, The Freudenthal–Springer–Tits constructions of exceptional Jordan algebras, *Trans. Amer. Math. Soc.* **139** (1969), 495–510. MR0238916
- [10] R. Parimala, R. Sridharan, and M. L. Thakur, A classification theorem for Albert algebras, *Trans. Amer. Math. Soc.* **350** (1998), no. 3, 1277–1284. MR1458310
- [11] R. Parimala, J.-P. Tignol and R. M. Weiss, The Kneser–Tits conjecture for groups with Tits-index $E_{8,2}^{66}$ over an arbitrary field, *Transform. Groups* **17** (2012), no. 1, 209–231. MR2891217
- [12] H. P. Petersson, Structure theorems for Jordan algebras of degree three over fields of arbitrary characteristic, *Comm. Algebra* **32** (2004), no. 3, 1019–1049. MR2063796
- [13] H. P. Petersson, A survey on Albert algebras, *Transform. Groups* **24** (2019), no. 1, 219–278. MR3916096
- [14] H. P. Petersson and M. L. Racine, Cubic subfields of exceptional simple Jordan algebras, *Proc. Amer. Math. Soc.* **91** (1984), no. 1, 31–36. MR0735558
- [15] H. P. Petersson and M. L. Racine, Springer forms and the first Tits construction of exceptional Jordan division algebras, *Manuscripta Math.* **45** (1984), no. 3, 249–272. MR0734841
- [16] H. P. Petersson and M. L. Racine, Reduced models of Albert algebras, *Math. Z.* **223** (1996), no. 3, 367–385. MR1417849
- [17] H. P. Petersson and M. L. Racine, Albert algebras, in *Jordan algebras (Oberwolfach, 1992)*, 197–207, de Gruyter, Berlin, 1994. MR1293320
- [18] H. P. Petersson and M. L. Racine, Jordan algebras of degree 3 and the Tits process, *J. Algebra* **98** (1986), no. 1, 211–243. MR0825144
- [19] H. P. Petersson and M. Thakur, The étale Tits process of Jordan algebras revisited, *J. Algebra* **273** (2004), no. 1, 88–107. MR2032452
- [20] T. A. Springer and F. D. Veldkamp, *Octonions, Jordan algebras and exceptional groups*, Springer Monogr. Math., Springer, Berlin, 2000. MR1763974
- [21] T. A. Springer, *Linear algebraic groups*, second edition, *Progr. Math.*, 9, Birkhäuser Boston, Inc., Boston, MA, 1998. MR1642713
- [22] M. Thakur, Automorphisms of Albert algebras and a conjecture of Tits and Weiss, *Trans. Amer. Math. Soc.* **365** (2013), no. 6, 3041–3068. MR3034458
- [23] M. Thakur, Automorphisms of Albert algebras and a conjecture of Tits and Weiss II, *Trans. Amer. Math. Soc.* **372** (2019), no. 7, 4701–4728. MR4009439
- [24] M. Thakur, On R -triviality of F_4 , *Israel Journal of Math.*, to appear; arXiv:1912.01433v2 [math.GR] (2019).
- [25] M. Thakur, Albert algebras and the Tits–Weiss conjecture, arXiv:1911.04976v2 [math.GR] (2021).
- [26] M. Thakur, The cyclicity problem for Albert algebras, *Israel J. Math.* **241** (2021), no. 1, 139–145. MR4242148
- [27] V. E. Voskresenskii, *Algebraic groups and their birational invariants*, translated from the Russian manuscript by Boris Kunyavski [Boris È. Kunyavskii], *Transl. Math. Monogr.*, 179, American Mathematical Society, Providence, RI, 1998. MR1634406

Received January 2, 2020; accepted June 9, 2021

Maneesh Thakur
 Indian Statistical Institute,
 Stat-Math.-Unit, 8th Mile Mysore Road,
 Bangalore-560059, India.
 E-mail: maneesh.thakur@gmail.com