Münster J. of Math. **14** (2021), 559–583
DOI 10.17879/06089640368
urn:nbn:de:hbz:6-06089640706

Münster Journal of Mathematics
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Classification of irrational Θ-deformed CAR C^*-algebras

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(Communicated by Joachim Cuntz)

Abstract. Given a skew-symmetric real $n \times n$ matrix Θ, we consider the universal enveloping C^*-algebra CAR_Θ of the $*$-algebra generated by $a_1, \dots, a_n$ subject to the relations

$$a_i^* a_i + a_i a_i^* = 1,$$
$$a_i^* a_j = e^{2\pi i \Theta_{i,j}} a_j a_i^*,$$
$$a_i a_j = e^{-2\pi i \Theta_{i,j}} a_j a_i.$$

We prove that CAR_Θ has a $C(K_n)$-structure, where $K_n = [0, \frac{1}{2}]^n$ is the hypercube, and describe the fibers. We classify irreducible representations of CAR_Θ in terms of irreducible representations of a higher-dimensional noncommutative torus. We prove that, for a given irrational skew-symmetric Θ_1, there are only finitely many Θ_2 such that $\mathsf{CAR}_{\Theta_1} \simeq \mathsf{CAR}_{\Theta_2}$. Namely, $\mathsf{CAR}_{\Theta_1} \simeq \mathsf{CAR}_{\Theta_2}$ implies $(\Theta_1)_{ij} = \pm(\Theta_2)_{\sigma(i,j)} \bmod \mathbb{Z}$ for a bijection σ of the set $\{(i,j) \mid i < j,\ i,j = 1, \dots, n\}$. For $n = 2$, we give a full classification: $\mathsf{CAR}_{\theta_1} \simeq \mathsf{CAR}_{\theta_2}$ if and only if $\theta_1 = \pm\theta_2 \bmod \mathbb{Z}$.

1. Introduction

One of the most well-studied examples of noncommutative manifolds are the noncommutative tori, see [12]. Given a real skew-symmetric $n \times n$ matrix $\Theta = (\Theta_{i,j})$, the noncommutative torus $C(\mathbb{T}^n_\Theta)$ is defined as the universal C^*-algebra generated by n unitaries $u_1, \dots, u_n$ subject to the relations

$$u_i u_j = e^{-2\pi i \Theta_{i,j}} u_j u_i.$$

The problem of classification of $C(\mathbb{T}^n_\Theta)$ up to C^*-isomorphism has been solved in [8] in the case when Θ is *irrational*. In particular, in the case $n = 2$, identifying Θ with $\Theta_{1,2} = \theta$, we have

$$C(\mathbb{T}^2_{\theta_1}) \simeq C(\mathbb{T}^2_{\theta_2}) \quad \text{if and only if} \quad \theta_1 = \pm\theta_2 \bmod \mathbb{Z}.$$

For rational Θ, the classification is given in [2].

In this paper, we study the universal enveloping C^*-algebra CAR_Θ of the $*$-algebra generated by $a_1, \dots, a_n$ subject to the relations

$$\begin{aligned} a_i^* a_i + a_i a_i^* &= 1, \\ a_i^* a_j &= e^{2\pi i \Theta_{i,j}} a_j a_i^*, \\ a_i a_j &= e^{-2\pi i \Theta_{i,j}} a_j a_i. \end{aligned}$$

The representation theory of CAR_Θ was studied in [10], and it appeared to be related to representation theory of a noncommutative torus. In this paper, we in particular explain and reprove the result by showing that CAR_Θ has a $C(K_n)$-structure for $K_n = [0, \frac{1}{2}]^n$ with fibers being isomorphic to matrix algebras over crossed products of noncommutative tori by finite groups. The description of CAR_Θ as a "noncommutative fiber bundle" allows us to establish a result about classification CAR_Θ up to isomorphism for irrational Θ which was the main motivation to pursue our study of the object.

Noncommutative tori have been playing a role of a training ground for testing various ideas in noncommutative geometry and topology. Such questions as classification up to C^*-isomorphism, classification of projective modules, classification up to Morita equivalence, construction of Dirac operators, study of quantum metric structures, construction of pseudodifferential calculi, study of index theory, generalizations of the notion of curvature and many others have been studied for $C(\mathbb{T}^n_\Theta)$. Because of the simplicity of the algebraic definition of CAR_Θ and the existence of a noncommutative fiber bundle structure on CAR_Θ with the fibers resembling noncommutative tori, it is natural to ask the same questions about its structure as for $C(\mathbb{T}^n_\Theta)$. In this paper, we are interested in the noncommutative topology of CAR_Θ, in particular, the classification of CAR_Θ. We prove that $\mathsf{CAR}_{\theta_1} \simeq \mathsf{CAR}_{\theta_2}$ for irrational θ_1, θ_2 and $n = 2$ if and only if $\theta_1 = \pm\theta_2 \bmod \mathbb{Z}$. Moreover, for general n and irrational Θ_1, Θ_2, we prove that $\mathsf{CAR}_{\Theta_1} \simeq \mathsf{CAR}_{\Theta_2}$ implies that $(\Theta_1)_{i,j} = \pm(\Theta_2)_{\sigma(i,j)} \bmod \mathbb{Z}$ for a bijection σ of the set $\{(i,j) \mid i < j,\, i,j = 1, \dots, n\}$.

The general idea for our analysis of CAR_Θ is to express it as Rieffel deformation of n tensor copies of CAR_1—the one-dimensional CAR-algebra, the structure of which is well-understood: it is a $C([0, \frac{1}{2}])$-C^*-algebra with well-known fibers. Then we use the fact that Rieffel deformation of a $C_0(X)$-C^*-algebra also has a $C_0(X)$-structure with fibers which are Rieffel deformations of the fibers of the undeformed C^*-algebra.

The structure of the article is the following. In Sections 2 and 3, we recall some relevant facts from the theory of $C_0(X)$-C^*-algebras and Rieffel deformations. In Section 4, we prove isomorphisms between Rieffel deformations of matrix algebras over a C^*-algebra A and matrix algebras over Rieffel deformations of A. Although this result has been known in the literature, here we construct an explicit isomorphism, which will be used in further sections. In Section 5, we show that the C^*-algebra CAR_Θ is isomorphic to Rieffel deformation of $\mathsf{CAR}_1^{\otimes n}$. In Section 6, we give an analysis of CAR_1—we describe its representation theory, show that it has a $C([0, \frac{1}{2}])$-structure and describe fibers with respect to this structure. In Section 7, we transfer the described structural features of CAR_1 first to $\mathsf{CAR}_1^{\otimes n}$ and then to its Rieffel deformation, that allows us to obtain an alternative proof for classification of irreducible representations of CAR_Θ (Theorem 7.5). In Section 8, we further exploit the noncommutative fiber bundle structure of CAR_Θ and prove the classification result (Theorem 8.8, Corollary 8.9).

We believe that the C^*-algebra CAR_Θ is a nice rich object to study other questions of noncommutative geometry, and this will be pursued elsewhere.

2. $C_0(X)$-STRUCTURE ON C^*-ALGEBRAS

Let X be a locally compact Hausdorff space, and let $C_0(X)$ be the C^*-algebra of continuous functions on X that vanish at infinity. For a C^*-algebra A, write $\mathcal{M}(A)$ to denote its multiplier algebra; let $Z(A)$ be its center. Furthermore, $C_0(X, A)$ will stand for the algebra of A-valued continuous functions on X that vanish at infinity.

Definition 2.1. A $C_0(X)$-structure on a C^*-algebra A is a monomorphism

$$\Phi : C_0(X) \to Z(\mathcal{M}(A))$$

such that the ideal $\Phi(C_0(X)) \cdot A$ is dense in A. In this case, we say that A is a $C_0(X)$-C^*-algebra.

Let A be a $C_0(X)$-C^*-algebra. For $x \in X$, consider the closed ideal

$$I_x^\Phi = \overline{\operatorname{span}\{\Phi(f) \cdot a,\, a \in A,\, f \in C_0(X) \text{ such that } f(x) = 0\}}.$$

The fiber $A^\Phi(x)$ of A over x is defined as $A^\Phi(x) = A/I_x^\Phi$, and the canonical quotient map $\mathrm{ev}_x^\Phi : A \to A^\Phi(x)$ will be called the evaluation map at x. When $C_0(X)$-structure Φ is evident from the context, we will simply write I_x, $A(x)$ and ev_x; we shall often write $a(x)$ instead of $\mathrm{ev}_x(a)$; it is also common to suppress mention of Φ and simply write $f \cdot a$ instead of $\Phi(f) \cdot a$.

Let G be a locally compact group, and let $\alpha : G \to \operatorname{Aut}(A)$ be a continuous (with respect to point norm topology) group homomorphism, which we call an action of G on A; thus (A, G, α) is a C^*-dynamical system.

Definition 2.2. Let A be a $C_0(X)$-C^*-algebra. We say that α is fiberwise if

$$\alpha_g(f \cdot a) = f \cdot (\alpha_g(a)), \quad g \in G,\, a \in A,\, f \in C_0(X).$$

If an action α is fiberwise, then it induces an action α^x of G on $A(x)$ for every $x \in X$ by letting $\alpha_g^x(a(x)) = \alpha_g(a)(x)$, $a \in A$, $g \in G$.

3. RIEFFEL DEFORMATION

We turn now to Rieffel's deformation [11], that will be essential for our consideration, and recall main constructions needed for the paper.

Given a C^*-dynamical system $(A, \mathbb{R}^n, \alpha)$, let A^∞ denote the set of all $a \in A$ such that $t \mapsto \alpha_t(a)$ is a C^∞-function. It is a dense $*$-subalgebra of A. Let Θ be a real skew-symmetric $n \times n$-matrix. To define Rieffel deformation, one keeps the involution unchanged and introduces on A^∞ the product defined by oscillatory integrals

$$a \cdot_\Theta b := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \alpha_{\Theta(x)}(a)\alpha_y(b)e^{2\pi i\langle x,y\rangle}\,dx\,dy,$$

where $\langle x,y\rangle$ is the inner product on $\mathbb{R}^n$. The $*$-algebra $(A^\infty, \cdot_\Theta)$ admits a C^*-completion A^Θ in a C^*-norm, defined by Hilbert module techniques. The action α leaves A^∞ invariant and extends to the action α^Θ on the C^*-algebra A^Θ. More generally, any equivariant $*$-homomorphism f between C^*-algebras A and B with actions α^A and α^B of $\mathbb{R}^n$ respectively (i.e. $f(\alpha^A_x(a)) = \alpha^B_x(f(a))$, $a\in A$, $x\in\mathbb{R}^n$) can be lifted to a $*$-homomorphism $f^\Theta : A^\Theta \to B^\Theta$, which is also equivariant. We refer the reader to [11] for these and other details concerning the construction. Through this section, we will keep notation α for the action that defines the Rieffel deformation.

The procedure of Rieffel deformation is invertible; the next statement follows from [7, Lem. 3.5].

Proposition 3.1. *The identity mapping extends to a $*$-isomorphism*

$$\mathrm{id} : A \to (A^\Theta)^{-\Theta}.$$

In nice situations, Rieffel deformation of a $C_0(X)$-algebra is also a $C_0(X)$-algebra.

Proposition 3.2 ([1, Prop. 4.4])**.** *Let α be a fiberwise action of $\mathbb{R}^n$ on a $C_0(X)$-C^*-algebra A. Then the Rieffel deformation A^Θ possesses a $C_0(X)$-structure such that $(A^\Theta)(x) \simeq (A(x))^\Theta$, $x\in X$.*

We will need to know how crossed product C^*-algebras are transformed under Rieffel deformation.

Given a C^*-dynamical system (A,G,σ), write $A\rtimes_\sigma G$ for the corresponding full or reduced crossed product C^*-algebra [14], and denote by A^σ the set of fixed points of A, i.e.

$$A^\sigma = \{a\in A \mid \sigma_g(a) = a \text{ for every } g\in G\}.$$

If α is an action of $\mathbb{R}^n$ on A such that

$$\sigma_g(\alpha_t(a)) = \alpha_t(\sigma_g(a)) \quad \text{for all } g\in G,\ t\in\mathbb{R}^n,\ a\in A, \tag{1}$$

then α extends to an action on $A\rtimes_\sigma G$ by letting

$$\alpha_t(f)(g) = \alpha_t(f(g)), \quad f\in C_c(G,A).$$

The next proposition identifies Rieffel deformations of A^σ and $A\rtimes_\sigma G$.

Proposition 3.3. *Let (A,G,σ) be a C^*-dynamical system, and let α be an $\mathbb{R}^n$-action on A which satisfies* (1) *and hence extends to the $\mathbb{R}^n$-action on $A\rtimes_\sigma G$ as above. Let Θ be a real skew-symmetric $n\times n$ matrix. Then $(a,g)\in (A^\Theta, G)\mapsto (\sigma_g)^\Theta(a)\in A^\Theta$ defines an action σ^Θ of G on A^Θ such that*

$$(A^\sigma)^\Theta \simeq (A^\Theta)^{\sigma^\Theta} \quad \textit{and} \quad (A\rtimes_\sigma G)^\Theta \simeq (A^\Theta)\rtimes_{\sigma^\Theta} G.$$

Proof. The identity maps $(A^\sigma)^\Theta \to (A^\Theta)^{\sigma^\Theta}$ and $(A \rtimes_\sigma G)^\Theta \to (A^\Theta) \rtimes_{\sigma^\Theta} G$ give the isomorphisms. The only nontrivial thing is to show the homomorphism property in the second case. Let $f, g \in C_c(G) \odot A^\infty$, the algebraic tensor product of $C_c(G)$ and A^∞. One has that f, g are smooth elements of $A \rtimes_\sigma G$, and writing the deformed product as convolution $*_\Theta$, we obtain

$$\begin{aligned}(f *_\Theta g)(s) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_G \alpha_{\Theta(x)}(f)(t)\sigma_t(\alpha_y(g)(t^{-1}s))e^{2\pi i\langle x,y\rangle}\,dt\,dx\,dy \\ &= \int_G\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \alpha_{\Theta(x)}(f(t))\alpha_y(\sigma_t(g(t^{-1}s)))e^{2\pi i\langle x,y\rangle}\,dx\,dy\,dt \\ &= \int_G f(t)\cdot_\Theta \sigma_t(g(t^{-1}s))\,dt,\end{aligned}$$

where the latter is the convolution determined by $(A^\Theta, \sigma^\Theta, G)$. □

In this paper, we will be interested in periodic actions of $\mathbb{R}^n$, i.e. we assume that α is an action of $\mathbb{T}^n$. Given a character $\chi \in \widehat{\mathbb{T}}^n \simeq \mathbb{Z}^n$, consider the associated spectral subspace

$$A_\chi = \{a \in A \mid \alpha_z(a) = \chi(z)a \text{ for every } z \in \mathbb{T}^n\}.$$

Then

$$A = \overline{\operatorname{span}\bigcup_{\chi\in\mathbb{Z}^n} A_\chi}$$

and $A_{\chi_1}\cdot A_{\chi_2} \subset A_{\chi_1+\chi_2}$, $A_\chi^* = A_{-\chi}$; hence A_χ, $\chi \in \mathbb{Z}^n$, can be treated as homogeneous components of the induced $\mathbb{Z}^n$-grading on A.

For $p = (p_1, \dots, p_n) \in \mathbb{Z}^n \simeq \widehat{\mathbb{T}}^n$, we will write χ_p for the character of $\mathbb{T}^n$ given by $\chi_p(z) = z_1^{p_1}\dots z_n^{p_n}$, $z = (z_1, \dots, z_n)$, and write A_p instead of A_{χ_p}.

For the action of $\mathbb{T}^n$, one has an explicit formula for the deformed product of homogeneous elements.

Proposition 3.4 ([11, Prop. 2.22])**.** *Suppose A is a C^*-algebra with a $\mathbb{T}^n$-action. Assume that $a \in A_p$, $b \in A_q$ for $p, q \in \mathbb{Z}^n$. Then $a \cdot_\Theta b = e^{2\pi i\langle\Theta(p),q\rangle} a\cdot b$.*

Consider a C^*-dynamical system $(A, \mathbb{T}^n, \alpha)$ and its covariant representation (π, U) on a Hilbert space $\mathcal{H}$, i.e. $\pi(\alpha_z(a)) = U_z\pi(a)U_z^*$, $a \in A$, $z \in \mathbb{T}^n$. For $p \in \mathbb{Z}^n$, consider the spectral space

$$\mathcal{H}_p = \{h \in \mathcal{H} \mid U_z h = \chi_p(z)h \text{ for all } z \in \mathbb{T}^n\}.$$

Then $\mathcal{H} = \bigoplus_{p\in\mathbb{Z}^n}\mathcal{H}_p$ (see [14]).

The next result describes a procedure how to lift the representation π of A to a representation of its Rieffel deformation.

Proposition 3.5 ([3, Thm. 2.8])**.** *Let (π, U) be a covariant representation of $(A, \mathbb{T}^n, \alpha)$ on a Hilbert space $\mathcal{H}$. Then π^Θ, given by*

$$\pi^\Theta(a)\xi = e^{2\pi i\langle\Theta(p),q\rangle}\pi(a)\xi,$$

for $\xi \in \mathcal{H}_q$, $a \in A_p$, $p, q \in \mathbb{Z}^n$, extends to a $*$-representation of A^Θ. Moreover, π^Θ is faithful if and only if π is faithful.

Remark 3.6. Notice that if the action of $\mathbb{R}^n$ on $A\otimes B$ is given by $\alpha = \mathrm{id}\otimes\alpha_B$, where α_B is an $\mathbb{R}^n$ action on B, then $(A\otimes B)^\Theta \simeq A\otimes B^\Theta$.

We have also the following invariance of K-groups under Rieffel deformation.

Proposition 3.7 ([7, Thm. 3.13])**.** *For a C^*-algebra $\mathcal{A}$, one has*

$$K_0(\mathcal{A}^\Theta) = K_0(\mathcal{A}) \quad \textit{and} \quad K_1(\mathcal{A}^\Theta) = K_1(\mathcal{A}).$$

4. Rieffel deformation of $M_n(A)$

In the sequel, we will need to work with Rieffel deformations of matrix algebras over a C^*-algebra A, which we will describe in this section.

Suppose $\mathbb{T}^k$ acts on $\mathbb{C}^n$ by unitaries U_z, $z\in\mathbb{T}^k$, i.e. $z\mapsto U_z$ is a strongly continuous representation of $\mathbb{T}^k$ on $\mathbb{C}^n$. It induces an action of $\mathbb{T}^k$ on M_n given by $\alpha_z(X)\xi = U_zXU_z^*\xi$, $z\in\mathbb{T}^k$, $X\in M_n$, $\xi\in\mathbb{C}^n$; thus $(M_n,\mathbb{T}^k,\alpha)$ is a C^*-dynamical system and (id,U) is its covariant representation on $\mathbb{C}^n$, where id is the identity representation of M_n when the latter is identified with $B(\mathbb{C}^n)$. These actions define $\mathbb{Z}^k$-gradings on $\mathbb{C}^n$ and M_n as in Section 3. The following lemma is a direct consequence of Proposition 3.5.

Lemma 4.1. *Let Θ be a real skew-symmetric $k\times k$ matrix, and let $\Psi : M_n^\Theta \to M_n$ be given by*

$$\Psi(a)\xi = e^{2\pi i\langle\Theta(p),q\rangle}a\xi,$$

where $a\in M_n$ is homogeneous of order $p\in\mathbb{Z}^k$ and $\xi\in\mathbb{C}^n$ is homogeneous of order $q\in\mathbb{Z}^k$. Then Ψ is an equivariant $*$-isomorphism from $(M_n^\Theta,\mathbb{T}^k,\alpha^\Theta)$ to $(M_n,\mathbb{T}^k,\alpha)$.

Let $(A,\mathbb{T}^m,\alpha^A)$ be a C^*-dynamical system, and consider the action of $\mathbb{T}^{k+m}$ on $M_n\otimes A$ given by $X\otimes a\mapsto \alpha_{z_1}(X)\otimes\alpha^A_{z_2}(a)$, $(z_1,z_2)\in\mathbb{T}^k\times\mathbb{T}^m$, $X\in M_n$, $a\in A$.

Let Θ be a real skew-symmetric matrix of size $k+m$, and consider its block partition

$$\Theta = \begin{pmatrix}\Theta_{1,1} & \Theta_{1,2}\\ \Theta_{2,1} & \Theta_{2,2}\end{pmatrix}, \quad \text{where } \Theta_{1,1}\in M_k \text{ and } \Theta_{2,2}\in M_m.$$

For $p\in\mathbb{Z}^m$, set

$$\omega_l(p) = e^{2\pi i\langle\Theta_{2,1}(\epsilon_l),p\rangle}\in\mathbb{T}, \quad l=1,\dots,k,$$
$$\omega(p) = (\omega_1(p),\dots,\omega_k(p))\in\mathbb{T}^k;$$

here $\{\epsilon_l\}_{l=1}^k$ is the standard orthonormal basis of $\mathbb{R}^k$. Each $\omega_l : \mathbb{Z}^m\to\mathbb{T}$ is clearly a character, and hence $\omega(p_1+p_2)=\omega(p_1)\omega(p_2)$, $p_1,p_2\in\mathbb{Z}^m$.

Theorem 4.2. *Let $\Theta\in M_{k+m}$ and $(M_n\otimes A,\mathbb{T}^{k+m},\alpha\otimes\alpha^A)$ be as above. Then*

$$(M_n\otimes A)^\Theta \simeq M_n\otimes A^{\Theta_{2,2}},$$

with the isomorphism given by

$$\Phi(X\otimes a) = \alpha_{\omega(-q)}(\Psi(X))U_{\omega(-2q)}\otimes a,$$

for $X\in M_n$, $a\in A$ homogeneous of order $q\in\mathbb{Z}^m$ and Ψ as defined in Lemma 4.1

Proof. Let (π, V) be a faithful covariant representation of $(A, \mathbb{T}^m, \alpha^A)$ on $\mathcal{H}$. Then $(\mathrm{id} \otimes \pi, U \otimes V)$ is a faithful covariant representation of

$$(M_n \otimes A, \mathbb{T}^{k+m}, \alpha \otimes \alpha^A)$$

on $\mathbb{C}^n \otimes \mathcal{H}$. Let $(\mathrm{id} \otimes \pi)^\Theta : (M_n \otimes A)^\Theta \to B(\mathbb{C}^n \otimes \mathcal{H})$ and $\pi^{\Theta_{2,2}} : A^{\Theta_{2,2}} \to B(\mathcal{H})$ be the $*$-representations defined as in Proposition 3.5.

As $(\mathrm{id} \otimes \pi)^\Theta$ and $\mathrm{id} \otimes \pi^{\Theta_{2,2}}$ are faithful representations of $(M_n \otimes A)^\Theta$ and $M_n \otimes A^{\Theta_{2,2}}$ respectively, to prove the theorem, it is enough to show that there exists a unitary operator $W \in B(\mathbb{C}^n \otimes \mathcal{H})$ such that

$$W^*(\mathrm{id} \otimes \pi)^\Theta(X \otimes a)W = \mathrm{id} \otimes \pi^{\Theta_{2,2}}(\Phi(X \otimes a)) \tag{2}$$

for all $X \in M_n$, $a \in A_p$ and $p \in \mathbb{Z}^m$; here A_p is the homogeneous component of order p with respect to $(A, \mathbb{T}^m, \alpha^A)$.

Recall the gradings on M_n and A which are determined by $(M_n, \mathbb{T}^k, \alpha)$ and $(A, \mathbb{T}^m, \alpha^A)$ respectively and the gradings on $\mathbb{C}^n$ and $\mathcal{H}$ determined by the representations (U_z) and (V_z) of $\mathbb{T}^k$ and $\mathbb{T}^m$ respectively. Let $X \in M_n$ and $a \in A$ be homogeneous of order $p \in \mathbb{Z}^k$ and $q \in \mathbb{Z}^m$ respectively, and let $\xi_1 \in \mathbb{C}^n$ and $\xi_2 \in \mathcal{H}$ be homogeneous of order $p_1 \in \mathbb{Z}^k$ and $q_1 \in \mathbb{Z}^m$. Then $X \otimes a$ is homogeneous of order $(p, q) \in \mathbb{Z}^k \times \mathbb{Z}^m$ with respect to $(M_n \otimes A, \mathbb{T}^{k+m}, \alpha \otimes \alpha^A)$, and $\xi_1 \otimes \xi_2$ is homogeneous of order $(r, s) \in \mathbb{Z}^k \times \mathbb{Z}^m$ with respect to $(U_z \otimes V_z)$. Furthermore, $\alpha_z(X) = \chi_p(z)X$ and $U_z\xi_1 = \chi_{p_1}(z)\xi_1$. Set $b = \binom{p}{q}$ and $c = \binom{r}{s}$. Then, by Proposition 3.5, we have

$$\begin{aligned}
&(\mathrm{id} \otimes \pi)^\Theta(X \otimes a)(\xi_1 \otimes \xi_2) \\
&\quad = e^{2\pi i \langle \Theta(b), c\rangle} X\xi_1 \otimes \pi(a)\xi_2 \\
&\quad = e^{2\pi i \langle \Theta_{1,1}p, r\rangle} e^{2\pi i \langle \Theta_{2,2}q, s\rangle} e^{2\pi i \langle \Theta_{2,1}p, s\rangle} e^{2\pi i \langle \Theta_{1,2}q, r\rangle} X\xi_1 \otimes \pi(a)\xi_2 \\
&\quad = e^{2\pi i \langle \Theta_{2,1}p, s\rangle} e^{-2\pi i \langle \Theta_{2,1}r, q\rangle} \Psi(X)\xi_1 \otimes \pi^{\Theta_{2,2}}(a)\xi_2 \\
&\quad = \chi_p(\omega(s))\chi_r(\omega(-q))\Psi(X)\xi_1 \otimes \pi^{\Theta_{2,2}}(a)\xi_2 \\
&\quad = \alpha_{\omega(s)}(\Psi(X))U_{\omega(-q)}\xi_1 \otimes \pi^{\Theta_{2,2}}(a)\xi_2.
\end{aligned}$$

Let W be a linear map defined on $\mathrm{span}\{\xi \otimes \eta \mid \xi \in \mathbb{C}^n,\ \eta \in \mathcal{H}_s,\ s \in \mathbb{Z}^m\} \subset \mathbb{C}^n \otimes \mathcal{H}$ by letting

$$W(\xi \otimes \eta) = U_{\omega(q)}\xi \otimes \eta, \quad \xi \in \mathbb{C}^n,\ \eta \in \mathcal{H}_q.$$

Any vector ζ in the span can be written as $\sum_{i=1}^l \xi_i \otimes \eta_i$, where $\xi_i \in \mathbb{C}^n$ and $\{\eta_i\}_{i=1}^l$ is an orthonormal set in $\mathcal{H}$ such that $\eta_i \in \mathcal{H}_{q_i}$, $q_i \in \mathbb{Z}^m$ (q_i can be equal for different i). We have

$$\begin{aligned}
\langle W\zeta, W\zeta\rangle &= \left\langle \sum_{i=1}^l U_{\omega(q_i)}\xi_i \otimes \eta_i, \sum_{i=1}^l U_{\omega(q_i)}\xi_i \otimes \eta_i \right\rangle \\
&= \sum_{i=1}^l \langle U_{\omega(q_i)}\xi_i, U_{\omega(q_i)}\xi_i\rangle\langle \eta_i, \eta_i\rangle = \sum_{i=1}^l \langle \xi_i, \xi_i\rangle\langle \eta_i, \eta_i\rangle = \langle \zeta, \zeta\rangle;
\end{aligned}$$

hence W can be extended to an isometry on $\mathbb{C}^n \otimes \mathcal{H}$; as the range of W is dense in $\mathbb{C}^n \otimes \mathcal{H}$, it is a unitary operator.

It is left to see that W satisfies (2). For X, a, ξ_1, ξ_2 as above, we have

$$\begin{aligned} &W^*(\mathrm{id}\otimes\pi)^{\Theta}(X\otimes a)W(\xi_1\otimes\xi_2)\\ &\quad= W^*(\mathrm{id}\otimes\pi)^{\Theta}(X\otimes a)(U_{\omega(s)}\xi_1\otimes\xi_2)\\ &\quad= W^*\big(\alpha_{\omega(s)}(\Psi(X))U_{\omega(-q)}U_{\omega(s)}\xi_1\otimes\pi^{\Theta_{2,2}}(a)\xi_2\big)\\ &\quad= U^*_{\omega(s+q)}\alpha_{\omega(s)}(\Psi(X))U_{\omega(-q)}U_{\omega(s)}\xi_1\otimes\pi^{\Theta_{2,2}}(a)\xi_2\\ &\quad= U_{\omega(-q)}\Psi(X)U_{\omega(-q)}\xi_1\otimes\pi^{\Theta_{2,2}}(a)\xi_2\\ &\quad= \alpha_{\omega(-q)}(\Psi(X))U_{\omega(-2q)}\xi_1\otimes\pi^{\Theta_{2,2}}(a)\xi_2\\ &\quad= \mathrm{id}\otimes\pi^{\Theta_{2,2}}(\Phi(X\otimes a))(\xi_1\otimes\xi_2).\end{aligned}$$

The result now follows by density arguments. □

We remark that the statements holds true if M_n is replaced by a subalgebra C of M_n such that $\alpha_z(C)=C$ for all $z\in\mathbb{T}^k$.

5. CAR_Θ as Rieffel deformation

In this section, we will show that our main object CAR_Θ can be seen as Rieffel deformation of a higher-dimensional CAR algebra. Recall that the one-dimensional algebra of canonical anti-commutation relations (CAR) is the $*$-algebra $\mathbb{C}\langle a,a^*\mid a^*a+aa^*=1\rangle$. We will denote its universal enveloping C^*-algebra by CAR_1; the latter algebra exists and is isomorphic to a C^*-subalgebra of the C^*-algebra of all continuous functions on the unit disk $\{z\mid |z|\le 1\}$ with values in M_2, see e.g. [9, Thm. 2.2]. Its other realization, which will be convenient for our purpose, will be described in the next section. The higher-dimensional CAR C^*-algebra is given by the tensor product $\mathsf{CAR}_1^{\otimes n}$. Note that CAR_1 is nuclear, and hence we do not need to specify the C^*-tensor product of its copies.

The C^*-algebra CAR_1 has a natural action of $\mathbb{T}$ given by

$$\alpha_w(a)=wa,\quad w\in\mathbb{T}, \tag{3}$$

where a is the generator of CAR_1. This $\mathbb{T}$-action will be always assumed on CAR_1 without mentioning it. It induces the action $\alpha^{\otimes n}$ of $\mathbb{T}^n$ on the tensor product $\mathsf{CAR}_1^{\otimes n}$ which we also fix through the paper. For this action, each generator $\tilde a_i := 1^{\otimes(i-1)}\otimes a\otimes 1^{\otimes(n-i)}$ is homogeneous of order $\delta_i\in\mathbb{Z}^n$, where $(\delta_i)_k=\delta_{i,k}$ is the Kronecker delta.

Fix now a real skew-symmetric matrix $\Theta=(\Theta_{i,j})_{i,j=1}^n$, and recall that CAR_Θ is the universal C^*-algebra generated by $a_1,\dots,a_n$ subject to the relations

$$\begin{aligned} a_i^*a_i+a_ia_i^*&=1,\\ a_i^*a_j&=e^{2\pi i\Theta_{i,j}}a_ja_i^*,\\ a_ia_j&=e^{-2\pi i\Theta_{i,j}}a_ja_i.\end{aligned}$$

We have the following isomorphism between CAR_Θ and a Rieffel deformation of $\mathsf{CAR}_1^{\otimes n}$.

Theorem 5.1. *Let Θ be a real skew-symmetric $n \times n$ matrix. Then*

$$\mathsf{CAR}_\Theta \simeq (\mathsf{CAR}_1^{\otimes n})^{\frac{\Theta}{2}}.$$

Proof. Consider

$$\varphi : \mathsf{CAR}_\Theta \to (\mathsf{CAR}_1^{\otimes n})^{\frac{\Theta}{2}}, \quad \varphi(a_i) = 1^{\otimes(i-1)} \otimes a \otimes 1^{\otimes(n-i)} := \tilde{a}_i, \quad i = 1, \ldots, n.$$

We shall see first that φ extends to a well-defined $*$-homomorphism. As $\tilde{a}_k$ and $\tilde{a}_k^*$ are homogeneous of order δ_k and $-\delta_k \in \mathbb{Z}^n$ respectively, by Proposition 3.4,

$$\begin{aligned}\varphi(a_k) \cdot_{\frac{\Theta}{2}} \varphi(a_k)^* &= e^{-\pi i \langle \Theta(\epsilon_k), \epsilon_k \rangle} \tilde{a}_k \tilde{a}_k^* \\ &= 1^{\otimes(k-1)} \otimes aa^* \otimes 1^{\otimes(n-k)} = \varphi(a_k a_k^*).\end{aligned}$$

Similarly, $\varphi(a_k)^* \cdot_{\frac{\Theta}{2}} \varphi(a_k) = \varphi(a_k^* a_k)$, and hence $\varphi(a_k)^* \cdot_{\frac{\Theta}{2}} \varphi(a_k) + \varphi(a_k) \cdot_{\frac{\Theta}{2}} \varphi(a_k)^* = 1$.

If $k < m$, then

$$\begin{aligned}\varphi(a_k) \cdot_{\frac{\Theta}{2}} \varphi(a_m) &= e^{\pi i \langle \Theta(\epsilon_k), \epsilon_m \rangle} \tilde{a}_k \tilde{a}_m \\ &= e^{-\pi i \Theta_{k,m}} 1^{\otimes(k-1)} \otimes a \otimes 1^{\otimes(m-k-1)} \otimes a \otimes 1^{\otimes(n-m)}, \\ \varphi(a_m) \cdot_{\frac{\Theta}{2}} \varphi(a_k) &= e^{\pi i \langle \Theta(\epsilon_m), \epsilon_k \rangle} \tilde{a}_m \tilde{a}_k \\ &= e^{\pi i \Theta_{k,m}} 1^{\otimes(k-1)} \otimes a \otimes 1^{\otimes m-k-1} \otimes a \otimes 1^{\otimes(n-m)}.\end{aligned}$$

Thus

$$\varphi(a_k) \cdot_{\frac{\Theta}{2}} \varphi(a_m) = e^{-2\pi i \Theta_{k,m}} \varphi(a_m) \cdot_{\frac{\Theta}{2}} \varphi(a_k).$$

Similar calculations give

$$\varphi(a_k)^* \cdot_{\frac{\Theta}{2}} \varphi(a_m) = e^{2\pi i \Theta_{k,m}} \varphi(a_m) \cdot_{\frac{\Theta}{2}} \varphi(a_k)^*,$$

so φ extends to a $*$-homomorphism, and φ is surjective as the $*$-algebra generated by $\tilde{a}_i$, $i = 1, \ldots, n$, is dense in $(\mathsf{CAR}_1^{\otimes n})^{\frac{\Theta}{2}}$.

Also, CAR_Θ has a natural $\mathbb{T}^n$-action determined by

$$\alpha_w(a_i) = w_i a_i, \quad w = (w_1, \ldots, w_n) \in \mathbb{T}^n,$$

and hence we can talk about its Rieffel deformation $(\mathsf{CAR}_\Theta)^{-\frac{\Theta}{2}}$. As above, for $\tilde{a}_k \in \mathsf{CAR}_1^{\otimes n}$, $k = 1, \ldots, n$, consider the map

$$\Psi : \mathsf{CAR}_1^{\otimes n} \to (\mathsf{CAR}_\Theta)^{-\frac{\Theta}{2}}, \quad \Psi(\tilde{a}_k) = a_k, \quad k = 1, \ldots, n.$$

As a_k and $a_k^* \in \mathsf{CAR}_\Theta$ are homogeneous of order δ_k and $-\delta_k \in \mathbb{Z}^n$ respectively, as above, we obtain that

$$\Psi(\tilde{a}_k) \cdot_{-\frac{\Theta}{2}} \Psi(\tilde{a}_m) = e^{\pi i \Theta_{k,m}} a_k a_m = e^{-\pi i \Theta_{k,m}} a_m a_k = \Psi(\tilde{a}_m) \cdot_{-\frac{\Theta}{2}} \Psi(\tilde{a}_k).$$

In a similar way, we get $\Psi(\tilde{a}_k)^* \cdot_{-\frac{\Theta}{2}} \Psi(\tilde{a}_m) = \Psi(\tilde{a}_m) \cdot_{-\frac{\Theta}{2}} \Psi(\tilde{a}_k)^*$, $m \neq k$, and $\Psi(\tilde{a}_k)^* \cdot_{-\frac{\Theta}{2}} \Psi(\tilde{a}_k) + \Psi(\tilde{a}_k) \cdot_{-\frac{\Theta}{2}} \Psi(\tilde{a}_k)^* = 1$. Thus Ψ extends to a $*$-homomorphism. It is clearly surjective. Moreover, one has the following commutative diagram:

$$\begin{tikzcd}
(\mathsf{CAR}_\Theta)^{-\frac{\Theta}{2}} \arrow[r, "\varphi^{-\frac{\Theta}{2}}"] & ((\mathsf{CAR}_1^{\otimes n})^{\frac{\Theta}{2}})^{-\frac{\Theta}{2}}. \\
\mathsf{CAR}_1^{\otimes n} \arrow[u, "\Psi"] \arrow[ur, equal, "\mathrm{id}"] &
\end{tikzcd}$$

Since Ψ is surjective, $\varphi^{-\frac{\Theta}{2}}$ is injective. Therefore, by Proposition 3.5, φ is injective too. $\square$

6. The C^*-algebra CAR_1

In this section, we recall the representation theory of the one-dimensional CAR $*$-algebra and describe its universal enveloping C^*-algebra as a subalgebra of $C([0,\frac{1}{2}], M_2(C(\mathbb{T})))$, showing that it has a $C([0,\frac{1}{2}])$-structure and that the action α of $\mathbb{T}$ on CAR_1 defined by (3) is fiberwise.

6.1. **Representation theory of CAR_1.** We will use the following classification of irreducible representations of CAR up to unitary equivalence:

- 2-dimensional:

$$\pi_{x,\varphi}(a) = e^{i\varphi}\begin{pmatrix} 0 & \sqrt{x} \\ \sqrt{1-x} & 0 \end{pmatrix}, \quad x \in \Big[0, \frac{1}{2}\Big), \varphi \in [0,\pi),$$

- 1-dimensional:

$$\rho_\varphi(a) = \frac{e^{i\varphi}}{\sqrt{2}}, \varphi \in [0, 2\pi).$$

Remark 6.2. These representations are unitary equivalent to the representations given in [9]. We note also that

$$\begin{aligned}
&\pi_{x,\varphi}(a) = W^*\pi_{x,\varphi-\pi}(a)W, \\
&\qquad\qquad x \in \Big[0, \frac{1}{2}\Big), \varphi \in [\pi, 2\pi), \qquad\qquad \text{where } W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
&\pi_{0,1}(a) = W(\varphi)^*\pi_{0,\varphi}(a)W(\varphi), \quad \varphi \in [\pi, 2\pi), \quad \text{where } W(\varphi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}, \\
&\pi_{\frac{1}{2},\varphi}(a) = \frac{e^{i\varphi}}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = V\frac{e^{i\varphi}}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}V^*, \quad \text{where } V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\end{aligned}$$

Hence any one-dimensional irreducible representation can be obtained by decomposing $\pi_{\frac{1}{2},\varphi}$, $\varphi \in [0,\pi)$, into irreducible ones. Also one has

$$\pi_{\frac{1}{2},\varphi} = W\pi_{\frac{1}{2},\varphi+\pi}W^*, \quad \varphi \in [0,\pi).$$

6.3. **Spatial picture of CAR_1.** In order to describe the universal enveloping C^*-algebra CAR_1, we recall the following version of the Stone–Weierstrass–Glimm theorem, see e.g. [6, Thm. 1.4], [13, Sec. 3].

Theorem 6.4. *Let Y be a compact Hausdorff space, and let $A \subseteq B$ be subalgebras of $C(Y, M_n)$. For every pair (y_1, y_2) of points in Y, define $A(y_1, y_2)$ as*

$$A(y_1, y_2) := \{(f(y_1), f(y_2)) \in M_n \times M_n \mid f \in A\},$$

and similarly $B(y_1, y_2)$. Then

$$A = B \iff A(y_1, y_2) = B(y_1, y_2) \quad \text{for all } y_1, y_2 \in Y.$$

For representations π_1, π_2 of a $*$-algebra $\mathcal{A}$ on Hilbert spaces $\mathcal{H}(\pi_1)$ and $\mathcal{H}(\pi_2)$ respectively, we write $\mathsf{Hom}(\pi_1, \pi_2)$ for the space of intertwining operators

$$\mathsf{Hom}(\pi_1, \pi_2) = \{c \in B(\mathcal{H}(\pi_2), \mathcal{H}(\pi_1)) \mid \pi_1(a)c = c\pi_2(a),\, a \in \mathcal{A}\}.$$

We remark that $\mathsf{Hom}(\pi_1, \pi_2) = \{0\}$ if and only if π_1, π_2 are disjoint, i.e. π_1, π_2 do not have unitary equivalent sub-representations.

For a $*$-algebra $\mathcal{A} \subset B(\mathcal{H})$, we denote by $\mathcal{A}'$ its commutant, i.e.

$$\mathcal{A}' = \{c \in B(\mathcal{H}) \mid ca = ac,\, a \in \mathcal{A}\}.$$

Let also $W(z) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & z \end{smallmatrix}\right)$, $z \in \mathbb{T}$, and retain the unitaries W and V from the previous subsection; in particular, $W = W(-1)$. Write $D_2 \subset M_2$ for the subalgebra of diagonal matrices.

Let $h : \mathsf{CAR}_1 \to C([0, \frac{1}{2}], M_2(C(\mathbb{T})))$ be a $*$-homomorphism given on the generator $a \in \mathsf{CAR}_1$ by

$$h(a)(x)(z) = z \begin{pmatrix} 0 & \sqrt{x} \\ \sqrt{1-x} & 0 \end{pmatrix}, \quad x \in \Big[0, \frac{1}{2}\Big],\, z \in \mathbb{T}. \tag{4}$$

The next proposition gives a desired realization of CAR_1.

Proposition 6.5. CAR_1 *is isomorphic to the C^*-algebra*

$$\begin{aligned} \mathcal{B} = \Big\{ f \in C\Big(\Big[0, \frac{1}{2}\Big], M_2(C(\mathbb{T}))\Big) \;\Big|\; & \\ & f(x)(z) = Wf(x)(-z)W^* \text{ for all } x \in \Big(0, \frac{1}{2}\Big), \\ & f(0)(z) = W(z)f(0)(1)W^*(z),\ f\Big(\frac{1}{2}\Big) \in V(D_2 \otimes C(\mathbb{T}))V^*, \\ & f\Big(\frac{1}{2}\Big)(z) = Wf\Big(\frac{1}{2}\Big)(-z)W^* \text{ for all } z \in \mathbb{T} \Big\}, \end{aligned} \tag{5}$$

with the map h which implements the $*$-isomorphism.

Proof. Set $\mathcal{A} = \mathsf{CAR}_1$. Observe first that $h(a)(x)(z) = \pi_{x,\varphi}(a)$ for $z = e^{i\varphi}$; hence h extends to a $*$-homomorphism of CAR_1. Moreover, it is easy to see that the image $h(\mathcal{A})$ is in $\mathcal{B}$. It follows from the classification of irreducible representations of $\mathcal{A}$ that $h : \mathcal{A} \to \mathcal{B}$ given by (4) is an isometry. Hence it is sufficient to see that h is surjective. Considering $\mathcal{B}$ as a C^*-subalgebra of $C([0, \frac{1}{2}] \times \mathbb{T}, M_2)$, by Theorem 6.4, it is enough to show that, for pairs $(x_1, x_2) \in [0, \frac{1}{2}]^2$ and $(z_1, z_2) \in \mathbb{T}^2$, $h(\mathcal{A})((x_1, z_1), (x_2, z_2)) = \mathcal{B}((x_1, z_1), (x_2, z_2))$.

We will follow the same scheme as in [9, Thm. 2.2] and prove the equality of the commutants

$$h(\mathcal{A})((x_1,z_1),(x_2,z_2))' = \mathcal{B}((x_1,z_1),(x_2,z_2))'.$$

For the notation simplicity, we will write $\pi_{x,z}$ instead of $\pi_{x,\varphi}$ if $z=e^{i\varphi}$.

Consider the equivalence relation on $[0,\frac{1}{2}]\times\mathbb{T}$ defined as follows:

$$(x_1,z_1)\sim(x_2,z_2) \text{ if either } x_1=x_2 \text{ and } z_1=\pm z_2 \text{ or } x_1=x_2=0,$$

and note that π_{x_1,z_1} and π_{x_2,z_2} are disjoint, and hence $\mathsf{Hom}(\pi_{x_1,z_1},\pi_{x_2,z_2})=\{0\}$ when $(x_1,z_1)\not\sim(x_2,z_2)$. Therefore, assuming $(x_1,z_1)\not\sim(x_2,z_2)$, we obtain that

$$h(\mathcal{A})((x_1,z_1),(x_2,z_2))' = h(\mathcal{A})((x_1,z_1))'\oplus h(\mathcal{A})((x_2,z_2))'.$$

As $h(\mathcal{A})\subset\mathcal{B}$, we have $\mathcal{B}((x_1,z_1),(x_2,z_2))'=\mathcal{B}((x_1,z_1))'\oplus\mathcal{B}((x_2,z_2))'$.

If $x_1=x_2=0$, then $\pi_{x_i,z_i}(b)=W(z_i)\pi_{0,1}(b)W(z_i)^*$, $b\in\mathcal{A}$, and one easily gets that

$$h(\mathcal{A})((x_1,z_1),(x_2,z_2))' = \{(\Lambda_{ij})\mid W(z_i)\Lambda_{ij}W(z_j)^*\in\pi_{0,1}(\mathcal{A})',\ i,j=1,2\},$$
$$\mathcal{B}((x_1,z_1),(x_2,z_2))' = \{(\Lambda_{ij})\mid W(z_i)\Lambda_{ij}W(z_j)^*\in\mathcal{B}((0,1))',\ i,j=1,2\}.$$

If $x_1=x_2$ and $z_1=-z_2$, similarly, we obtain that

$$h(\mathcal{A})((x_1,z_1),(x_2,z_2))' = \{(\Lambda_{ij})\mid \Lambda_{i1},W^*\Lambda_{1j}W\in\pi_{x_1,z_1}(\mathcal{A})',\ i,j=1,2\},$$
$$\mathcal{B}((x_1,z_1),(x_2,z_2))' = \{(\Lambda_{ij})\mid \Lambda_{i1},W^*\Lambda_{1j}W\in\mathcal{B}((x_1,z_1))',\ i,j=1,2\}.$$

If $x_1=x_2$ and $z_1=z_2$, then

$$h(\mathcal{A})((x_1,z_1),(x_2,z_2))' = (I\otimes h(\mathcal{A})((x_1,z_1)))' = M_2\otimes(h(\mathcal{A})((x_1,z_1)))'$$

and similarly $\mathcal{B}((x_1,z_1),(x_2,z_2))'=M_2\otimes(\mathcal{B}((x_1,z_1)))'$. Therefore, to prove the statement, it is enough to see that

$$\pi_{x,z}(\mathcal{A})'\subset\mathcal{B}((x,z))' \tag{6}$$

for any $(x,z)\in[0,\frac{1}{2}]\times\mathbb{T}$.

We consider two cases: $x\neq\frac{1}{2}$ and $x=\frac{1}{2}$.

Case 1: $x\neq\frac{1}{2}$. In this case, $\pi_{x,z}$ is irreducible, and hence $\pi_{x,z}(\mathcal{A})'=\mathbb{C}I_2$; the inclusion (6) holds trivially.

Case 2: $x=\frac{1}{2}$. In this case, we have $C\in\pi_{x,z}(\mathcal{A})'$ if and only if $V^*CV\in D$, where D is the subalgebra of the diagonal matrices. It follows from the definition of $\mathcal{B}$ that any such C is in $\mathcal{B}(\frac{1}{2},z)$. This completes the proof. □

6.6. **CAR_1 as a fixed point subalgebra.** It will be useful to see CAR_1 as a fixed point subalgebra of a larger C^*-subalgebra of $C([0,\frac{1}{2}],M_2(C(\mathbb{T})))$ with a $\mathbb{Z}_2$-action defined on it.

Given a C^*-algebra $\mathcal{A}$, the C^*-algebra $C(\mathbb{T},\mathcal{A})$ has a natural $\mathbb{T}$-action which will be always denoted by β:

$$\beta_w(f)(z)=f(wz),\quad f\in C(\mathbb{T},\mathcal{A}),\ z\in\mathbb{T},\ w\in\mathbb{T}.$$

Considering $\mathbb{Z}_2$ as the subgroup $\{1,-1\}$ of $\mathbb{T}$, we shall denote by β also the restriction of it to $\mathbb{Z}_2$.

We keep the notation of Subsection 6.3 and consider the C^*-algebra

$$\mathcal{C} = \Big\{ f \in C\Big(\Big[0, \frac{1}{2}\Big], M_2(C(\mathbb{T}))\Big) \;\Big|\; f(0)(z) = W(z)f(0)(1)W^*(z), \\ f\Big(\frac{1}{2}\Big) \in V(D_2 \otimes C(\mathbb{T}))V^* \text{ for all } z \in \mathbb{T} \Big\}.$$

Let σ be the action of $\mathbb{Z}_2$ on $M_2(C(\mathbb{T})) \simeq M_2 \otimes C(\mathbb{T})$ given by

$$\sigma_w = \mathsf{Ad}(W(w)) \otimes \beta_w, \quad w \in \mathbb{Z}_2,$$

where we write $\mathsf{Ad}(v)$ for the inner automorphism $T \mapsto vTv^*$ of M_2. Let Σ be the action of $\mathbb{Z}_2$ on $\mathcal{C}$ given by

$$\Sigma_w(f)(x) = \sigma_w(f(x)), \quad f \in \mathcal{C},\, w \in \mathbb{Z}_2.$$

Theorem 6.7. *Let $\mathcal{B}$ be the C^*-algebra given by* (5)*. Then $\mathcal{B} \simeq \mathcal{C}^\Sigma$.*

Proof. The only condition to be checked is that the 0-fiber is stable under Σ:

$$\begin{aligned}\Sigma_{-1}(f)(0)(z) &= W(-1)f(0)(-z)W(-1)^* \\ &= W(-1)W(-z)f(0)(1)W(-z)^*W(-1)^* \\ &= W(z)f(0)(1)W(z)^* = f(0)(z).\end{aligned}$$

□

6.8. **$C([0, \frac{1}{2}])$-structure on CAR_1.** Let $\mathcal{B}$ be as in Proposition 6.5. Since $\mathsf{CAR}_1 \simeq \mathcal{B}$, it has a $C([0, \frac{1}{2}])$-structure induced by the natural $C([0, \frac{1}{2}])$-structure on $\mathcal{B}$ given by

$$\Phi(g)(f)(x) = g(x)f(x), \quad g \in C\Big(\Big[0, \frac{1}{2}\Big]\Big),\, f \in \mathcal{B}, \tag{7}$$

so that $\mathsf{CAR}_1(x) \simeq \mathcal{B}(x)$ with the isomorphism defined by $b(x) \mapsto h(b)(x)$, $b \in \mathsf{CAR}_1$, $x \in [0, \frac{1}{2}]$.

We next identify the fibers $\mathsf{CAR}_1(x)$, $x \in [0, \frac{1}{2}]$. We note first that, with β as in the previous subsection, we have that $w \mapsto \beta(w)(f)(x) := (\beta_w(f(x))$, $f \in C([0, \frac{1}{2}], M_2(C(\mathbb{T})))$, $w \in \mathbb{T}$, is a fiberwise action of $\mathbb{T}$. Moreover, the isomorphism h of Proposition 6.5 is equivariant in the sense that

$$h(\alpha_w(b))(x) = \beta_w(h(b)(x)), \quad b \in \mathsf{CAR}_1,\, w \in \mathbb{T},\, x \in \Big[0, \frac{1}{2}\Big].$$

In particular, it implies that α is fiberwise with the induced action α^x on $\mathsf{CAR}_1(x)$ given by $\alpha^x_w(a(x)) = wa(x)$, where a is the generator of CAR_1.

Let $C(\mathbb{T}) \rtimes_\beta \mathbb{Z}_2$ be the crossed product C^*-algebra corresponding to the dynamical system $(C(\mathbb{T}), \beta, \mathbb{Z}_2)$. It is the universal C^*-algebra generated by unitaries u and v satisfying the relations $uv = -vu$, $v^2 = 1$; the action β of $\mathbb{T}$ on $C(\mathbb{T})$, $\beta_w(f)(z) = f(wz)$, $w, z \in \mathbb{T}$, induces a $\mathbb{T}$-action τ on $C(\mathbb{T}) \rtimes_\beta \mathbb{Z}_2$, given by $\tau_w(u) = wu$ and $\tau_w(v) = v$, $w \in \mathbb{T}$.

In what follows, it will be convenient to use the generators of Clifford algebra. We recall that the Clifford C^*-algebra Cl_2 is

$$\mathrm{Cl}_2 = C^*(e \mid e^2 = 0,\, ee^* + e^*e = 1).$$

Clearly, $M_2 \simeq \mathrm{Cl}_2$, where the isomorphism is given by $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \mapsto e$.

Theorem 6.9. *One has the following isomorphisms of fibers of* CAR_1*:*

$$\begin{aligned}
&\mathsf{CAR}_1(0) \simeq M_2 \simeq \mathrm{Cl}_2, \quad \psi_0 : h(a)(0) \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto e,\\
&\mathsf{CAR}_1(x) \simeq C(\mathbb{T}) \rtimes_\beta \mathbb{Z}_2,\\
&\qquad \psi_x : h(a)(x) \mapsto \frac{u}{2}\big((\sqrt{1-x}+\sqrt{x})1 + (\sqrt{1-x}-\sqrt{x})v\big), \quad 0 < x < \frac{1}{2},\\
&\mathsf{CAR}_1\Big(\frac{1}{2}\Big) \simeq C(\mathbb{T}), \quad \psi_{\frac{1}{2}} : h(a)\Big(\frac{1}{2}\Big)(z) \mapsto \frac{1}{\sqrt{2}} z.
\end{aligned}$$

Moreover, the isomorphisms are $\mathbb{T}$*-equivariant when* M_2*,* $C(\mathbb{T}) \rtimes_\beta \mathbb{Z}_2$*,* $C(\mathbb{T})$ *are equipped with the* $\mathbb{T}$*-action given by*

$$\begin{aligned}
w \curvearrowright T &= W(w) T W(w)^*, && T \in M_2,\\
w \curvearrowright T &= \tau_w(T), && T \in C(\mathbb{T}) \rtimes_\beta \mathbb{Z}_2,\\
w \curvearrowright f &= \beta_w(f), && f \in C(\mathbb{T}).
\end{aligned}$$

Proof. For $x \in (0, \frac{1}{2})$, by Theorem 6.7, we have

$$\mathsf{CAR}_1(x) \simeq (\mathcal{C}^\Sigma)(x) \simeq (\mathcal{C}(x))^\sigma = M_2(C(\mathbb{T}))^\sigma,$$

with the natural $C([0, \frac{1}{2}])$-structure on $\mathcal{C}$ given as in (7). By the duality theorem (see e.g. [14, Sec. 7.1]), $M_2 \otimes C(\mathbb{T}) \simeq (C(\mathbb{T}) \rtimes_\beta \mathbb{Z}_2) \rtimes_{\hat{\beta}} \mathbb{Z}_2$, where $\hat{\beta}$ is the dual action of $\widehat{\mathbb{Z}_2} \simeq \mathbb{Z}_2$ on $C(\mathbb{T}) \rtimes_\beta \mathbb{Z}_2$; the double dual action $\hat{\hat{\beta}}$ on $(C(\mathbb{T}) \rtimes_\beta \mathbb{Z}_2) \rtimes_{\hat{\beta}} \mathbb{Z}_2)$ is carried by the isomorphism into $\tilde{\beta}$ on $M_2 \otimes C(\mathbb{T})$ given by $\tilde{\beta}_w = \mathrm{Ad}(W(w)) \otimes \beta_w = \sigma_w$, $w \in \mathbb{Z}_2$ ($w \mapsto W(w)$ is unitary equivalent to the left regular representation of $\mathbb{Z}_2$); hence, using the fixed point theorem, we obtain

$$M_2(C(\mathbb{T}))^\sigma \simeq ((C(\mathbb{T}) \rtimes_\beta \mathbb{Z}_2) \rtimes_{\hat{\beta}} \mathbb{Z}_2)^{\hat{\hat{\beta}}} \simeq C(\mathbb{T}) \rtimes_\beta \mathbb{Z}_2.$$

In particular, the isomorphism maps the generators u and v of $C(\mathbb{T}) \rtimes_\beta \mathbb{Z}_2$ to $z\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ in $M_2(C(\mathbb{T}))^\sigma$ respectively, and from which one can easily see that the element $\frac{u}{2}((\sqrt{1-x}+\sqrt{x})1 + (\sqrt{1-x}-\sqrt{x})v)$ maps to $z\left(\begin{smallmatrix} 0 & \sqrt{x} \\ \sqrt{1-x} & 0 \end{smallmatrix}\right)$.

If $x = 0$, then

$$\mathsf{CAR}_1(0) \simeq \{ f \in M_2(C(\mathbb{T})) \mid f(z) = W(z) f(1) W(z)^*, \, z \in \mathbb{T} \} \simeq M_2(\mathbb{C}),$$

with the isomorphism given by $f \mapsto f(1)$.

For $x = \frac{1}{2}$, one has the isomorphism $\mathcal{C}(\frac{1}{2}) \simeq C(\mathbb{T}) \oplus C(\mathbb{T})$ given by $\phi : f \mapsto V^* f V$, where $V = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$. Let $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and let $\widetilde{\sigma}$ be the action of $\mathbb{Z}_2$ on $M_2(C(\mathbb{T}))$ given by

$$\widetilde{\sigma}(f)(z) = X f(-z) X^*, \quad z \in \mathbb{T}.$$

Then

$$\phi(\sigma(f)) = \widetilde{\sigma}(\phi(f)), \quad f \in M_2(C(\mathbb{T})).$$

Notice that $\widetilde{\sigma}$ acts on $C(\mathbb{T}) \oplus C(\mathbb{T})$ as

$$\widetilde{\sigma}(f, g)(z) = (g, f)(-z), \quad z \in \mathbb{T}.$$

Hence

$$\mathsf{CAR}_1\Big(\frac{1}{2}\Big) \simeq (C(\mathbb{T}) \oplus C(\mathbb{T}))^{\widetilde{\sigma}} \simeq C(\mathbb{T}).$$

The formula for $\psi_{\frac{1}{2}}$ can be easily derived. That the isomorphisms are $\mathbb{T}$-equivariant is straight-forward. □

Remark 6.10. In what follows, we shall also use the isomorphism $\mathsf{CAR}_1(0) \simeq M_2$ given by $h(a)(0) \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. The isomorphism is $\mathbb{T}$-equivariant when M_2 is given the $\mathbb{T}$-action $w \curvearrowright T = W(w)^* T W(w)$, $T \in M_2$, $w \in \mathbb{T}$.

7. CAR_Θ as $C(K_n)$-algebra and its fibers

Let $K_n = [0, \frac{1}{2}]^n$. We shall now use our knowledge about CAR_1 to describe a $C(K_n)$-structure on CAR_Θ and the corresponding fibers. For this, we will use the fact that CAR_Θ is a Rieffel deformation of the tensor product of n copies of CAR_1 (Theorem 5.1). The result will allow us in particular to obtain a classification of all irreducible representations of CAR_Θ providing an alternative proof of [10, Thm. 3].

Let Θ be a real skew-symmetric $n \times n$ matrix, and let α be the action of $\mathbb{T}$ on CAR_1 given by (3). Since α is fiberwise with respect to the $C([0, \frac{1}{2}])$-structure on CAR_1, we get an action on $\mathsf{CAR}_1(x)$. Similarly, $\alpha^{\otimes n}$ is fiberwise with respect to the natural $C(K_n)$-structure on $\mathsf{CAR}_1^{\otimes n}$. Thus, by Theorem 5.1 and Proposition 3.2, one has the following statement.

Proposition 7.1. *There exists a $C(K_n)$-structure on CAR_Θ such that*

$$\mathsf{CAR}_\Theta(x) \simeq (\mathsf{CAR}_1^{\otimes n}(x))^{\frac{\Theta}{2}}.$$

Next we shall give a more explicit description of the fibers.

Given $x = (x_1, \dots, x_n) \in K_n$, let

$$L_x = \{i \in \mathbb{N}_n \mid x_i = 0\},$$
$$M_x = \Big\{i \in \mathbb{N}_n \,\Big|\, 0 < x_i < \frac{1}{2}\Big\},$$
$$R_x = \Big\{i \in \mathbb{N}_n \,\Big|\, x_i = \frac{1}{2}\Big\}.$$

For $S = \{S(1), \dots, S(m)\} \subset \{1, \dots, n\}$, we let Θ_S be the $m \times m$ matrix such that $(\Theta_S)_{i,j} = \Theta_{S(i),S(j)}$. For a set Y, we write $Y^S = \{(a_i)_{i \in S} \mid a_i \in Y\}$. If Y is a group, then so is Y^S with respect to coordinate-wise multiplication; similarly, Y^S is a Hilbert space if so is Y with natural linear operations and scalar product on it. Set

$$\mathrm{Cl}_S = C^*(e_i,\, i \in S \mid e_i^2 = 0,\, e_i e_i^* + e_i^* e_i = 1,\, e_i e_j = e_j e_i) \simeq \mathrm{Cl}_2^{\otimes |S|}$$

and write $C(\mathbb{T}^S_{\Theta_S})$ for the non-commutative torus:

$$C^*(u_k,\, k \in S \mid u_k u_l = e^{-2\pi i \Theta_{k,l}} u_l u_k,\, u_k u_k^* = u_k^* u_k = 1).$$

If $S = \{1, \dots, n\}$, we write simply Cl_{2n} and $C(\mathbb{T}^n_\Theta)$; if $n = 2$, identifying Θ with $\theta := \Theta_{1,2} \in \mathbb{R}$, we denote the non-commutative torus by $C(\mathbb{T}^2_\theta)$.

For $x \in K_n$, let β^x_Θ be the action of $\mathbb{Z}_2^{M_x}$ on $C(\mathbb{T}^{M_x \sqcup R_x}_{\Theta_{M_x \sqcup R_x}})$ given by

$$\beta^x_\Theta(\omega)(u_l) = \omega_l u_l, \quad l \in M_x, \quad \text{and} \quad \beta^x_\Theta(\omega)(u_l) = u_l, \quad l \in R_x,$$

for $\omega = (\omega_k)_{k \in M_x} \in \mathbb{Z}_2^{M_x}$. Then

$$C(\mathbb{T}^{M_x \sqcup R_x}_{\Theta_{M_x \sqcup R_x}}) \rtimes_{\beta^x_\Theta} \mathbb{Z}_2^{M_x}$$

is generated by u_i, $i \in M_x \sqcup R_x$, which satisfy the relations in $C(\mathbb{T}^{M_x \sqcup R_x}_{\Theta_{M_x \sqcup R_x}})$, and by selfadjoint unitaries v_i, $i \in M_x$, such that

$$v_i v_j = v_j v_i \quad \text{and} \quad v_i^* u_j v_i = \beta^x_\Theta(\omega^i)(u_j),$$

$i \in M_x$, $j \in M_x \sqcup R_x$, where $\omega^i_i = -1$ and $\omega^i_k = 1$ otherwise.

Proposition 7.2. *Let $x = (x_1, \ldots, x_n) \in K_n$. Then*

$$\mathsf{CAR}_\Theta(x) \simeq \mathrm{Cl}_{L_x} \otimes C(\mathbb{T}^{M_x \sqcup R_x}_{\Theta_{M_x \sqcup R_x}}) \rtimes_{\beta^x_\Theta} \mathbb{Z}_2^{M_x}.$$

The isomorphism is given by

$$\begin{aligned}
h^x_\Theta(a_i(x)) &= \prod_{k \in L_x} (e_k e_k^* + e^{\pi i \Theta_{i,k}} e_k^* e_k) e_i \otimes 1, \quad i \in L_x, \\
h^x_\Theta(a_i(x)) &= \prod_{k \in L_x} (e_k e_k^* + e^{2\pi i \Theta_{i,k}} e_k^* e_k) \\
&\qquad \otimes \frac{u_i}{2}\big((\sqrt{1-x_i} + \sqrt{x_i}) + (\sqrt{1-x_i} - \sqrt{x_i}) v_i\big), \quad i \in M_x, \\
h^x_\Theta(a_i(x)) &= \prod_{k \in L_x} (e_k e_k^* + e^{2\pi i \Theta_{i,k}} e_k^* e_k) \otimes \frac{u_i}{\sqrt{2}}, \quad i \in R_x.
\end{aligned}$$

Proof. By Proposition 7.1, we have

$$\mathsf{CAR}_\Theta(x) \simeq (\mathsf{CAR}_1^{\otimes n}(x))^{\frac{\Theta}{2}} \simeq \left(\bigotimes_{i=1}^n \mathsf{CAR}_1(x_i)\right)^{\frac{\Theta}{2}}.$$

By Theorem 6.9,

$$\mathsf{CAR}_\Theta(x) \simeq \big(\mathrm{Cl}_{L_x} \otimes (C(\mathbb{T}^{M_x}) \rtimes_\beta \mathbb{Z}_2^{M_x}) \otimes C(\mathbb{T}^{R_x})\big)^{\frac{\Theta}{2}},$$

where the action of $\mathbb{T}^{L_x \sqcup M_x \sqcup R_x}$, which determines the latter Rieffel deformation, is given by the corresponding product of $\mathbb{T}$-actions on $\mathrm{Cl}_2 \simeq M_2$, $C(\mathbb{T}) \rtimes_\beta \mathbb{Z}_2$ and $C(\mathbb{T})$ given in Remark 6.10 (for the action on M_2) and Theorem 6.9. Identify Cl_{L_x} with $\bigotimes_{k \in L_x} M_2$ through $e_k \mapsto \bigotimes_{l \in L_x} a_l$, where $a_k = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $a_l = I_2$ otherwise. The action $\mathbb{T}^{L_x}$ on Cl_{L_x}, given by

$$\alpha_{L_x} : (z_i)_{i \in L_x} \curvearrowright e_i = z_i e_i,$$

is implemented by the unitary representation (U_z) of $\mathbb{T}^{L_x}$ on $(\mathbb{C}^2)^{L_x}$ given by

$$U_z\left(\bigotimes_{i \in L_x} \begin{pmatrix} \xi_i^1 \\ \xi_i^2 \end{pmatrix}\right) = \bigotimes_{x \in L_x} \begin{pmatrix} \xi_i^1 \\ \overline{z_i} \xi_i^2 \end{pmatrix}, \quad z = (z_i)_{i \in L_x} \in \mathbb{T}^{L_x}.$$

By Theorem 4.2 and the remark after it, we have

$$\mathsf{CAR}_\Theta(x) \simeq \mathrm{Cl}_{L_x} \otimes \big((C(\mathbb{T}^{M_x}) \rtimes_\beta \mathbb{Z}_2^{M_x}) \otimes C(\mathbb{T}^{R_x})\big)^{\frac{\Theta_{M_x \sqcup R_x}}{2}}.$$

Since $\tilde{\beta} := \beta \otimes \mathrm{id} : \mathbb{Z}_2{}^{M_x \sqcup R_x} \to \mathrm{Aut}(C(\mathbb{T}^{M_x}) \otimes C(\mathbb{T}^{R_x}))$ commutes with the action that defines the Rieffel deformation, by Remark 3.6 and Proposition 3.3, we have

$$\begin{aligned}\mathsf{CAR}_\Theta(x) &\simeq \mathrm{Cl}_{L_x} \otimes \big(C(\mathbb{T}^{M_x \sqcup R_x}) \rtimes_{\tilde{\beta}} \mathbb{Z}_2^{M_x}\big)^{\frac{\Theta_{M_x \sqcup R_x}}{2}} \\ &\simeq \mathrm{Cl}_{L_x} \otimes \big(C(\mathbb{T}^{M_x \sqcup R_x})\big)^{\frac{\Theta_{M_x \sqcup R_x}}{2}} \rtimes_{\beta^x_\Theta} \mathbb{Z}_2^{M_x} \\ &\simeq \mathrm{Cl}_{L_x} \otimes \big(C(\mathbb{T}^{M_x \sqcup R_x}_{\Theta_{M_x \sqcup R_x}}) \rtimes_{\beta^x_\Theta} \mathbb{Z}_2^{M_x}\big).\end{aligned}$$

To see the formulas, recall the maps Φ and Ψ from Section 4. If $i \in L_x$, then

$$h^x_\Theta(a_i(x)) = \Phi(e_i \otimes 1) = \Psi(e_i) \otimes 1.$$

Let $\xi \in (\mathbb{C}^2)^{L_x}$ be homogeneous of order $q = (b_i)_{i \in L_x}$ with $b_i \in \{0, -1\}$, i.e. $\xi = \bigotimes_{i \in L_x} f_{b_i}$, where $\big\{f_0 = \binom{1}{0},\ f_{-1} = \binom{0}{1}\big\}$ is the standard basis in $\mathbb{C}^2$. If $b_k = -1$, then $e_k^* e_k(\xi) = \xi$, $e_k e_k^*(\xi) = 0$. Since e_j is homogeneous of order $p = \delta_j \in \mathbb{Z}^{L_x}$, one has

$$\begin{aligned}\Psi(e_i)\xi &= e^{2\pi i \langle \Theta_{L_x}(\delta_i)/2, q\rangle} e_i \xi = e^{2\pi i \sum_{k \in L_x} \frac{1}{2}\Theta_{k,i} b_k} e_i \xi \\ &= \prod_{k : b_k = -1} e^{-\pi i \Theta_{k,i}} e_i \xi = \prod_{k \in L_x} (e_k e_k^* + e^{-\pi i \Theta_{k,i}} e_k^* e_k) e_i \xi.\end{aligned}$$

Let $i \in R_x$. Since u_i is homogeneous of order $\delta_i \in \mathbb{Z}^{M_x \sqcup R_x}$, by Theorem 4.2,

$$h^x_\Theta(a_i(x)) = (\alpha_{L_x})_{\omega(-\delta_i)}(\Psi(1)) U_{\omega(-2\delta_i)} \otimes \frac{1}{\sqrt{2}} u_i = U_{\omega(-2\delta_i)} \otimes \frac{1}{\sqrt{2}} u_i.$$

Notice that

$$U_z = \prod_{k \in L_x} (e_k e_k^* + \overline{z_k} e_k^* e_k), \quad z = (z_i)_{i \in L_x},$$

and $\omega_j(-\delta_i) = e^{-2\pi i \Theta_{i,j}}$. Thus

$$h^x_\Theta(a_i(x)) = \prod_{k \in L_x} (e_k e_k^* + e^{2\pi i \Theta_{i,k}} e_k^* e_k) \otimes \frac{1}{\sqrt{2}} u_i.$$

If $i \in M_x$, then similar calculations give

$$\begin{aligned}h^x_\Theta(a_i(x)) = \prod_{k \in L_x} &(e_k e_k^* + e^{2\pi i \Theta_{i,k}} e_k^* e_k) \\ &\otimes \frac{u_i}{2}\big((\sqrt{1 - x_i} + \sqrt{x_i}) + (\sqrt{1 - x_i} - \sqrt{x_i}) v_i\big).\end{aligned} \qquad \square$$

Now, having a description of fibers of CAR_Θ, we can classify all irreducible representations of CAR_Θ. The following lemma reduces the procedure to the classification of irreducible representations of the fibers.

Lemma 7.3 ([14, Prop. C.5]). *Suppose a C^*-algebra $\mathcal{A}$ is equipped with a $C_0(X)$-structure. Then any irreducible representation of $\mathcal{A}$ factors through an irreducible representation of a fiber $\mathcal{A}(x)$ for some $x \in X$.*

Lemma 7.3 and Proposition 7.2 reduce the classification of all irreducible representations of CAR_Θ to that of the C^*-algebra $\mathrm{Cl}_{2k} \otimes C(\mathbb{T}_\Theta^{n+m}) \rtimes_{\beta_\Theta} \mathbb{Z}_2^n$. We will next derive explicit formulas reducing further the classification to the classification of irreducible representations of a non-commutative torus.

As in the proof of Proposition 7.2, we write e_i for the image of $e_i \in \mathrm{Cl}_S$ in $\bigotimes_{k\in S} M_2$ for $S = L_x$ and $S = M_x$, $x \in [0, \frac{1}{2}]^{M_x}$, i.e. $e_i = \bigotimes_{k\in S} a_k$ with $a_i = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $a_k = I_2$ otherwise.

Lemma 7.4. *Any irreducible representation of* CAR_Θ *is unitary equivalent to a subrepresentation of the representation* $\rho_x \circ \mathrm{ev}_x$, $x = (x_i)_{i\in M_x} \in [0, \frac{1}{2}]^{M_x}$, $\mathrm{ev}_x : \mathsf{CAR}_\Theta \to \mathsf{CAR}_\Theta(x)$, $a \mapsto a(x)$ *and* ρ_x *is the representation of* $\mathsf{CAR}_\Theta(x)$ *on* $(\bigotimes_{k\in L_x} \mathbb{C}^2) \otimes (\bigotimes_{k\in M_x} \mathbb{C}^2) \otimes H$ *given by*

$$
\begin{aligned}
\rho_x(a_i(x)) &= \prod_{k\in L_x} (e_k e_k^* + e^{\pi i \Theta_{i,k}} e_k^* e_k) e_i \otimes 1 \otimes 1_H, && i \in L_x,\\
\rho_x(a_i(x)) &= \prod_{k\in L_x} (e_k e_k^* + e^{2\pi i \Theta_{i,k}} e_k^* e_k) \\
&\qquad \otimes (\sqrt{x_i} e_i + \sqrt{1-x_i} e_i^*) \otimes u_i, && i \in M_x,\\
\rho_x(a_i(x)) &= \prod_{k\in L_x} (e_k e_k^* + e^{2\pi i \Theta_{i,k}} e_k^* e_k) \otimes 1 \otimes \frac{1}{\sqrt{2}} u_i, && i \in R_x.
\end{aligned} \tag{8}
$$

where $\{u_i \mid i \in M_x \sqcup R_x\}$ *is an irreducible representation of* $C(\mathbb{T}^{M_x\sqcup R_x}_{\Theta_{M_x\sqcup R_x}})$ *on the Hilbert space* H.

Proof. Proposition 7.2 and the duality arguments for crossed products as in the proof of Theorem 6.9 give

$$
\begin{aligned}
\mathsf{CAR}_\Theta(x) &\simeq \mathrm{Cl}_{L_x} \otimes \big(M_{2|M_x|} \otimes C(\mathbb{T}^{M_x\sqcup R_x}_{\Theta_{M_x\sqcup R_x}})\big)^{\tilde\beta_{\Theta_{M_x\sqcup R_x}}} \\
&\subset \Big(\bigotimes_{k\in L_x} M_2\Big) \otimes \Big(\bigotimes_{k\in M_x} M_2\Big) \otimes C(\mathbb{T}^{M_x\sqcup R_x}_{\Theta_{M_x\sqcup R_x}}),
\end{aligned}
$$

where $\tilde\beta_{\Theta_{M_x\sqcup R_x}}$ is defined by

$$
\tilde\beta_{\Theta_{M_x\sqcup R_x}}(w) = \mathrm{Ad}(W(w_i))^{\otimes|M_x|} \otimes \beta^x_\Theta(w), \quad w = (w_i)_{i\in M_x} \in \mathbb{Z}_2^{M_x}.
$$

The imbedding is given by (8) with $(u_i)_{i\in M_x\sqcup R_x}$ the generators of $C(\mathbb{T}^{M_x\sqcup R_x}_{\Theta_{M_x\sqcup R_x}})$. As any irreducible representation of

$$
\Big(\bigotimes_{k\in L_x} M_2\Big) \otimes \Big(\bigotimes_{k\in M_x} M_2\Big) \otimes C(\mathbb{T}^{M_x\sqcup R_x}_{\Theta_{M_x\sqcup R_x}})
$$

is unitary equivalent to $\mathrm{id} \otimes \mathrm{id} \otimes \pi$, where π is a representation of $C(\mathbb{T}^{M_x\sqcup R_x}_{\Theta_{M_x\sqcup R_x}})$, the statement now follows from [5, Prop. 2.10.2]. □

The next result was proved in [10], but here we present its alternative proof that uses essentially a new approach employing $C(K_n)$-structure of CAR_Θ. The representations in the list below are unitary equivalent to those given in [10, Thm. 3].

Theorem 7.5. *Any irreducible representation of* CAR_Θ *is unitary equivalent to a representation* τ_x, $x \in [0, \frac{1}{2}]^{M_x}$, *given on* $(\bigotimes_{k\in L_x} \mathbb{C}^2) \otimes (\bigotimes_{k\in M_x} \mathbb{C}^2) \otimes H$ *by*

$$
\begin{aligned}
\tau_x(a_i) &= \prod_{k\in L_x} (e_k e_k^* + e^{\pi i \Theta_{i,k}} e_k^* e_k) e_i \otimes 1 \otimes 1_H, && i \in L_x,\\
\tau_x(a_i) &= \prod_{k\in L_x} (e_k e_k^* + e^{\pi i \Theta_{i,k}} e_k^* e_k)\\
&\quad \otimes \Big(\Big(\prod_{k\in M_x, k<i} (e_k^* e_k + e^{2\pi i \Theta_{i,k}} e_k e_k^*) \otimes 1_H \Big)\\
&\qquad \times \Big(\sqrt{x_i} \prod_{k\in M_x, k\geq i} (e_k^* e_k + e^{4\pi i \Theta_{i,k}} e_k e_k^*) e_i \otimes v_i\\
&\qquad\qquad + \sqrt{1-x_i}\, e_i^* \otimes 1_H \Big) \Big), && i \in M_x,\\
\tau_x(a_i) &= \prod_{k\in L_x} (e_k e_k^* + e^{\pi i \Theta_{i,k}} e_k^* e_k)\\
&\quad \otimes \prod_{k\in M_x} (e_k^* e_k + e^{2\pi i \Theta_{i,k}} e_k e_k^*) \otimes \frac{v_i}{\sqrt{2}}, && i \in R_x.
\end{aligned}
$$

where $(v_i)_{i\in M_x \sqcup R_x}$ *defines an irreducible representation of* $C(\mathbb{T}_\Sigma^{M_x\sqcup R_x})$ *on* H, *where*

$$
\Sigma_{i,j} = \begin{cases} 4\Theta_{i,j}, & i,j \in M_x,\\ 2\Theta_{i,j}, & (i,j) \text{ or } (j,i) \in M_x \times R_x,\\ \Theta_{i,j}, & i,j \in R_x. \end{cases}
$$

Moreover, two such irreducible representations τ_x *and* τ_y *are unitary equivalent if and only if* $x = y$ *and the corresponding representations of* $C(\mathbb{T}_\Sigma^{M_x\sqcup R_x})$ *are unitary equivalent.*

Proof. Recall the representation ρ_x from Lemma 7.4, and consider the unitary operators on $(\bigotimes_{k\in L_x} \mathbb{C}^2) \otimes (\bigotimes_{k\in M_x} \mathbb{C}^2) \otimes H$ given by

$$
V_k = 1 \otimes (e_k e_k^* \otimes 1 + e_k^* e_k \otimes u_k), \quad k \in M_x.
$$

Set $V = V_{i_1} \dots V_{i_{|M_x|}}$, where $M_x = \{i_1, \dots, i_{|M_x|}\}$ and $i_k < i_l$ if $k < l$. Then

$$
\begin{aligned}
V\rho_x(a_i)V^* &= \rho_x(a_i) = \prod_{k\in L_x} (e_k e_k^* + e^{\pi i \Theta_{i,k}} e_k^* e_k) e_i \otimes 1 \otimes 1_H, && i \in L_x,\\
V\rho_x(a_i)V^* &= \prod_{k\in L_x} (e_k e_k^* + e^{\pi i \Theta_{i,k}} e_k^* e_k)\\
&\quad \otimes \Big(\Big(\prod_{k\in M_x, k<i} (e_k^* e_k + e^{2\pi i \Theta_{i,k}} e_k e_k^*) \otimes 1_H \Big)\\
&\qquad \times \Big(\sqrt{x_i} \prod_{k\in M_x, k\geq i} (e_k^* e_k + e^{4\pi i \Theta_{i,k}} e_k e_k^*) e_i \otimes u_i^2\\
&\qquad\qquad + \sqrt{1-x_i}\, e_i^* \otimes 1_H \Big) \Big), && i \in M_x,\\
V\rho_x(a_i)V^* &= \prod_{k\in L_x} (e_k e_k^* + e^{\pi i \Theta_{i,k}} e_k^* e_k)\\
&\quad \otimes \prod_{k\in M_x} (e_k^* e_k + e^{2\pi i \Theta_{i,k}} e_k e_k^*) \otimes \frac{1_H}{\sqrt{2}}, && i \in R_x.
\end{aligned}
$$

It is easy to see that the family $\{u_i^2 \mid i \in M_x\} \cup \{u_i \mid i \in R_x\}$ forms a representation of $C(\mathbb{T}_\Sigma^{M_x \sqcup R_x})$. Moreover, any such family with v_i instead of u_i^2, $i \in M_x$, and v_i instead of u_i, $i \in R_x$, where $(v_i)_{i \in M_x \sqcup R_x}$ defines a representation of $C(\mathbb{T}_\Sigma^{M_x \sqcup R_x})$, is a representation of CAR_Θ.

Fix $x \in [0, \frac{1}{2}]^{M_x}$, and let C be an operator intertwining the representations corresponding to families $\mathbb{V} = (v_i)_{i \in M_x \sqcup R_x}$ and $\mathbb{W} = (w_i)_{i \in M_x \sqcup R_x}$ acting on $H_\mathbb{V}$ and $H_\mathbb{W}$ respectively. Denote them by $\tau_\mathbb{V}$ and $\tau_\mathbb{W}$ respectively; we have $C\tau_\mathbb{V}(a) = \tau_\mathbb{W}(a)C$, $a \in \mathsf{CAR}_\Theta$, i.e. $C \in \operatorname{Hom}(\tau_\mathbb{V}, \tau_\mathbb{W})$. In particular,

$$C\tau_\mathbb{V}(a_i^* a_i) = \tau_\mathbb{W}(a_i^* a_i)C, \quad i \in L_x \sqcup M_x \sqcup R_x. \tag{9}$$

We have

$$\tau_\mathbb{V}(a_i^* a_i) = \begin{cases} e_i^* e_i \otimes 1 \otimes 1_{H_\mathbb{V}}, & i \in L_x, \\ 1 \otimes ((1 - x_i) e_i e_i^* + x_i e_i^* e_i) \otimes 1_{H_\mathbb{V}}, & i \in M_x, \\ 1 \otimes 1 \otimes 1_{H_\mathbb{V}}, & i \in R_x. \end{cases} \tag{10}$$

Therefore, it is easy to see that (9) implies that $C = \sum_i p_i \otimes C_i$, where $p_i = \prod_{k \in L_x \sqcup M_x} q_k^i$ with

$$q_k^i \in \{e_k e_k^*, e_k^* e_k\}_{k \in L_x \sqcup M_x} \subset \bigotimes_{k \in L_x \sqcup M_x} M_2 \ \Big(= \Big(\bigotimes_{k \in L_x} M_2 \Big) \otimes \Big(\bigotimes_{k \in M_x} M_2 \Big) \Big),$$

$C_i \in \mathcal{B}(H_\mathbb{V}, H_\mathbb{W})$; the summation is over all possible products $p_i = \prod_{k \in L_x \sqcup M_x} q_k^i$.

The condition $C\tau_\mathbb{V}(a_k) = \tau_\mathbb{W}(a_k)C$ for $k \in L_x$ is equivalent to

$$\sum_i \alpha_{i,k} p_i e_k \otimes C_i = \sum_i \alpha_{i,k} e_k p_i \otimes C_i$$

for some nonzero $\alpha_{i,k}$. As

$$p_i e_k = \begin{cases} 0 & \text{if } q_k^i = e_k^* e_k, \\ e_k p_{\sigma_k(i)} & \text{if } q_k^i = e_k e_k^*, \end{cases}$$

here $q_k^{\sigma_k(i)} = e_k^* e_k$ if $q_k^i = e_k e_k^*$ and vice versa, and $q_j^{\sigma_k(i)} = q_j^i$ otherwise (i.e. we swap the projection q_k^i to the other possible value for the kth factor), we obtain $C_i = C_{\sigma_k(i)}$ for all $k \in L_x$. Similarly, the condition $C\tau_\mathbb{V}(a_k) = \tau_\mathbb{W}(a_k)C$ for $k \in M_x$ implies first that $C_i = C_{\sigma_k(i)}$, $k \in M_x$, giving now that all C_i's are equal; call the common value C' and get $C = 1 \otimes C'$. Then we obtain that $C' v_k = w_k C'$ for all $k \in M_x$. The condition $C\tau_\mathbb{V}(a_k) = \tau_\mathbb{W}(a_k)C$ for $k \in R_x$ gives $C' v_k = w_k C'$. Therefore, we have a bijection $\operatorname{Hom}(\tau_\mathbb{V}, \tau_\mathbb{W}) \to \operatorname{Hom}(\mathbb{V}, \mathbb{W})$, $1 \otimes C \mapsto C$. From this, it easily follows that $\tau_\mathbb{V}$ is irreducible if and only if $\mathbb{V}$ defines an irreducible representation $C(\mathbb{T}_\Sigma^{M_x \sqcup R_x})$.

That τ_x and τ_y are not unitary equivalent for $x \neq y$ follows from the fact that the spectrum of $\tau_x(a_i^* a_i)$ is in $\{1, x_i, 1 - x_i\}$ if $i \in M_x$, see (10). $\square$

8. Classification of CAR_Θ

This section contains the main result of the paper and concerns the classification of CAR_Θ up to isomorphism. To obtain the result, we will employ another $C(K)$-structure coming from the center of CAR_Θ and relate it to the $C([0,\frac{1}{2}])$-structure on the algebra. We will then use K-theoretical arguments applied to the fibers to derive the result.

Let Θ_1 and Θ_2 be skew-symmetric real $n\times n$ matrices. Suppose $\varphi\colon \mathsf{CAR}_{\Theta_1}\to \mathsf{CAR}_{\Theta_2}$ is an isomorphism. It induces an isomorphism of the centers and a homeomorphism $\alpha:\operatorname{spec} Z(\mathsf{CAR}_{\Theta_2})\to \operatorname{spec} Z(\mathsf{CAR}_{\Theta_1})$ of their Gelfand spectrum. Let $Z_{\Theta_i}=\operatorname{spec} Z(\mathsf{CAR}_{\Theta_i})$, $i=1,2$. We have a natural $C(Z_{\Theta_i})$-structure on CAR_{Θ_i} given by the inverse of the Gelfand transform $\hat{g}\mapsto g$, $g\in Z(\mathsf{CAR}_{\Theta_i})$, $i=1,2$: $\Phi_i(\hat{g})\cdot a = ga$, $a\in \mathsf{CAR}_{\Theta_i}$. Letting

$$I_z^\Theta=\{ga\mid a\in \mathsf{CAR}_\Theta,\ \hat{g}(z)=0\},\quad z\in Z_\Theta,$$

we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & I^{\Theta_1}_{\alpha(z)} \arrow[r] \arrow[d, "\varphi|_{I^{\Theta_1}_{\alpha(z)}}"] & \mathsf{CAR}_{\Theta_1} \arrow[r] \arrow[d, "\varphi"] & \mathsf{CAR}_{\Theta_1}(\alpha(z)) \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & I^{\Theta_2}_{z} \arrow[r] & \mathsf{CAR}_{\Theta_2} \arrow[r] & \mathsf{CAR}_{\Theta_2}(z) \arrow[r] & 0,
\end{tikzcd}$$

which gives the isomorphisms $\mathsf{CAR}_{\Theta_1}(\alpha(z))\simeq \mathsf{CAR}_{\Theta_2}(z)$ for every $z\in Z_{\Theta_2}$.

The $C(K_n)$-structure on CAR_Θ induces an injective homomorphism from $C(K_n)$ to $C(Z_\Theta)$ and hence a canonical continuous surjection $\pi: Z_\Theta\to K_n$. We also have for all $z\in Z_\Theta$ that $I_{\pi(z)}$ is an ideal in I_z^Θ and hence

$$\mathsf{CAR}_\Theta(z)\simeq \mathsf{CAR}_\Theta/I_z^\Theta\simeq (\mathsf{CAR}_\Theta/I_{\pi(z)})/(I_z^\Theta/I_{\pi(z)})$$

so that $\mathsf{CAR}_\Theta(z)$ is a quotient of $\mathsf{CAR}_\Theta(\pi(z))$.

Definition 8.1. Recall L_x, M_x and R_x, $x\in K_n$, from Section 7, and for each $z\in Z_\Theta$, define the face signature to be $\mathsf{face}(z)=(|L_{\pi(z)}|,|M_{\pi(z)}|,|R_{\pi(z)}|)$.

Definition 8.2. We say that a real skew-symmetric $n\times n$ matrix Θ is *irrational* if, whenever $p\in\mathbb{Z}^n$ satisfies $e^{2\pi i\langle p,\Theta(q)\rangle}=1$ for all $q\in\mathbb{Z}^n$, then $p=0$.

We note that some authors choose to call such Θ non-degenerate, see e.g. [8]. We now give a description of the fibers of CAR_Θ over Z_Θ using the above connection with $C(K_n)$-structure and the description of fibers given in Proposition 7.2.

Let Θ be an irrational skew-symmetric $n\times n$-matrix. For $z\in Z_\Theta$, set $x=\pi(z)\in K_n$ and $l=|L_x|$, $m=|M_x|$, $r=|R_x|$. The description splits in the following four cases.

(i) If $m+r\geq 2$, then $\mathsf{CAR}_\Theta(x)\simeq \mathrm{Cl}_{2l}\otimes C(\mathbb{T}^{m+r}_{\Theta_{M_x\sqcup R_x}})\rtimes_{\beta^x_\Theta}\mathbb{Z}_2^m$. Since $\Theta_{M_x\sqcup R_x}$ is irrational, $Z(\mathsf{CAR}_\Theta(x))\simeq\mathbb{C}$. From this, one can easily derive that $I_{\pi(z)}=I_z^\Theta$ and hence

$$\mathsf{CAR}_\Theta(z)\simeq \mathrm{Cl}_{2l}\otimes C(\mathbb{T}^{m+r}_{\Theta_{M_x\sqcup R_x}})\rtimes_{\beta^x_\Theta}\mathbb{Z}_2^m.$$

(ii) If $l=n-1$, $m=1$, then $\mathsf{CAR}_\Theta(z)$ is a quotient of $\mathsf{CAR}_\Theta(x)\simeq \mathrm{Cl}_{2n-2}\otimes C(\mathbb{T})\rtimes_{\beta^x_\Theta}\mathbb{Z}_2$. As $C(\mathbb{T})\rtimes_\beta\mathbb{Z}_2\simeq M_2(C(\mathbb{T}))$ (see e.g. [4, Prop. 3.4]), we have $\mathsf{CAR}_\Theta(x)\simeq \mathrm{Cl}_{2n}\otimes C(\mathbb{T})$ with all quotients being of the form $\mathrm{Cl}_{2n}\otimes C(K)$ for some closed subset $K\subset\mathbb{T}$.

(iii) If $l = n-1$, $r = 1$, then $\mathsf{CAR}_\Theta(z)$ is a quotient of $\mathsf{CAR}_\Theta(x) \simeq \mathrm{Cl}_{2n-2} \otimes C(\mathbb{T})$. All such quotients have the form $\mathrm{Cl}_{2n-2} \otimes C(K)$ for a closed subset $K \subset \mathbb{T}$.

(iv) If $l = n$, then $\mathsf{CAR}_\Theta(x) \simeq \mathrm{Cl}_{2n} \simeq \mathsf{CAR}_\Theta(z)$.

To prove the main result, we need the following auxiliary lemmas.

Lemma 8.3. *Let Θ be irrational and $\sigma \in \mathrm{Aut}(C(\mathbb{T}^n_\Theta))$, given by $\sigma(u_1) = -u_1$, $\sigma(u_k) = u_k$, $k > 1$. Then*

$$C(\mathbb{T}^n_\Theta)^\sigma \simeq C(\mathbb{T}^n_{\Theta^{(1)}}),$$

where $\Theta^{(1)}_{i,j} = 2\Theta_{i,j}$ if either i or $j = 1$ and $\Theta^{(1)}_{i,j} = \Theta_{i,j}$ otherwise.

Proof. We note first that $C(\mathbb{T}^n_\Theta)^\sigma = \{x + \sigma(x) \mid x \in C(\mathbb{T}^n_\Theta)\}$ from which it is easy to see using approximation arguments that $C(\mathbb{T}^n_\Theta)^\sigma$ equals the C^*-subalgebra $C^*(u_1^2, u_2, \ldots, u_n)$, generated by u_1^2, $u_2, \ldots, u_n$. Furthermore, the map $u_1 \mapsto u_1^2$, $u_k \mapsto u_k$, $k > 1$, extends to a surjective $*$-homomorphism from $C(\mathbb{T}^n_{\Theta^{(1)}})$ to $C^*(u_1^2, u_2, \ldots, u_n)$. The statement now follows from the simplicity of $C(\mathbb{T}^n_{\Theta^{(1)}})$, see e.g. [8, Thm. 1.9]. ☐

For a skew-symmetric real matrix Θ of size $n = m + r$, let Σ be given by

$$\Sigma_{i,j} = \begin{cases} 4\Theta_{i,j}, & i,j \le m, \\ 2\Theta_{i,j}, & \text{either } i \le m \text{ or } j \le m, \\ \Theta_{i,j}, & i,j > m. \end{cases}$$

Define $\beta_\Theta : \mathbb{Z}_2^m \to \mathrm{Aut}(C(\mathbb{T}^{m+r}_\Theta))$ by

$$\beta_\Theta(\omega)(u_k) = \begin{cases} \omega_k u_k, & k \le m, \\ u_k, & k > m, \end{cases}$$

where $\omega = (\omega_1, \ldots, \omega_m)$.

Lemma 8.4. *Let Θ, Σ and β_Θ be as above. Then*

$$C(\mathbb{T}^{m+r}_\Theta) \rtimes_{\beta_\Theta} \mathbb{Z}_2^m \simeq \mathrm{Cl}_{2m} \otimes C(\mathbb{T}^{m+r}_\Sigma).$$

Proof. Let first $m = 1$ and write σ for β_Θ. The arguments as in Theorem 6.9 show that

$$C(\mathbb{T}^{1+r}_\Theta) \rtimes_\sigma \mathbb{Z}_2 \simeq (M_2(C(\mathbb{T}^{1+r}_\Theta)))^{\tilde\sigma},$$

where $\tilde\sigma = \mathrm{Ad}\, W \otimes \sigma$ and $W = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Furthermore, if $U = \left(\begin{smallmatrix} 0 & 1 \\ u_1 & 0 \end{smallmatrix}\right)$, then

$$U M_2(C(\mathbb{T}^{1+r}_\Theta))^{\tilde\sigma} U^* = M_2(C(\mathbb{T}^{1+r}_\Theta)^\sigma),$$

as

$$M_2(C(\mathbb{T}^{1+r}_\Theta))^{\tilde\sigma} = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \,\middle|\, A, D \in C(\mathbb{T}^{1+r}_\Theta)^\sigma,\, B, C \in C(\mathbb{T}^{1+r}_\Theta)^\sigma(-1) \right\},$$

where $\mathcal{A}^\sigma(-1) = \{a \in \mathcal{A} \mid \sigma(a) = -a\}$. This together with Lemma 8.3 yields the statement for $m = 1$. To see it for general m, we note first that

$$C(\mathbb{T}^{m+r}_\Theta) \rtimes_{\beta_\Theta} \mathbb{Z}_2^m \simeq (C(\mathbb{T}^{1+(m-1+r)}_\Theta) \rtimes_\sigma \mathbb{Z}_2) \rtimes_{\beta'_\Theta} \mathbb{Z}_2^{m-1}$$

which together with the previous result and simple calculations gives

$$C(\mathbb{T}_\Theta^{m+r}) \rtimes_{\beta_\Theta} \mathbb{Z}_2^m \simeq \mathrm{Cl}_2 \otimes C(\mathbb{T}_{\Theta^{(1)}}^{m+r}) \rtimes_{\beta_\Theta^{(1)}} \mathbb{Z}_2^{m-1},$$

where $\Theta^{(1)}$ is as in Lemma 8.3, β'_Θ acts as β_Θ on $C(\mathbb{T}_\Theta^{1+(m-1+r)})$ and identically on the generator of $\mathbb{Z}_2$, and $\beta_\Theta^{(1)} : \mathbb{Z}_2^{m-1} \to \mathrm{Aut}(C(\mathbb{T}_{\Theta^{(1)}}^{m+r})$ is given by $\beta_\Theta^{(1)}(\omega)(u_i) = \omega_i u_i$ if $2 \le i \le m$ and $\beta_\Theta^{(1)}(\omega)(u_1) = u_1$ for $\omega = (\omega_2, \dots, \omega_m)$. The statement now follows by the successive application of the above argument. □

Lemma 8.5. *For $z \in Z_\Theta$, set $m = |M_{\pi(z)}|$ and $r = |R_{\pi(z)}|$. If $m + r > 1$ and Θ is irrational, then*

$$K_0(\mathsf{CAR}_\Theta(z)) \simeq \mathbb{Z}^{2^{m+r-1}}.$$

Proof. If $m + r > 1$, then

$$\mathsf{CAR}_\Theta(z) \simeq \mathrm{Cl}_{2l} \otimes C(\mathbb{T}^{m+r}_{\Theta_{M_x \sqcup R_x}}) \rtimes_{\beta_\Theta^x} \mathbb{Z}_2^m,$$

and by Lemma 8.4, $\mathsf{CAR}_\Theta(z) \simeq \mathrm{Cl}_{2l+2m} \otimes C(\mathbb{T}_\Sigma^{m+r})$. Thus, by Proposition 3.7,

$$K_0(\mathsf{CAR}_\Theta(z)) \simeq K_0(C(\mathbb{T}_\Sigma^{m+r})) \simeq K_0(C(\mathbb{T}^{m+r})) \simeq \mathbb{Z}^{2^{m+r-1}}. \qquad \square$$

Lemma 8.6. *Let $\theta_1, \theta_2, \theta_3 \in \mathbb{R} \setminus \mathbb{Q}$. The C^*-algebras*

$$\mathrm{Cl}_{2n-4} \otimes C(\mathbb{T}^2_{\theta_1}), \quad \mathrm{Cl}_{2n-4} \otimes C(\mathbb{T}^2_{\theta_2}) \rtimes_{\beta_1} \mathbb{Z}_2, \quad \mathrm{Cl}_{2n-4} \otimes C(\mathbb{T}^2_{\theta_3}) \rtimes_{(\beta_1 \times \beta_2)} \mathbb{Z}_2^2$$

are mutually nonisomorphic.

Proof. It is known that $C(\mathbb{T}^2_\theta)$ and $\bigotimes_{k \in S} M_{n(k)} \otimes C(\mathbb{T}^2_\theta)$ are C^*-algebras with unique normalized trace which we denote by tr. By a result of Rieffel ([12, Thm. 1.2, Prop. 1.3]), $\mathrm{tr}(\mathcal{P}(M_n \otimes C(\mathbb{T}^2_\theta))) = n^{-1}(\mathbb{Z} + \theta\mathbb{Z}) \cap [0,1]$, where $\mathcal{P}(A)$ is the set of projections of A. Therefore,

$$\begin{aligned}
\mathrm{tr}(\mathcal{P}(\mathrm{Cl}_{2n-4} \otimes C(\mathbb{T}^2_{\theta_1}))) &= \frac{1}{2^{n-2}} \mathrm{tr}(\mathcal{P}(C(\mathbb{T}^2_{\theta_1}))) \\
&= \frac{1}{2^{n-2}} (\mathbb{Z} + \theta_1\mathbb{Z}) \cap [0,1], \\
\mathrm{tr}(\mathcal{P}(\mathrm{Cl}_{2n-4} \otimes C(\mathbb{T}^2_{\theta_2}) \rtimes_{\beta_1} \mathbb{Z}_2)) &\overset{\text{Lem. 8.4}}{=} \mathrm{tr}(\mathcal{P}(\mathrm{Cl}_{2n-2} \otimes C(\mathbb{T}^2_{2\theta_2})) \\
&= \frac{1}{2^{n-1}} (\mathbb{Z} + 2\theta_2\mathbb{Z}) \cap [0,1], \\
\mathrm{tr}(\mathcal{P}(\mathrm{Cl}_{2n-4} \otimes C(\mathbb{T}^2_{\theta_3}) \rtimes_{(\beta_1 \times \beta_2)} \mathbb{Z}_2^2)) &\overset{\text{Lem. 8.4}}{=} \mathrm{tr}(\mathcal{P}(\mathrm{Cl}_{2n} \otimes C(\mathbb{T}^2_{4\theta_3}))) \\
&= \frac{1}{2^n} (\mathbb{Z} + 4\theta_3\mathbb{Z}) \cap [0,1],
\end{aligned}$$

showing that the C^*-algebras $\mathrm{Cl}_{2n-4} \otimes C(\mathbb{T}^2_{\theta_1})$, $\mathrm{Cl}_{2n-4} \otimes (C(\mathbb{T}^2_{\theta_2}) \rtimes_{\beta_1} \mathbb{Z}_2)$ and $\mathrm{Cl}_{2n-4} \otimes (C(\mathbb{T}^2_{\theta_3}) \rtimes_{(\beta_1 \times \beta_2)} \mathbb{Z}_2^2)$ are mutually nonisomorphic. □

Lemma 8.7. *Let Θ_1 and Θ_2 be irrational skew-symmetric $n \times n$ matrices, and let $\varphi : \mathsf{CAR}_{\Theta_1} \to \mathsf{CAR}_{\Theta_2}$ be an isomorphism with the induced homeomorphism $\alpha : Z_{\Theta_2} \to Z_{\Theta_1}$. If $z \in Z_{\Theta_2}$ satisfies $\mathsf{face}(z) = (n-2, 0, 2)$, then $\mathsf{face}(z) = \mathsf{face}(\alpha(z))$.*

Proof. We observe first that if $z \in Z_\Theta$ is such that $m = |M_{\pi(z)}|$ and $r = |R_{\pi(z)}|$ satisfy $m + r \leq 1$, then z is either of type (ii), (iii) or (iv), and hence $\mathsf{CAR}_\Theta(z)$ is a C^*-algebra of the form $M_n(C(X))$, which is either finite-dimensional or non-simple, while if $m + r > 1$, then $\mathsf{CAR}_\Theta(z)$ is infinite-dimensional and simple. From this, we can conclude that if Θ_1 and Θ_2 are irrational, then $|M_{\pi(\alpha(z))}| + |R_{\pi(\alpha(z))}| \leq 1$ when $|M_{\pi(z)}| + |R_{\pi(z)}| \leq 1$. Therefore, if $\mathsf{face}(z) = (n-2, 0, 2)$, then $|M_{\pi(\alpha(z))}| + |R_{\pi(\alpha(z))}|$ is necessarily larger than 1, and by Lemma 8.5 must be exactly 2. This gives that the possible values of $\mathsf{face}(\alpha(z))$ are $(n-2, 0, 2)$, $(n-2, 1, 1)$ and $(n-2, 2, 0)$. Hence, as $\mathsf{CAR}_{\Theta_1}(\alpha(z)) \simeq \mathsf{CAR}_{\Theta_2}(z)$, to prove the statement, it is enough to see that $\mathsf{CAR}_\Theta(z)$ are nonisomorphic for different z with $(m, r) \in \{(0,2), (1,1), (2,0)\}$. But for $(m, r) = (0,2), (1,1), (2,0)$, $\mathsf{CAR}_\Theta(z)$ is isomorphic to

$$\mathrm{Cl}_{2n-4} \otimes C(\mathbb{T}^2_\theta), \quad \mathrm{Cl}_{2n-4} \otimes C(\mathbb{T}^2_\theta) \rtimes_{\beta_1} \mathbb{Z}_2, \quad \mathrm{Cl}_{2n-4} \otimes C(\mathbb{T}^2_\theta) \rtimes_{(\beta_1 \times \beta_2)} \mathbb{Z}_2^2$$

respectively. Thus Lemma 8.5 concludes the proof. □

A matrix $P = (p_{i,j})_{i,j=1}^n \in M_n$ is called a signed permutation matrix if there exists $(\sigma, b) \in S_n \times \{0,1\}^n$ such that $p_{i,j} = (-1)^{b_i} \delta_{j,\sigma(i)}$. We are now ready to prove our main results.

Theorem 8.8. *Let Θ_1 and Θ_2 be irrational $n \times n$-matrices.*
(i) *If P is a signed permutation matrix, $\Theta_1 = P\Theta_2 P^t$ implies $\mathsf{CAR}_{\Theta_1} \simeq \mathsf{CAR}_{\Theta_2}$.*
(ii) *If $\mathsf{CAR}_{\Theta_1} \simeq \mathsf{CAR}_{\Theta_2}$, then $(\Theta_2)_{i,j} = \pm(\Theta_1)_{\sigma(i,j)} \bmod \mathbb{Z}$ for a bijection σ of the set $\{(i,j) \mid i < j,\ i,j = 1, \ldots, n\}$.*

Proof. (i) If P is a signed permutation matrix which corresponds to a signed permutation $(\sigma, b) \in S_n \times \{1, *\}^n$, then the corresponding isomorphism is given by $\psi_P(a_i) = a_{\sigma(i)}^{b_i}$.

(ii) Let z be the unique element of Z_{Θ_2} such that $\pi(z) = \frac{1}{2}(\delta_i + \delta_j)$, $i < j$. Since $\mathsf{face}(z) = (n-2, 0, 2)$, by Lemma 8.7, $\mathsf{face}(\alpha(z)) = (n-2, 0, 2)$ and hence $\alpha(z) = \frac{1}{2}(\delta_k + \delta_l)$, where $(k, l) = \sigma(i, j)$ for a bijection σ of the set $\{(i,j) \mid i < j,$ $i, j = 1, \ldots, n\}$. Thus

$$\mathsf{CAR}_{\Theta_2}(z) \simeq \mathrm{Cl}_{2n-4} \otimes C(\mathbb{T}^2_{(\Theta_2)_{i,j}}) \simeq \mathsf{CAR}_{\Theta_1}(\alpha(z)) \simeq \mathrm{Cl}_{2n-4} \otimes C(\mathbb{T}^2_{(\Theta_1)_{\sigma(i,j)}}),$$

and by [12, Thm. 3], $(\Theta_2)_{i,j} = \pm(\Theta_1)_{\sigma(i,j)} \bmod \mathbb{Z}$. □

For $\theta \in \mathbb{R}$, write simply CAR_θ for CAR_Θ if $n = 2$ and $\Theta_{1,2} = \theta$. In this case, we have the full classification similar to the classification of two-dimensional non-commutative tori.

Corollary 8.9. *If θ_1, θ_2 are irrational numbers, then $\mathsf{CAR}_{\theta_1} \simeq \mathsf{CAR}_{\theta_2}$ if and only if $\theta_1 = \pm\theta_2 \bmod \mathbb{Z}$.*

Acknowledgments. We wish to express our gratitude to Daniil Proskurin and Magnus Goffeng for helpful discussions. We would also like to thank the anonymous referee for valuable suggestions that led to improvements in the paper.

References

[1] F. Belmonte and M. Măntoiu, Covariant fields of C^*-algebras under Rieffel deformation, SIGMA Symmetry Integrability Geom. Methods Appl. **8** (2012), Paper 091, 12 pp. MR3007268

[2] B. Brenken, A classification of some noncommutative tori, Rocky Mountain J. Math. **20** (1990), no. 2, 389–397. MR1065837

[3] D. Buchholz, G. Lechner, and S. J. Summers, Warped convolutions, Rieffel deformations and the construction of quantum field theories, Comm. Math. Phys. **304** (2011), no. 1, 95–123. MR2793931

[4] M.-D. Choi and F. Latrémolière, C^*-crossed-products by an order-two automorphism, Canad. Math. Bull. **53** (2010), no. 1, 37–50. MR2583209

[5] J. Dixmier, *C^*-algebras*, translated from the French by Francis Jellett, North-Holland Math. Libr., 15, North-Holland Publishing Co., Amsterdam, 1977. MR0458185

[6] J. M. G. Fell, The structure of algebras of operator fields, Acta Math. **106** (1961), 233–280. MR0164248

[7] P. Kasprzak, Rieffel deformation via crossed products, J. Funct. Anal. **257** (2009), no. 5, 1288–1332. MR2541270

[8] N. C. Phillips, Every simple higher dimensional noncommutative torus is an AT algebra, arXiv:math/0609783v1 [math.OA] (2006).

[9] D. Proskurin, Y. Savchuk, and L. Turowska, On C^*-algebras generated by some deformations of CAR relations, in *Noncommutative geometry and representation theory in mathematical physics*, 297–312, Contemp. Math., 391, American Mathematical Society, Providence, RI, 2005. MR2184031

[10] D. P. Proskurin and K. M. Sukretnyi, On $*$-representations of deformations of canonical anticommutation relations, Ukrainian Math. J. **62** (2010), no. 2, 227–240; translated from Ukraïn. Mat. Zh. **62** (2010), no. 2, 203–214. MR2888593

[11] M. A. Rieffel, Deformation quantization for actions of $\mathbf{R}^d$, Mem. Amer. Math. Soc. **106** (1993), no. 506, x+93 pp. MR1184061

[12] M. A. Rieffel, C^*-algebras associated with irrational rotations, Pacific J. Math. **93** (1981), no. 2, 415–429. MR0623572

[13] N. B. Vasil'ev, C^*-algebras with finite-dimensional irreducible representations, Uspehi Mat. Nauk **21** (1966), no. 1 (127), 135–154; translated in Russian Math. Surveys **21** (1966), no. 1, 137–155. MR0201994

[14] D. P. Williams, *Crossed products of C^*-algebras*, Math. Surveys Monogr., 134, American Mathematical Society, Providence, RI, 2007. MR2288954

Received October 26, 2020; accepted June 12, 2021

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