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A preperfectoid approach to Robba rings

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Abstract

We construct so-called preperfectoid spaces $\mathfrak{X}^{\mathrm{perf}}(r)$ and $\mathfrak{X}^{\mathrm{perf}}(r,s)$ based on the character variety $\mathfrak{X}$ and define a "preperfectoid version" of the Robba ring $\mathcal{R}_L(\mathfrak{X}^{\mathrm{perf}})$ as well as associated rings of bounded functions $\mathcal{E}_L^{\dagger}(\mathfrak{X}^{\mathrm{perf}})$ and $\mathcal{E}_L^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}})$ based on these spaces. We show that base change from étale φ-modules over $\mathcal{E}_L^{\dagger}(\mathfrak{X}^{\mathrm{perf}})$ to étale φ-modules over $\mathcal{R}_L(\mathfrak{X}^{\mathrm{perf}})$ is an equivalence of categories. We also discuss the construction of the preperfectoid open unit disk and of the associated Robba ring.

Wir konstruieren sogenannte präperfektoide Räume $\mathfrak{X}^{\mathrm{perf}}(r)$ und $\mathfrak{X}^{\mathrm{perf}}(r,s)$ basierend auf der Charaktervarietät $\mathfrak{X}$ und definieren eine "präperfektoide Version" des Robba-Rings sowie assoziierte Ringe beschränkter Funktionen $\mathcal{E}_L^{\dagger}(\mathfrak{X}^{\mathrm{perf}})$ und $\mathcal{E}_L^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}})$ basierenden auf diesen Räumen. Wir zeigen, dass Basiswechsel von étalen φ-Moduln über $\mathcal{E}_L^{\dagger}(\mathfrak{X}^{\mathrm{perf}})$ zu étalen φ-Modulen über $\mathcal{R}_L(\mathfrak{X}^{\mathrm{perf}})$ eine Kategorienäquivalenz ist. Wir diskutieren außerdem die Konstruktion der präperfektoiden offenen Einheitskreisscheibe und des assoziierten Robba-Rings.

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Contents

Introduction

Let L be a finite extension of $\mathbb{Q}_p$ with ring of integers o_L and absolute Galois group G_L. Considering a Lubin-Tate formal group LT for a uniformizer $\pi \in L$, we obtain an infinite Galois extension L_∞ of L by adjoining the torsion points of LT. The Galois group of L_∞/L is denoted by $\Gamma = \mathrm{Gal}(L_\infty/L)$, and is isomorphic to $o_L^\times$ via a natural group isomorphism $\chi_L : \Gamma \to o_L^\times$.
The Robba ring $\mathcal{R}_L(\mathfrak{B})$ over the open unit disk $\mathfrak{B}_{/L}$ over L is the ring of formal power series $\sum_{i\in\mathbb{Z}} a_i T^i$ with coefficients in L which converge on an open annulus with outer radius 1 and an inner radius $r < 1$. We also have the subring $\mathcal{E}_L^\dagger(\mathfrak{B})$ of those series in $\mathcal{R}_L(\mathfrak{B})$ with bounded coefficients. We have an action of $o_L \setminus \{0\}$ on the rings $\mathcal{R}_L(\mathfrak{B})$ and $\mathcal{E}_L^\dagger(\mathfrak{B})$ which is given by the endomorphisms of LT. In this way, we obtain a notion of (φ, Γ)-modules over these rings. There are equivalences of categories between L-linear continuous representations of G_L and certain (φ, Γ)-modules over $\mathcal{E}_L^\dagger(\mathfrak{B})$, and between L-linear L-analytic representations of G_L and certain L-analytic (φ, Γ)-modules over $\mathcal{R}_L(\mathfrak{B})$ (see [28] and [5]).
Let K/L be complete. Consider the open unit disk $\mathfrak{B}_{/\mathbb{Q}_p}$ over $\mathbb{Q}_p$ with its $\mathbb{Z}_p \setminus \{0\}$-action. It is naturally isomorphic to the space of locally $\mathbb{Q}_p$-analytic characters on $\mathbb{Z}_p$ via the bijection $z \in \mathfrak{B}(K) \mapsto (\kappa_z : a \mapsto (1+z)^a)$ on K-points. In the paper [35], the authors construct a rigid-analytic variety $\mathfrak{X}$ whose K-points parametrize locally L-analytic K-valued characters $o_L \to K^\times$. The varieties $\mathfrak{B}$ and $\mathfrak{X}$ are not isomorphic, but they become isomorphic after base change to $\mathbb{C}_p$. The rings $\mathcal{R}_K(\mathfrak{X})$ and $\mathcal{E}_K^\dagger(\mathfrak{X})$ are defined. They carry an action of $o_L \setminus \{0\}$ which comes from an $o_L \setminus \{0\}$-action on the rigid-analytic variety $\mathfrak{X}$. The isomorphism $\mathfrak{X}_{/\mathbb{C}_p} \cong \mathfrak{B}_{/\mathbb{C}_p}$ leads to an isomorphism $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}) \cong \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B})$. Moreover, there is a "standard" action of G_L on these rings such that $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{B})^{G_L} = \mathcal{R}_L(\mathfrak{B})$ and a "twisted" action of G_L such that $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{B})^{G_L,*} = \mathcal{R}_L(\mathfrak{X})$. In [6], the theory of (φ, Γ)-modules over these rings is developed. Using the isomorphism $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}) \cong \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B})$ and the G_L-actions, one can compare (φ, Γ)-modules over $\mathcal{R}_L(\mathfrak{X})$ and $\mathcal{R}_L(\mathfrak{B})$. It turns out that there is an equivalence of categories between the category of L-analytic (φ, Γ)-modules over $\mathcal{R}_L(\mathfrak{B})$ and the category of L-analytic (φ, Γ)-modules over $\mathcal{R}_L(\mathfrak{X})$.

The main idea of this project was to transport the space $\mathfrak{X}$ into the world of perfectoid spaces. We construct preadic spaces $\mathfrak{X}^{\mathrm{perf}}(r)$ whose K-points are locally analytic K-valued characters $L \to K^\times$, and define rings $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ and $\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})$ which obtain an action of $o_L \setminus \{0\}$. Then we think about φ-modules and (φ, Γ)-modules over these rings.
In the first chapter, we summarize facts about Huber rings and Huber pairs as well

as about preadic and adic spaces. We explain basic constructions for use in the following chapters. Moreover, we briefly explain certain aspects of the theory of perfectoid and preperfectoid spaces.

In the second chapter, we give an overview over the construction of $\mathfrak{X}$ as well as the isomorphism $\mathfrak{X}_{/\mathbb{C}_p} \cong \mathfrak{B}_{/\mathbb{C}_p}$. Then we explain the construction of the rings $\mathcal{R}_K(\mathfrak{X})$ and $\mathcal{E}_K^\dagger(\mathfrak{X})$.

In the third chapter, we discuss in detail the construction of the open and closed preperfectiod unit disks. The open unit disk $\mathfrak{B}_{/K}$ has a covering by affinoids $\mathfrak{B}(r)_{/K}$ which are closed disk of radius $r < 1$. We construct Tate rings $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r))$ by forming the inductive limit over the rings $\mathcal{O}_K(\mathfrak{B}(r^{1/q^i}))$ where the transition maps are given by the action of π. We show that these rings are stably uniform and therefore define adic spaces $\mathfrak{B}^{\mathrm{perf}}(r)$ in such a way that there are open immersions $\mathfrak{B}^{\mathrm{perf}}(r) \subseteq \mathfrak{B}^{\mathrm{perf}}(s)$ for $r < s$. Glueing of the spaces $\mathfrak{B}^{\mathrm{perf}}(r)$ gives the preperfectoid reps. perfectoid open unit disk.
Similarly, we construct rings $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$ which come from affinoid annuli $\mathfrak{B}(r_1, r_2)_{/K}$. We obtain the ring $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})$ which is the union of the rings $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r := \varprojlim(\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$ where the maps in the projective limit are given by restriction maps. Since the latter rings are projective limits of Banach spaces and therefore Fréchet spaces, we may endow $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})$ with the locally convex inductive limit topology. We define a continuous action of $o_L \setminus \{0\}$ on $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})$.
Another way to construct preperfectoid and perfectoid Robba rings over the unit disk is via certain completions of Witt vectors over the tilt $K^\flat$ of a perfectoid K. This approach can be found in the papers [3] or [27]. We explain it and discuss how both constructions are related.

In the fourth chapter, we introduce the Tate rings $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r))$ and $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2))$. The variety $\mathfrak{X}$ has an open covering by affinoids $\mathfrak{X}(r)$ which over $\mathbb{C}_p$ become isomorphic to closed disks $\mathfrak{B}(s)_{/\mathbb{C}_p}$. Moreover, there are affinoids $\mathfrak{X}(r_1, r_2)$ which over $\mathbb{C}_p$ are isomorphic to affinoid annuli. The rings $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r))$ and $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2))$ are obtained by forming the inductive limit of the rings $\mathcal{O}_K(\mathfrak{X}(r^{1/p^i}))$ respective $\mathcal{O}_K(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i}))$ where the transitions maps are given by the action of p. This leads to preperfectoid spaces $\mathfrak{X}^{\mathrm{perf}}(r)$ and $\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)$. We then imitate the construction of $\mathcal{R}_K(\mathfrak{X})$ to construct the ring $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ as the inductive limit of Fréchet spaces $\varprojlim \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2))$. We discuss its topological properties. Moreover, we construct the ring of bounded functions $\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})$ and define a norm $\|\cdot\|_1$ on it. We define an action of $o_L \setminus \{0\}$ on these rings. We show that there are isomorphisms $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}) \cong \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})$ and $\mathcal{E}_{\mathbb{C}_p}^\dagger(\mathfrak{X}^{\mathrm{perf}}) \cong \mathcal{E}_{\mathbb{C}_p}^\dagger(\mathfrak{B}^{\mathrm{perf}})$.

Originally the goal was to construct a preperfectoid character variety $\mathfrak{X}^{\mathrm{perf}}$ which would be obtained by glueing together the spaces $\mathfrak{X}^{\mathrm{perf}}(r)$. Unfortunately, this turned out to be an unexpected difficult problem due to topological issues. The rings $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r))$ are uniform by construction (i.e. the power-bounded elements $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r))^\circ$ are a ring of definition), but we do not know whether they are stably uniform, i.e. whether all rational localizations remain uniform. This implies that we do not know whether $\mathfrak{X}^{\text{perf}}(r) \subseteq \mathfrak{X}^{\text{perf}}(s)$ is an open immersion of preadic spaces. If these rings were stably uniform, this problem disappeared, and we would obtain an adic space $\mathfrak{X}^{\text{perf}}$. We explain at the end of Chapter 4 how a conjecture of Kedlaya and Hansen ([21]) implies this result.

In the fifth chapter, we discuss φ-modules over the rings constructed in the fourth chapter. We start with some general statements about φ-modules over a ring R, i.e. finite projective R-modules M with a semilinear action of φ such that the linearized map $R \otimes_{R,\varphi} M \to M$ is an isomorphism. Then we prove the main theorem

Theorem. The base change functor from étale φ-modules over $\mathcal{E}_K^\dagger(\mathfrak{X}^{\text{perf}})$ to étale φ-modules over $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})$ is an equivalence of categories.

We discuss the relation between φ-modules over $\mathcal{R}_K(\mathfrak{X})$ and $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})$. Then we look at (φ, Γ)-modules over $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})$. Unfortunately, we cannot define a base change functor from (φ, Γ)-modules over $\mathcal{E}_K^\dagger(\mathfrak{X}^{\text{perf}})$ to (φ, Γ)-modules over $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})$ since we do not know whether the induced $o_L^\times$-action is continuous. We discuss the arising problems and explain certain cases in which such a functor exists.

Notation

Rings are commutative with 1. If K is a nonarchimedean field, then we denote its ring of integers by o_K. We fix a prime p and an algebraic closure $\mathbb{Q}_p^{alg}$ of $\mathbb{Q}_p$. Let $\mathbb{C}_p$ denote its completion. Let L be a finite extension of $\mathbb{Q}_p$ in $\mathbb{C}_p$ with ring of integers o_L and a fixed prime element $\pi = \pi_L \in o_L$, and residue field $k = o_L/\pi$. Let $d = [L : \mathbb{Q}_p]$ and e be the ramification index of $L/\mathbb{Q}_p$. We assume that the absolute value $|\cdot|$ on $\mathbb{C}_p$ is normalized by $|p| = p^{-1}$.

Let A be a perfect k-algebra. We denote by $W(A)_L$ the ramified Witt vectors over A. The Witt polynomials are denoted by Φ_n for $n \geq 0$, i.e.

$$\Phi_n(X_0, ..., X_{n+1}) = X_0^{q^n} + \pi X_1^{q^{n-1}} + ... + \pi^n X_n.$$

We denote the Teichmüller map $A \to W(A)_L$ by τ and the Witt vector Frobenius by Fr. Moreover, we denote by V the Verschiebung $V : W(A)_L \to W(A)_L$, and denote by $V_m(A)_L$ the image of V^m for $m \geq 1$.

1 Adic spaces

Adic spaces have been introduced by Roland Huber in his papers [22] and [23]. This chapter gives a brief overview of the theory of adic spaces and collects some results for later use. We mainly follow the presentation in [41] and [7], see also [29].

1.1 Huber rings and Huber pairs

In this section, we introduce Huber rings and Huber pairs.

Let A be a topological ring.

Definition 1.1. 1. *A is called* adic *if there is an ideal $I \subseteq A$ such that $(I^n)_{n\in\mathbb{N}}$ is a fundamental system of neighbourhoods of 0 in A. The ideal I is called an* ideal of definition.

2. *A is called a* Huber ring *if there is an open adic subring $A_0 \subseteq A$ with finitely generated ideal of definition I. We call A_0 a* ring of definition.

If A is a Huber ring with ring of definition A_0 and ideal of definition I, we say that (A_0, I) is a *pair of definition* of A. Since $A_0 \subseteq A$ is open, the ideals $(I^n)_{n\in\mathbb{N}}$ of A_0 are a fundamental system of neighbourhoods of 0 for A.

For a subset $T \subseteq A$ and $n \geq 1$ we write

$$T(n) := \{t_1 \cdot ... \cdot t_n \,|\, t_1, ..., t_n \in T\}.$$

Definition 1.2. 1. *We call a subset $T \subseteq A$* bounded *if for every neighbourhood U of 0 in A there is a neighbourhood V of 0 in A such that the set $\{vt \,|\, v \in V, t \in T\}$ is contained in U.*

2. *A subset $T \subseteq A$ is* power-bounded *if the set $\bigcup_{n\geq 1} T(n)$ is bounded.*

3. *A subset $T \subseteq A$ is* topologically nilpotent *if for every neighbourhood U of 0 in A, there is an $n_0 \geq 1$ such that $T(n) \subseteq U$ for all $n \geq n_0$.*

An element $a \in A$ is called power-bounded if $\{a\}$ is power-bounded, i.e. if $\{a^n \,|\, n \geq 1\}$ is bounded. It is called topologically nilpotent if $\{a\}$ is topologically nilpotent.
We use the following notation for the set of power-bounded resp. topologically nilpotent elements:

$$A^\circ := \{a \in A |\, a \text{ is power-bounded}\}, \quad \text{and}$$
$$A^{\circ\circ} := \{a \in A |\, a \text{ is topologically nilpotent}\}.$$

Remark 1.3. Let A be a Huber ring. A subring of A is a ring of definition if and only if it is open and bounded ([29, Lemma II.1.1.7]). The set of power-bounded elements A° of A is the union of all rings of definition of A. It is an integrally closed subring containing $A^{\circ\circ}$ which is a radical ideal of A° ([29, Proposition II.1.2.4]).

All Huber rings occurring in this work belong to the following type of Huber rings:

Definition 1.4. *A Huber ring A is called a* Tate ring *if A has a topologically nilpotent element t which is a unit in A.*

We always assume that $t \in A_0$ (by changing to t^n for big enough n if $t \notin A_0$). The sets $(t^n A_0)_{n\in\mathbb{N}}$ form a fundamental system of neighbourhoods of 0.

Remark 1.5. Let $f : A \to B$ a continuous ring homomorphism between Tate rings A and B. For any topologically nilpotent unit $t \in A$, the element $f(t) \in B$ is a topologically nilpotent unit as well. Let A_0 respective B_0 be rings of definition of A resp. B. Then $A_0' := A_0 \cap f^{-1}(B_0)$ is a ring of definition of A since it is open and bounded in A. If B_0' denotes the subring of B generated by $f(A_0)$ and B_0, then $B_0 \subseteq B_0'$ and $f(t)^n B_0' \subseteq B_0$ for sufficiently big n, so B_0' is open and bounded in B and therefore a ring of definition.

A convenient feature of Tate rings is that they are seminormed rings:

Lemma 1.6. *If A is a Tate ring with topologically nilpotent unit t, then A is a seminormed ring with the submultiplicative seminorm*

$$|a| := 2^{\inf(n\in\mathbb{Z};\, t^n a\in A_0)} \text{ for } a \in A, \text{ where } 2^{-\infty} := 0.$$

Proof. Let $a, b \in A$. We have $|a + b| \leq \max(|a|, |b|)$ and $|a + b| = \max(|a|, |b|)$ if $|a| \neq |b|$. We have $|0| = 0$. This shows that $|\cdot|$ is a seminorm which is clearly submultiplicative. Since

$$t^n A_0 = \{a \in A |\, |a| \leq 2^{-n}\},$$

the given topology on A and the topology induced by $|\cdot|$ have equal neighbourhood systems of 0, and therefore they coincide. □

Conversely, every semi-normed ring $(A, |\cdot|)$ with topologically nilpotent unit t is a Tate ring with ring of definition $\{a \in A \,|\, |a| \leq 1\}$.

Remark 1.7. If A is a Tate ring with topologically nilpotent unit t, then we have $A = A_0[1/t]$.

Proof. The ideals $(t^n), n \in \mathbb{N}$, in A_0 form a fundamental system of neighbourhoods of 0 for the topology on A. If $a \in A$ is any element, then multiplication with a is continuous. Thus $t^n a \in A_0$ for sufficiently large n and hence $a \in A_0[1/t]$. □

The next step in defining adic spaces is to consider Huber rings together with a so-called ring of integral elements:

Definition 1.8. 1. *A* Huber pair *is a pair (A, A^+), where A is a Huber ring and $A^+ \subseteq A^\circ$ is an open subring of A that is integrally closed in A. Then A^+ is called a* ring of integral elements *of A.*

2. *A morphism of Huber pairs $(A, A^+) \to (B, B^+)$ is a continuous ring homomorphism $f : A \to B$ such that $f(A^+) \subseteq B^+$. It is called* adic *if there are rings of definition A_0 and B_0 of A and B and an ideal of definition $I \subseteq A_0$ such that $f(A_0) \subseteq B_0$ and $f(I)B_0$ is an ideal of definition of B_0.*

Definition 1.9. *If (A, A^+) is a Huber pair with A Tate, we say that (A, A^+) is a Tate-Huber pair.*

Example 1.10. Let $L \subseteq K \subseteq \mathbb{C}_p$ be complete. If A is a affinoid K-algebra (i.e. a quotient of some Tate algebra $K\langle T_1, ..., T_n\rangle$) which is reduced, then A is a Huber ring with ring of definition A° and ideal of definition (π). The pair (A, A°) is a Tate-Huber pair.

Remark 1.11. Any morphism of Tate-Huber pairs $(A, A^+) \to (B, B^+)$ is adic.

Proof. See [41, Proposition 6.25]. □

1.1.1 Completion of a Huber pair

We define the completion of a Huber ring and of a Huber pair. We mainly follow the presentation in [29]. Firstly, we recall some definitions and facts from general topological algebra.

Definition 1.12. 1. *Let X be a set. A* filter *on X is a nonempty collection of subsets $\mathcal{F}$ of X which is stable under finite intersections and such that $A \in \mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$ where A and B are subsets of X.*

2. *If X is a topological space and $x \in X$ is an element, then x is a* limit *of a filter $\mathcal{F}$ on X if every neighbourhood of x is in $\mathcal{F}$.*

A topological space X is Hausdorff if and only if every filter has at most one limit. If X is a metric space, then every sequence $(x_n)_{n\in\mathbb{N}}$ has an associated filter consisting of the sets $A \subseteq X$ such that there is an $n_0 \in \mathbb{N}$ with $x_n \in A$ for all $n \geq n_0$.

Definition 1.13. *Let G be an abelian topological group.*

1. *A filter $\mathcal{F}$ on G is called a* Cauchy filter *if for every neighbourhood of zero U, there is a $V \in \mathcal{F}$ such that $x - y \in U$ for all $x, y \in V$.*

2. *G is called complete if it is Hausdorff and if every Cauchy filter has a limit.*

For every abelian topological group G there is a complete abelian topological group $\widehat{G}$ and a map $\iota : G \to \widehat{G}$ which fulfils the following universal property: For every complete abelian topological group H and every continuous homomorphism $f : G \to H$, there is a unique continuous map $\widehat{f} : \widehat{G} \to H$ such that $\widehat{f} \circ \iota = f$. Then $(\widehat{G}, \iota)$ is unique up to unique isomorphism. We call $\widehat{G}$ the completion of G. The analogous statements for topological rings are true. If G is a metric space, then a sequence in G is a Cauchy sequence if and only if the associated filter is a Cauchy filter.

Lemma 1.14. *Let A_0 be a ring and I a finitely generated ideal of A_0. Let M be an A_0-module. Set $\widehat{M} := \varprojlim_n M/I^n M$. We denote by $\iota : M \to \widehat{M}$ the canonical map. Then the abelian group $\widehat{M}$ is Hausdorff and complete for the $\iota(I)\widehat{M}$-topology. Moreover, for every $n \geq 1$, the map ι induces an isomorphism $M/I^n M \to \widehat{M}/\iota(I)^n \widehat{M}$.*

Proof. See [1, Tag 05GG]. □

If A_0 is an adic ring with finitely generated ideal of definition I, then $\widehat{A_0} = \varprojlim_n A_0/I^n$ is the completion of A_0. This does not depend on the choice of the ideal of definition. To complete a general Huber ring, we look at the following lemma:

Lemma 1.15 (Huber)**.** *Let A be a Huber ring with ring of definition A_0 and ideal of definition I. Set $\widehat{A} = \varprojlim_n A/I^n$ (as abelian groups; I is the ideal in A_0, not in A).*

1. *The canonical map $\widehat{A_0} \to \widehat{A}$ is injective (so we may identify $\widehat{A_0}$ with its image in $\widehat{A}$).*

2. *The diagram*

$$\begin{array}{ccc} A_0 & \longrightarrow & \widehat{A_0} \\ \downarrow & & \downarrow \\ A & \longrightarrow & \widehat{A} \end{array}$$

   is cocartesian.

3. *We endow $\widehat{A}$ with the unique topology such that $\widehat{A}$ is a topological group and such that $\widehat{A_0}$ is an open subgroup. Then $\widehat{A}$ is a complete topological group.*

4. *There is a unique multiplication on $\widehat{A}$ which makes the canonical map $A \to \widehat{A}$ continuous, and $\widehat{A}$ is a topological ring.*

5. *$\widehat{A}$ is a Huber ring, $\widehat{A_0}$ is a ring of definition of $\widehat{A}$, and $I \cdot \widehat{A_0}$ is an ideal of definition of $\widehat{A_0}$. The canonical map $A \to \widehat{A}$ is adic. Moreover, the canonical map $\widehat{A_0} \otimes_{A_0} A \to \widehat{A}$ is an isomorphism.*

Proof. 1. This follows from the facts that for every n, the map $A_0/I^n \to A/I^n$ is injective, and that taking projective limits is left exact.

2. Denote by $\iota : A \to \widehat{A}$ the canonical map. We show $\iota(A) \cap \widehat{A_0} = \iota(A_0)$. Let $a \in A$ be an element such that $\iota(a) \in \widehat{A_0}$. Then for every $n \geq 1$ there is a $b_n \in A_0$ such that $a \in b_n + I^n$. This means that a is in the closure of A_0 in A. But since A_0 is an open subgroup of A, it is also closed, so $a \in A_0$. The other inclusion is clear.

3. The existence and uniqueness of the topology follow from [10, Chapter III, §1.2 Proposition 1]. Then $\widehat{A}$ is Hausdorff since $\widehat{A_0}$ is a Hausdorff neighbourhood of 0. Let $\mathcal{F}$ be a Cauchy filter on $\widehat{A}$. Note that $\mathcal{F}$ does not contain $\emptyset$ since $\widehat{A}$ is Hausdorff. There is an $F \in \mathcal{F}$ such that $x - y \in \widehat{A_0}$ for all $x, y \in F$. Choose an element $x_0 \in F$. We define a Cauchy filter $\mathcal{F}_0$ on $\widehat{A_0}$ by declaring $G \in \mathcal{F}_0$ if and only if $x_0 + G \in \mathcal{F}$. Note that $\mathcal{F}_0$ is not empty since $F - x_0 \in \mathcal{F}_0$. Since $\widehat{A}_0$ is complete, there is a limit a of $\mathcal{F}_0$, and hence $a + x_0$ is a limit of $\mathcal{F}$.

4. A_0 is dense in $\widehat{A_0}$. Therefore A is dense in $\widehat{A}$. This implies the uniqueness of the multiplication. The existence follows from [10, Chapter III, §6, 5].

5. See [22, Lemma 1.6].

□

Definition 1.16. *The Huber ring $\widehat{A}$ is the* completion *of A.*

If A is Tate, then A is a seminormed ring (Lemma 1.6) and its completion coincides with the completion of A as a seminormed ring.

Lemma 1.17. *Let G be a topological group and $\widehat{G}$ its completion. Denote by $\iota : G \to \widehat{G}$ the canonical map. Then*

$$H \mapsto \overline{\iota(H)} \text{ (the closure of } \iota(H) \text{ in } \widehat{G}),$$
$$\iota^{-1}(H') \leftarrow H'$$

defines a bijection between open subgroups of G and open subgroups of $\widehat{G}$.

Proof. The closure $\overline{\iota(H)}$ of $\iota(H) \subseteq \widehat{G}$ is canonically isomorphic to the completion $\widehat{H}$ ([10, Chapter II, §3, 9]). We have $\ker \iota = \overline{\{0\}}$ and $\iota(G)$ is dense in $\widehat{G}$. Let H be an open, hence closed subgroup of G. Then H contains $\ker \iota$ and we have $H = \iota^{-1}(\overline{\iota(H)})$. On the other hand, if H' is an open subgroup of $\widehat{G}$, then $H' \cap \iota(G)$ is dense in H'. It follows that H' is the closure of $\iota(\iota^{-1}(H'))$. □

Proposition 1.18 (Huber)**.** *Let A be a Huber ring. Under the bijection of the previous lemma, the following open subgroups correspond to each other:*

1. *A° and $(\widehat{A})^\circ$,*

2. *$A^{\circ\circ}$ and $(\widehat{A})^{\circ\circ}$,*

3. *rings of definition of A and rings of definition of $\widehat{A}$,*

4. *rings of integral elements of A and rings of integral elements of $\widehat{A}$.*

Proof. Let $\iota : A \to \widehat{A}$ denote the canonical map.

1. Lemma 1.18 implies that a subset $E \subseteq A$ is bounded if and only of $\iota(E) \subseteq \widehat{A}$ is bounded after Lemma 1.17, and hence $\iota^{-1}((\widehat{A})^\circ) \subseteq A^\circ$. This implies $(\widehat{A})^\circ \subseteq \widehat{A^\circ}$. On the other hand, we have $\iota(A^\circ) \subseteq (\widehat{A})^\circ \subseteq \widehat{A^\circ}$. Therefore $(\widehat{A})^\circ$ is dense in $\widehat{A^\circ}$. As $(\widehat{A})^\circ$ is open in $\widehat{A}$, it is closed, so $\widehat{A^\circ} = (\widehat{A})^\circ$.

2. By Lemma 1.18, an element $a \in A$ is topologically nilpotent if and only if $\iota(a) \in \widehat{A}$ is topologically nilpotent. The argument is then analogue to 1.

3. Let H be an open subgroup of A which correspondents to $H' \subseteq \widehat{A}$ under the bijection in Lemma 1.18. Then, using the density of $\iota(H)$ in H' and that $\iota(A)$ is a subring of $\widehat{A}$, we see that H is a ring if and only if H' is a ring. We already know that H is bounded if and only if $\iota(H)$ is bounded. Therefore H is a ring of definition if H' is. On the other hand, let H be a bounded subring of A. Then $\iota(H)$ is bounded. We can find an open bounded subgroup $U \subseteq H'$. We then have $H' = \iota(H) + U$ because $\iota(H)$ is dense in H'. Therefore H' is bounded.

4. We have already shwon that $K \subseteq A^\circ$ if and only if $H' \subset (\widehat{A})^\circ$. Let H be an open subgroup of A which correspondents to $H' \subseteq \widehat{A}$ under the bijection in Lemma 1.18. Assume that H is an open and integrally closed subring of A. We show that H' is integrally closed in $\widehat{A}$. Let $b \in \widehat{A}$ be integral over H'. Then there are elements $a_0, ..., a_{n-1} \in \widehat{A}$ such that $b^n + a_{n-1}b^{n-1} + ... + a_0 = 0$. We find a $\tilde{b} \in A$ such that $b - \iota(\tilde{b}) \in H'$, and $\tilde{a}_{n-1}, ..., \tilde{a}_0 \in H$ such that

$$(b^n + a_{n-1}b^{n-1} + ... + a_0) - (\iota(\tilde{b^n}) + \iota(\tilde{a_{n-1}})\iota(\tilde{b}^{n-1}) + ... + \iota(\tilde{a_0})) \in H'.$$

This implies $\tilde{b}^n + \tilde{a_{n-1}}\tilde{b}^{n-1} + ... + \tilde{a_0} \in H$. But H is integrally closed over A and hence we have $\tilde{b} \in H$. Then $b = (b - \iota(\tilde{b})) + \iota(\tilde{b}) \in H'$. On the other hand, let H' be an open and integrally closed subring of $\widehat{A}$. If $x \in A$ is integral over H, then $\iota(x)$ is integral over H', hence $\iota(x) \in H'$ and $x \in H$. Hence H is integrally closed.

□

Corollary 1.19. *If $A^+ \subseteq A$ is a ring of integral elements, then the closure $\widehat{A^+}$ of of the image of A^+ in $\widehat{A}$ is a ring of integral elements of $\widehat{A}$.*

Definition 1.20. *Let (A, A^+) be a Huber pair. The pair $(A, A^+)^\wedge := (\widehat{A}, \widehat{A^+})$ is a Huber pair called the* completion *of (A, A^+).*

1.1.2 Localization of a Huber pair

Let A be a Huber ring with pair of definition (A_0, I) and let $\emptyset \neq T = \{t_1, ..., t_m\} \subseteq A$ be a finite subset such that $T \cdot A$ is open in A and let $s \in A$ be an element. Note that if A is a Tate ring, then $T \cdot A$ is open in A if and only if T generates the unit ideal in A. Consider the localization $S^{-1}A = A[s^{-1}]$ of A at $S = \{1, s, s^2, ...\}$. We set

$$D := A_0[\frac{t_1}{s}, ..., \frac{t_m}{s}] \subseteq A[s^{-1}],$$

i.e. D is the subring of $A[s^{-1}]$ generated by A_0 and $\{\frac{t_1}{s}, ..., \frac{t_m}{s}\}$. We define a topology on $A[s^{-1}]$ by taking the sets $(I^n D)_{n \in \mathbb{N}} \subseteq D$ as a fundamental system of neighbourhoods of 0. We denote the resulting topological ring by $A(\frac{T}{s})$.

Definition 1.21. *Let $\emptyset \neq T = \{t_1, ..., t_m\} \subseteq A$ be a finite subset such that $T \cdot A$ is open in A and $s \in A$. Then $A(\frac{T}{s})$ is a Huber ring with pair of definition (D, ID). The completion of $A(\frac{T}{s})$ is denoted by $A\langle\frac{T}{s}\rangle$.*

In $A(\frac{T}{s})$ resp. $A\langle\frac{T}{s}\rangle$, the elements $\frac{t_i}{s}, i = 1, ..., m$ are now power-bounded. The canonical maps $\varphi : A \to A(\frac{T}{s})$ resp. $\widehat{\varphi} : A \to A\langle\frac{T}{s}\rangle$ are continuous and universal with respect to all continuous morphisms $f : A \to B$ of Huber rings (resp. complete Huber rings) such that $f(s)$ is invertible in B and such that the set $\{\frac{f(t_i)}{f(s)} \mid i = 1, ..., m\}$ is power-bounded in B. This means that for every Huber ring (resp. complete Huber ring) and for every continuous ring map $f : A \to B$ such that $f(s)$ is invertible in B and such that the set $\{\frac{f(t_i)}{f(s)} \mid i = 1, ..., m\}$ is power-bounded in B, there is a unique continuous ring map $g : A(\frac{T}{s}) \to B$ (resp. $g : A\langle\frac{T}{s}\rangle \to B$) such that $f = g \circ \varphi$ (resp. $f = g \circ \widehat{\varphi}$).

Note that $A(\frac{T}{s}) = A(\frac{T \cup s}{s})$, so we can always assume $s \in T$.

Definition 1.22. *Let (A, A^+) be a Huber pair. Let $\emptyset \neq T = \{t_1, ..., t_m\} \subseteq A$ be a finite subset such that $T \cdot A$ is open in A and $s \in A$. Let $A(\frac{T}{s})^+$ be the integral closure of $A^+[\frac{t_1}{s}, ..., \frac{t_m}{s}]$ in $A(\frac{T}{s})$. The pair $(A(\frac{T}{s}), A(\frac{T}{s})^+)$ is a Huber pair. Its completion is denoted by $(A\langle\frac{T}{s}\rangle, A\langle\frac{T}{s}\rangle^+)$. We call $(A\langle\frac{T}{s}\rangle, A\langle\frac{T}{s}\rangle^+)$ a* rational localization *of (A, A^+).*

The canonical map of Huber pairs $(A, A^+) \to (A(\frac{T}{s}), A(\frac{T}{s})^+)$ resp. $(A, A^+) \to (A\langle\frac{T}{s}\rangle, A\langle\frac{T}{s}\rangle^+)$ is adic. It is universal for maps of Huber pairs (resp. complete Huber pairs) $f : (A, A^+) \to (B, B^+)$ such that $f(s)$ is invertible in B, and $\frac{f(t_i)}{f(s)} \in B^+$ for $i = 1, ..., m$.

Remark 1.23. Let $L \subseteq K \subseteq \mathbb{C}_p$ be complete and let A be an affinoid algebra over K. We have the Tate-Huber pair (A, A°). If $T = \{t_1, ..., t_n\}$ and s are as in Definition 1.21, then the integral closure in $A\langle T/s\rangle$ of $A^\circ\langle T/s\rangle =$ π-completion of $A^\circ[t_1/s, ..., t_n/s]$ is equal to $A\langle T/s\rangle^\circ$. This can be seen as follows (see lecture 12 of Brian Conrads notes [17]): The completed localization $A\langle T/s\rangle$ is isomorphic to the quotient

$$A\langle T_1, ..., T_n\rangle/(sT_i - t_i)$$

of a relative Tate algebra $A\langle T_1, ..., T_n\rangle$. The power-bounded elements of $A\langle T_1, ..., T_n\rangle$ are given by $A^\circ\langle T_1, ..., T_n\rangle$. Then the map $A^\circ\langle T_1, ..., T_n\rangle \to A\langle T/s\rangle^\circ$ is integral which follows from the finiteness of $A\langle T_1, ..., T_n\rangle \to A\langle T/s\rangle$ and [31, 6.3.5/ Theorem 1]. But $A^\circ\langle T_1, ..., T_n\rangle \to A\langle T/s\rangle^\circ$ factors through $A^\circ\langle T/s\rangle$, so the map $A^\circ\langle T/s\rangle \to A\langle T/s\rangle^\circ$ is integral as well. The integral closure of $A^\circ\langle T/s\rangle$ in $A\langle \frac{T}{s}\rangle$ coincides with $A\langle \frac{T}{s}\rangle^+$. It follows that $A\langle \frac{T}{s}\rangle^+ = A\langle \frac{T}{s}\rangle^\circ$.

1.1.3 Tensor products

Let $(A, A^+), (B, B^+), (C, C^+)$ be Huber pairs and let $f : (C, C^+) \to (A, A^+), g : (C, C^+) \to (B, B^+)$ be adic morphisms. Assume that $f(C_0) \subseteq A_0$ and $g(C_0) \subseteq B_0$ for the resp. rings of definition. Then we can regard the tensor product $A \otimes_C B$ as a Huber ring with ring of definition $D :=$ image of $A_0 \otimes_{C_0} B_0$ in $A \otimes_C B$, and ideal of definition ID where I is an ideal of definition of C_0. Let $(A \otimes_C B)^+$ be the integral closure of the image of $A^+ \otimes_{C^+} B^+$ in $A \otimes_C B$. Then $(A \otimes_C B, (A \otimes_C B)^+)$ is a Huber pair.

If the Huber pairs $(A, A^+), (B, B^+), (C, C^+)$ are Tate and if $t \in C$ is a topologically nilpotent unit, then $f(t) \in A$ is a topologically nilpotent unit, too. The tensor product $(A \otimes_C, B, (A \otimes_C B)^+)$ is again a Tate-Huber pair with topologically nilpotent unit $f(t) \otimes 1$. The tensor product seminorm on $A \otimes_C B$ coming from the seminorms on A and B as in Lemma 1.6 coincides with the seminorm on $A \otimes_C B$ as in Lemma 1.6. This is because an element $\sum_i x_i \otimes y_i \in A \otimes_C B$ lies in $(f(t) \otimes 1)^n \cdot D$ for $n \geq 0$ if and only if it has a presentation $\sum_i x_i' \otimes y_i'$ such that $x_i' \otimes y_i' \in (f(t) \otimes 1)^n \cdot D$ for all i.

Let K be a nonarchimedean field and let K'/K be complete. Let A be a reduced affinoid algebra over K. The algebra $A_{K'} := A\widehat{\otimes}_K K'$ (where the completion is taken with respect to the tensor product norm coming from the supremum norm on A and the norm on K') is an affinoid algebra over K' which is again reduced ([16, Lemma 3.3.1(1)]). The canonical map $A \to A_{K'}$ is an isometry for the supremum norm on both sides because both are the only complete power-multiplicative norms on A resp. $A_{K'}$ (see [31, Lemma 3.8.3/3 and Thm. 6.2.4/1]). The tensor product norm on $A_{K'}$ is equivalent to the supremum norm, but they may not coincide.
The corresponding Tate-Huber pair (A, A°) gives rise to a Tate-Huber pair $(A \otimes_K K', (A \otimes_K K')^+)$ as in the previous remark.
Denote the image of $A^\circ \otimes_{o_K} o_{K'}$ in $A \otimes_K K'$ by D. The topology on $A_{K'}$ as a Huber ring coincides with the topology induced by the tensor product norm. Moreover, the integral closure of the image of $A^\circ\widehat{\otimes}_{o_K} o_{K'}$ in $A_{K'}$ is equal to $A^\circ_{K'}$ (which are the elements with supremum norm ≤ 1). This can be seen as follows (see lecture

12 of Brian Conrad's notes [17]):
If $A = T_n = K\langle T_1, ..., T_n\rangle$ is a Tate algebra, then the power-bounded elements are $o_K\langle T_1, ..., T_n\rangle$. The power-bounded elements of $T_n\widehat{\otimes}_K K'$ are given by $o_{K'}\langle T_1, ..., T_n\rangle = T_n^\circ\widehat{\otimes}_{o_K} o_{K'}$ and the claim follows directly. If A is a quotient of T_n, then $A_{K'}$ is a quotient of $T_n\widehat{\otimes}_K K'$, and the quotient map $T_n\widehat{\otimes}_K K' \to A_{K'}$ is finite and hence integral. Then the map $T_n^\circ\widehat{\otimes}_{o_K} o_{K'} \to A_{K'}^\circ$ is integral ([31, 6.3.5/ Theorem 1]). We have a commutative diagram

$$\begin{array}{ccc} T_n^\circ\widehat{\otimes}_{o_K} o_{K'} & \xrightarrow{=} & T_n^\circ\widehat{\otimes}_{o_K} o_{K'} \\ \downarrow & & \downarrow \\ A^\circ\widehat{\otimes}_{o_K} o_{K'} & \longrightarrow & A_{K'}^\circ \end{array}$$

The map $T_n^\circ\widehat{\otimes}_{o_K} o_{K'} \to A_{K'}^\circ$ is integral. Then the map $A^\circ\widehat{\otimes}_{o_K} o_{K'} \to A_{K'}^\circ$ is integral as well [1, Tag 02JM]. The completion of $(A \otimes_{K'} K, (A \otimes_K K')^+)$ is given by $(A_{K'}, A_{K'}^\circ)$.

1.1.4 Uniformity

Definition 1.24. 1. *We say that a Huber ring A is* uniform *if the set of power-bounded elements A° is bounded.*

2. *A is* stably uniform *if $A\langle\frac{T}{s}\rangle$ is uniform for every $T \subseteq A, s \in A$ as in Definition 1.21.*

Remark 1.25. For a uniform Huber ring A, the power-bounded elements A° are a ring of definition because A° is open and bounded.

Definition 1.26. *We call a Huber pair (A, A^+)* uniform *(*stably uniform*) if A is uniform (stably uniform). This is independent of the choice of the A^+.*

Lemma 1.27. *A Huber ring is uniform if and only if its completion is uniform. Similarly, a Huber pair is uniform if and only if its completion is uniform.*

Proof. This follows from Lemma 1.18. □

Lemma 1.28. *Let A be a Tate ring which is a Banach algebra over a complete nonarchimedean field K, and let K' be a field extension of K which is complete. Put $A_{K'} = A\widehat{\otimes}_K K'$. Then A is stably uniform if $A_{K'}$ is stably uniform. The converse does not hold.*

Proof. This is Remark 2.8.12 in [27]. □

Lemma 1.29. *A Tate-Huber pair (A, A^+) is uniform if and only if A^+ is a ring of definition.*

Proof. Let $t \in A$ be a topologically nilpotent unit. We have $tA^\circ \subseteq A^{\circ\circ}$. Note that $A^{\circ\circ} \subseteq A^+$ as for every $a \in A^{\circ\circ}$ there is an n such that $a^n \in A^+$ and hence $a \in A^+$ by integral closeness. This means $tA^\circ \subseteq A^{\circ\circ} \subseteq A^+ \subseteq A^\circ$. Therefore A° is bounded if and only if A^+ is bounded. □

Remark 1.30. A reduced affinoid algebra is stably uniform.

Proof. Every reduced affinoid algebra is uniform ([31, Theorem 6.2.4/1]), and a rational localization of a reduced affinoid algebra is again reduced ([31, Corollary 7.3.2/10]). □

Let (A, A^+) be a Tate-Huber pair with topologically nilpotent unit $t \in A$. The sets $t^n A^+, n \in \mathbb{N}$ form a set of subgroups of $(A, +)$ which fulfil the following conditions

1. $t^n A^+ \subseteq t^n A^+ \cap t^m A^+$ if $m \leq n$,
2. for all $x \in A$ and all $n \in N$ there is an m such that $t^m x \subseteq t^n A^+$,
3. we have $t^n A^+ \cdot t^n A^+ \subseteq t^n A^+$.

Therefore there is a unique topology on A such that the $t^n A^+, n \in \mathbb{N}$ form a fundamental system of neighbourhoods of 0 and such that A is a topological ring ([10, III, §6.3]). We define the *uniformization* (A_u, A_u^+) of (A, A^+) to be (A, A^+) but with the topology for which $t^n A^+, n \in \mathbb{N}$ is a fundamental system of neighbourhoods of zero. This makes A^+ a ring of definition. The Tate-Huber pair (A_u, A_u^+) is uniform (Lemma 1.29).

In the following we describe uniform Tate-Huber pairs purely algebraically (see [7, Chapter 7, 7.2.6]). We write $\underline{\text{Tate}}$ for the category of Tate-Huber pairs and $\underline{\text{Tate}}_u$ for its full subcategory of uniform Tate-Huber pairs.
The inclusion $\underline{\text{Tate}}_u \to \underline{\text{Tate}}$ has a left adjoint

$$\begin{aligned} L : \underline{\text{Tate}} &\to \underline{\text{Tate}}_u, \\ (A, A^+) &\mapsto L(A, A^+), \end{aligned}$$

where $L(A, A^+) = (A_u, A_u^+)$. We regard $\underline{\text{Tate}}_u$ as a localization of $\underline{\text{Tate}}$. The functor L inverts all maps $(A, A^+) \to (B, B^+)$ which define bijections $A^+ \cong B^+$.
Let $\underline{\text{Tate}}_{alg}$ be the category of pairs (R, I) where R is a commutative ring, $I \subseteq R$ is the radical of an ideal generated by a non zero divisor x, and R is integrally closed in $R[1/x]$. A morphism $(R, I) \to (S, J)$ in this category is a map $f : R \to S$ such that $\mathrm{Rad}(f(I)S) = J$ where $\mathrm{Rad}(f(I)S)$ is the radical of $f(I)S$. We have a functor $\underline{\text{Tate}} \to \underline{\text{Tate}}_{alg}, (A, A^+) \mapsto (A^+, A^{\circ\circ})$. It restricts to a functor $\underline{\text{Tate}}_u \to \underline{\text{Tate}}_{alg}$.
On the other hand, given a pair $(R, I) \in \underline{\text{Tate}}_{alg}$ with a generator (up to radicals) x of I, we can form a Tate-Huber pair $(R[1/x], R)$ with couple of definition (R, x). This defines a functor $\underline{\text{Tate}}_{alg} \to \underline{\text{Tate}}_u$ which is quasi-inverse to the functor $\underline{\text{Tate}}_u \to \underline{\text{Tate}}_{alg}$.

Remark 1.31. We have an equivalence of categories $\underline{\mathrm{Tate}}_u \cong \underline{\mathrm{Tate}}_{alg}$.

Remark 1.32. Let (A, A^+) be a uniform Tate-Huber pair with topologically nilpotent unit $t \in A$. The completion of (A, A^+) is computed as $(\widehat{A^+}[1/t], \widehat{A^+})$, where $\widehat{A^+}$ is the t-adic completion of A^+, and $(\widehat{A^+}, t)$ is a pair of definition.

Proof. $\widehat{A^+}$ is the closure of A^+ in $\widehat{A^+}[1/t]$ and the completion of A is equal to $\widehat{A^+}[1/t]$. To see that $\widehat{A^+}$ is integrally closed in $\widehat{A^+}[1/t]$, see Proposition 1.18. □

We briefly explain uniformity from the viewpoint of semi-normed rings.

Definition 1.33. *Let A be a semi-normed ring with semi-norm $|\cdot|$. Then the* spectral semi-norm *is defined as* $|a|_{spec} := \lim_{n\to\infty} |a^n|^{1/n}, a \in A$.

This is a power-multiplicative semi-norm on A and we have $|\cdot|_{spec} \leq |\cdot|$ (see [31, 1.3.2/1]).

Let (A, A^+) be a Tate-Huber pair with topologically nilpotent unit $t \in A$. Let $|\cdot|$ be a semi-norm on A which induces the given topology of the Huber ring A (e.g. the semi-norm defined in Lemma 1.6). Then the topology on the uniformization A_u which has the sets $t^n A^+$ as a neighbourhood basis of 0 is equal to the one induced by the spectral semi-norm $|\cdot|_{spec}$ coming from the semi-norm $|\cdot|$ on A. This can be seen as follows: The topology induced by $|\cdot|_{spec}$ is given by the neighbourhood basis of 0 consisting of $\{a \in A \,|\, |a|_{spec} \leq \varepsilon\}$ for $\varepsilon > 0$. For this topology, A^+ is open and bounded: Every $b \in A^+$ is power-bounded in the original topology on A, then b is power-bounded in the topology induced by $|\cdot|_{spec}$, i.e we have $A^+ \subseteq \{a \in A \,|\, |a|_{spec} \leq 1\}$. This shows that A^+ is bounded for $|\cdot|_{spec}$. We find an $0 < \varepsilon < 1$ such that $\{a \in A \,|\, |a| \leq \varepsilon\} \subseteq A^+$ because A^+ is open in the original topology on A. Let $a \in A$ such that $|a|_{spec} \leq \varepsilon/2$. Then for big n, we have $|a^n|^{1/n} \leq \varepsilon$, so $|a^n| \leq \varepsilon$ and hence $a^n \in A^+$, and by integral closeness, we have $a \in A^+$. This shows that A^+ is open in the topology induced by $|\cdot|_{spec}$. It follows that we may regard (A, A^+) with the topology induced by $|\cdot|_{spec}$ as a Tate-Huber pair such that A^+ is a ring of definition. It then coincides with the uniformization (A_u, A_u^+).

Regarding the completion $(\widehat{A_u}, \widehat{A_u^+})$, note that $\widehat{A_u}$ is equal to the completion of A for the spectral seminorm coming from the seminorm on A, and $\widehat{A_u^+}$ is equal to the closure of the image of A_u^+ in $\widehat{A_u}$. Note that the uniform completion (i.e. the completion with respect to the spectral seminorm) of A is equal to the uniform completion of $\widehat{A}$.

Lemma 1.34 (Definition 2.8.1 (and Errata) of [27])**.** *Let A be a Banach ring with norm $|\cdot|$ and such that there is a topologically nilpotent unit $z \in A$ such that$|z| \cdot |z^{-1}| = 1$. Then the following conditions are equivalent:*

1. *The norm on A is equivalent (in the sense of [27, Definition 2.1.1]) to some power-multiplicative norm.*

2. *There is a $c > 0$ such that $|a^2| \geq c|a|^2$ for all $a \in A$.*

3. *The norm on A is equivalent (in the sense of [27, Definition 2.1.1]) to its spectral semi-norm, i.e. the semi-norm $|a|_{spec} := \lim_{n\to\infty} |a^n|^{1/n}, a \in A$.*

4. *The power-bounded elements A° are bounded.*

The lemma applies especially if A is a Banach algebra over a nonarchimedean field.

1.2 Adic spaces

1.2.1 The valuation spectrum of a Huber pair

Definition 1.35. *Let A be a topological ring and Γ a totally ordered abelian group (written multiplicatively).*

1. *A* valuation *on A is a map*

$$|\cdot| : A \to \Gamma \cup \{0\},$$

such that

$$|0| = 0, |1| = 1, |ab| = |a| \cdot |b|, \text{ and } |a + b| \leq \max\{|a|, |b|\}.$$

for all $a, b \in A$.

2. *The valuation $|\cdot|$ is* continuous *if the set $\{a \in A \,|\, |a| < \gamma\}$ is open in A for all $\gamma \in \Gamma$.*

3. *If $|\cdot|'$ is another valuation on A, then $|\cdot|$ and $|\cdot|'$ are called* equivalent *if for all $a, b \in A$ we have*

$$|a| \geq |b| \Leftrightarrow |a|' \geq |b|'.$$

Every valuation $|\cdot|$ on A determines a prime ideal $\mathfrak{p}_x := |\cdot|^{-1}(0)$ of A (the *support* of $|\cdot|$) and an integral domain $A/\mathfrak{p}_x$ with fraction field $\kappa(\mathfrak{p}_x)$.

Definition 1.36. *Let (A, A^+) be a Huber pair. We define $\mathrm{Spa}(A, A^+)$ to be the set consisting of all equivalence classes of continuous valuations $|\cdot|$ on A such that $|f| \leq 1$ for all $f \in A^+$.*

For elements $f \in A$ and $x \in \mathrm{Spa}(A, A^+)$ we often write $|f(x)|$ instead of $x(f)$. Let $f, g \in A$. We define a topology on $\mathrm{Spa}(A, A^+)$ by taking the sets

$$\{x \in \mathrm{Spa}(A, A^+) \mid |f(x)| \leq |g(x)| \neq 0\}$$

as basic open subsets.

Theorem 1.37 (Huber)**.** *Let (A, A^+) be a Huber pair. Then $X := \mathrm{Spa}(A, A^+)$ is a spectral space (i.e. $\mathrm{Spa}(A, A^+) \cong \mathrm{Spec}(R)$ for some ring R). A basis of the topology consisting of open quasi-compact subsets is given by the subsets*

$$X(\frac{T}{s}) := \{x \in X \mid \forall t \in T \mid |t(x)| \leq |s(x)| \neq 0\},$$

with $s \in A$ and $T \subseteq A$ as in Definition 1.21. We call these subsets rational subsets. *Finite intersections of rational subsets are again rational subsets.*

Let $f : (A, A^+) \to (B, B^+)$ be a map of Huber pairs. By composition, we get a well defined continuous map $\mathrm{Spa}(f) : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$. If f is adic, then the preimage under $\mathrm{Spa}(f)$ of a rational subset of $\mathrm{Spa}(A, A^+)$ is a rational subset of $\mathrm{Spa}(B, B^+)$.

Proposition 1.38. *Let (A, A^+) be a Huber pair. The canonical map*

$$\mathrm{Spa}(\widehat{A}, \widehat{A}^+) \to \mathrm{Spa}(A, A^+)$$

is a homeomorphism. It identifies rational subsets.

Proof. This is Proposition 3.9 in [22].

□

Lemma 1.39. *Let (A, A^+) be a Huber pair and $T \subseteq A, s \in A$ as in Definition 1.21. Set $X = \mathrm{Spa}(A, A^+)$.*

1. *The canonical map $\iota : (A, A^+) \to (A\langle \frac{T}{s} \rangle, A\langle \frac{T}{s} \rangle^+)$ induces an open immersion of topological spaces $\mathrm{Spa}(A\langle \frac{T}{s} \rangle, A\langle \frac{T}{s} \rangle^+) \to X$ with image $X(\frac{T}{s})$. Under this map, rational subsets in $X(\frac{T}{s})$ correspond to rational subsets in X that are contained in $X(\frac{T}{s})$.*

2. *For every continuous map $f : (A, A^+) \to (B, B^+)$ to a complete Huber pair (B, B^+) such that $\mathrm{Spa}(f) : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ factors through $X(\frac{T}{s})$, there is a unique continuous map $g : A\langle \frac{T}{s} \rangle \to B$ such that $g \circ \iota = f$, and we have $g(A\langle \frac{T}{s} \rangle^+) \subseteq B^+$.*

3. *Let $T' \subseteq A$ and $s' \in A$ be another finite set and element as in Definition 1.21. If $X(\frac{T'}{s'}) \subseteq X(\frac{T}{s})$, then there is a unique continuous map $h : A\langle \frac{T}{s} \rangle \to A\langle \frac{T'}{s'} \rangle$ such that $\iota' = h \circ \iota$, where $\iota' : A \to A\langle \frac{T'}{s'} \rangle$ is the canonical map.*

Proof. See [29, Proposition III.6.1.1]. □

Corollary 1.40. *Let $T, T' \subseteq A$ and $s, s' \in A$ as in Definition 1.21. If $X(\frac{T}{s}) = X(\frac{T'}{s'})$, then there is a canonical isomorphism of Huber pairs $(A\langle \frac{T}{s} \rangle, A\langle \frac{T}{s} \rangle^+) \to A\langle \frac{T'}{s'} \rangle, A\langle \frac{T'}{s'} \rangle^+))$ such that the diagram*

$$\begin{array}{ccc} A & \longrightarrow & A\langle \frac{T}{s} \rangle \\ & \searrow \quad \swarrow & \\ & A\langle \frac{T'}{s'} \rangle & \end{array}$$

commutes.

Now we want to define a presheaf on $\mathrm{Spa}(A, A^+)$.

Definition 1.41. *Let (A, A^+) be a Huber pair, and let $X = \mathrm{Spa}(A, A^+)$. We define a presheaf $\mathcal{O}_X$ on X with values in the category of complete topological rings in the following way*

- *if $X \supseteq U = X(\frac{T}{s})$ is a rational subset, then $\mathcal{O}_X(U) := A\langle\frac{T}{s}\rangle$,*
- *if $V \subseteq X$ is any open subset, then*

$$\mathcal{O}_X(V) = \varprojlim_{V \subseteq U \text{ rational}} \mathcal{O}_X(U),$$

where U ranges over the rational subsets which are contained in V, and $\mathcal{O}_X(V)$ has the projective limit topology. The restriction maps are given by Lemma 1.39.

Definition 1.42. *In the situation of the previous definition, we define a sub-presheaf $\mathcal{O}_X^+$ of $\mathcal{O}_X$ by setting*

$$\mathcal{O}_X^+ = \{f \in \mathcal{O}_X(U) \mid |f(x)| \leq 1 \, \forall x \in U\}$$

for an open subset $U \subseteq X$.

This is a presheaf of complete topological rings as well. Note that if $X(\frac{T}{s})$ is a rational subset, then

$$(\mathcal{O}_X(X(\frac{T}{s})), \mathcal{O}_X^+(X(\frac{T}{s}))) = (A\langle\frac{T}{s}\rangle, A\langle\frac{T}{s}\rangle^+).$$

Especially we have $(\mathcal{O}_X(X), \mathcal{O}_X^+(X)) = (A, A^+)$.

Let $f^\flat : (A, A^+) \to (B, B^+)$ be a map of Huber pairs, set $X = \mathrm{Spa}(A, A^+)$ and $Y = \mathrm{Spa}(B, B^+)$, and let $f = \mathrm{Spa}(f^\flat) : Y \to X$ be the induced map. If $U \subseteq X$ and $V \subseteq Y$ are rational subsets such that $f(V) \subseteq U$, then we have a continuous ring map $\mathcal{O}_X(U) \to \mathcal{O}_Y(V)$ because of Lemma 1.39. If $U \subseteq X$ is any open subset, then we have a map $f_U^\flat : \mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}(U))$ which defines a map of presheaves such that $\mathcal{O}_X^+$ is sent to $f_*\mathcal{O}_Y^+$.

For a point $x \in \mathrm{Spa}(A, A^+)$ we let

$$\mathcal{O}_{X,x} := \varinjlim_{x \in U \text{ open}} \mathcal{O}_X(U) = \varinjlim_{x \in U \text{ rational}} \mathcal{O}_X(U)$$

be the stalk at x (in the category of rings). We deduce from Lemma 1.39 that for every rational subset $U \ni x$ the valuation $x : A \to \Gamma_x \cup \{0\}$ extends uniquely to a continuous valuation $\nu_U : \mathcal{O}_X(U) \to \Gamma_x \cup \{0\}$. We pass to the inductive limit and get a valuation $\nu_x : \mathcal{O}_{X,x} \to \Gamma_x \cup \{0\}$. One can show that for $f \in \mathcal{O}_X(U)$ such that $|f(x)| \neq 0$ the image of f in $\mathcal{O}_{X,x}$ is a unit, and therefore $\mathcal{O}_{X,x}$ is a local ring whose maximal ideal is the support of ν_x.

Definition 1.43. *We define a category $\mathcal{V}^{pre}$ whose objects $X = (|X|, \mathcal{O}_X, \{\nu_x\}_{x\in|X|})$ are triples of*

1. *a topological space $|X|$,*

2. *a presheaf $\mathcal{O}_X$ of complete topological rings such that the stalk $\mathcal{O}_{X,x}$ is a local ring for every point $x \in |X|$,*

3. *for every $x \in |X|$ an equivalence class ν_x of valuations on $\mathcal{O}_{X,x}$ whose support is the maximal ideal of $\mathcal{O}_{X,x}$.*

A morphism $f : X \to Y$ in $\mathcal{V}^{pre}$ is a pair $(f, f^\flat)$ where $f : |X| \to |Y|$ is a continuous map and $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of presheaves of topological rings such that for every $x \in |X|$ the induced ring map $f^\flat_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ fulfils $\nu_{f(x)} = \nu_x \circ f^\flat_x$.

If $X = (|X|, \mathcal{O}_X, \{\nu_x\}_{x\in|X|})$ is an object in $\mathcal{V}^{pre}$ and $|U| \subseteq |X|$ is an open subset, then $(|U|, \mathcal{O}_{X|U}, (\nu_x)_{x\in U})$ is again an object in $\mathcal{V}^{pre}$.

Definition 1.44. *An* open immersion *in $\mathcal{V}^{pre}$ is a morphism*

$$(f, f^\flat) : X = (|X|, \mathcal{O}_X, \{\nu\}_{x\in|X|}) \to Y = (|Y|, \mathcal{O}_Y, \{\nu\}_{x\in|Y|})$$

such that $f : |X| \to |Y|$ is a homeomorphism onto an open subset $|U| \subseteq |Y|$ and the induced morphism $(|X|, \mathcal{O}_X, \{\nu\}_{x\in|X|}\}) \to (|U|, \mathcal{O}_{Y|_U}, \{\nu\}_{x\in|U|}\})$ is an isomorphism.

Remark 1.45. Let (A, A^+) be a Huber pair and T, s as in Definition 1.21. Then the canonical map $\operatorname{Spa}(A\langle\frac{T}{s}\rangle, A\langle\frac{T}{s}\rangle^+) \to \operatorname{Spa}(A, A^+)$ is an open immersion.

Definition 1.46 (See Remark/Definition 8.10 in [41])**.** *An affinoid pre-adic space is an object of $\mathcal{V}^{pre}$ that is isomorphic to* $\operatorname{Spa}(A, A^+)$ *for a Huber pair (A, A^+).*
Let X be an object in $\mathcal{V}^{pre}$ such that there is an open covering $(U_i)_i$ such that $(U_i, \mathcal{O}_{X|U_i}, (\nu_x)_{x\in|U_i|})$ is an affinoid pre-adic space. We call an open subset $|U| \subseteq |X|$ an open affinoid subspace *if $(|U|, \mathcal{O}_{X|U}, (\nu_x)_{x\in|U|})$ is an affinoid pre-adic space. Then the sets $|U|$ where U is an open affinoid subspace form a basis of the topology of $|X|$. If the presheaf $\mathcal{O}_X$ is adapted to the basis of open affinoid subspaces (i.e. if the restrictions $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ for $V \subseteq |X|$ open, U an open affinoid subspace, and $U \subseteq V$ give an isomorphism $\mathcal{O}_X(V) \cong \varprojlim_U \mathcal{O}_X(U)$), we call X a* pre-adic space*. The* category of pre-adic spaces *is the full subcategroy of $\mathcal{V}^{pre}$ consisting of pre-adic spaces.*

If (A, A^+) is a Huber pair, then $\operatorname{Spa}(A, A^+)$ is an affinoid pre-adic space. Let X be a pre-adic space. As for affinoid pre-adic spaces, we define $\mathcal{O}^+_X = \{f \in \mathcal{O}_X(U) \,|\, |f(x)| \leq 1 \text{ for all } x \in U\}$ for every open subset $|U| \subseteq |X|$, endowed with the subspace topology from $\mathcal{O}_X(U)$. This is a sub-presheaf of topological rings of $\mathcal{O}_X$. For every $x \in |X|$ denote by $\mathcal{O}^+_{X,x}$ the stalk of $\mathcal{O}^+_X$ at x.

Lemma 1.47. *Let X and Y be pre-adic spaces. Let $(f, f^\flat) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a pair such that $f : X \to Y$ is continuous and such that $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a local map of presheaves of topological rings. Then $(f, f^\flat)$ is a morphism in $\mathcal{V}^{pre}$ if and only if the following conditions are satisfied:*

1. *$f^\flat(\mathcal{O}_Y^+) \subseteq f_*(\mathcal{O}_X^+)$.*

2. *The induced morphism $\mathcal{O}_Y^+ \to f_*\mathcal{O}_X^+$ is a local morphism of presheaves of rings.*

Proof. See [41, Lemma 8.14]. □

Let $\mathcal{V}$ be the full subcategory of $\mathcal{V}^{pre}$ consisting of those objects $(X, \mathcal{O}_X, (v_x)_{x\in X})$ of $\mathcal{V}^{pre}$ such that $\mathcal{O}_X$ is a sheaf of topological rings.

Definition 1.48.

1. *A Huber pair (A, A^+) is called* sheafy *if $\mathcal{O}_{\mathrm{Spa}(A,A^+)}$ is a sheaf of topological rings.*

2. *An* affinoid adic space *is an object in $\mathcal{V}$ which is isomorphic to* $\mathrm{Spa}(A, A^+)$ *for a (sheafy) Huber pair (A, A^+).*

3. *An* adic space *is an object $(|X|, \mathcal{O}_X, (v_x)_{x\in|X|})$ of $\mathcal{V}$ such that there is an open covering $(U_i)_{i\in I}$ of X such that $(|U_i|, \mathcal{O}_{|U_i}, (v_x)_{x\in|U_i|})$ is a affinoid adic space for all $i \in I$.*

Remark 1.49. We have a functor $(A, A^+) \mapsto \mathrm{Spa}(A, A^+)$ from the category of sheafy Huber pairs to the category of adic spaces. The functor $(A, A^+) \mapsto \mathrm{Spa}(A, A^+)$ from the category of complete sheafy Huber pairs to the category of adic spaces is fully faithful.

Proposition 1.50. *The canonical map* $\mathrm{Spa}(\widehat{A}, \widehat{A}^+) \to \mathrm{Spa}(A, A^+)$ *is an isomorphism of (pre-)adic spaces.*

Proof. This follows from Lemma 1.5 in [23]. □

In general, it is not easy to determine whether a given Huber pair is sheafy. However, there are some conditions which ensure sheafiness:

An affinoid pre-adic space $X = \mathrm{Spa}(A, A^+)$ is called stably uniform if A is stably uniform, that it, for every rational subset $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is uniform. (This is independent of the A^+.)

Theorem 1.51 (Buzzard-Verberkmoes, Mihara)**.** *If A is stably uniform, then (A, A^+) is sheafy.*

Proof. This is Theorem 7 in [12]. □

1.2.2 The functor r_K from rigid-analytic spaces to adic spaces

Adic spaces can be considered as a generalization of rigid-analytic spaces. Let K be a complete nonarchimedean field. If A is an affinoid K-algebra, then $\mathrm{Sp}(A)$ denotes the associated affinoid rigid-analytic space (see e.g. [8]). In Huber's book [24, 1.1.] a functor

$$r_K : \{\text{Rigid analytic spaces over } \mathrm{Sp}(K)\} \to \{\text{Adic spaces over } \ \mathrm{Spa}(K, o_K)\}$$

from the category of rigid analytic spaces over $\mathrm{Sp}(K)$ to the category of adic spaces over $\mathrm{Spa}(K, o_K)$ with the following properties is constructed:

1. If $X = \mathrm{Sp}(A)$ is an affinoid rigid analytic space, then $r_K(X) = \mathrm{Spa}(A, A^\circ)$.
2. If $f : X \to Y$ is an open immersion of rigid analytic spaces, then
$$r_K(f) : r_K(X) \to r_K(Y)$$
is an open immersion of adic spaces.
3. A family $\{X_i\}_{i\in I}$ of admissible open subsets of a rigid analytic space X is an admissible covering of X if and only if $r(X) = \bigcup_{i\in I} r(X_i)$ is an open covering.
4. r_K is fully faithful.

If X is a rigid analytic space over K, we write $r_K(X) = X^{\mathrm{ad}}$.

Remark 1.52. Let A and B be affinoid K-algebras and

$$f : X = \mathrm{Sp}(A) \to Y = \mathrm{Sp}(B)$$

be a map between rigid-analytic spaces over (K, o_K). We have Tate-Huber pairs (A, A°) and (B, B°) over K. The corresponding map $f^\flat : (B, B^\circ) \to (A, A^\circ)$ is adic.

If $U \subseteq Y$ is a rational subset, then the preimage $f^{-1}(U) \subseteq X$ is a rational subset as well. More precisely, if $U = Y(\frac{T}{s})$, then

$$f^{-1}(U) = X(\frac{f^\flat(T)}{f^\flat(s)}) = \mathrm{Sp}(A\langle\frac{f^\flat(T)}{f^\flat(s)}\rangle).$$

The analogous statement is true for $r_K(f) : X^{\mathrm{ad}} \to Y^{\mathrm{ad}}$ (here we use that $A\langle\frac{f^\flat(T)}{f^\flat(s)}\rangle^\circ = A\langle\frac{f^\flat(T)}{f^\flat(s)}\rangle^+$ as in Remark 1.23), and we see that

$$\begin{aligned} r_K(f^{-1}(U)) &= r_K(\mathrm{Sp}((A\langle\frac{f^\flat(T)}{f^\flat(s)}\rangle)) \\ &= \mathrm{Spa}(A\langle\frac{f^\flat(T)}{f^\flat(s)}\rangle, A\langle\frac{f^\flat(T)}{f^\flat(s)}\rangle^\circ) \\ &= r_K(f)^{-1}(r_K(U)) \end{aligned}$$

According to the Gerritzen-Grauert theorem (see e.g. [8, 3.3, Theorem 20]) every affinoid subdomain of an affinoid K-space is a finite union of rational subsets.
Let $U = U_1 \cup ... \cup U_n \subseteq Y$ now be an affinoid subdomain which is the union of rational subsets $U_1, ..., U_n$. This is an admissible covering of U, as well as $f^{-1}(U_1) \cup ... \cup f^{-1}(U_n)$ is an admissible covering of $f^{-1}(U)$, therefore we have $r_K(U) = r_K(U_1) \cup ... r_K(U_n)$ and $r_K(f^{-1}(U)) = r_K(f^{-1}(U_1)) \cup ... \cup r_K(f^{-1}(U_n))$. Then

$$\begin{aligned} r_K(f)^{-1}(r_K(U)) &= r_K(f)^{-1}(r_K(U_1) \cup ... \cup r_K(U_n)) \\ &= r_K(f)^{-1}(r_K(U_1)) \cup ... \cup r_K(f)^{-1}(r_K(U_n)) \\ &= r_K(f^{-1}(U_1)) \cup ... \cup r_K(f^{-1}(U_n)) \\ &= r_K(f^{-1}(U_1) \cup ... \cup f^{-1}(U_1)) \\ &= r_K(f^{-1}(U)). \end{aligned}$$

1.2.3 Fibre products of adic spaces

Definition 1.53. *Let $f : X \to Y$ be a map of adic spaces.*

(i) f *is called* adic *if for every $x \in X$ there are open affinoid neighbourhoods U of x and V of $f(x)$ such that $f(U) \subseteq V$ and such that the induced map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is adic.*

(ii) f *is called* locally of weakly finite type *if for every $x \in X$ there are open affinoid subspaces U and V of X resp. Y such that $x \in U, f(U) \subseteq V$, and the induced morphism of Huber rings $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of topologically finite type.*

(iii) f *is called* locally of finite type *if for every $x \in X$ there are open affinoid subspaces U and V of X resp. Y such that $x \in U, f(U) \subseteq V$, and the morphism $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ of Huber pairs is of topologically finite type.*

Proposition 1.54. *Let $f : X \to Z$ and $g : Y \to Z$ be morphisms of adic spaces. The fibre product $X \times_Z Y$ of f and g in the category of adic spaces exists in the following cases:*

1. *f is locally of finite type,*
2. *f is locally of weakly finite type and g is adic.*

Proof. Proposition 1.2.2 in [24]. □

In the second case, and if $Z = \mathrm{Spa}(C, C^+), X = \mathrm{Spa}(A, A^+), Y = \mathrm{Spa}(B, B^+)$ are affinoid, the space $X \times_Z Y$ is given by $\mathrm{Spa}(A\widehat{\otimes}_C B, (A\widehat{\otimes}_C B)^+)$.

1.2.4 Inductive limits of uniform Tate-Huber pairs

In general, when forming the inductive limit of a system of Tate rings in the category of rings, there is no obvious topology which makes it into a Tate ring. Therefore we restrict ourselves to the case of uniform Tate rings resp. uniform Tate-Huber pairs. Then we can always take the power-bounded elements are ring of definition.

Proposition 1.55 (See Chapter 7, 7.4.10 in [7]). *Let $(A_i, A_i^+)_{i\in I}$ be a filtered inductive system of uniform Tate-Huber pairs with maps $\varphi_{ij} : A_i \to A_j$ for $j \geq i$.*

1. *The inductive limit (A, A^+) of $(A_i, A_i^+)_{i\in I}$ exists in the category of uniform Tate-Huber pairs. The A is identified as $\varinjlim_{i\in I} A_i$ and the A^+ is identified as $\varinjlim_{i\in I} A_i^+$, computed in the category of rings.*
2. *The natural map $|\mathrm{Spa}(A, A^+)| \to \varprojlim_{i\in I} |\mathrm{Spa}(A_i, A_i^+)|$ is a homeomorphism. Moreover, each rational subset of $|\mathrm{Spa}(A, A^+)|$ is pulled back from a rational subset of $|\mathrm{Spa}(A_i, A_i^+)|$ for some $i \in I$.*

Proof. We may assume that I has a minimal element i_0. We choose a topological nilpotent unit $t \in A_{i_0}^+$. Each $A_i^+ \subseteq A_i$ is a ring of definition with ideal of definition (t) where t denotes by abuse of notation the image of t under the map $A_{i_0} \to A_i$. We set $A = \varinjlim_{i\in I} A_i$. We view A as a Huber ring with ring of definition $A^+ := \varinjlim_{i\in I} A_i^+$ and ideal of definition (t). Then (A, A^+) is uniform because the ring of integral elements A^+ is by definition a ring of definition.
We show that (A, A^+) is in fact the inductive of $(A_i, A_i^+)_{i\in I}$ in the category of uniform Tate-Huber pairs. Let $f_i : (A_i, A_i^+) \to (B, B^+)$ be a compatible system of maps with (B, B^+) being a uniform Tate-Huber pair. Then there is a unique map $f : A \to B$ with $f(A^+) \subseteq B^+$ in the category of rings. Since B is uniform, B^+ is a ring of definition for B. We have $f^{-1}(t^n B^+) \supseteq t^n A^+$, so f is continuous. Therefore we get an continuous map $(A, A^+) \to (B, B^+)$ of uniform Tate-Huber pairs. It is clearly unique.
We have a valuation ring $R_x \subseteq \kappa(\mathfrak{p}_x)$ for every point $x \in \mathrm{Spa}(A, A^+)$. Then $(\kappa(\mathfrak{p}_x), R_x)$ is a uniform Tate-Huber pair for the valuation topology on $\kappa(\mathfrak{p}_x)$. The point x determines and is determined by a map $(A, A^+) \to (\kappa(\mathfrak{p}_x), R_x)$ of Tate-Huber pairs (see [7, Proposition 7.3.7]), and it also determines points $x_i \in \mathrm{Spa}(A_i, A_i^+)$ together with maps $(A_i, A_i^+) \to (\kappa(\mathfrak{p}_{x_i}), R_{x_i})$. Then $(\kappa(\mathfrak{p}_x), R_x) = \varinjlim_{i\in I}(\kappa(\mathfrak{p}_{x_i}), R_{x_i})$ as uniform Tate-Huber pairs. We see that

$$\mathrm{Spa}(A, A^+) \to \varprojlim_{i\in I} \mathrm{Spa}(A_i, A_i^+)$$

is a continuous bijection. Since $A = \varinjlim_{i\in I} A_i$ as rings, the defining open subsets $\mathrm{Spa}(A, A^+)(\frac{f}{g})$ with $f, g \in A$ arise via pullback from $\mathrm{Spa}(A_i, A_i^+)$. □

Remark 1.56. Let $(A_i, A_i^+)_{i\in I}$ and (A, A^+) be as in the previous proposition, and for each i denote by $\alpha_i : A_i \to A$ the canonical map. As mentioned in Remark

1.6 the topology on A as a Huber ring is induced by the seminorm on A given by $|x| = 2^{\inf(n\in\mathbb{Z};\, t^n x\in A^+)}$ for $x \in A$, and similarly for the A_i. This seminorm on A coincides with the inductive limit seminorm coming from the seminorms on the A_i (see Definition 6.7 in the Appendix) which is given by $\inf_{i\in I, x_i\in\alpha_i^{-1}(x)} 2^{\inf(n\in\mathbb{Z};\, t^n x_i\in A_i^+)}$.

In the situation of the previous proposition we now assume additionally that all A_i are stably uniform. Let s, T be as in Definition 1.21 and consider the rational localization $(A\langle\frac{T}{s}\rangle, A\langle\frac{T}{s}\rangle^+)$. We may assume that $s \in T$. We then have the following lemma:

Lemma 1.57. *Let $(A_i, A_i^+)_{i\in I}$ be an inductive system of stably uniform Tate-Huber pairs with maps $\varphi_{ij} : A_i \to A_j$ for $j \geq i$. Denote by (A, A^+) its inductive limit in the category of uniform Tate-Huber pairs. Then $(A\langle\frac{T}{s}\rangle_u, A\langle\frac{T}{s}\rangle_u^+)$ is the t-adic completion of the inductive limit of rational localizations of the (A_i, A_i^+).*
Conversely, if we consider a rational localization $(A_{i_0}\langle\frac{T_{i_0}}{s_{i_0}}\rangle, A_{i_0}\langle\frac{T_{i_0}}{s_{i_0}}\rangle^+)$ of $(A_{i_0}, A_{i_0}^+)$ for an i_0, then the completed inductive limit of the induced inductive system is given $(A\langle\frac{T}{s}\rangle_u, A\langle\frac{T}{s}\rangle_u^+)$ (where T and s are the images of T_{i_0} and s_{i_0} in A).

Proof. Let $i_0 \in I$ such that we can choose a preimage $x_{k,i_0} \in A_{i_0}$ for every $x_k \in T = \{x_1, ..., x_n\}$ and a preimage $s_{i_0} \in A_{i_0}$ of s, and such that the x_{k,i_0} generate the unit ideal in A_{i_0}. Let T_{i_0} be the (finite) set consisting of the x_{k,i_0}. We choose a topologically nilpotent unit $t \in A_{i_0}$.
Denote the image of T_{i_0} respective s_{i_0} under the map $\varphi_{i_0 i}$ in A_i by $T_i = \{x_{1,i}, ..., x_{n,i}\}$ respective s_i for $i \geq i_0$. Note that T_i generates the unit ideal in A_i for all $i \geq i_0$, hence we have uniform Tate-Huber pairs $(A_i(\frac{T_i}{s_i}), A_i(\frac{T_i}{s_i})^+)$. We have induced continuous maps

$$\varphi_{ij} : A_i(\frac{T_i}{s_i}) \to A_j(\frac{T_j}{s_j}),$$

which fulfil $\varphi_{ij}(A_i(\frac{T_i}{s_i})^+) \subseteq A_j(\frac{T_j}{s_j})^+$. The $(A_i(\frac{T_i}{s_i}), A_i(\frac{T_i}{s_i})^+)$ together with the maps φ_{ij} for $i, j \geq i_0$ form an inductive system in the category of uniform Tate-Huber pairs.

We have an isomorphism of (abstract) rings

$$f : \varinjlim_{i\geq i_0} A_i(\frac{T_i}{s_i}) = \varinjlim_{i\geq i_0} A_{i,s} \to A = A_s$$

as inductive limits commute with localizations. Moreover, we have

$$\varinjlim_{i\geq i_0} A_i^+[\frac{x_{1,i}}{s_i}, ..., \frac{x_{n,i}}{s_i}] = A^+[\frac{x_1}{s}, ..., \frac{x_n}{s}].$$

As forming inductive limits commutes with taking the integral closure, the image of $\varinjlim_{i\geq i_0}(A_i(\frac{T_i}{s_i})^+)$ under f is $A(\frac{T}{s})^+$, i.e. the integral closure of $A^+[\frac{x_1}{s}, ..., \frac{x_n}{s}]$ in $A(\frac{T}{s})$. We obtain an isomorphism

$$(\varinjlim_{i\geq i_0}(A_i(\frac{T_i}{s_i})^+), \mathrm{Rad}((t))) \cong (A(\frac{T}{s})^+, \mathrm{Rad}((t)))$$

in the category $\underline{\mathrm{Tate}}_{alg}$ (see Remark 1.31). Therefore we get an isomorphism

$$(A(\frac{T}{s})_u, A(\frac{T}{s})_u^+) \cong (\varinjlim_{i\geq i_0} A_i(\frac{T_i}{s_i}), \varinjlim_{i\geq i_0} A_i(\frac{T_i}{s_i})^+).$$

of uniform Tate-Huber pairs. Passing to the completions gives an isomorphism

$$(A\langle\frac{T}{s}\rangle_u, A\langle\frac{T}{s}\rangle_u^+) \cong (\widehat{\varinjlim}_{i\geq i_0} A_i(\frac{T_i}{s_i}), \widehat{\varinjlim}_{i\geq i_0} A_i(\frac{T_i}{s_i})^+).$$

The latter is isomorphic to $(\widehat{\varinjlim}_{i\geq i_0} A_i\langle\frac{T_i}{s_i}\rangle, \widehat{\varinjlim}_{i\geq i_0} A_i\langle\frac{T_i}{s_i}\rangle^+)$. Therefore

$$(A\langle\frac{T}{s}\rangle_u, A\langle\frac{T}{s}\rangle_u^+) \cong (\widehat{\varinjlim}_{i\geq i_0} A_i\langle\frac{T_i}{s_i}\rangle, \widehat{\varinjlim}_{i\geq i_0} A_i\langle\frac{T_i}{s_i}\rangle^+).$$

Conversely, if we have a rational localization $A_{i_0}\langle\frac{T_{i_0}}{s_{i_0}}\rangle$ of some A_{i_0}, then we denote the image of T_{i_0} and s_{i_0} in A by T respective s, and with the same reasoning as before we get an isomorphism

$$(\widehat{\varinjlim}_{i\geq i_0} A_i\langle\frac{T_i}{s_i}\rangle, \widehat{\varinjlim}_{i\geq i_0} A_i\langle\frac{T_i}{s_i}\rangle^+) \cong (A\langle\frac{T}{s}\rangle_u, A\langle\frac{T}{s}\rangle_u^+),$$

□

Remark 1.58. Assume now that all A_i are stably uniform and that the limit $\varinjlim_{i\in I} A_i$ is also stably uniform (so that we do not need to uniformize as in the previous lemma).
Let $U = \mathrm{Spa}(A, A^+)(\frac{T}{s}) \subseteq \mathrm{Spa}(A, A^+)$ be a rational subset which arises via pullback from $U_{i_0} \subseteq \mathrm{Spa}(A_{i_0}, A_{i_0}^+)$, and let $U_i \subseteq \mathrm{Spa}(A_i, A_i^+)$ be the preimage of U_{i_0} under the map $\mathrm{Spa}(A_i, A_i^+) \to \mathrm{Spa}(A_{i_0}, A_{i_0}^+)$ which is a rational subset (since it is a preimage of a rational subset under an adic map). We have

$$\mathcal{O}_{\mathrm{Spa}(A,A^+)}(U) = \widehat{\varinjlim}_{i\geq i_0} \mathcal{O}_{\mathrm{Spa}(A_i,A_i^+)}(U_i)$$

as topological rings because of Lemma 1.57. If $V \subseteq U$ is another rational subset of $\mathrm{Spa}(A, A^+)$ with preimage $V_i \subseteq \mathrm{Spa}(A_i A_i^+)$, the diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathrm{Spa}(A_i,A_i^+)}(U_i) & \longrightarrow & \mathcal{O}_{\mathrm{Spa}(A_j,A_j^+)}(U_j) \\ \Big\downarrow{\scriptstyle \mathrm{res}} & & \Big\downarrow{\scriptstyle \mathrm{res}} \\ \mathcal{O}_{\mathrm{Spa}(A_i,A_i^+)}(V_i) & \longrightarrow & \mathcal{O}_{\mathrm{Spa}(A_j,A_j^+)}(V_j) \end{array}$$

commutes for all $j \geq i \geq i_0$. By passing to the inductive limit and then to the completion (by continuity), we get maps

$$\mathrm{res} : \widehat{\varinjlim}_{i\geq i_0} \mathcal{O}_{\mathrm{Spa}(A_i,A_i^+)}(U_i) \to \widehat{\varinjlim}_{i\geq i_0} \mathcal{O}_{\mathrm{Spa}(A_i,A_i^+)}(V_i),$$

so we get a presheaf $U \mapsto \widehat{\varinjlim}_{i\geq i_0} \mathcal{O}_{\mathrm{Spa}(A_i,A_i^+)}(U_i)$ on the basis of rational subsets of $|\mathrm{Spa}(A,A^+)|$. One checks that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}_{\mathrm{Spa}(A,A^+)}(U) & \xrightarrow{\cong} & \widehat{\varinjlim}_{i\geq i_0} \mathcal{O}_{\mathrm{Spa}(A_i,A_i^+)}(U_i) \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\ \mathcal{O}_{\mathrm{Spa}(A,A^+)}(V) & \xrightarrow{\cong} & \widehat{\varinjlim}_{i\geq i_0} \mathcal{O}_{\mathrm{Spa}(A_i,A_i^+)}(V_i) \end{array}$$

We see that $U \mapsto \widehat{\varinjlim}_{i\geq i_0} \mathcal{O}_{\mathrm{Spa}(A_i,A_i^+)}(U_i)$ is a presheaf of complete topological rings on the basis of rational subsets which coincides with the sheaf $\mathcal{O}_{\mathrm{Spa}(A,A^+)}$ on the basis of rational subsets. Therefore it is a sheaf on the basis of rational subsets and extends uniquely to a sheaf on $|\mathrm{Spa}(A,A^+)|$ (see [1, 6.30]). It is isomorphic to $\mathcal{O}_{\mathrm{Spa}(A,A^+)}$.

1.3 Preperfectoid and perfectoid spaces

In this section we give a brief overview of the theory of perfectoid and preperfectoid spaces. Perfectoid spaces were introduced by Scholze in [36]. The main sources for this section are [36], [34], and [27]. See also [30] and [43].

1.3.1 Perfectoid fields

Definition 1.59. *A nonarchimedean field of residue characteristic p is called* perfectoid *if*

(i) the absolute value $|\cdot|$ on K is not discrete,

(ii) K is complete,

(iii) the Frobenius on $o_K/(p)$ is surjective.

Remark 1.60. Every element of the value group of a perfectoid field K is a p-th power.

Proof. This is Remark 1.4.3 in [34]. □

Let $L \subseteq K \subseteq \mathbb{C}_p$ be a perfectoid field. We construct the *tilt* $K^\flat$ of K which is a perfectoid field of characteristic p. Let $\mathfrak{m}_K$ denote the maximal ideal of o_K. We choose an element $\varpi \in \mathfrak{m}_K$ such that $|\varpi| \geq |\pi|$. We define

$$o_{K^\flat} := \varprojlim_{(\cdot)^q} o_K/(\varpi).$$

This is a perfect k-algebra.

Let $\alpha = (\alpha_0, \alpha_1, ...) \in o_{K^\flat}$. Choose an $a_i \in o_K$ such that $\alpha_i = a_i \bmod (\varpi)$ for $i \geq 0$. We have $a_{i+1}^q = a_i \bmod (\varpi)$ and hence $a_{i+1}^{q^{i+1}} = a_i^{q^i} \bmod (\varpi^{i+1})$ by (a generalization of) [34, Lemma 1.1.1]. We form the limit

$$\alpha^\sharp := \lim_{i\to\infty} a_i^{q^i} \in o_K.$$

We get a well defined multiplicative map

$$\begin{aligned} o_{K^\flat} &\to o_K, \\ \alpha &\mapsto \alpha^\sharp, \end{aligned}$$

which fulfils $\alpha^\sharp \mod (\varpi) = \alpha_0$.

Lemma 1.61. *We have a multiplicative bijection*

$$\begin{aligned} \varprojlim_{(\cdot)^q} o_K &\to o_{K^\flat}, \\ (a_0, a_1, ...) &\mapsto (a_0 \mod (\varpi), a_1 \mod (\varpi), ...). \end{aligned}$$

In particular, $o_{K^\flat}$ is independent of the choice of ϖ.

Proof. This is Lemma 1.4.5 in [34]. □

Lemma 1.62. *The map*

$$\begin{aligned} |\cdot|_\flat : o_{K^\flat} &\to \mathbb{R}_{\geq 0}, \\ \alpha &\mapsto |\alpha^\sharp| \end{aligned}$$

is a nonarchimedean absolute value. We have

1. $|o_K| = |o_{K^\flat}|_\flat$;
2. $\alpha o_{K^\flat} \subseteq \beta o_{K^\flat}$ *if and only if* $|\alpha|_\flat \leq |\beta|_\flat$ *for* $\alpha, \beta \in o_{K^\flat}$;
3. $\mathfrak{m}_{K^\flat} := \{\alpha \in o_{K^\flat} \,|\, |\alpha|_\flat < 1\}$ *is the unique maximal ideal in* $o_{K^\flat}$;
4. *let* $\varpi^\flat \in o_{K^\flat}$ *be any element such that* $|\varpi^\flat|_\flat = |\varpi|$, *then the projection map which sends* $(\alpha_0, ...)$ *to* α_0 *induces an isomorphism of rings* $o_{K^\flat}/(\varpi^\flat) \cong o_K/(\varpi)$. *We have* $o_{K^\flat}/\mathfrak{m}_{K^\flat} \cong o_K/\mathfrak{m}_K$.

Proof. This is Lemma 1.4.6 in [34]. □

We now fix an element $\varpi^\flat \in o_{K^\flat}$ such that $|\varpi^\flat|_\flat = |\varpi|$. We deduce from the previous lemma that $o_{K^\flat}$ is an integral domain, and that every element in its field of fractions $K^\flat$ can be written as $(\frac{\alpha}{\varpi^\flat})^n$ for $\alpha \in o_{K^\flat}$ and $n \geq 0$. We can extend $|\cdot|_\flat$ by multiplicativity to $K^\flat$. The the value groups of K and $K^\flat$ coincide, and $o_{K^\flat}$ is the ring of integers of $K^\flat$ for $|\cdot|_\flat$.

Proposition 1.63. *$K^\flat$ is a perfect and complete nonarchimedean field of characteristic p. It is called the* tilt *of K.*

Proof. This is Proposition 1.4.7 in [34]. □

Example 1.64. $\mathbb{C}_p$ is a perfectoid field. Its tilt $\mathbb{C}_p^\flat$ is algebraically closed.

Lemma 1.65. *If $L \subseteq K \subseteq \mathbb{C}_p$ is complete with dense value group such that for an element $\varpi \in K$ we have $|p| \leq |\varpi| < 1$ and $(o_K/(\varpi))^q = o_K/(\varpi)$, then K is perfectoid.*

Proof. This is Lemma 1.4.11 in [34]. □

A complete nonarchimedean field of characteristic p is perfectoid if and only if it is perfect.

Theorem 1.66 (Scholze)**.** *Let $L \subseteq K \subseteq \mathbb{C}_p$ be a perfectoid field.*

(i) Let K' be a finite extension of K. Then K' (with its natural topology as a finite-dimensional K-vector space) is a perfectoid field.

(ii) Let $K^\flat$ be the tilt of K. The tilting functor $K' \mapsto (K')^\flat$ induces an equivalence of categories between the category of finite extensions of K and the category of finite extensions of $K^\flat$. This equivalence preserves degrees.

Proof. This is Theorem 3.7 in [36] □

1.3.2 Perfectoid Tate rings and perfectoid spaces

In the following let K be a perfectoid field and let $\varpi \in \mathfrak{m}_K$ with $|\varpi| \geq |p|$. We fix an element $\varpi^\flat \in K^\flat$ such that $|\varpi| = |\varpi^\flat|_\flat$.

Definition 1.67. *A Tate ring A over K is called perfectoid if it is complete and uniform, and if the Frobenius $A^\circ/(\varpi) \to A^\circ/(\varpi)$ is surjective.*

Remark 1.68. Let A be a perfectoid Tate ring over K. The Frobenius $A^\circ/(p) \to A^\circ/(p)$ is surjective if and only if the Frobenius $A^\circ/(\varpi) \to A^\circ/(\varpi)$ is surjective.

Proof. If $(A^\circ/(p))^p = A^\circ/(p)$, then also $(A/(\varpi))^p = A^\circ/(p)$. Suppose we have $(A^\circ/(\varpi))^p = A^\circ/(\varpi)$. Let $\varpi_1 \in K$ be an element such that $|\varpi|^{1/p} \leq |\varpi_1| < 1$. Then $\varpi A^\circ \subseteq \varpi_1^p A^\circ$ and then $(A^\circ/(\varpi_1^p))^p = A^\circ/(\varpi_1^p)$. Let $x \in A^\circ$. Then we inductively find elements $(y_n)_n$ and $(x_n)_n$ in A° such that

$$\begin{aligned} x &= y_0^p + \varpi_1^p x_1 \\ x_1 &= y_1^p + \varpi_1^p x_2 \\ &\vdots \\ x_n &= y_n^p + \varpi_1^p x_{n+1} \\ &\vdots \end{aligned}$$

We see hat there are elements $(z_n)_n$ in A° such that

$$x \equiv z_n^p + \varpi_1^{p(n+1)} x_{n+1} \mod (p) \quad \text{for any } n \geq 0.$$

But as we have $|\varpi_1^{p(n+1)}| \leq |p|$ and hence $(\varpi_1^{p(n+1)}) \subseteq (p)$ for sufficiently large n, we see that $(A^\circ/(p))^p = A^\circ/(p)$. □

Tilting for perfectoid Tate rings works in much the same way as for perfectoid fields:

Definition 1.69. *Let A be a perfectoid Tate ring over K. We define*

$$A^{\flat\circ} := \varprojlim_{x \mapsto x^p} A^\circ/(\varpi)$$

and give it the inverse limit topology.

It is a topological $o_{K^\flat}$-algebra. Set $A^\flat = A^{\flat\circ}[1/\varpi^\flat]$. This is a perfectoid $K^\flat$-algebra. We call it the *tilt* of A. We have a well-defined multiplicative map

$$A^{\flat\circ} \to A^\circ,$$
$$(a_0 \mod (\varpi), a_1 \mod (\varpi), ...) \mapsto \alpha^\sharp := \lim_{i\to\infty} a_i^{q^i}$$

which fulfils $\alpha^\sharp \equiv a_i^{q^i} \mod (\varpi)$. There is an isomorphism

$$\varprojlim_{x \mapsto x^p} A^\circ \to A^{\flat\circ},$$
$$(a_0, a_1, ...) \mapsto (a_0 \mod (\varpi), a_1 \mod (\varpi), ...)$$

of multiplicative monoids.

Definition 1.70. *Let (A, A^+) be a Tate-Huber pair over (K, o_K) (i.e. there is a map $(K, o_K) \to (A, A^+)$). Then (A, A^+) is called perfectoid if A is a perfectoid Tate ring.*

Let (A, A^+) be a perfectoid Tate-Huber pair. There is a bijection between the set of rings of integral elements of A and the set of rings of integral elements of $A^\flat$. The bijection is given by $A^+ \mapsto A^{\flat+}$, where $A^{\flat+} = \varprojlim_{x\mapsto x^p} A^+$.

Theorem 1.71 (Scholze)**.** *There is an equivalence of categories between perfectoid K-algebras and perfectoid $K^\flat$-algebras.*

Proof. Theorem 5.2 in [36] □

Theorem 1.72 (Scholze)**.** *Let (A, A^+) be a perfectoid Tate-Huber pair over (K, o_K), and let $X = \mathrm{Spa}(A, A^+)$ with associated presheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$. Let $(A^\flat, A^{\flat+})$ be the tilt of A, and $X^\flat = \mathrm{Spa}(A^\flat, A^{\flat+})$.*

1. *We have a homeomorphism* $|X| \cong |X^\flat|$, *given by mapping* $x \in X$ *to the valuation* $x^\flat \in X^\flat$ *defined by* $|f(x^\flat)| = |f^\sharp(x)|$. *This homeomorphism identifies rational subsets.*

2. *For any rational subset* $U \subseteq X$ *with tilt* $U^\flat \subseteq X^\flat$, *the complete Tate-Huber pair* $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ *is perfectoid, with tilt* $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$.

3. *The presheaves* $\mathcal{O}_X$ *and* $\mathcal{O}_{X^\flat}$ *are sheaves.*

Proof. Theorem 6.3 in [36]. □

Definition 1.73. *An* affinoid perfectoid space *over a perfectoid field* K *is an adic space* $\mathrm{Spa}(A, A^+)$ *for some perfectoid Tate-Huber pair* (A, A^+) *over* (K, o_K). *A* perfectoid space *over* K *is an adic space over* K *which is locally isomorphic to an affinoid perfectoid space. Morphisms of perfectoid spaces are morphisms of adic spaces.*

Tilting gives a functor $X \mapsto X^\flat$ from perfectoid spaces over K to perfectoid spaces over $K^\flat$.

1.3.3 Preperfectoid algebras and spaces

We have the following definition which is a variant of the definitions 2.3.4 and 2.3.9 in [37].

Definition 1.74. 1. *Let* $L \subseteq K \subseteq \mathbb{C}_p$ *be a perfectoid field and let* X *be a (pre-)adic space over* $\mathrm{Spa}(K, o_K)$. *Then* X *is called* preperfectoid *if there is a covering of* X *by open affinoid spaces* $U_i = \mathrm{Spa}(A_i, A_i^+) \subseteq X$ *such that* $(\hat{A}_{i,u}, \hat{A}_{i,u}^+)$ *is a perfectoid Tate-Huber pair over* (K, o_K), *where we take the uniform completion, i.e. the completion with respect to the topology on* A_i *giving* A_i^+ *the* π*-adic topology.*

2. *Let* $X = \mathrm{Spa}(A, A^+)$ *for a Tate-Huber pair* (A, A^+) *over* (L, o_L). *Then* X *is* L-preperfectoid *if there is a perfectoid field* $L \subseteq K \subseteq \mathbb{C}_p$ *such that the uniform completion* $((A\widehat{\otimes}_L K)_u, (A\widehat{\otimes}_L K)_u^+)^\wedge$ *is perfectoid, so* $\mathrm{Spa}(((A\widehat{\otimes}_L K)_u, (A\widehat{\otimes}_L K)_u^+)$ *is a perfectoid space.*

3. *Let* X *be a (pre-)adic space over* $\mathrm{Spa}(L, o_L)$. *Then* X *is called* L*-preperfectoid if there is an open covering* $(U_i)_i$ *of* X *and a perfectoid field* $L \subseteq K \subseteq \mathbb{C}_p$ *such that* $U_i = \mathrm{Spa}(A_i, A_i^+)$ *for a Tate-Huber pair* (A_i, A_i^+) *over* (L, o_L) *and the uniform completion* $((A_i\widehat{\otimes}_L K)_u, (A_i\widehat{\otimes}_L K)_u^+)^\wedge$ *is perfectoid for every* i.

Remark 1.75. In [27] and [30], the authors use a different definition of preperfectoidness. There, a Banach algebra A over $\mathbb{Q}_p$ is said to be preperfectoid if there is a perfectoid $K/\mathbb{Q}_p$ such that $A\widehat{\otimes}_{\mathbb{Q}_p} K$ is perfectoid. The main difference is that in this definition the topology coming from the tensor product is required to be uniform so that one does not need to uniformize it. It proves to be too restrictive for our purpose. However, it ensures sheafiness.

Proposition 1.76. *Let (A, A^+) be a Tate-Huber pair over (L, o_L). Assume that there is a perfectoid field K over L such that $A\widehat{\otimes}_L K$ is perfectoid. Then (A, A^+) is stably uniform and therefore sheafy.*

Proof. For $L = \mathbb{Q}_p$, this is Proposition 6.3.3 in [43]. Let $L \subseteq K \subseteq \mathbb{C}_p$ be a perfectoid field such that $A' = A\widehat{\otimes}_L K$ is perfectoid. We want to show that (A, A^+) is stably uniform. Then it is sheafy by Theorem 1.51. Set $X = \mathrm{Spa}(A, A^+)$ and let $X' = \mathrm{Spa}(A', A'^+)$, A'^+ being the integral closure of $A^+\widehat{\otimes}_{o_L} o_K$ in A'. Let $U \subseteq X$ be a rational subset, and let V be its preimage under the map $X' \to X$. It is also a rational subset. We then have $\mathcal{O}_{X'}(V) = \mathcal{O}_X(U)\widehat{\otimes}_{o_L} o_K$ and $\mathcal{O}_X(U)$ is a topological subring of $\mathcal{O}_{X'}(V)$.
$\mathcal{O}_{X'}(V)$ is perfectoid because A' is perfectoid, and hence it is uniform. Therefore we can take the power-bounded elements $\mathcal{O}_{X'}(V)^\circ$ as a ring of definition. If $t \in A$ is a topologically nilpotent unit, then it is a topologically nilpotent unit in $\mathcal{O}_X(U)$ and $\mathcal{O}_{X'}(V)$ as well, and $t^n(\mathcal{O}_{X'}(V)^\circ \cap \mathcal{O}_X(U))$ for $n \geq 0$ is a basis of neighbourhoods of 0 in $\mathcal{O}_X(U)$. Then $\mathcal{O}_{X'}(V)^\circ \cap \mathcal{O}_X(U)$ is open and bounded in $\mathcal{O}_X(U)$ and therefore a ring of definition. Thus it is contained in $\mathcal{O}_X(U)^\circ$. Since $\mathcal{O}_X(U)^\circ \subseteq \mathcal{O}_{X'}(V)^\circ$, we conclude that $\mathcal{O}_X(U)^\circ = \mathcal{O}_X(U) \cap \mathcal{O}_{X'}(V)^\circ$ which is bounded in $\mathcal{O}_X(U)$. Therefore $\mathcal{O}_X(U)$ is uniform. □

2 The character variety

Remember that $L \subseteq \mathbb{C}_p$ is a finite extension of $\mathbb{Q}_p$ of degree d with ring of integers o_L and uniformizer $\pi_L = \pi$. Let $L \subseteq K \subseteq \mathbb{C}_p$ be a complete intermediate field.
In this chapter, we summarize the construction of the rigid-analytic character variety $\mathfrak{X}$ and of the corresponding Robba ring $\mathcal{R}_K(\mathfrak{X})$ as developed in [35] and [6]. Let A be an affinoid algebra over K. If the base extension $A_{\mathbb{C}_p} := A\widehat{\otimes}_K\mathbb{C}_p$ is reduced, then A is reduced as well, and the map $A \to A_{\mathbb{C}_p}$ is isometric for the supremum norm (see [31, Lemma 3.8.3/3 and Theorem 6.2.4/1]).

If X is a rigid-analytic variety over K, then we denote by $\mathcal{O}_K(X)^{bd}$ the elements $f \in \mathcal{O}_K(X)$ which are bounded, i.e. for which there is a constant C such that $|f(x)| \leq C$ for all $x \in X(\mathbb{C}_p)$.

2.1 The character variety $\mathfrak{X}$

We denote by $G = o_L$ the additive group o_L viewed as a locally L-analytic group. Let G_0 be the locally $\mathbb{Q}_p$-analytic group obtained from G via restriction of scalars. Furthermore, let $\widehat{G}(K)$ resp. $\widehat{G}_0(K)$ denote the group of K-valued locally analytic characters of G resp. G_0, i.e. group homomorphisms $G \to K^\times$ resp. $G_0 \to K^\times$ which are locally given by a power series.
Denote by $\mathfrak{B}_1$ the rigid $\mathbb{Q}_p$-analytic open unit disk around $1 \in \mathbb{Q}_p$. We have a bijection

$$\mathfrak{B}_1(K) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o_L, \mathbb{Z}_p) \to \widehat{G}_0(K), \tag{2.1}$$

$$z \otimes \beta \mapsto \chi_{z\otimes\beta}(g) := z^{\beta(g)}, \tag{2.2}$$

where we define $z^a := \sum_{n\geq 0} \binom{a}{n}(z-1)^n$ for $z \in \mathfrak{B}_1(K)$ and $a \in \mathbb{Z}_p$. Set

$$\mathfrak{X}_0 := \mathfrak{B}_1 \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o_L, \mathbb{Z}_p).$$

This is a rigid-analytic group variety which is (noncanonically) isomorphic to a d-dimensional open unit polydisk. On the level of K-points we have $\mathfrak{X}_0(K) = \widehat{G}_0(K)$. In this sense $\mathfrak{X}_0$ "represents" the character group $\widehat{G}_0$. It is shown in [35] that one can define a one-dimensional rigid-analytic variety $\mathfrak{X}$ which "represents" the character group $\widehat{G}$. This variety $\mathfrak{X}$ is constructed via explicit equations as a subvariety of $\mathfrak{X}_0$. Namely, if $t_1, ..., t_d$ is a $\mathbb{Z}_p$-basis of o_L, then $\mathfrak{X}$ is defined by the equations

$$(\beta(t_i) - t_i \cdot \beta(1)) \cdot \log(z) = 0 \qquad \text{for } 1 \leq i \leq d.$$

Let $\beta_1, ..., \beta_d$ be the basis of $\mathrm{Hom}_{\mathbb{Z}_p}(o_L, \mathbb{Z}_p)$ dual to $t_1, ..., t_d$. Identifying $\mathfrak{X}_0$ with $\mathfrak{B}_1^d$, we get an identification of $\mathfrak{X}$ with

$$\begin{aligned}&\{(z_1, ..., z_d) \in \mathfrak{B}^d_{1/L} \mid \sum_j \beta_j(1)\log(z_j) = \frac{1}{t_i}\log(z_i) \text{ for } 1 \leq i \leq d\}\\ &= \{(z_1, ..., z_d) \in \mathfrak{B}^d_{1/L} \mid \log(z_i) = \frac{t_i}{t_1}\log(z_1) \text{ for } 1 \leq i \leq d\}.\end{aligned}$$

Restriction of functions gives a surjective ring homomorphism $\mathcal{O}_K(\mathfrak{X}_0) \to \mathcal{O}_K(\mathfrak{X})$ which restricts to an injective ring homomorphism $\mathcal{O}_K(\mathfrak{X}_0)^{bd} \to \mathcal{O}_K(\mathfrak{X})^{bd}$ between the rings of bounded functions (see [6, Lemma 1.15] for the injectivity of the latter map).

We denote by $\mathfrak{B}$ the rigid $\mathbb{Q}_p$-analytic open unit disk around $0 \in \mathbb{Q}_p$. For any $r \in (0,1) \cap p^{\mathbb{Q}}$ we denote by $\mathfrak{B}_1(r)$ resp. $\mathfrak{B}(r)$ the $\mathbb{Q}_p$-affinoid disk of radius r around 1, resp. 0. We put

$$\mathfrak{X}(r) := \mathfrak{X} \cap (\mathfrak{B}_1(r) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o_L, \mathbb{Z}_p))_{/L}.$$

This is an affinoid subgroup of $\mathfrak{X}$.

Lemma 2.1. *For any $r \in (0, p^{-\frac{1}{p-1}}) \cap p^{\mathbb{Q}}$ the map*

$$\begin{aligned}\mathfrak{B}(r) &\to \mathfrak{X}(r)\\ y &\mapsto \chi_y(g) := \exp(gy)\end{aligned}$$

is an isomorphism of L-affinoid groups.

Proof. This is Lemma 1.16 in [6]. □

2.2 The isomorphism $\mathfrak{X}_{/\mathbb{C}_p} \cong \mathfrak{B}_{/\mathbb{C}_p}$

2.2.1 Brief overwiev of Lubin-Tate theory

For a detailed presentation of Lubin-Tate theory see e.g. [34].

Definition 2.2. *A (one-dimensional) commutative formal group law over o_L is a formal power series $F \in o_L[|X, Y|]$ in two variables such that*

1. *$F(X, Y) = X + Y+$ terms of higher degree,*
2. *$F(X, F(Y, Z)) = F(F(X, Y), Z)$,*
3. *$F(X, Y) = F(Y, X)$,*
4. *there is a unique formal power series $\iota_F(X) \in X o_L[|X|]$ such that $F(X, \iota_F(X)) = 0$.*

We have $F(X,0) = X$ and $F(0,Y) = Y$. A homomorphism of formal group laws $h : F \to G$ over o_L is a formal power series $h \in Xo_L[|X|]$ such that $h(F(X,Y)) = G(h(X), h(Y))$. For any formal group laws F and G over o_L, the set of homomorphisms $\mathrm{Hom}_{o_L}(F,G)$ is an abelian group with addition $h_1 + h_2 := G(h_1(X), h_2(X))$. The abelian group $\mathrm{End}_{o_L}(F)$ of endomorphisms of F is a ring under the multiplication $f \circ g$.

Now let $\varphi(X) \in o_L[|X|]$ be a Frobenius power series for π, i.e.

$$\varphi(X) = \pi X + \text{terms of higher degree},$$
$$\varphi(X) = X^q \mod \pi o_L[|X|].$$

Proposition 2.3. *For any Frobenius power series φ for π, there is a unique formal group law $F_\varphi(X,Y)$ with coefficients in o_L such that $\varphi \in End_{o_L}(F_\varphi)$. We call it the Lubin-Tate formal group law of φ. Moreover, if ψ is another Frobenius power series for π, then for every $a \in o_L$ there is a unique $[a]_{\varphi,\psi} \in o_L[|X|]$ such that*

$$[a]_{\varphi,\psi}(X) = aX + \text{ higher terms},$$
$$\varphi \circ [a]_{\varphi,\psi} = [a]_{\varphi,\psi} \circ \psi.$$

Such an $[a]_{\varphi,\psi}$ is a homomorphism $F_\psi \to F_\varphi$.

Let φ be a Frobenius power series for π. By taking $[a]_\varphi := [a]_{\varphi,\varphi}$ we obtain a unique injective group homomorphism

$$o_L \to \mathrm{End}_{o_L}(F_\varphi),$$
$$a \mapsto [a]_\varphi(X) = aX + \text{terms of higher order}$$

such that $[\pi]_\varphi(X) = \varphi(X)$.
If ψ is another Frobenius power series for π, then there is an isomorphism of formal groups $F_\varphi \cong F_\psi$. In fact, every $u \in o_L^\times$ gives an isomorphism of formal groups $[u]_{\varphi,\psi}$.

Let $L \subseteq K \subseteq \mathbb{C}_p$ be a complete intermediate field. The set $\mathfrak{B}(K) = \{z \in K \,|\, |z| < 1\}$ with the addition $z_1 +_{F_\varphi} z_2 := F_\varphi(z_1, z_2)$ is an abelian group. Any endomorphism $h : F_\varphi \to F_\varphi$ defines a group homomorphism $(\mathfrak{B}(K), +_{F_\varphi}) \to (\mathfrak{B}(K), +_{F_\varphi}), z \mapsto h(z)$. We define an o_L-module structure on $(\mathfrak{B}(K), +_{F_\varphi})$ by $(a,z) \mapsto [a]_\varphi(z), z \in \mathfrak{B}(K), a \in o_L$.

Every $u \in o_L^\times$ defines a module isomorphism

$$[u]_{\varphi,\psi} : (\mathfrak{B}(K), +_{F_\psi}) \to (\mathfrak{B}(K), +_{F_\varphi}).$$

2.2.2 The LT-isomorphism

In the following,we fix a Frobenius power series φ for π and write $[\cdot] = [\cdot]_\varphi$. Denote by LT the corresponding Lubin-Tate formal group law. By identifying LT with the open unit disk $\mathfrak{B}_{/L}$ over L, we view the rigid variety $\mathfrak{B}_{/L}$ as an o_L-module object; we get a global coordinate T on LT, and the action of o_L on $\mathfrak{B}_{/L}$ is via $(a,z) \mapsto [a](z)$. In particular, we get an action of the multiplicative monoid $o_L \setminus \{0\}$ on $\mathfrak{B}_{/L}$. For $r \in (0,1) \cap p^{\mathbb{Q}}$, the L-affinoid disk $\mathfrak{B}(r)_{/L}$ is an o_L-submodule object of $\mathfrak{B}_{/L}$.

Remark 2.4. We may assume that, up to isomorphism, the action of π is given by $[\pi](z) = \pi z + z^q$.

Let $\mathcal{F}_n$ be the set of $[\pi^n]$-torsion points of $\mathfrak{B}(L^{alg})$. Consider the o_L-module

$$T := \varprojlim_{[\pi],n} (\mathcal{F}_n).$$

This is the Tate module of LT. It is a free o_L-module of rank one. The action of $\mathrm{Gal}(L^{alg}/L)$ on $T = \varprojlim_{[\pi],n} \mathcal{F}_n$ is given by a continuous character $\chi_{LT} : \mathrm{Gal}(L^{alg}/L) \to o_L^\times$. Let T' be the Tate module of the p-divisible group dual to LT. By Lubin-Tate theory, this is again a free o_L-module of rank one, and the Galois action on T' is given by the continuous character $\tau := \chi_{cyc} \cdot \chi_{LT}^{-1}$, where χ_{cyc} is the cyclotomic character.
By Cartier duality, we can identify T' with the group of homomorphisms of formal groups over $o_{\mathbb{C}_p}$ from LT to the formal multiplicative group. We get a natural pairing

$$\langle \cdot, \cdot \rangle : T' \otimes_{o_L} \mathfrak{B}(\mathbb{C}_p) \to \mathfrak{B}_1(\mathbb{C}_p),$$

which is invariant for the Galois and the o_L-action. Fix a generator t_0' of the o_L-module T'.

Theorem 2.5 (Schneider, Teitelbaum)**.** *There is an isomorphism*

$$\kappa : \mathfrak{B}_{/\mathbb{C}_p} \to \mathfrak{X}_{/\mathbb{C}_p}$$

of rigid varieties over $\mathbb{C}_p$.

Proof. This is Theorem 3.6 in [35]. □

On $\mathbb{C}_p$-points, this isomorphism is given by

$$\begin{aligned} \mathfrak{B}(\mathbb{C}_p) &\to \mathfrak{X}(\mathbb{C}_p) = \widehat{G}(\mathbb{C}_p), \\ z &\mapsto \kappa_z(g) := \langle t_0', [g](z) \rangle. \end{aligned}$$

Corollary 2.6. *The ring $\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X})$ is isomorphic to the ring $\mathcal{O}_{\mathbb{C}_p}(\mathfrak{B})$.*

Definition 2.7. *Put $\omega := p^{1/e(q-1)-1/(p-1)}$ and*

$$R_n := p^{\mathbb{Q}} \cap [p^{-q/e(q-1)}, p^{-1/e(q-1)})^{1/q^{en}}) \quad \textit{for } n \geq 0,$$
$$S_0 := R_0\omega = p^{\mathbb{Q}} \cap [p^{-1/e-1/(p-1)}, p^{-1/(p-1)}) \subseteq p^{\mathbb{Q}} \cap [p^{-p/(p-1)}, p^{-1/(p-1)}), \textit{ and}$$
$$S_n := S_0^{1/p^n} \quad \textit{for } n \geq 0.$$

The R_n as well as the S_n are pairwise disjoint and any sequence $(r_n)_{n\geq 0}$ with $r_n \in R_n$ respective $(s_n)_{n\geq 0}$ with $s_n \in S_n$ converges to 1. We have an order preserving bijection

$$\begin{aligned} S_n &\to R_n, \\ s &\mapsto s^{1/p^{(d-1)n}} \omega^{-1/p^{dn}} \end{aligned}$$

for $n \geq 0$.

Lemma 2.8. *(i) For any $r \in p^{\mathbb{Q}}$ such that $p^{q/e(q-1)} \leq r < 1$ we have*

$$[\pi]^{-1}(\mathfrak{B}(r)) = \mathfrak{B}(r^{1/q}) \qquad \text{and} \qquad [p]^{-1}(\mathfrak{B}(r)) = \mathfrak{B}(r^{1/q^e}),$$

and in this situation, the map $[\pi^n] : \mathfrak{B}(r^{1/p^n}) \to \mathfrak{B}(r)$ is a finite étale affinoid map for $n \in \mathbb{N}$.

(ii) For any $r \in p^{\mathbb{Q}}$ such that $p^{-p/(p-1)} \leq r < 1$ we have

$$\mathfrak{X}(r^{1/p}) = \{\chi \in \mathfrak{X} \mid \chi^p \in \mathfrak{X}(r)\},$$

and in this situation the map $p^n : \mathfrak{X}(r^{1/p^n}) \to \mathfrak{X}(r), \chi \mapsto \chi^{p^n}$ is a finite étale affinoid map for $n \in \mathbb{N}$.

Proof. This is Lemma 3.2 and 3.3 in [35]. □

Proposition 2.9. *The restriction of the isomorphism in Theorem 2.5,*

$$\kappa : \mathfrak{B}(s^{1/p^{(d-1)n}} \omega^{-1/p^{dn}})_{/\mathbb{C}_p} \to \mathfrak{X}(s)_{/\mathbb{C}_p}$$

is a rigid analytic isomorphism for $n \geq 0$ and $s \in S_n$.

Proof. This is [6, Proposition 1.20]. □

For every $n \in \mathbb{N}$ and $r \in R_0$ we have commutative diagrams

$$\begin{array}{ccc} \mathfrak{B}(r^{1/q^{en}})_{/\mathbb{C}_p} & \xrightarrow{\kappa} & \mathfrak{X}((r\omega)^{1/p^n})_{/\mathbb{C}_p} \\ {\scriptstyle [p^n]}\downarrow & & \downarrow{\scriptstyle p^n} \\ \mathfrak{B}(r)_{/\mathbb{C}_p} & \xrightarrow[\kappa]{} & \mathfrak{X}(r\omega)_{/\mathbb{C}_p} \end{array}$$

where the horizontal maps are rigid isomorphisms.

If $n \geq 1$, then we write $\mathfrak{B}_n := \mathfrak{B}(p^{-1/e(q-1)q^{en-1}})$ and $\mathfrak{X}_n := \mathfrak{X}(p^{-(1+e/(p-1))/ep^n})$. Then $\mathfrak{B}_n$ and $\mathfrak{X}_n$ corresponded to each other under the above bijection.

Proposition 2.9 implies that the rings $\mathcal{O}_K(\mathfrak{X}(s))$ for $s \in S_n$, and $\mathcal{O}_K(\mathfrak{X})$ are integral domains. On $\mathcal{O}_K(\mathfrak{X}(s))$ we have the supremum norm

$$||f||_{\mathfrak{X}(s)} = \sup_{\chi \in \mathfrak{X}(s)(\mathbb{C}_p)} (f(\chi)).$$

The multiplicativity of the supremum norm $\|\cdot\|_{\mathfrak{B}(s^{1/p^{(d-1)n}}\omega^{-1/p^{dn}})}$ on $\mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(s^{1/p^{(d-1)n}}\omega^{-1/p^{dn}}))$ implies that $\|\cdot\|_{\mathfrak{X}(s)}$ is multiplicative as well. The power-bounded elements of $\mathcal{O}_K(\mathfrak{X}(s))$ are given by

$$\mathcal{O}_K^{\circ}(\mathfrak{X}(s)) = \mathcal{O}_K^{\leq 1}(\mathfrak{X}(s)) = \{f \in \mathcal{O}(\mathfrak{X}(s)) \,|\, ||f||_{\mathfrak{X}(s)} \leq 1\}.$$

Then $(\mathcal{O}_K(\mathfrak{X}(s)), \mathcal{O}_K^{\leq 1}(\mathfrak{X}(s)))$ is a uniform Tate-Huber pair with ideal of definition (π) and topologically nilpotent unit π.

2.2.3 The action of $o_L \setminus \{0\}$

Let $a \in o_L$. The map $a \mapsto ag$ on G is locally L-analytic. This induces an action of the multiplicative monoid $o_L \setminus \{0\}$ on the vector space of locally analytic functions $C^{an}(G, K) \subseteq C^{an}(G_0, K)$. It is given by $f \mapsto a^*(f)(g) := f(ag)$. For every character $\chi \in \widehat{G}(K)$ respective $\chi \in \widehat{G}_0(K)$, the function $a^*(\chi)$ is also a character in $G(K)$ respective $\widehat{G}_0(K)$. Therefore we have an action of o_L on the groups $\widehat{G}(K)$ and $\widehat{G}_0(K)$. Under the bijection 2.1, this action correspondents to the action on $\mathrm{Hom}_{\mathbb{Z}_p}(o_L, \mathbb{Z}_p)$ defined by $f \mapsto a^*(f)(g) := f(ag)$ for $f \in \mathrm{Hom}_{\mathbb{Z}_p}(o_L, \mathbb{Z}_p)$. We see that the action on $\widehat{G}(K)$ respective $\widehat{G}_0(K)$ comes from an o_L-action on the rigid analytic varieties $\mathfrak{X}$ respective $\mathfrak{X}_0$. It respects the affinoids $\mathfrak{X}(r)$. We obtain an $o_L \setminus \{0\}$-action on the rings $\mathcal{O}_K(\mathfrak{X})$ and $\mathcal{O}_K(\mathfrak{X}(r))$ which we will denote by $(a, f) \mapsto a_*(f)$.
Note that the isomorphism $\kappa : \mathfrak{B}/\mathbb{C}_p \to \mathfrak{X}/\mathbb{C}_p$ is equivariant for the o_L-action, because we have $\kappa_{[a](z)} = a^*(\kappa_z)$ for any $a \in o_L, z \in \mathfrak{B}(\mathbb{C}_p)$. This implies that the isomorphism $\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}) \cong \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B})$ is equivariant for the $o_L \setminus \{0\}$-action as well.

We often denote the action of $\pi \in o_L \setminus \{0\}$ by φ.

Remark 2.10. The action of π induces a surjection $\varphi : \mathfrak{X}(\mathbb{C}_p) \to \mathfrak{X}(\mathbb{C}_p)$ on $\mathbb{C}_p$-points.

Proof. We may assume that the action of π on $\mathfrak{B}(\mathbb{C}_p)$ is given by $[\pi](z) = \pi z + z^q$ for $z \in \mathfrak{B}(\mathbb{C}_p)$ (Remark 2.4). Then we see that $\varphi : \mathfrak{B}(\mathbb{C}_p) \to \mathfrak{B}(\mathbb{C}_p)$ is surjective. We deduce with Theorem 2.5 that $\varphi : \mathfrak{X}(\mathbb{C}_p) \to \mathfrak{X}(\mathbb{C}_p)$ is surjective as well. □

Remark 2.11. For any $r \in p^{\mathbb{Q}}$ such that $p^{-p/(p-1)} \leq r < 1$, the map

$$p_*^n : \mathcal{O}_K(\mathfrak{X}(r)) \to \mathcal{O}_K(\mathfrak{X}(r^{1/p^n}))$$

is isometric for the supremum norms $\|\cdot\|_{\mathfrak{X}(r^{1/p^n})}$ respective $\|\cdot\|_{\mathfrak{X}(r)}$.

Proof. Since p^n is equal to a power of φ times an automorphism, it is surjective on $\mathfrak{X}(\mathbb{C}_p)$. Therefore we see with Lemma 2.8 that the map

$$p^n : \mathfrak{X}(r^{1/p^n})(\mathbb{C}_p) \to \mathfrak{X}(r)(\mathbb{C}_p)$$

is surjective as well, and we have

$$\begin{aligned}\|f\|_{\mathfrak{X}(r)} &= \sup_{x\in\mathfrak{X}(r)(\mathbb{C}_p)} |f(x)| \\ &= \sup_{x\in\mathfrak{X}(r^{1/p^n})(\mathbb{C}_p)} |f(p^n(x))| \\ &= \|p^n_*(f)\|_{\mathfrak{X}(r^{1/p^n})}.\end{aligned}$$

□

2.3 The Robba rings $\mathcal{R}_K(\mathfrak{B})$ and $\mathcal{R}_K(\mathfrak{X})$

The complement of an affinoid domain in an affinoid rigid-analytic space is an admissible open subset. That means that for any $r > r_0 \in (0,1) \cap p^{\mathbb{Q}}$ we have admissible open subsets $\mathfrak{B}(r) \setminus \mathfrak{B}(r_0)$ in $\mathfrak{B}(r)$. We have an admissible covering $\{\mathfrak{B}(r)\}_r$ of $\mathfrak{B}$, so a subset S of $\mathfrak{B}$ is admissible open if and only if $S \cap \mathfrak{B}(r)$ is admissible open in $\mathfrak{B}(r)$ for all r. Therefore $\mathfrak{B} \setminus \mathfrak{B}(r)$ is an admissible open subset of $\mathfrak{B}$ and we can form the ring $\mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r))$ and the ring $\mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r))^{bd}$. We define the *Robba ring* over $\mathfrak{B}$ over K as

$$\mathcal{R}_K(\mathfrak{B}) := \bigcup_r \mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r)).$$

This is the ring of all formal power series $f(T) = \sum_{n\in\mathbb{Z}} a_n T^n, a_n \in K$ which converge on $(\mathfrak{B} \setminus \mathfrak{B}(r))(\mathbb{C}_p)$ for some $r > 0$ (depending on f). The $o_L^\times$-action on $\mathfrak{B}$ respects $\mathfrak{B}(r)$ and hence $\mathfrak{B}\setminus\mathfrak{B}(r)$, and therefore we have an action of $o_L^\times$ on $\mathcal{R}_K(\mathfrak{B})$. Moreover, since $[\pi](\mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r))) \subseteq \mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r^{1/q}))$ (Lemma 2.8), we have an action of the full multiplicative monoid $o_L \setminus \{0\}$ on $\mathcal{R}_K(\mathfrak{B})$.

We also define the ring

$$\mathcal{E}_K^\dagger(\mathfrak{B}) := \bigcup_r \mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r))^{bd}.$$

By the maximum modulus principle, the ring $\mathcal{O}_K(\mathfrak{B})^{bd}$ is the ring of all formal power series $f(T) = \sum_{n\in\mathbb{N}} a_n T^n, a_n \in K$ such that $\sup_{n\geq 0}|a_n| < \infty$, and we have

$$||f||_{\mathfrak{B}} = \sup_{z\in\mathfrak{B}(\mathbb{C}_p)}|f(z)| = \sup_{n\geq 0}|a_n|.$$

Similarly, the ring $\mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r))^{bd}$ consists of formal power series $\sum_{n\in\mathbb{Z}} a_n T^n$ with bounded coefficients which converge for $r < |z| < 1$. We define a norm $\|\cdot\|_1$ on $\mathcal{E}_K^\dagger(\mathfrak{B})$ by setting $\|f\|_1 = \lim_{r\to 1} \|f\|_{\mathfrak{B}\setminus\mathfrak{B}(r)}$. This is a multiplicative norm and we have $\|f\|_1 = \sup_{n\in\mathbb{Z}} |a_n|$ if $f = \sum_{n\in\mathbb{Z}} a_n T^n$.

Likewise we define the Robba ring over $\mathfrak{X}$ as

$$\mathcal{R}_K(\mathfrak{X}) := \bigcup_r \mathcal{O}_K(\mathfrak{X} \setminus \mathfrak{X}(r))$$

and

$$\mathcal{E}_K^\dagger(\mathfrak{X}) := \bigcup_r \mathcal{O}_K(\mathfrak{X} \setminus \mathfrak{X}(r))^{bd}.$$

Next, we want to describe the rings $\mathcal{O}_K(\mathfrak{X} \setminus \mathfrak{X}(r))$ as projective limits of affinoids. In the following all radii will be understood to lie in $(0,1) \cap p^{\mathbb{Q}}$. Denote by $\mathfrak{B}_1^-(r)$ the open $\mathbb{Q}_p$-affinoid disk of radius r around 1. Similarly, denote by $\mathfrak{B}(r)^-$ the open disk of radius r around 0. We define the subsets

$$\begin{aligned}
\mathfrak{X}_0(r) &:= \mathfrak{B}_1(r) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o_L, \mathbb{Z}_p), \\
\mathfrak{X}_0^-(r) &:= \mathfrak{B}_1^-(r) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o_L, \mathbb{Z}_p), \\
\mathfrak{X}_0(r_1, r_2) &:= \mathfrak{X}_0(r_2) \setminus \mathfrak{X}_0^-(r_1) \text{ for } r_1 \leq r_2
\end{aligned}$$

of $\mathfrak{X}_0$. Note that $\mathfrak{X}_0(r)$ and $\mathfrak{X}_0^-(r)$ are admissible open subsets. We also define the affinoid subdomains

$$\mathfrak{X}_0^{(i)}(r_1, r_2) := \{x \in \mathfrak{X}_0(r_2) \,|\, |z_i(x)| \geq r_1\} \subseteq \mathfrak{X}_0(r_2)$$

for $i = 1, ..., d$ with z_i being coordinate functions on $\mathfrak{X}_0$. Then

$$\mathfrak{X}_0(r_1, r_2) = \bigcup_i \mathfrak{X}_0^{(i)}(r_1, r_2).$$

As a finite union of affinoid subdomains, $\mathfrak{X}_0(r_1, r_2)$ is admissible open in $\mathfrak{X}_0(r_2)$ and hence in $\mathfrak{X}_0$ ([31, Cor. 9.1.4/4]). As described in [6, 2.1], we have an admissible covering

$$\mathfrak{X}_0 \setminus \mathfrak{X}_0(r_0) = \bigcup_{r_0 < r_1 \leq r_2 < 1} \mathfrak{X}_0(r_1, r_2).$$

Now we put

$$\begin{aligned}
\mathfrak{X}^-(r) &:= \mathfrak{X} \cap \mathfrak{X}_0^-(r)_{/L} \qquad \text{and} \\
\mathfrak{X}(r_1, r_2) &:= \mathfrak{X} \cap \mathfrak{X}_0(r_1, r_2)_{/L} = \mathfrak{X}(r_2) \setminus \mathfrak{X}(r_1)^-.
\end{aligned}$$

Then $\mathfrak{X}(r_1, r_2)$ is a finite union of affinoid subdomains and admissible open in $\mathfrak{X}(r_2)$. We have

$$\mathcal{O}_K(\mathfrak{X} \setminus \mathfrak{X}(r_0)) = \varprojlim_{r_0 < r_1 \leq r_2 < 1} \mathcal{O}_K(\mathfrak{X}(r_1, r_2)).$$

Since $\mathfrak{X}$ and hence each $\mathfrak{X}(r)$ are connected, smooth, and one-dimensional rigid-analytic varieties, finite unions of affinoid subdomains in $\mathfrak{X}(r)$ are again affinoid subdomains. In particular each $\mathcal{O}_K(\mathfrak{X}(r_1, r_2))$ is a K-affinoid algebra which is a Banach algebra with respect to the supremum norm.

Remark 2.12. Let $p^{-(1+e/(p-1))/ep^n} < s_1 \leq s_2 < 1$ with $s_1 = (\omega r_1)^{1/p^n} \in S_n$ and $s_2 = (\omega r_2)^{1/p^m} \in S_m$ for some $r_1, r_2 \in R_0$ and $m \geq n$. Note that

$$\mathfrak{X}^-(s_1) = \cup_{s<s_1}\mathfrak{X}(s) = \cup_{s<s_1, s\in S_n}\mathfrak{X}(s).$$

By Theorem 2.5 we have

$$\begin{aligned}\mathfrak{X}^-(s_1)_{/\mathbb{C}_p} &= \cup_{s<s_1,s\in S_n}\mathfrak{X}(s)_{/\mathbb{C}_p}\\ &= \cup_{r<r_1^{1/q^{en}}, r\in R_n}\mathfrak{B}(r)_{/\mathbb{C}_p}\\ &= \mathfrak{B}^-(r_1^{1/q^{en}})_{/\mathbb{C}_p}.\end{aligned}$$

This implies that $\mathfrak{X}(s_1, s_2)$ is isomorphic to $\mathfrak{B}(r_1^{1/q^{en}}, r_2^{1/q^{en}})$ over $\mathbb{C}_p$. Especially we have

$$\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}(s_1, s_2)) \cong \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(r_1^{1/q^{en}}, r_2^{1/q^{en}})).$$

The ring $\mathcal{R}_K(\mathfrak{X}) = \varinjlim_n \mathcal{O}_K(\mathfrak{X}\setminus\mathfrak{X}_n)$ is the inductive limit of the Fréchet spaces $\mathcal{O}_K(\mathfrak{X}\setminus\mathfrak{X}_n)$. We endow $\mathcal{R}_K(\mathfrak{X})$ with the locally convex final topology with respect to the inclusions $\mathcal{O}_K(\mathfrak{X}\setminus\mathfrak{X}_n) \to \mathcal{R}_K(\mathfrak{X})$, that is the locally convex inductive limit topology.

Proposition 2.13. *1. $\mathcal{R}_K(\mathfrak{X})$ is a regular inductive limit.*

2. $\mathcal{R}_K(\mathfrak{X})$ is Hausdorff, complete, nuclear, and reflexive.

Proof. Proposition 2.6 and Proposition 2.7 in [6]. □

Writing $\mathcal{R}_K(\mathfrak{B}) = \varinjlim_i \mathcal{O}_K(\mathfrak{B}\setminus\mathfrak{B}_n)$, the isomorphism in Theorem 2.5 gives an isomorphism $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}) \cong \mathcal{R}_{\mathbb{C}_p}(\mathfrak{X})$.

2.3.1 The twisted Galois action

Set $G_K = \mathrm{Gal}(K^{alg}/K)$. On $\mathcal{O}_{\mathbb{C}_p}(\mathfrak{B})$ we have the standard Galois action defined by

$$\begin{aligned}G_K \times \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}) &\to \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}),\\ (\sigma, f = \sum_i a_i T^i) &\mapsto {}^{\sigma}f := \sum_i \sigma(a_i) T^i\end{aligned}$$

and the twisted Galois action

$$\begin{aligned}G_K \times \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}) &\to \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}),\\ (\sigma, f = \sum_i a_i T^i) &\mapsto {}^{\sigma *}f := ({}^{\sigma}f)([\tau(\sigma^{-1})](\cdot))\end{aligned}$$

where $\tau = \chi_{cyc} \cdot \chi_{LT}^{-1}$.

Proposition 2.14. *We have*

$$\mathcal{O}_{\mathbb{C}_p}(\mathfrak{B})^{G_K,*} = \mathcal{O}_K(\mathfrak{X})$$

Proof. See the discussion after Remark 1.22 in [6] or [35, Corollary 3.8]. □

Remark 2.15. Let $r_1 \in R_n$ for some n and $r_2 \in \bigcup_{m \geq n} R_m$. The twisted Galois action is well-defined on $\mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(r_1, r_2))$ because the action of every element $a \in o_L^\times$ preserves $\mathfrak{B}(r_1, r_2)$. It follows similarly to the above proposition that

$$\mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(r_1, r_2))^{G_K,*} = \mathcal{O}_K(\mathfrak{X}(a_1, a_2))$$

for certain radii a_1, a_2.

We have an isomorphism $\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X} \setminus \mathfrak{X}_n) \cong \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B} \setminus \mathfrak{B}_n)$. It restricts to an isometric isomorphism between the resp. rings of bounded functions. The twisted Galois action is well defined on $\mathcal{O}_{\mathbb{C}_p}(\mathfrak{B} \setminus \mathfrak{B}_n)$. Moreover the standard Galois action on $\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X} \setminus \mathfrak{X}_n)$ corresponds to the twisted Galois action on $\mathcal{O}_{\mathbb{C}_p}(\mathfrak{B} \setminus \mathfrak{B}_n)$. We may pass to the inductive limit and define the twisted and standard Galois action on $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}) \cong \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B})$. Then $\mathcal{R}_K(\mathfrak{X}) = \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B})^{G_K,*}$. We may restrict both actions to to $\mathcal{E}^\dagger_{\mathbb{C}_p}(\mathfrak{X}) \cong \mathcal{E}^\dagger_{\mathbb{C}_p}(\mathfrak{B})$. On $\mathcal{E}^\dagger_K(\mathfrak{B})$ we already defined the $\|\cdot\|_1$-norm. Similarly, since $\|\cdot\|_{\mathfrak{X}\setminus\mathfrak{X}(r_2)} \leq \|\cdot\|_{\mathfrak{X}\setminus\mathfrak{X}(r_1)}$ if $r_2 \geq r_1$ for any $r_1, r_2 \in (0,1) \cap p^{\mathbb{Q}}$, we may define

$$\|f\|_1 = \lim_{r \to 1} \|f\|_{\mathfrak{X}\setminus\mathfrak{X}(r)}$$

for $f \in \mathcal{E}^\dagger_K(\mathfrak{X})$. The identification $\mathcal{E}^\dagger_{\mathbb{C}_p}(\mathfrak{X}) \cong \mathcal{E}^\dagger_{\mathbb{C}_p}(\mathfrak{B})$ and the twisted Galois action are isometric for $\|\cdot\|_1$. Since $\|\cdot\|_1$ is a multiplicative norm on $\mathcal{E}^\dagger_K(\mathfrak{B})$, this is also the case for $\mathcal{E}^\dagger_K(\mathfrak{X})$. We denote by $\mathcal{E}_K(\mathfrak{B})$ reps. $\mathcal{E}_K(\mathfrak{X})$ the completion of $\mathcal{E}^\dagger_K(\mathfrak{B})$ and $\mathcal{E}^\dagger_K(\mathfrak{X})$ for $\|\cdot\|_1$.

2.3.2 The monoid action on $\mathcal{R}_K(\mathfrak{X})$

Every element $u \in o_L^\times$ preserves $\mathfrak{X}(r)$ and $\mathfrak{X}(r_1, r_2)$. It also preserves the admissible open subset $\mathfrak{X} \setminus \mathfrak{X}(r)$. Therefore the action of $o_L^\times$ extends to the rings $\mathcal{O}_K(\mathfrak{X} \setminus \mathfrak{X}(r))$ and $\mathcal{O}_K(\mathfrak{X} \setminus \mathfrak{X}(r))^{bd}$. Moreover, every $u \in o_L^\times$ acts isometrically on $\mathcal{O}_K(\mathfrak{X}(r), \mathcal{O}_K(\mathfrak{X}(r_1, r_2)$, and $\mathcal{O}_K(\mathfrak{X} \setminus \mathfrak{X}(r))^{bd}$ (in the respective supremum norm). Set $\mathcal{E}^{\dagger,\leq 1}_K(\mathfrak{X}) = \{f \in \mathcal{E}^\dagger_K(\mathfrak{X}) \,|\, \|f\|_1 \leq 1\}$. The action of $o_L^\times$ extends to the rings $\mathcal{R}_K(\mathfrak{X}), \mathcal{E}^{\dagger,\leq 1}_K(\mathfrak{X}), \mathcal{E}^\dagger_K(\mathfrak{X})$, and (being isometric in the $\|\cdot\|_1$-norm) to $\mathcal{E}_K(\mathfrak{X})$.

Lemma 2.16. *For any $r \in [p^{-1/e-1/(p-1)}, 1) \cap p^{\mathbb{Q}}$ and $r_1 \in S_n, r_2 \in \bigcup_{m \geq n} S_m$ with $r_2 \geq r_1$, the $o_L^\times$-action on the rings $\mathcal{O}_K(\mathfrak{X} \setminus \mathfrak{X}(r))$ and $\mathcal{O}_K(\mathfrak{X}(r_1, r_2))$ is continuous.*

Proof. Lemma 2.10 and text after Lemma 2.18 in [6]. □

To obtain an action of the full monoid $o_L \setminus \{0\}$ on the rings $\mathcal{R}_K(\mathfrak{X}), \mathcal{E}^\dagger_K(\mathfrak{X}), \mathcal{E}^{\dagger,\leq 1}_K(\mathfrak{X})$, and $\mathcal{E}_K(\mathfrak{X})$ we need the following lemma:

Lemma 2.17. *For any $r \in [p^{-p/(p-1)}, 1) \cap p^{\mathbb{Q}}$ we have*

$$(\pi_L^*)^{-1}(\mathfrak{X} \setminus \mathfrak{X}(r)) \supseteq \mathfrak{X} \setminus \mathfrak{X}(r^{1/p}).$$

Proof. Lemma 2.11 in [6] □

We conclude that the action of π extends to the rings $\mathcal{R}_K(\mathfrak{X}), \mathcal{E}_K^\dagger(\mathfrak{X}), \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X})$, and $\mathcal{E}_K(\mathfrak{X})$.

Lemma 2.18. *The $o_L \setminus \{0\}$-action on $\mathcal{R}_K(\mathfrak{X})$ is continuous.*

Proof. Lemma 2.12 in [6] □

Similarly, the $o_L \setminus \{0\}$-action on $\mathcal{R}_K(\mathfrak{B})$ is continuous.

3 The Robba ring over $\mathfrak{B}^{\mathrm{perf}}$

In this chapter we introduce the perfectoid and the preperfectoid unit disk and explain the construction of the Robba ring over the perfectoid and the preperfectoid unit disk. In the following, let $L \subseteq K \subseteq \mathbb{C}_p$ be a complete intermediate field.

3.1 The perfectoid unit disk

Let $K\langle T\rangle$ be the Tate algebra over K in one variable. We have the K-algebra homomorphism

$$\begin{aligned} \varphi : K\langle T\rangle &\to K\langle T\rangle, \\ T &\mapsto \pi T + T^q. \end{aligned}$$

Consider the open unit disk $\mathfrak{B}_{/K}$ with the action of π as in Chapter 2. If we assume that the chosen Lubin-Tate formal group law is the special one, i.e. that the action of π on $\mathfrak{B}_{/K}$ is given by $[\pi](z) = \pi z + z^q$, then π acts on the closed unit disk $\overline{\mathfrak{B}}_{/K}$ as well and the corresponding map on $\mathcal{O}_K(\overline{\mathfrak{B}}) = K\langle T\rangle$ is given by φ.

The pair $(K\langle T\rangle, o_K\langle T\rangle)$ is a uniform Tate-Huber pair with pair of definition $(o_K\langle T\rangle, (\pi))$. Since $\varphi : K\langle T\rangle \to K\langle T\rangle$ is continuous and fulfils $\varphi(o_K\langle T\rangle) \subseteq o_K\langle T\rangle$, it defines a morphism of Tate-Huber pairs. We have an inductive system of Tate-Huber pairs

$$(K\langle T\rangle, o_K\langle T\rangle) \xrightarrow{\varphi} (K\langle T\rangle, o_K\langle T\rangle) \xrightarrow{\varphi} \dots \xrightarrow{\varphi} (K\langle T\rangle, o_K\langle T\rangle) \xrightarrow{\varphi} \dots$$

We form the inductive limit $(\varinjlim_\varphi K\langle T\rangle, \varinjlim_\varphi o_K\langle T\rangle)$ in the category of uniform Tate-Huber pairs as in Proposition 1.55. We complete it and get a complete uniform Tate-Huber pair

$$(\widehat{\varinjlim}_\varphi K\langle T\rangle, \widehat{\varinjlim}_\varphi o_K\langle T\rangle).$$

Proposition 3.1. *If K is perfectoid, then $\widehat{\varinjlim}_\varphi K\langle T\rangle$ is a perfectoid K-algebra.*

Proof. Uniformity is given by construction. Note that the subring of power-bounded elements is $\widehat{\varinjlim}_\varphi o_K\langle T\rangle$. We show that every element in $\widehat{\varinjlim}_\varphi o_K\langle T\rangle$ has a p-th root modulo (p). We only need to consider the dense subset $\varinjlim_\varphi o_K\langle T\rangle$. Moreover, it is enough to consider a finite sum $\sum_{k=0}^n a_n T^n$ which lies in a copy of $o_K\langle T\rangle$ in $\varinjlim_\varphi o_K\langle T\rangle$. Since K is perfectoid, taking the the q-th power on $o_K/(\pi)$ is surjective.

Therefore we need to ensure that the image of $T \in o_K\langle T\rangle$ in $\varinjlim_\varphi o_K\langle T\rangle$ has a q-th root in $\varinjlim_\varphi o_K\langle T\rangle$ modulo (π). The map

$$\overline{\varphi} := \varphi \bmod (\pi) : o_K\langle T\rangle/(\pi) \to o_K\langle T\rangle/(\pi)$$

is equal to the map induced by $T \bmod (\pi) \mapsto T^q \bmod (\pi)$ on $o_K\langle T\rangle/(\pi)$, so by passing to the inductive limit and noting that inductive limits commute with quotients, we see that the image of T has a q-th root in $(\varinjlim_\varphi o_K\langle T\rangle)/(\pi)$. This implies that taking the q-th power is surjective on $\widehat{\varinjlim_\varphi} o_K\langle T\rangle/(\pi)$. It follows that taking the p-th power is surjective on $\widehat{\varinjlim_\varphi} o_K\langle T\rangle/(\pi)$. With Remark 1.68 we see that taking the p-th power on

$$(\widehat{\varinjlim_\varphi} o_K\langle T\rangle)/(p) \to (\widehat{\varinjlim_\varphi} o_K\langle T\rangle)/(p)$$

is surjective as well. □

So, if K is perfectoid, the Tate-Huber pair $(\widehat{\varinjlim_\varphi} K\langle T\rangle, \widehat{\varinjlim_\varphi} o_K\langle T\rangle)$ is perfectoid and the associated affinoid adic space is a perfectoid space, namely the *closed perfectoid unit disk* over K:

Definition 3.2. *Let K be perfectoid. The perfectoid space*

$$\overline{\mathfrak{B}}_K^{\text{perf}} := \operatorname{Spa}(\widehat{\varinjlim_\varphi} K\langle T\rangle, \widehat{\varinjlim_\varphi} o_K\langle T\rangle)$$

is the closed perfectoid unit disk *over K.*

Remark 3.3. Denote by q the K-algebra map $q : K\langle T\rangle \to K\langle T\rangle, T \mapsto T^q$. Instead of Definition 3.2, one can define the perfectoid Tate algebra as the π-adic completion of $\varinjlim_q K\langle T\rangle$, and the perfectoid unit disk as $\operatorname{Spa}(\widehat{\varinjlim_q} K\langle T\rangle, \widehat{\varinjlim_q} o_K\langle T\rangle)$. But in our case, we choose to use φ to match the definition of the closed perfectoid unit disk with the definition of the open unit disk later in this chapter.

If K is a perfectoid field, then $\widehat{\varinjlim_q} K\langle T\rangle$ is a perfectoid K-algebra. This can be seen with the same arguments as in the proof of Proposition 3.1. The tilts of $\widehat{\varinjlim_\varphi} K\langle T\rangle$ and $\widehat{\varinjlim_q} K\langle T\rangle$ coincide since both rings coincide modulo π. In fact, we have

$$(\widehat{\varinjlim_\varphi} K\langle T\rangle)^\flat = (\widehat{\varinjlim_q} K\langle T\rangle)^\flat = \widehat{\varinjlim_q} K^\flat\langle T\rangle.$$

Hence $\widehat{\varinjlim_\varphi} K\langle T\rangle$ and $\widehat{\varinjlim_q} K\langle T\rangle$ are isomorphic (Theorem 1.71).

The map $\varphi : K\langle T\rangle \to K\langle T\rangle$ is an isometry for the supremum norm $\|\cdot\|_{\overline{\mathfrak{B}}}$ on $K\langle T\rangle$ because the corresponding map $\overline{\mathfrak{B}}(\mathbb{C}_p) \to \overline{\mathfrak{B}}(\mathbb{C}_p), z \mapsto z^q + \pi z$ on $\mathbb{C}_p$-points is surjective. We equip $\varinjlim_\varphi K\langle T\rangle$ with the inductive limit norm coming from the supremum norm $\|\cdot\|_{\overline{\mathfrak{B}}}$ on each $K\langle T\rangle$ as defined in Definition 6.7 in the Appendix. The topology induced by this norm coincides with the π-adic topology on

$\varinjlim_{\varphi} K\langle T\rangle$, i.e. the topology of $\varinjlim_{\varphi} K\langle T\rangle$ as a Huber ring. We extend the norm to the completion $\widehat{\varinjlim}_{\varphi} K\langle T\rangle$. We denote this norm by $\|\cdot\|_{\overline{\mathfrak{B}}^{\mathrm{perf}}}$. Then $\widehat{\varinjlim}_{\varphi} K\langle T\rangle$ with $\|\cdot\|_{\overline{\mathfrak{B}}^{\mathrm{perf}}}$ is a normed K-vector space. Let $f \in \varinjlim_{\varphi} K\langle T\rangle$ with preimage f_{i_0} in the i_0-th copy of $K\langle T\rangle$ under the canonical map $K\langle T\rangle \to \varinjlim_{\varphi} K\langle T\rangle$. Then $\|f\|_{\overline{\mathfrak{B}}^{\mathrm{perf}}} = \|f_{i_0}\|_{\overline{\mathfrak{B}}}$. This is because the transition maps $\varphi : K\langle T\rangle \to K\langle T\rangle$ in the inductive limit are isometries. Next, we want to describe the elements of $\widehat{\varinjlim}_{\varphi} K\langle T\rangle$ explicitly.

In the following, let $(V, \|\cdot\|)$ be a normed vector space over K and I an index set of at most countable cardinality.

Definition 3.4. *1. A* topological generating system *of V over K is a set $\{v_i\}_{i\in I}$ of elements of V such that each $v \in V$ can be written as a convergent series*

$$v = \sum_{i\in I} c_i v_i, \quad c_i \in K.$$

2. If the sequence $\{c_i\}_{i\in I}$ is uniquely determined by v, then $\{v_i\}_{i\in I}$ is called a Schauder basis *of V over K.*

Remark 3.5 (see 2.6.1 in [31])**.** Fix an element ρ in the value group of K with $\rho > 1$. For each $v \in V \setminus \{0\}$ we can find a $c \in K^\times$ such that $1 \le \|cv\| \le \rho$. Hence, given a basis of V, we can always pass to a basis $\{v_i\}_{i\in I}$ of V such that $1 \le \|v_i\| \le \rho$ for all $i \in I$. We call such a set *bounded.*

Definition 3.6 (2.6.1/3 in [31])**.** *Let α be a positive real number. A bounded family $\{v_i\}_{i\in I}$ of V with $v_i \neq 0$ for all $i \in I$ is called* α-cartesian *if*

$$\max_{i\in I}\{\|a_i v_i\|\} \le \alpha \|\sum_{i\in I} a_i v_i\|$$

for every linear combination $\sum_{i\in I} a_i v_i$ such that $a_i = 0$ for all but finitely many i.

Proposition 3.7. *Set $T_i :=$ image of the i-th copy of $T \in K\langle T\rangle$ in $\varinjlim_{\varphi_i} K\langle T\rangle$ for $i \in \mathbb{N}$. The K-vector space $\widehat{\varinjlim}_{\varphi} K\langle T\rangle$ has a Schauder basis consisting of the elements $T_i^{j_i}$ with $i, j_i \in \mathbb{N}$ such that $q \nmid j_i$ for $i > 0$. Moreover, if*

$$f = \sum_{i\ge 0, j_i \ge 0} a_{i,j_i} T_i^{j_i} \in \widehat{\varinjlim}_{\varphi} K\langle T\rangle$$

is any element, we have

$$\|f\|_{\overline{\mathfrak{B}}^{\mathrm{perf}}} = \max_{i,j_i}\{|a_{i,j_i}|\}.$$

Proof. To avoid confusion, we write $\varphi_i : K\langle T_i\rangle \to K\langle T_i\rangle$ for the map $K\langle T_i\rangle \to K\langle T_i\rangle, T_i \mapsto T_i^q + \pi T_i$ on the i-th copy of $K\langle T\rangle$ in the inductive limit, i.e. $\varphi_i(T_i)$ denotes the polynomial $\pi T_i + T_i^q$ and similarly for powers of T_i and φ_i.

Firstly, we show that the set consisting of the $T_i^{j_i}$ with $i, j_i \in \mathbb{N}$ such that $q \nmid j_i$ for $i > 0$ is α-cartesian for $\alpha = 1$. We have $\|T_i^{j_i}\|_{\overline{\mathfrak{B}}^{\mathrm{perf}}} = 1$, hence the set consisting of the $T_i^{j_i}$ is bounded. In $\varinjlim_\varphi K\langle T\rangle$, we have the equalities

$$T_i = T_{i+1}^q + \pi T_{i+1} = \varphi_{i+1}(T_{i+1}), \text{ and generally } T_i = \varphi_n^{n-i}(T_n) \quad \text{for } n \geq 1, i \leq n.$$

We write

$$T_i^{j_i} = \varphi_n^{n-i}(T_n^{j_i}) = T_n^{q^{n-i}j_i} + R_{i,j_i},$$

where R_{i,j_i} is a polynomial in T_n such that $\|R_{i,j_i}\|_{\overline{\mathfrak{B}}^{\mathrm{perf}}} \leq |\pi|$, note that $\|T_i^{j_i}\|_{\overline{\mathfrak{B}}^{\mathrm{perf}}} = \|T_n^{q^{n-i}j_i}\|_{\overline{\mathfrak{B}}^{\mathrm{perf}}} = 1$. For a finite sum

$$\sum_{0\leq i\leq n, 0\leq j_i\leq m} a_{i,j_i} T_i^{j_i} \in \varinjlim_\varphi K\langle T\rangle,$$

we have

$$\begin{aligned}\|\sum_{0\leq i\leq n, 0\leq j_i\leq m} a_{i,j_i} T_i^{j_i}\|_{\overline{\mathfrak{B}}^{\mathrm{perf}}} &= \|\sum_{0\leq i\leq n, 0\leq j_i\leq m} a_{i,j_i} \varphi_n^{n-i}(T_n^{j_i})\|_{\overline{\mathfrak{B}}^{\mathrm{perf}}} \\ &= \|\sum_{0\leq i\leq n, 0\leq j_i\leq m} a_{i,j_i}(T_n^{j_i q^{n-i}} + R_{i,j_i})\|_{\overline{\mathfrak{B}}^{\mathrm{perf}}}.\end{aligned}$$

Note that $j_i q^{n-i} = j'_{i'} q^{n-i'}$ implies $i = i'$ and then $j'_{i'} = j_i$ because of the condition on i and j_i. Then computing

$$\|\sum_{0\leq i\leq n, 0\leq j_i\leq m} a_{i,j_i} T_n^{j_i q^{n-i}}\|_{\overline{\mathfrak{B}}^{\mathrm{perf}}} = \|\sum_{0\leq i\leq n, 0\leq j_i\leq m} a_{i,j_i} T_n^{j_i q^{n-i}}\|_{\overline{\mathfrak{B}}}$$

in the n-th copy $K\langle T_n\rangle$ using the Maximum Modulus Principle gives

$$\|\sum_{0\leq i\leq n, 0\leq j_i\leq m} a_{i,j_i} T_n^{j_i q^{n-i}}\|_{\overline{\mathfrak{B}}} = \max_{i,j_i}\{\|a_{i,j_i} T_n^{j_i q^{n-i}}\|_{\overline{\mathfrak{B}}}\} = \max_{i,j_i}\{|a_{i,j_i}|\}.$$

This implies

$$\|\sum_{0\leq i\leq n, 0\leq j_i\leq m} a_{i,j_i} T_n^{j_i q^{n-i}}\|_{\overline{\mathfrak{B}}^{\mathrm{perf}}} > \|\sum_{0\leq i\leq n, 0\leq j_i\leq m} a_{i,j_i} R_{i,j_i}\|_{\overline{\mathfrak{B}}^{\mathrm{perf}}},$$

and hence

$$\begin{aligned}\|\sum_{0\leq i\leq n, 0\leq j_i\leq m} a_{i,j_i}(T_n^{j_i q^{n-i}} + R_{i,j_i})\|_{\overline{\mathfrak{B}}^{\mathrm{perf}}} &= \|\sum_{0\leq i\leq n, 0\leq j_i\leq m} a_{i,j_i} T_n^{j_i q^{n-i}}\|_{\overline{\mathfrak{B}}^{\mathrm{perf}}} \\ &= \max_{i,j_i}\{|a_{i,j_i}|\} \\ &= \max_{i,j_i}\{\|a_{i,j_i} T_i^{j_i}\|_{\overline{\mathfrak{B}}^{\mathrm{perf}}}\}.\end{aligned}$$

We see that the set consisting of the $T_i^{j_i}, q \nmid j_i$ if $i > 0$, is α-cartesian for $\alpha = 1$. Note that this implies that it is linearly independent.

The set A consisting of finite sums $\sum_{0 \le i \le n, 0 \le j_i \le m} a_{i,j_i} T_i^{j_i}, q \nmid j_i$ if $i > 0$, is a K-subspace of $\widehat{\varinjlim_\varphi} K\langle T\rangle$ of countable dimension with a basis consisting of the $T_i^{j_i}, q \nmid j_i$ if $i > 0$. We claim that these sums are dense in $\widehat{\varinjlim_\varphi} K\langle T\rangle$. For this, we show that A contains all powers of T_i for every $i \in \mathbb{N}$. Note that $T_i^q = T_{i-1} - \pi T_i$. Let $n \in \mathbb{N}$ and write

$$T_i^{qn} = (T_{i-1} - \pi T_i)^n = \sum_{k=0}^{n} \binom{n}{k} T_{i-1}^k \cdot \pi^{n-k} T_i^{n-k}.$$

The last term in this sum is T_{i-1}^n. For $k < n$, we compute

$$\begin{aligned} T_{i-1}^k \cdot T_i^{n-k} &= (T_i^q + \pi T_i)^k \cdot T_i^{n-k} \\ &= \sum_{l=0}^{k} \binom{k}{l} T_i^{ql} \cdot \pi^{k-l} T_i^{k-l} \cdot T_i^{n-k} \\ &= \sum_{l=0}^{k} \binom{k}{l} \pi^{k-l} T_i^{ql+k-l+n-k} \\ &= \sum_{l=0}^{k} \binom{k}{l} \pi^{k-l} T_i^{ql-l+n}. \end{aligned}$$

If q divides $n - l$, then write $n - l = aq$ for an integer a. Then $a < n - l$ and $ql - l + n = q(l + a) < q(l + n - l) = qn$. If q does not divide $n - l$, then q does not divide $ql - l + n$. All in all, we can write T_i^{qn} as a sum of terms which are either of the form T_{i-1}^n, or of the form T_i^m such that q does not divide m, or of the form T_i^{qm} such that $m < n$. By repeating this process, we arrive after finitely many steps at a sum of the desired form, i.e. a sum which lies in A.
Therefore A is dense in $\varinjlim_\varphi K\langle T\rangle$, and hence is dense in $\widehat{\varinjlim_\varphi} K\langle T\rangle$ as well. The $T_i^{j_i}, q \nmid j_i$ if $i > 0$ form a topological generating system of $\widehat{\varinjlim_\varphi} K\langle T\rangle$. Then [31, 2.7.2/3] shows that the set consisting of the $T_i^{j_i}$ with $i, j_i \in \mathbb{N}$ such that $q \nmid j_i$ for $i > 0$ is a Schauder basis for $\widehat{\varinjlim_\varphi} K\langle T\rangle$.
If $f = \sum_{i \ge 0, j_i \ge 0} a_{i,j_i} T_i^{j_i} \in \widehat{\varinjlim_\varphi} K\langle T\rangle$ is any element, then any sequence of partial sums converges to f ([31, 1.1.8/ Proposition 2]), and therefore there is an m and an n such that

$$\begin{aligned} \|f\|_{\mathfrak{B}^{\text{perf}}} &= \| \sum_{0 \le i \le n, 0 \le j_i \le m} a_{i,j_i} T_i^{j_i} \|_{\mathfrak{B}^{\text{perf}}} \\ &= \max_{0 \le i \le n, 0 \le j_i \le m} \{ \|a_{i,j_i} T_i^{j_i}\|_{\mathfrak{B}^{\text{perf}}} \} \\ &= \max_{i \ge 0, j_i \ge 0} \{ \|a_{i,j_i} T_i^{j_i}\|_{\mathfrak{B}^{\text{perf}}} \}. \end{aligned}$$

□

Remark 3.8. Let K be perfectoid and let $(K\langle T\rangle\langle\frac{U}{s}\rangle, K\langle T\rangle\langle\frac{U}{s}\rangle^+)$ be a rational localization of $K\langle T\rangle$, U is a finite set generating the unit ideal in $K\langle T\rangle$ and $s \in K\langle T\rangle$. We have induced inductive systems

$$K\langle T\rangle\langle\frac{U}{s}\rangle \xrightarrow{\varphi} K\langle T\rangle\langle\frac{\varphi(U)}{\varphi(s)}\rangle \xrightarrow{\varphi} K\langle T\rangle\langle\frac{\varphi^2(U)}{\varphi^2(s)}\rangle \xrightarrow{\varphi} \dots.$$

and

$$K\langle T\rangle\langle\frac{U}{s}\rangle^+ \xrightarrow{\varphi} K\langle T\rangle\langle\frac{\varphi(U)}{\varphi(s)}\rangle^+ \xrightarrow{\varphi} K\langle T\rangle\langle\frac{\varphi^2(U)}{\varphi^2(s)}\rangle^+ \xrightarrow{\varphi} \dots.$$

Note that $\varphi^n(U)$ still generates the unit ideal in $K\langle T\rangle$ for all n. Then the π-adic completion of the inductive limit in the category of uniform Tate-Huber pairs is isomorphic to the rational localization

$$(\widehat{\varinjlim}_{\varphi}(K\langle T\rangle)\langle\frac{U}{s}\rangle, \widehat{\varinjlim}_{\varphi}(K\langle T\rangle)\langle\frac{U}{s}\rangle^+)$$

of $(\widehat{\varinjlim}_{\varphi} K\langle T\rangle, \widehat{\varinjlim}_{\varphi} K\langle T\rangle^+)$ (where we denote by U and s the image of $U \subseteq K\langle T\rangle$ and $s \in K\langle T\rangle$ in $\varinjlim_{\varphi} K\langle T\rangle$ by abuse of notation). This follows from Lemma 1.57 (note that $\widehat{\varinjlim}_{\varphi} K\langle T\rangle$ and therefore also $\varinjlim_{\varphi} K\langle T\rangle$ are stably uniform because $\widehat{\varinjlim}_{\varphi} K\langle T\rangle$ is perfectoid). Moreover, the π-adic completion $\widehat{\varinjlim}_{\varphi}(K\langle T\rangle)\langle\frac{U}{s}\rangle$ is a perfectoid K-algebra because it comes from a rational localization of the perfectoid $(\widehat{\varinjlim}_{\varphi} K\langle T\rangle, \widehat{\varinjlim}_{\varphi} K\langle T\rangle^+)$ (see Lemma 1.72).

3.1.1 The preperfectoid closed unit disk

If we apply the above construction to $(L\langle T\rangle, o_L\langle T\rangle)$, we get the uniform Tate-Huber pair $(\widehat{\varinjlim}_{\varphi} L\langle T\rangle, \widehat{\varinjlim}_{\varphi} o_L\langle T\rangle)$.

Proposition 3.9. *Let K be perfectoid. Then $\widehat{\varinjlim}_{\varphi} L\langle T\rangle \widehat{\otimes}_L K \cong \widehat{\varinjlim}_{\varphi} K\langle T\rangle$ is a perfectoid K-algebra.*

Proof. We have an isometric isomorphism (for the tensor product resp. inductive limit norms)

$$(\widehat{\varinjlim}_{\varphi} L\langle T\rangle)\widehat{\otimes}_L K = \widehat{\varinjlim}_{\varphi}(L\langle T\rangle \widehat{\otimes}_L K)$$

(Corollary 6.14 in the appendix, we take the completions with respect to these norms). Moreover, we have an isometric isomorphism $L\langle T\rangle\widehat{\otimes}_L K \cong K\langle T\rangle$ for the tensor product resp. supremum norm ([8, Appendix B, Proposition 5]). This gives an isomorphism

$$\widehat{\varinjlim}_{\varphi} L\langle T\rangle\widehat{\otimes}_L K \cong \widehat{\varinjlim}_{\varphi} K\langle T\rangle.$$

The latter is a perfectoid K-algebra as we have already seen. □

Together with Proposition 1.76 this shows that $(\widehat{\varinjlim_{\varphi}} L\langle T\rangle, \widehat{\varinjlim_{\varphi}} o_L\langle T\rangle)$ is sheafy. We define the *closed preperfectoid unit disk* over L to be the corresponding affinoid adic space:

$$\overline{\mathfrak{B}}_L^{\mathrm{perf}} := \mathrm{Spa}(\widehat{\varinjlim_{\varphi}} L\langle T\rangle, \widehat{\varinjlim_{\varphi}} o_L\langle T\rangle).$$

Corollary 3.10. $\overline{\mathfrak{B}}_L^{\mathrm{perf}}$ *is an L-preperfectoid space.*

3.1.2 The preperfectoid open unit disk

Next, we explain the construction of the open perfectoid and preperfectoid unit disk as in [42, 2.2] respective [33]. The open disks are not affinoid but are instead obtained by glueing together affinoid disks of radius $r < 1$.

We briefly recall the definition of a formal scheme and the adic generic fibre, as in [8, Chapter 7].

Definition 3.11. *Let A be a complete adic ring with ideal of definition I. We define the* formal spectrum $\mathrm{Spf}(A)$ *as the set of all open prime ideals $\mathfrak{p} \subseteq A$. Then* $\mathrm{Spf}(A)$ *is canonically identified with Spec$(A/I) \subseteq$ Spec(A) (as sets).*

The Zariski topology on $\mathrm{Spec}(A)$ induces a topology on $\mathrm{Spf}(A)$. Let $f \in A$. If $D(f)$ denotes the open subset of $\mathrm{Spf}(A)$ where f does not vanish, then

$$D(f) \mapsto A\langle f^{-1}\rangle = \varprojlim_n (A/I^n[f^{-1}])$$

defines a sheaf $\mathcal{O}_{\mathrm{Spf}(A)}$ of topological rings on the category of subsets $D(f) \subseteq \mathrm{Spf}(A)$ for $f \in A$ which extends to the category of all Zariski open subsets of $\mathrm{Spf}(A)$. The set $\mathrm{Spf}(A)$ together with the sheaf $\mathcal{O}_{\mathrm{Spf}(A)}$ forms a locally ringed space $(\mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)})$ which is called the *affine formal scheme* of A (and denoted by $\mathrm{Spf}(A)$). A *formal scheme* is a locally ringed space which is locally isomorphic to an affine formal scheme.

Now let A be an o_K-algebra which is complete with respect to the topology induced by a finitely generated ideal. Then we have $A = A^\circ$ and (A, A) is a Huber pair over (o_K, o_K). The generic point of $\mathrm{Spa}(o_K, o_K)$ is $\eta = \mathrm{Spa}(K, o_K)$, and the *adic generic fibre* of $\mathrm{Spf}(A)$ is $(\mathrm{Spa}(A, A))_\eta = (\mathrm{Spa}(A, A)) \setminus \{\pi = 0\}$.

Lemma 3.12 (Lemma 2.2.1 in [42])**.** *The adic generic fibre of* $\mathrm{Spf}(A)$ *has a cover by rational subsets* $\mathrm{Spa}(A, A)(f_1/\pi, ..., f_n/\pi)$ *where* $(f_1, ..., f_n)$ *runs through tuples of elements generating an ideal of definition of A.*

Proof. Let $f_1, ..., f_n \in A$ elements which generate an ideal of definition if A. Then for every $x \in (\mathrm{Spa}(A, A)) \setminus \{\pi = 0\}$ and each $i = 0, ..., n$ we have

$$|f_i(x)|^m \to 0 \quad \text{if } m \to \infty.$$

Then there is an $N \geq 1$ such that

$$|f_i(x)|^N \leq |\pi(x)| \neq 0,$$

and we conclude $x \in \mathrm{Spa}(A, A)(f_1^N/\pi, ..., f_n^N/\pi)$. Note that $(f_1^N, ..., f_n^N)$ is open since it contains $(f_1, ..., f_n)^{n \cdot N}$. □

Definition 3.13. *The adic open unit disk over L is the adic generic fibre of* $\mathrm{Spf}(o_L[[T]])$.

The adic open unit disk has a cover by rational subsets of the form

$$U_n := \mathrm{Spa}(o_L[|T|], o_L[|T|])(\{T^n, \pi\}/\pi) = \{x \in \mathrm{Spa}(o_L[|T|], o_L[|T|]) \mid |T^n(x)| \leq |\pi(x)| \neq 0\}$$

for $n \geq 1$. Set $A_n := o_L[|T|][T^n/\pi]$. Then $(A_n[1/\pi], A_n^+)$ is a Tate-Huber pair, where A_n^+ is the integral closure of A_n in $A_n[1/\pi]$. The completion $(\widehat{A_n}[1/\pi], \widehat{A_n^+})$ coincides with

$$(L\langle T\rangle\langle T^n/\pi\rangle, L\langle T\rangle\langle T^n/\pi\rangle^+) = (\mathcal{O}_L(\mathfrak{B}(r)), \mathcal{O}_L(\mathfrak{B}(r))^\circ), r = |\pi|^{1/n}.$$

We have $U_n = \mathrm{Spa}(\widehat{A_n}[1/\pi], \widehat{A_n^+})$. The adic open unit disk is therefore the union of closed disks of radius $|\pi|^{1/n}$ for $n \geq 1$.

In the following we fix a Frobenius power series $\varphi \in o_L[|T|]$ for π. We have an inductive system of rings

$$o_L[|T|] \xrightarrow{\varphi} o_L[|T|] \xrightarrow{\varphi} ... \xrightarrow{\varphi} o_L[|T|] \xrightarrow{\varphi} ....$$

To construct the preperfectoid open adic unit disk over L, we set

$$R_L := \varinjlim_{\varphi} o_L[[T]],$$

and denote by $\widehat{R}_L$ be the (π, T)-adic completion of R_L. Then (R_L, R_L) and $(\widehat{R}_L, \widehat{R}_L)$ are Huber pairs with ring of definition R_L resp. $\widehat{R}_L$ and ideal of definition (π, T).

Lemma 3.14 (cf. Remark 1.3 in [33])**.** *$\widehat{R}_L$ is π-adically complete and Hausdorff.*

Proof. The ring $\widehat{R}_L$ is Hausdorff for the (π, T)-adic topology, and we have

$$\bigcap_n \pi^n \widehat{R}_L \subseteq \bigcap_n (\pi, T)^n \widehat{R}_L = \{0\}.$$

Hence $\widehat{R}_L$ is Hausdorff for the π-adic topology.
For completeness, let $(a_i)_{i\in\mathbb{N}} \in \widehat{R}_L$ be a Cauchy sequence for the π-adic topology, then $(a_i)_{i\in\mathbb{N}}$ is also a Cauchy sequence for the (π, T)-adic topology which has a (π, T)-adic limit $a \in \widehat{R}_L$. We claim that a is the π-adic limit of $(a_i)_{i\in\mathbb{N}}$. We replace the original sequence by a subsequence such that

$$a_{i+1} - a_i \in \pi^i \widehat{R}_L \quad \text{for any } i \geq 0.$$

Let $a_{i+1} - a_i = \pi^i z_i$ for $z_i \in \widehat{R}_L$. For $j \geq 0$ we compute

$$\begin{aligned} a_{i+j} - a_i &= (a_{i+j} - a_{i+j-1}) + ... + (a_{i+1} - a_i) \\ &= \pi^i(\pi^{j-1} z_{i+j-1} + ... + \pi z_{i+1} + z_i). \end{aligned}$$

Set $y_{i,j} := \pi^{j-1} z_{i+j-1} + ... + \pi z_{i+1} + z_i$. Then we have

$$y_{i,j} - y_{i,m} = \pi^{j-1} z_{i+j-1} + ... + \pi^m z_{i+m} \in \pi^m \widehat{R}_L$$

for $j > m$. Hence the sequence $(y_{i,j})_{j\in\mathbb{N}}$ has a (π, T)-adic limit y_i. Then we compute

$$a - a_i = \lim_{j\to\infty} (a_{i+j} - a_i) = \pi^i \lim_{j\to\infty} y_{i,j} = \pi^i y_i \in \pi^i \widehat{R}_L,$$

which shows that $a_i \to a$ in the π-adic topology. □

Remark 3.15 (Remark 2.1 in [33])**.** $\widehat{R}_L$ is a flat o_L-algebra.

Proof. Let $f, g \in o_L[|T|]$. If $Tf = \pi g$, then $f \in \pi o[|T|]$ and $g \in T o_L[|T|]$. This is still true in R_L which can be checked using the defining properties of a Frobenius power series. We consider the following commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & K_{n+1} & \longrightarrow & R_L/(\pi^{n+1}, T^{n+1}) & \xrightarrow{\pi} & R_L/(\pi^{n+1}, T^{n+1}) & \xrightarrow{\text{pr}} & R_L/(\pi, T^{n+1}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \text{pr}} & & \downarrow{\scriptstyle \text{pr}} & & \downarrow{\scriptstyle \text{pr}} & & \\ 0 & \longrightarrow & K_n & \longrightarrow & R_L/(\pi^n, T^n) & \xrightarrow{\pi} & R_L/(\pi^n, T^n) & \xrightarrow{\text{pr}} & R_L/(\pi, T^n) & \longrightarrow & 0 \end{array}$$

where K_n, K_{n+1} are the kernels of the multiplication by π. Passing to the projective limit, we get an exact sequence

$$0 \to \varprojlim_n R_L/(\pi^n, T^n) \xrightarrow{\pi} \varprojlim_n R_L/(\pi^n, T^n) \to \varprojlim_n R_L/(\pi, T^n) \to 0.$$

This is because countable projective systems with zero transition maps have zero projective limits and zero $\varprojlim^1$-term (Mittag-Leffler). Note that $\widehat{R}_L = \varprojlim_n R_L/(\pi^n, T^n)$. We have

$$\varprojlim_n R_L/(\pi, T^n) = \varprojlim_n (R_L/\pi R_L)/(T^n) = \varprojlim_n (\varinjlim_n k[|T|])/(T^n)).$$

We get the short exact sequence

$$0 \to \widehat{R}_L \xrightarrow{\pi} \widehat{R}_L \to \varprojlim_n (\varinjlim_{q,n} k[|T|])/(T^n)) \to 0.$$

We see that $\widehat{R}_L$ is π-torsion free and therefore flat over o_L. □

Definition 3.16. *The* preperfectoid open unit disk $\mathfrak{B}_L^{\text{perf}}$ *over L is the adic generic fibre of* $\text{Spf}(\widehat{R}_L)$.

To describe $\mathfrak{B}_L^{\mathrm{perf}}$ explicitly, set

$$R_{L,n} := R_L[\frac{T^n}{\pi}] = R_L[X]/(T^n - \pi X)$$

for $n \geq 1$, i.e. the subalgebra of $R_L[1/\pi]$ generated by $\frac{T^n}{\pi}$. Then $R_{L,n}[1/\pi]$ is a Huber ring with pair of definition $(R_{L,n}, (\pi))$[1]. We have the Tate-Huber pair $(R_{L,n}[1/\pi], R_{L,n}^+)$ with topologically nilpotent unit π where $R_{L,n}^+ = R_L(T^n/\pi)^+$ is the integral closure of $R_{L,n}$ in $R_{L,n}[1/\pi]$. We form the completion $(\widehat{R}_{L,n}[1/\pi], \widehat{R}_{L,n}^+)$.

Lemma 3.17. *$(R_{L,n}[1/\pi], R_{L,n}^+)$ and $(\widehat{R}_{L,n}[1/\pi], \widehat{R}_{L,n}^+)$ are uniform Tate-Huber pairs.*

Proof. We only have to consider $(R_{L,n}[1/\pi], R_{L,n}^+)$ since a Tate-Huber pair is uniform if and only if its completion is uniform (Lemma 1.27). It is enough to show that $R_{L,n}^+$ is a ring of definition (Lemma 1.29). Since $R_{L,n}^+$ is open, we only have to show that it is bounded.
Note that we have $o_L[|T|][T^{nq^i}/\pi] = o_L[|T|][\varphi^i(T^n)/\pi]$ as rings for every $i \in \mathbb{N}$. Then, as rings, we compute

$$\begin{aligned} R_{L,n} &= R_L[T^n/\pi] \\ &= (\varinjlim_{\varphi} o_L[|T|])[T^n/\pi] \\ &= \varinjlim_{\varphi,i}(o_L[|T|][\varphi^i(T^n)/\pi]) \\ &= \varinjlim_{\varphi,i}(o_L[|T|][T^{nq^i}/\pi]). \end{aligned}$$

Since taking integral closures commutes with taking inductive limits, the abstract ring $R_{L,n}^+$ is given by the inductive limit of the integral closure $o_L[|T|](T^{nq^i}/\pi)^+$ of $o_L[|T|][T^{nq^i}/\pi]$ in $o_L[|T|][T^{nq^i}/\pi][1/\pi]$, i.e. we have

$$\begin{aligned} R_{L,n}^+ &= (\varinjlim_{\varphi} o_L[|T|])(T^n/\pi)^+ \\ &= \varinjlim_{\varphi,i}(o_L[|T|](T^{nq^i}/\pi)^+). \end{aligned}$$

Set $r := |\pi|^{1/n}$. The ring $o_L[|T|](T^{nq^i}/\pi)^+ \subseteq \mathcal{O}_L(\mathfrak{B}(r^{1/q^i}))^{\leq 1}$ consists of elements $g \in \mathcal{O}_L(\mathfrak{B}(r^{1/q^i}))$ with $\|g\|_{\mathfrak{B}(r^{1/q^i})} \leq 1$ where $\|\cdot\|_{\mathfrak{B}(r^{1/q^i})}$ denotes the supremum norm on $\mathcal{O}_L(\mathfrak{B}(r^{1/q^i}))$. Let $g = \sum_{k\geq 0} a_k T^k \in o_L[|T|](T^{nq^i}/\pi)^+ \subseteq \mathcal{O}_L(\mathfrak{B}(r^{1/q^i}))$ be an element such that

$$\|g\|_{\mathfrak{B}(r^{1/q^i})} = \sup_{k\in\mathbb{N}}\{|a_k||\pi|^{k/nq^i}\} \leq |\pi|.$$

[1] The ideal in R_L generated by T^n is not open, but the ideal (π, T^n) is open in R_L. In writing $R_L[T^n/\pi]$, we suppress the fraction $\pi/\pi = 1$. We have $R_L, n[1/\pi] = R_L(U/\pi)$ for $U = (T^n, \pi)$.

If $k \leq nq^i$, then

$$|a_k| \leq \frac{|\pi|}{|\pi^{k/nq^i}|} = |\pi|^{(nq^i-k)/nq^i} \leq 1$$

and hence $a_kT^k \in o_L[|T|]$. If $k > nq^i$, then write

$$a_kT^k = \frac{T^{nq^i}}{\pi} \cdot T^{k-nq^i}\pi^{(2nq^i-k)/nq^i} \cdot u$$

for $\pi^{(2nq^i-k)/nq^i}, u \in o_{\mathbb{C}_p}$; and we have $\|T^{k-nq^i}\pi^{(2nq^i-k)/nq^i} \cdot u\|_{\mathfrak{B}(r^{1/q^i})} \leq |\pi|$. If $\|\pi^{(2nq^i-k)/nq^i} \cdot u\|_{\mathfrak{B}(r^{1/q^i})} > 1$, we may repeat this. By iterating this process we eventually arrive in the case $nq^i \geq k$. This shows that every monomial a_kT^k can be written as $a_kT^k = (\frac{T^{nq^i}}{\pi})^{j_k} f_{j_k}$ for some j_k and $f_{j_k} \in o_L[|T|]$. Note that the coefficients a_k are bounded and that the f_{j_k} converge to 0 for $\|\cdot\|_{\mathfrak{B}(r^{1/q^i})}$ and hence in the (π, T)-adic topology on $o_L[|T|]$. Therefore we find an m such that we can write

$$g = \sum_{j=0}^{m} (\frac{T^{nq^i}}{\pi})^j g_j$$

with $g_j \in o_L[|T|]$. We conclude that $g \in o_L[|T|][T^{nq^i}/\pi]$. This shows that

$$\pi o_L[|T|](T^{nq^i}/\pi)^+ \subseteq o_L[|T|][T^{nq^i}/\pi]$$

for every i. Passing to the limit, we see that

$$\pi R^+_{L,n} \subseteq R_{L,n}.$$

Hence $R^+_{L,n}$ is bounded. □

Now we look at the rings $\widehat{R}_{L,n}$ from a slightly different angle. Let $L \subseteq K \subseteq \mathbb{C}_p$ be a complete intermediate field. Let $r = |\varpi|^{1/n}$ where $\varpi \in K$ is a topologically nilpotent unit and such that $r \in p^{\mathbb{Q}} \cap [p^{-q/e(q-1)}, 1)$. For the rest of this section, we always assume that radii r are of this form (unless stated otherwise). Consider the Tate-Huber pairs $(\mathcal{O}_K(\mathfrak{B}(r^{1/q^i})), \mathcal{O}^+_K(\mathfrak{B}(r^{1/q^i})))$ for any $i \in \mathbb{N}$, where

$$\mathcal{O}^+_K(\mathfrak{B}(r^{1/q^i})) = \mathcal{O}_K(\mathfrak{B}(r^{1/q^i}))^{\leq 1} = \{f \in \mathcal{O}_K(\mathfrak{B}(r^{1/q^i})) \mid \|f\|_{\mathfrak{B}(r^{1/q^i})} \leq 1\}$$

are the power-bounded elements. Here $\|\cdot\|_{\mathfrak{B}(r^{1/q^i})}$ is the supremum norm of $\mathcal{O}_K(\mathfrak{B}(r^{1/q^i}))$. The ring $\mathcal{O}_K(\mathfrak{B}(r^{1/q^i}))$ is reduced, hence $(\mathcal{O}_K(\mathfrak{B}(r^{1/q^i})), \mathcal{O}^+_K(\mathfrak{B}(r^{1/q^i})))$ is stably uniform (Remark 1.30). Note that $\mathcal{O}_K(\mathfrak{B}(r^{1/q^i})) = K\langle T\rangle\langle T^{nq^i}/\varpi\rangle$.

We have inductive systems of rings

$$\mathcal{O}_K(\mathfrak{B}(r)) \xrightarrow{\varphi} \mathcal{O}_K(\mathfrak{B}(r^{1/q})) \xrightarrow{\varphi} \ldots \xrightarrow{\varphi} \mathcal{O}_K(\mathfrak{B}(r^{1/q^i})) \xrightarrow{\varphi} \mathcal{O}_K(\mathfrak{B}(r^{1/q^{i+1}})) \xrightarrow{\varphi} \ldots$$

and

$$\mathcal{O}_K^+(\mathfrak{B}(r)) \xrightarrow{\varphi} \mathcal{O}_K^+(\mathfrak{B}(r^{1/q})) \xrightarrow{\varphi} \ldots \xrightarrow{\varphi} \mathcal{O}_K^+(\mathfrak{B}(r^{1/q^i})) \xrightarrow{\varphi} \mathcal{O}_K^+(\mathfrak{B}(r^{1/q^{i+1}})) \xrightarrow{\varphi} \ldots$$

Set

$$\breve{\mathcal{O}}_K(\mathfrak{B}^{\text{perf}}(r)) := \varinjlim_{\varphi,i} \mathcal{O}_K(\mathfrak{B}(r^{1/q^i})), \quad \text{and}$$
$$\breve{\mathcal{O}}_K^+(\mathfrak{B}^{\text{perf}}(r)) := \varinjlim_{\varphi,i} \mathcal{O}_K^+(\mathfrak{B}(r^{1/q^i})).$$

Then $(\breve{\mathcal{O}}_K(\mathfrak{B}^{\text{perf}}(r)), \breve{\mathcal{O}}_K^+(\mathfrak{B}^{\text{perf}}(r))$ is the inductive limit of the inductive system consisting of the maps

$$\varphi : (\mathcal{O}_K(\mathfrak{B}(r^{1/q^i}), \mathcal{O}_K^+(\mathfrak{B}(r^{1/q^i})) \to (\mathcal{O}_K(\mathfrak{B}(r^{1/q^{i+1}}), \mathcal{O}_K^+(\mathfrak{B}(r^{1/q^{i+1}}))$$

for every i in the category of uniform Tate-Huber pairs as in Proposition 1.55. We denote by $\mathcal{O}_K^+(\mathfrak{B}^{\text{perf}}(r))$ the completion of $\breve{\mathcal{O}}_K^+(\mathfrak{B}^{\text{perf}}(r))$. Then

$$\mathcal{O}_K(\mathfrak{B}^{\text{perf}}(r)) := \mathcal{O}_K^+(\mathfrak{B}^{\text{perf}}(r)) \otimes_{\breve{\mathcal{O}}_K^+(\mathfrak{B}^{\text{perf}}(r))} \breve{\mathcal{O}}_K(\mathfrak{B}^{\text{perf}}(r))$$

is the completion of $\breve{\mathcal{O}}_K(\mathfrak{B}^{\text{perf}}(r))$, and $(\mathcal{O}_K(\mathfrak{B}^{\text{perf}}(r)), \mathcal{O}_K^+(\mathfrak{B}^{\text{perf}}(r)))$ is a complete uniform Tate-Huber pair.

The ring $\breve{\mathcal{O}}_K(\mathfrak{B}^{\text{perf}}(r))$ carries the inductive limit seminorm coming from the supremum norms on the $\mathcal{O}_K(\mathfrak{B}(r^{1/q^i}))$. It is a norm since the transition map

$$\varphi : \mathcal{O}_K(\mathfrak{B}(r^{1/q^i})) \to \mathcal{O}_K(\mathfrak{B}(r^{1/q^{i+1}}))$$

is an isometry for every i (Lemma 2.8). Denote the continuous extension of this norm to $\mathcal{O}_K(\mathfrak{B}^{\text{perf}}(r))$ by $\|\cdot\|_{\mathfrak{B}^{\text{perf}}(r)}$. The norm topology coincides with the topology of $\mathcal{O}_K(\mathfrak{B}^{\text{perf}}(r))$ as a Huber ring.

Remark 3.18. Sometimes it is convenient to assume that the φ is given by $\varphi(T) = \pi T + T^q$, i.e. the chosen Lubin-Tate group law is the special one. If ψ is another Frobenius power series for π, then there is an isomorphism of formal groups $[1]_{\psi,\varphi} : F_\varphi \to F_\psi$. If we want to discriminate between different choices of the Frobenius power series, then we write T_φ resp. T_ψ for the global coordinates, and moreover $\widehat{R}_{L,\varphi}$ and $\widehat{R}_{L,\psi}$ or $\mathcal{O}_K(\mathfrak{B}^{\text{perf}}(r))_\varphi$ and $\mathcal{O}_K(\mathfrak{B}^{\text{perf}}(r))_\psi$ for the respective constructions with φ resp. ψ.
There is a continuous (for the (π, T)-adic topology) isomorphism of rings

$$[1]_{\psi,\varphi} : o_K[|T_\varphi|] \to o_K[|T_\psi|],$$
$$T_\varphi \mapsto T_\psi + \text{higher degree terms in } o_L[|T_\psi|].$$

with continuous inverse $[1]_{\varphi,\psi}$. This extends to a homeomorphism between $\varinjlim_{\varphi} o_K[|T_\varphi|]$ and $\varinjlim_{\psi} o_K[|T_\psi|]$ and to a homeomorphism between the respective completions. Furthermore, the power series $[1]_{\psi,\varphi}$ gives an isometric (for the supremum norm) isomorphism

$$\mathcal{O}_K(\mathfrak{B}(r))_\varphi \to \mathcal{O}_K(\mathfrak{B}(r))_\psi$$

which respects the $o_L \setminus \{0\}$ action on both sides. This isomorphism extends to an isometric isomorphism

$$\mathcal{O}_K(\mathfrak{B}^{\text{perf}}(r))_\varphi \to \mathcal{O}_K(\mathfrak{B}^{\text{perf}}(r))_\psi.$$

Lemma 3.19. *Let K'/K be complete. The Tate-Huber pair*

$$(\mathcal{O}_K(\mathfrak{B}^{\text{perf}}(r))\widehat{\otimes}_K K', (\mathcal{O}_K(\mathfrak{B}^{\text{perf}}(r))\widehat{\otimes}_K K')^+)$$

is uniform.

Proof. We have to show that $\mathcal{O}_K(\mathfrak{B}^{\text{perf}}(r))\widehat{\otimes}_K K'$ is uniform. It is enough to think about the dense subset $\breve{\mathcal{O}}_K(\mathfrak{B}^{\text{perf}}(r)) \otimes_K K'$ because uniformity is preserved under completion (Lemma 1.27). Let $(\breve{\mathcal{O}}_K(\mathfrak{B}^{\text{perf}}(r)) \otimes_K K')^+$ be the integral closure of the image of $\breve{\mathcal{O}}_K^+(\mathfrak{B}^{\text{perf}}(r)) \otimes_{o_K} o_{K'}$ in $\breve{\mathcal{O}}_K(\mathfrak{B}^{\text{perf}}(r)) \otimes_K K'$. We show that

$$\pi(\breve{\mathcal{O}}_K(\mathfrak{B}^{\text{perf}}(r)) \otimes_K K')^+ \subseteq \text{Im}(\breve{\mathcal{O}}_K^+(\mathfrak{B}^{\text{perf}}(r)) \otimes_{o_K} o_{K'} \to \breve{\mathcal{O}}_K(\mathfrak{B}^{\text{perf}}(r)) \otimes_K K').$$

We start with a general observation. Let $s < 1$ and consider the affinoid algebra $\mathcal{O}_K(\mathfrak{B}(s))\widehat{\otimes}_K K' \cong \mathcal{O}_{K'}(\mathfrak{B}(s))$ with supremum norm $\|\cdot\|_{\mathfrak{B}(s)}$. The integral closure of the image of $\mathcal{O}_K(\mathfrak{B}(s))^{\leq 1}\widehat{\otimes}_{o_K} o_{K'}$ in $\mathcal{O}_K(\mathfrak{B}(s))\widehat{\otimes}_K K'$ is given by the power-bounded elements

$$(\mathcal{O}_L(\mathfrak{B}(s))\widehat{\otimes}_K K')^\circ = \mathcal{O}_{K'}(\mathfrak{B}(s))^{\leq 1} = \mathcal{O}_{K'}(\mathfrak{B}(s))^\circ.$$

Let $f = \sum_i f_i \otimes c_i \in \mathcal{O}_K(\mathfrak{B}(s)) \otimes_K K'$ such that $\|f\|_{\mathfrak{B}(s)} \leq |\pi|$. Write $f_i = \sum_k a_{k_i} T^k$. The image of f under the isomorphism $\mathcal{O}_K(\mathfrak{B}(s))\widehat{\otimes}_K K' \cong \mathcal{O}_{K'}(\mathfrak{B}(s))$ is $f = \sum_k (T^k \cdot \sum_i a_{k_i} c_i)$. Then

$$\|f\|_{\mathfrak{B}(s)} = \max_{k\in\mathbb{N}}\{|\sum_i a_{k_i} c_i| s^k\} \leq |\pi|.$$

We find an $n \in \mathbb{N}$ such that $|\pi^n c_i| \leq 1$ for all (finitely many) i. Then let k_0 be large enough such that $\|a_{k_i} T^k\|_{\mathfrak{B}(s)} \leq |\pi|^n$ for all i and $k > k_0$. We write

$$f = \underbrace{\sum_i (\sum_{k=0}^{k_0} a_{k_i} T^k \otimes c_i)}_{=:f_1} + \underbrace{\sum_i (\sum_{k>k_0} a_{k_i} T^k \otimes c_i)}_{=:f_2}.$$

Regarding the second summand, we write

$$f_2 = \sum_i (\sum_{k>k_0} a_{k_i} T^k \otimes c_i) = \sum_i (\sum_{k>k_0} \pi^{-n} \cdot a_{k_i} T^k \otimes \pi^n \cdot c_i) \in \mathcal{O}_K(\mathfrak{B}(s))^{\leq 1} \otimes_{o_K} o_{K'}.$$

The first summand $f_1 = \sum_{k=0}^{k_0} \sum_i a_{k_i} c_i T^k$ consists of finitely many terms $\sum_i a_{k_i} c_i T^k$. Set $b_k = \sum_i a_{k_i} c_i$.

For every b_k we find a minimal $j_k \in \mathbb{Z}$ such that $|b_k \pi^{j_k}| \leq 1$. Then $|b_k \pi^{j_k - 1}| > 1$ and $|b_k \pi^{j_k}| > |\pi|$. On the other hand we have $|b_k| s^k = |b_k \pi^{j_k} \pi^{-j_k}| s^k \leq |\pi|$. Hence

$$|\pi^{-j_k}| s^k \leq |\pi| / |b_k \pi^{j_k}| \leq 1.$$

We have $f_1 = \sum_{k \geq 0} b_k \pi^{j_k} \cdot \pi^{-j_k} T^k$. We see that f_1 correspondents to

$$\sum_{k=0}^{k_0} \pi^{-j_k} T^k \otimes b_k \pi^{j_k} \in \mathcal{O}_K(\mathfrak{B}(s))^{\leq 1} \otimes_{o_K} o_{K'},$$

and conclude that

$$\pi(\mathcal{O}_K(\mathfrak{B}(s)) \otimes_K K')^+ \subseteq \mathrm{Im}(\mathcal{O}_K(\mathfrak{B}(s))^{\leq 1} \otimes_{o_K} o_{\mathbb{C}_p} \to \mathcal{O}_K(\mathfrak{B}(s)) \otimes_K \mathbb{C}_p)$$

where $(\mathcal{O}_K(\mathfrak{B}(s)) \otimes_K K')^+$ is the integral closure of the image of $\mathcal{O}_K(\mathfrak{B}(s))^{\leq 1} \otimes_{o_K} o_{K'}$ in $\mathcal{O}_K(\mathfrak{B}(s)) \otimes_K K'$. This implies

$$\pi(\breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r)) \otimes_K K')^+ \subseteq \mathrm{Im}(\breve{\mathcal{O}}_K^+(\mathfrak{B}^{\mathrm{perf}}(r)) \otimes_{o_K} o_{K'} \to \breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r)) \otimes_K K')$$

by applying the computation to $s = r^{1/q^i}$ for every i and passing to the limit.

□

Corollary 3.20. *We have an isomorphism of Tate-Huber pairs*

$$(\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r)) \widehat{\otimes}_K K', (\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r)) \widehat{\otimes}_K K')^+) \cong (\mathcal{O}_{K'}(\mathfrak{B}^{\mathrm{perf}}(r)), \mathcal{O}_K^+(\mathfrak{B}^{\mathrm{perf}}(r))).$$

Proof. As abstract rings, we have isomorphisms

$$(\varinjlim_{\varphi,i} \mathcal{O}_K(\mathfrak{B}(r^{1/q^i}))) \otimes_K K' \cong \varinjlim_{\varphi,i} (\mathcal{O}_K(\mathfrak{B}(r^{1/q^i})) \otimes_K K')$$

and

$$(\varinjlim_{\varphi,i} \mathcal{O}_K(\mathfrak{B}(r^{1/q^i}))^{\leq 1}) \otimes_K K')^+ \cong \varinjlim_{\varphi,i} (\mathcal{O}_K(\mathfrak{B}(r^{1/q^i}))^{\leq 1} \otimes_K K')^+.$$

With Remark 1.31 this gives an isomorphism between the respective (uniform) Tate-Huber pairs. Then passing to the π-adic completion and using that $\mathcal{O}_K(\mathfrak{B}(r^{1/q^i})) \widehat{\otimes}_K K' \cong \mathcal{O}_{K'}(\mathfrak{B}(r^{1/q^i}))$ and $(\mathcal{O}_K(\mathfrak{B}(r^{1/q^i})) \otimes_K K')^+ \cong \mathcal{O}_{K'}(\mathfrak{B}(r^{1/q^i}))^{\leq 1}$ (see the section about tensor products in the first chapter) gives isomorphisms

$$(\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r)) \widehat{\otimes}_K K')^+ \cong \mathcal{O}_{K'}^+(\mathfrak{B}^{\mathrm{perf}}(r))$$

and

$$\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r)) \widehat{\otimes}_K K' \cong \mathcal{O}_{K'}(\mathfrak{B}^{\mathrm{perf}}(r)).$$

□

Lemma 3.21. *Let K'/K be perfectoid. The K'-algebra $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r))\widehat{\otimes}_K K'$ is perfectoid.*

Proof. If $\varphi(T) = \pi T + T^q$, then $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r))\widehat{\otimes}_K K' = \mathcal{O}_{K'}(\mathfrak{B}^{\mathrm{perf}}(r))$ is a rational localization of the perfectoid K' algebra $\widehat{\varinjlim_\varphi K'\langle T\rangle}$ (see Remark 3.8) and hence perfectoid. Otherwise, use the isomorphism $\mathcal{O}_{K'}(\mathfrak{B}^{\mathrm{perf}}(r))_\varphi \cong \mathcal{O}_{K'}(\mathfrak{B}^{\mathrm{perf}}(r))_\psi$. $\square$

From the viewpoint of normed rings, Lemma 3.19 implies that the tensor product norm on $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r))\widehat{\otimes}_K K'$ is equivalent to the norm $\|\cdot\|_{\mathfrak{B}^{\mathrm{perf}}(r)}$ on $\mathcal{O}_{K'}(\mathfrak{B}^{\mathrm{perf}}(r))$. This can be seen as follows: The tensor product norm on $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r))\widehat{\otimes}_K K'$ is equal to the completion of the inductive limit norm on $\varinjlim_{\varphi,i}(\mathcal{O}_K(\mathfrak{B}(r^{1/q^i}))\otimes_K K')$ coming from the tensor product norms $\|\cdot\|_{\mathfrak{B}(r^{1/q^i}),K}\otimes|\cdot|$ on the $\mathcal{O}_K(\mathfrak{B}(r^{1/q^i}))\otimes_K K'$ (Lemma 6.13). We denote this norm for the moment by $\|\cdot\|_{\mathfrak{B}^{\mathrm{perf}}(r),K}\otimes|\cdot|$. For every i, the tensor product norm on $\mathcal{O}_K(\mathfrak{B}(r^{1/q^i}))\widehat{\otimes}_K K'$ is equivalent to the supremum norm $\|\cdot\|_{\mathfrak{B}(r^{1/q^i}),K'}$ on $\mathcal{O}_{K'}(\mathfrak{B}(r^{1/q^i}))$. However, they do not need to be equal. Therefore $\|\cdot\|_{\mathfrak{B}^{\mathrm{perf}}(r),K}\otimes|\cdot|$ is not automatically equivalent to the norm $\|\cdot\|_{\mathfrak{B}^{\mathrm{perf}}(r),K'}$ on $\mathcal{O}_{K'}(\mathfrak{B}^{\mathrm{perf}}(r))$. But in the proof of Lemma 3.19 we have seen that for every i we have

$$\pi\mathcal{O}_{K'}(\mathfrak{B}(r^{1/q^i}))^{\leq 1} \subseteq \mathrm{Im}(\mathcal{O}_K(\mathfrak{B}(r^{1/q^i})^{\leq 1}\widehat{\otimes}_{o_K} o_{K'} \to \mathcal{O}_K(\mathfrak{B}(r^{1/q^i}))\widehat{\otimes}_K K')).$$

This implies that

$$|\pi^2|\cdot\|\cdot\|_{\mathfrak{B}(r^{1/q^i}),K}\otimes|\cdot| \leq \|\cdot\|_{\mathfrak{B}(r^{1/q^i}),K'}.$$

On the other hand, we have $\|\cdot\|_{\mathfrak{B}(r^{1/q^i}),K'} \leq \|\cdot\|_{\mathfrak{B}(r^{1/q^i}),K}\otimes|\cdot|$ by the theory of affinoid algebras (see e.g. [8, Chapter 3.2, Proposition 9]). Loosely speaking, the tensor product norms on the $\mathcal{O}_K(\mathfrak{B}(r^{1/q^i}))\widehat{\otimes}_K K'$ are equivalent to the respective supremum norms in an uniform way for every i. This implies the equivalence of $\|\cdot\|_{\mathfrak{B}^{\mathrm{perf}},K'}$ on $\breve{\mathcal{O}}_{K'}(\mathfrak{B}^{\mathrm{perf}}(r))$ and $\|\cdot\|_{\mathfrak{B}^{\mathrm{perf}}(r),K}\otimes|\cdot|$ on $\breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r))\otimes_K K'$ and on the resp. completions.

For $K = L$ and $r = |\pi|^{1/n}, n \geq 1$ we have an embedding of rings with dense image

$$R_{L,n}[1/\pi] = \varinjlim_{\varphi,i}(o_L[|T|][T^{nq^i}/\pi])[1/\pi] \hookrightarrow \breve{\mathcal{O}}_L(\mathfrak{B}^{\mathrm{perf}}(r)) = \varinjlim_{\varphi,i}\mathcal{O}_L(\mathfrak{B}(r^{1/q^i})),$$

and

$$R^+_{L,n} = \varinjlim_{\varphi,i}(o_L[|T|](T^{nq^i}/\pi)^+) \hookrightarrow \breve{\mathcal{O}}^+_L(\mathfrak{B}^{\mathrm{perf}}(r)) = \varinjlim_{\varphi,i}\mathcal{O}_L(\mathfrak{B}(r^{1/q^i}))^{\leq 1}.$$

Passing to the completion leads to isomorphisms

$$\widehat{R}_{L,n}[1/\pi] \cong \mathcal{O}_L(\mathfrak{B}^{\mathrm{perf}}(r)), \quad \text{and} \quad \widehat{R}^+_{L,n} \cong \mathcal{O}^+_L(\mathfrak{B}^{\mathrm{perf}}(r)).$$

Note that the (π, T)-adic topology on $R_{L,n}$ is the π-adic one since (π^n, T^n) is contained in (π) (and of course (π) is contained in (π, T)).

Corollary 3.22. *The spaces* $\mathrm{Spa}(\widehat{R}_{L,n}[1/\pi], \widehat{R}_{L,n})$ *are* L*-preperfectoid.*

Lemma 3.23. *We have open immersions*

$$\mathrm{Spa}(\widehat{R}_{L,n}[1/\pi], \widehat{R}^+_{L,n}) \hookrightarrow \mathrm{Spa}(\widehat{R}_{L,n+1}[1/\pi], \widehat{R}^+_{L,n+1})$$

for $n \geq 1$.

Proof. We show that $\mathrm{Spa}(\widehat{R}_{L,n}[1/\pi], \widehat{R}^+_{L,n})$ is a rational subset of $\mathrm{Spa}(\widehat{R}_{L,n+1}[1/\pi], \widehat{R}^+_{L,n+1})$. Then the lemma follows from Remark 1.45. Set $r = |\pi|^{1/n}$ and $r' = |\pi|^{1/n+1}$. Note that $\mathcal{O}_L(\mathfrak{B}(r^{1/q^i})) = \mathcal{O}_L(\mathfrak{B}((r')^{1/q^i}))\langle \frac{T^{nq^i}}{\pi} \rangle$ and $\mathcal{O}_L(\mathfrak{B}(r^{1/q^i}))^{\leq 1} = \mathcal{O}_L(\mathfrak{B}((r')^{1/q^i}))\langle \frac{T^{nq^i}}{\pi} \rangle^+$ for every i. Using Lemma 1.57 and noting that $\mathcal{O}_L(\mathfrak{B}^{\mathrm{perf}}(r'))$ is stably uniform (Proposition 1.76), we have

$$\begin{aligned} \widehat{R}_{L,n}[1/\pi] &= \mathcal{O}_L(\mathfrak{B}^{\mathrm{perf}}(r)) \\ &= \widehat{\varinjlim}_{\varphi,i} \mathcal{O}_L(\mathfrak{B}(r^{1/q^i})) \\ &= \widehat{\varinjlim}_{\varphi,i} (\mathcal{O}_L(\mathfrak{B}((r')^{1/q^i}))\langle \frac{T^{nq^i}}{\pi} \rangle) \\ &= (\widehat{\varinjlim}_{\varphi,i} \mathcal{O}_L(\mathfrak{B}((r')^{1/q^i})))\langle \frac{T^n}{\pi} \rangle. \end{aligned}$$

Likewise, we have $\mathcal{O}^+_L(\mathfrak{B}^{\mathrm{perf}}(r)) = \mathcal{O}_L(\mathfrak{B}^{\mathrm{perf}}(r'))\langle T^n/\pi \rangle^+$. □

Glueing the $\mathrm{Spa}(\widehat{R}_{L,n}[1/\pi], \widehat{R}_{L,n})$ together along these open immersions, we obtain the preperfectoid open adic unit disk:

$$\mathfrak{B}^{\mathrm{perf}}_L = \varinjlim_{n \geq 1} \mathrm{Spa}(\widehat{R}_{L,n}[1/\pi], \widehat{R}_{L,n})$$

over L.

For general complete $L \subseteq K \subseteq \mathbb{C}_p$, we set $R_K = \varinjlim_{\varphi} o_K[|T|]$ and denote by $\widehat{R}_K$ its (π, T)-adic completion. Likewise, we define $\widehat{R}_{K,n}$ and $\mathfrak{B}^{\mathrm{perf}}_K$ in the same way as for $K = L$. Then we have

$$\widehat{R}_{K,n}[1/\pi] = \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r)).$$

Definition 3.24. *Let* K *be perfectoid. Set* $r = |\pi|^{1/n}, n \geq 1$. *The space*

$$\mathfrak{B}^{\mathrm{perf}}_K(r) := \mathrm{Spa}(\widehat{R}_{K,n}[1/\pi], \widehat{R}^+_{K,n})$$

is the perfectoid disk of radius r *inside of* $\mathfrak{B}^{\mathrm{perf}}_K$.

If $\varphi(T) = \pi T + T^q$, then for any n, $\mathfrak{B}^{\mathrm{perf}}_K(r)$ is a rational subset of the closed disk $\overline{\mathfrak{B}}^{\mathrm{perf}}_K$. Therefore we have an open immersion $\mathfrak{B}^{\mathrm{perf}}_K \hookrightarrow \overline{\mathfrak{B}}^{\mathrm{perf}}_K$.

Remark 3.25. Note that we have an isomorphism of rings

$$\varinjlim_{\varphi} o_L[|T|] \otimes_{o_L} o_K = \varinjlim_{\varphi} (o_L[|T|] \otimes_{o_L} o_K).$$

On both sides, the topology is given by taking the ideals $(T \otimes 1, 1 \otimes \pi)^n$ as a neighbourhood basis of 0. Completing gives a topological isomorphism

$$\widehat{R}_L \widehat{\otimes}_{o_L} o_K = \widehat{R}_K.$$

Proposition 3.26. *Assume that $r > p^{-q/e(q-1)}$ and that φ is given by $\varphi(T) = \pi T + T^q$. Set*

$$T_i := \text{ image of } T \in \mathcal{O}_K(\mathfrak{B}(r^{1/q^i})) \text{ in } \check{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r)) = \varinjlim_{\varphi, i} \mathcal{O}_K(\mathfrak{B}(r^{1/q^i}))$$

for $i \in \mathbb{N}$. The K-vector space $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r)) = \widehat{\varinjlim}_{\varphi,i} \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r^{1/p^i}))$ has a Schauder basis consisting of the elements $T_i^{j_i}$ with $i, j_i \in \mathbb{N}$ such that $q \nmid j_i$ for $i > 0$.

Proof. The proof is basically the same as the proof of Proposition 3.7, but uses that $\|\pi T\|_{\mathfrak{B}(r^{1/q^i})} < \|T^q\|_{\mathfrak{B}(r^{1/q^i})}$ in the supremum norm on $\mathcal{O}_K(\mathfrak{B}(r^{1/q^i}))$ for $i > 0$, hence we need the condition on r. Again, we write $\varphi_i(T_i)$ for the polynomial $\pi T_i + T_i^q$ and similarly for powers of T_i and φ_i.
For a finite sum $\sum_{0 \le i \le n, 0 \le j_i \le m} a_{i,j_i} T_i^{j_i}$, we write

$$\sum_{0 \le i \le n, 0 \le j_i \le m} a_{i,j_i} T_i^{j_i} = \sum_{0 \le i \le n, 0 \le j_i \le m} a_{i,j_i} \varphi_n^{n-i}(T_n^{j_i})$$

The latter sum lies in the image of $\mathcal{O}_K(\mathfrak{B}(r^{1/q^n}))$ in $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r))$. We write

$$\varphi_n^{n-i}(T_n^{j_i}) = T_n^{j_i q^{n-i}} + R_{i,j_i}$$

We claim that $\|T_n^{j_i q^{n-i}}\|_{\mathfrak{B}^{\mathrm{perf}}(r)} > \|R_{i,j_i}\|_{\mathfrak{B}^{\mathrm{perf}}(r)}$ and hence

$$\|T_i^{j_i}\|_{\mathfrak{B}^{\mathrm{perf}}(r)} = \|\varphi_n^{n-i}(T_n^{j_i})\|_{\mathfrak{B}^{\mathrm{perf}}(r)} = \|T_n^{q^{n-i} j_i}\|_{\mathfrak{B}^{\mathrm{perf}}(r)}.$$

We can compute this in $\mathcal{O}_K(\mathfrak{B}(r^{1/q^n}))$. Generally, let $s \ge r$. Then we have the map $\varphi^m : \mathcal{O}_K(\mathfrak{B}(s)) \to \mathcal{O}_K(\mathfrak{B}(s^{1/q^m}))$ for every m. This is an isometry which follows from Lemma 2.8. Since $\|T^{q-1}\|_{\mathfrak{B}(s^{1/q})} = s^{q-1/q} > |\pi|$, we have $\|T^q\|_{\mathfrak{B}(s^{1/q})} > \|\pi T\|_{\mathfrak{B}(s^{1/q})}$. Write $\varphi(T^k) = (\pi T + T^q)^k = T^{kq} + R$. Then we have $\|T^{kq}\|_{\mathfrak{B}(s^{1/q})} > \|R\|_{\mathfrak{B}(s^{1/q})}$. Furthermore, if we already know that $\varphi^m(T) = T^{q^m} + R'$ with $\|R'\|_{\mathfrak{B}(s^{1/q^m})} < \|T^{q^m}\|_{\mathfrak{B}(s^{1/q^m})}$, then $\varphi^{m+1}(T) = \varphi(T^{q^m} + R') = \varphi(T^{q^m}) + \varphi(R')$, and together with the previous computation it follows that we may write $\varphi^{m+1}(T) = T^{q^{m+1}} + R''$ with $\|R''\|_{\mathfrak{B}(s^{1/q^{m+1}})} < \|T^{q^{m+1}}\|_{\mathfrak{B}(s^{1/q^{m+1}})}$. Moreover, we also have $\varphi^{m+1}(T^k) = (T^{q^{m+1}} + R'')^k = T^{kq^{m+1}} + R'''$ with $\|R'''\|_{\mathfrak{B}(s^{1/q^{m+1}})} < \|T^{kq^{m+1}}\|_{\mathfrak{B}(s^{1/q^{m+1}})}$. This proves the claim (take $s = r^{1/q^i}$).

Hence

$$\begin{aligned}\|\sum_{0\le i\le n, 0\le j_i\le m} a_{i,j_i}T_i^{j_i}\|_{\mathfrak{B}^{\mathrm{perf}}(r)} &= \|\sum_{0\le i\le n, 0\le j_i\le m} a_{i,j_i}(T_n^{j_i q^{n-i}} + R_{i,j_i})\|_{\mathfrak{B}^{\mathrm{perf}}(r)}\\ &= \|\sum_{0\le i\le n, 0\le j_i\le m} a_{i,j_i}T_n^{j_i q^{n-i}}\|_{\mathfrak{B}^{\mathrm{perf}}(r)}\\ &= \max\{\|a_{i,j_i}T_n^{j_i q^{n-i}}\|_{\mathfrak{B}^{\mathrm{perf}}(r)}\}\\ &= \max\{\|a_{i,j_i}\varphi^{n-i}(T_n^{j_i})\|_{\mathfrak{B}^{\mathrm{perf}}(r)}\}\\ &= \max\{\|a_{i,j_i}T_i^{j_i}\|_{\mathfrak{B}^{\mathrm{perf}}(r)}\}.\end{aligned}$$

We have $\|T_i^{j_i}\|_{\mathfrak{B}^{\mathrm{perf}}(r)} = r^{j_i/q^i}$. We multiply each $T_i^{j_i}$ with a suitable $b_{i,j_i} \in K$ such that $1 \le \|b_{i,j_i}T_i^{j_i}\|_{\mathfrak{B}^{\mathrm{perf}}(r)} \le C$ where $C \in K$ is a fixed constant. Then the set consisting of the $b_{i,j_i}T_i^{j_i}$ is bounded. The above computation then shows that this set is α-cartesian for $\alpha = 1$. Then the rest of the proof goes through as in Proposition 3.7. □

In the situation of Remark 3.18, we cannot compute a Schauder basis for $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r))_\psi$ as in Lemma 3.26, but, using the isometric isomorphism between $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r))_\varphi$ and $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r))_\psi$ (Remark 3.18), we still find a Schauder basis consisting of elements in $\varinjlim_\psi o_L[|T_\psi|]$.

Remark 3.27. Let $r = |\pi|^{1/n}, n \ge 1$. We have an injection $o_K[|T|] \hookrightarrow \mathcal{O}_L(\mathfrak{B}(r^{1/q^i}))$ for each i which commutes with φ, and hence a map

$$R_K = \varinjlim_{\varphi} o_K[|T|] \hookrightarrow \check{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r)) = \varinjlim_{\varphi,i} \mathcal{O}_K(\mathfrak{B}(r^{1/q^i})).$$

Remember that on $\check{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r))$ we have the norm $\|\cdot\|_{\mathfrak{B}^{\mathrm{perf}}(r)}$ which is the inductive limit norm coming from the supremum norms on the $\mathcal{O}_K(\mathfrak{B}(r^{1/q^i}))$. The restriction of the norm $\|\cdot\|_{\mathfrak{B}^{\mathrm{perf}}(r)}$ to R_K defines a topology on R_K which coincides with the (π, T)-adic one. This can be seen as follows:
The sets

$$\pi^m(\check{\mathcal{O}}_K^+(\mathfrak{B}^{\mathrm{perf}}(r))) = \{f \in \check{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r)) \mid \|f\|_{\mathfrak{B}^{\mathrm{perf}}(r)} \le |\pi|^m\}$$

form a neighbourhood basis of 0. We obviously have

$$(\pi, T)^k \subseteq \pi^m(\check{\mathcal{O}}_K^+(\mathfrak{B}^{\mathrm{perf}}(r))) \cap R_K$$

for $k \ge mn$. Now use the notation T_i and $T = T_0$ an in Proposition 3.26. Let $i \in \mathbb{N}$ be fixed and

$$f = \sum_{k\ge 0} a_k T_i^k \in o_K[|T_i|] \subseteq \mathcal{O}_K(\mathfrak{B}(r^{1/q^i})) \subseteq \check{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r))$$

be an element such that $\|f\|_{\mathfrak{B}^{\mathrm{perf}}(r)} \leq |\pi^2|$. Note that T_i^m lies in (π, T) for large m which can be seen by writing $T_i^q = T_{i-1} + \pi T_i{}^2$ which lies in the ideal (π, T_{i-1}) (so that we arrive in (π, T) for $m \geq q^i$). Therefore we only have to consider finitely many terms $a_k T^k$. We have $|a_k|(r^{1/q^i})^k \leq |\pi^2|$ for each k. This means that either $a_k \in \pi o_K$, or $k \geq q^i$ which both implies that $a_k T_i^k \in (\pi, T)$. Thus

$$\pi^2(\breve{\mathcal{O}}_K^+(\mathfrak{B}^{\mathrm{perf}}(r))) \cap R_K \subseteq (\pi, T).$$

Completion then gives an isometric embedding

$$\widehat{R}_K \hookrightarrow \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r)).$$

In particular, $\widehat{R}_K$ is an integral domain.

3.2 The Robba ring $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})$

3.2.1 The Robba ring defined geometrically for $\mathfrak{B}_K^{\mathrm{perf}}$

We cannot define the Robba ring over $\mathfrak{B}_K^{\mathrm{perf}}$ in exactly the same way as over $\mathfrak{B}$ since we are working over adic spaces instead of rigid-analytic spaces. The subsets $\mathfrak{B}_K^{\mathrm{perf}} \setminus \mathfrak{B}_K^{\mathrm{perf}}(r)$ cannot be open for any $0 < r < 1$ since $\mathfrak{B}_K^{\mathrm{perf}}(r)$ is open and $\mathfrak{B}_K^{\mathrm{perf}}$ is connected.

Lemma 3.28. *Let $r_1, r_2 \in p^{\mathbb{Q}} \cap (p^{-q/e(q-1)}, 1)$ and $r_1 \leq r_2$. The preimage of $\mathfrak{B}(r_1, r_2)(\mathbb{C}_p) = \{z \in \mathbb{C}_p \,|\, |z| \in [r_1, r_2]\}$ in $\mathfrak{B}(\mathbb{C}_p)$ under φ is $\mathfrak{B}(r_1^{1/q}, r_2^{1/q})(\mathbb{C}_p) = \{z \in \mathbb{C}_p \;\, |z| \in [r_1^{1/q}, r_2^{1/q}]\}$.*

Proof. We may assume $\varphi(z) = z^q + \pi z$. Let z be an element with $|z| = t^{1/q} \in [r_1^{1/q}, r_2^{1/q}]$. Since by assumption $t > p^{-q/e(q-1)}$, we have $|z|^{q-1} = t^{(q-1)/q} > p^{-1/e} = |\pi|$, and therefore

$$\begin{aligned} |\varphi(z)| &= |\pi z + z^q| \\ &= |\pi z + z \cdot z^{q-1}| \\ &= |z^q| = t. \end{aligned}$$

On the other hand, let $z \in \mathfrak{B}(\mathbb{C}_p)$ be an element with $|\varphi(z)| = t \in [r_1, r_2]$. If $|\pi z| = |z^q|$, then we have $|z^{q-1}| = |\pi| = p^{-1/e}$ and $|z| = p^{-1/e(q-1)}$, and hence

$$\begin{aligned} |\varphi(z)| &= |\pi z + z^q| \\ &= \max\{|\pi z|, |z^q|\} \\ &= |z^q| \end{aligned}$$

[2] Or $T_i^q = T_{i-1} + \pi T_i + \pi \cdot$ higher terms if the chosen group law is not the special one.

since otherwise $t = |\varphi(z)| < |z|^q = p^{-q/e(q-1)}$ which contradicts the assumption on the radii. If $|\pi z| \neq |z^q|$, then $|z^q| > |\pi z|$ since otherwise $|z^{q-1}| < |\pi|$ and $|z| < |\pi|^{1/(q-1)} = p^{-1/e(q-1)}$, and therefore

$$t = |\pi||z| < p^{-1/e} \cdot p^{-1/e(q-1)} = p^{-q/e(q-1)}.$$

We see that in any case $|\varphi(z)| = |z^q| = t$ and therefore $|z| = t^{1/q} \in [r_1^{1/q}, r_2^{1/q}]$. $\square$

Note that if $r_1 = |\varpi|^{1/m}$ and $r_2 = |\varpi|^{1/n}$ for a topologically nilpotent unit $\varpi \in K$, then we have

$$\mathcal{O}_K(\mathfrak{B}(r_1, r_2)) = K\langle T\rangle\langle \frac{T^n}{\varpi}, \frac{\varpi}{T^m}\rangle$$

and if $\varpi = \pi$, then

$$\begin{aligned}\mathcal{O}_K(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i})) &= K\langle T\rangle\langle \frac{T^{nq^i}}{\pi}, \frac{\pi}{T^{mq^i}}\rangle = K\langle T\rangle\langle \frac{\varphi^i(T)^n}{\pi}, \frac{\pi}{\varphi^i(T)^m}\rangle \\ &= \{\sum_{i\in\mathbb{Z}} a_i T^i \mid \sum_{i\in\mathbb{Z}} a_i z^i \text{ converges for } z \in [r_1^{1/q^i}, r_2^{1/q^i}]\}.\end{aligned}$$

For the rest of this chapter, we assume that all radii like r_1, r_2, s_1, s_2, r_0 lie in $p^{\mathbb{Q}} \cap (p^{-q/e(q-1)}, 1)$.

Let $r_1 \leq r_2$. Lemma 3.28 implies that for any $i \in \mathbb{N}$ we have maps of uniform Tate-Huber pairs

$$\begin{aligned}\varphi : (\mathcal{O}_K(\mathfrak{B}(r_1^{1/p^i}, r_2^{1/p^i})), \mathcal{O}_K(\mathfrak{B}(r_1^{1/p^i}, r_2^{1/p^i}))^{\leq 1}) \to \\ (\mathcal{O}_K(\mathfrak{B}(r_1^{1/p^{i+1}}, r_2^{1/p^{i+1}})), \mathcal{O}_K(\mathfrak{B}(r_1^{1/p^{i+1}}, r_2^{1/p^{i+1}}))^{\leq 1}).\end{aligned}$$

We get an inductive system of uniform Tate-Huber pairs and set

$$\breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)) := \varinjlim_{\varphi, i} \mathcal{O}_K(\mathfrak{B}(r_1^{1/p^i}, r_2^{1/p^i}))$$

and

$$\breve{\mathcal{O}}_K^+(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)) := \varinjlim_{\varphi, i} \mathcal{O}_K(\mathfrak{B}(r_1^{1/p^i}, r_2^{1/p^i}))^{\leq 1}$$

Then $(\breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)), \breve{\mathcal{O}}_K^+(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)))$ is the inductive limit in the category of uniform Tate-Huber pairs. Denote by $\mathcal{O}_K^+(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$ the π-adic completion of $\breve{\mathcal{O}}_K^+(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$. Then

$$\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)) := \mathcal{O}_K^+(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)) \otimes_{\breve{\mathcal{O}}_K^+(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))} \breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$$

is the completion of $\breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2))$, and $(\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)), \mathcal{O}_K^+(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2))$ is a complete uniform Tate-Huber pair. Set

$$\mathfrak{B}_K^{\mathrm{perf}}(r_1,r_2) := \mathrm{Spa}(\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)), \mathcal{O}_K^+(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2))).$$

Later we will see that $\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)$ is an adic space.

If r_1,r_2,s_1,s_2 such that $r_1 \le s_1 \le s_2 \le r_2$, then we have continuous restrictions

$$\mathrm{res}: \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i})) \to \mathcal{O}_K(\mathfrak{B}(s_1^{1/q^i}, s_2^{1/q^i}))$$

for any i. Since they commute with φ, we extend them first to restrictions

$$\mathrm{res}: \breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)) \to \breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(s_1,s_2)),$$

and then by continuity to

$$\mathrm{res}: \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)) \to \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(s_1,s_2)).$$

Since the restriction $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)) \to \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(t_1,t_2))$ coincides with the composition of the restrictions $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)) \to \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(s_1,s_2)) \to \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(t_1,t_2))$ if $r_1 \le s_1 \le t_1 \le t_2 \le s_2 \le r_2$ we just write res without reference to the exact radii.

Next, we introduce a norm on the rings $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2))$.

Lemma 3.29. *For every i the map*

$$\varphi: \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i})) \to \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^{i+1}}, r_2^{1/q^{i+1}}))$$

is an isometry for the supremum norms on $\mathcal{O}_K(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i}))$ respective $\mathcal{O}_K(\mathfrak{B}(r_1^{1/q^{i+1}}, r_2^{1/q^{i+1}}))$.

Proof. We may assume $\varphi(z) = z^q + \pi z$, $z \in \mathfrak{B}(\mathbb{C}_p)$. The map $\varphi: \mathfrak{B}(\mathbb{C}_p) \to \mathfrak{B}(\mathbb{C}_p)$ is surjective. Let $f \in \mathcal{O}_K(\mathfrak{B}(r^{1/q^i}, s^{1/q^i}))$. We use Lemma 3.28 to compute

$$\begin{aligned}\|f\|_{\mathfrak{B}(r^{1/q^i},s^{1/q^i})} &= \sup_{x\in\mathfrak{B}(r^{1/q^i},s^{1/q^i})(\mathbb{C}_p)} |f(x)| \\ &= \sup_{x\in\varphi^{-1}(\mathfrak{B}(r^{1/q^i},s^{1/q^i})(\mathbb{C}_p))} |f(\varphi(x))| \\ &= \|\varphi(f)\|_{\mathfrak{B}(r^{1/q^{i+1}},s^{1/q^{i+1}})}.\end{aligned}$$

□

Definition 3.30. *We denote by $\|\cdot\|_{\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)}$ the continuous extension to $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2))$ of the inductive limit seminorm on $\breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)) = \varinjlim_{\varphi,i} \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i}))$ coming from the supremum norms on the $\mathcal{O}_K(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i}))$.*

It is actually a norm because the transition maps φ in the inductive limit are isometries thanks to the previous lemma. It induces the π-adic topology for which the $\pi^n\mathcal{O}_K^+(\mathfrak{B}^{\text{perf}}(r_1, r_2))$ form a neighbourhood basis of 0. Note that we can compute the norm of an element $f \in \breve{\mathcal{O}}_K(\mathfrak{B}^{\text{perf}}(r_1, r_2))$ on a preimage in some $\mathcal{O}_K(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i}))$.

In the setting of Remark 3.18, we have isometric isomorphisms

$$\mathcal{O}_K(\mathfrak{B}^{\text{perf}}(r_1, r_2))_\varphi \to \mathcal{O}_K(\mathfrak{B}^{\text{perf}}(r_1, r_2))_\psi.$$

Remark 3.31. Let r_1, r_2, t such that $r_1 \leq t \leq r_2$. We have the restriction map

$$\text{res} : \mathcal{O}_K(\mathfrak{B}(r_1, r_2)) \to \mathcal{O}_K(\mathfrak{B}(t, t)).$$

The ring $\mathcal{O}_K(\mathfrak{B}(t,t))$ has the supremum norm $\|\cdot\|_{\mathfrak{B}(t,t)}$ which can be computed as $\|f\|_{\mathfrak{B}(t,t)} = \sup_j(|a_j|t^j)$ if $f = \sum_{j\in\mathbb{Z}} a_j T^j \in \mathcal{O}_K(\mathfrak{B}(t,t))$ and which is multiplicative. Similarly, we define a multiplicative seminorm $\|\cdot\|_t$ on $\mathcal{O}_K(\mathfrak{B}^{\text{perf}}(r_1, r_2))$ by setting

$$\|\cdot\|_t := \|\cdot\|_{\mathfrak{B}^{\text{perf}}(t,t)} \circ \text{res}$$

where $\text{res} : \mathcal{O}_K(\mathfrak{B}^{\text{perf}}(r_1, r_2)) \to \mathcal{O}_K(\mathfrak{B}^{\text{perf}}(t,t))$ is the restriction. Let $f \in \breve{\mathcal{O}}_K(\mathfrak{B}^{\text{perf}}(r_1, r_2))$ with preimage $f_{i_0} = \sum_{j\in\mathbb{Z}} a_j T^j$ in $\mathcal{O}_K(\mathfrak{B}(r_1^{1/q^{i_0}}, r_2^{1/q^{i_0}}))$ under the canonical map $\mathcal{O}_K(\mathfrak{B}(r_1^{1/q^{i_0}}, r_2^{1/q^{i_0}})) \to \breve{\mathcal{O}}_K(\mathfrak{B}^{\text{perf}}(r_1, r_2))$ for some i_0. We have

$$\|f\|_t = \sup_{j\in\mathbb{Z}}(|a_j|t^{j/q^{i_0}}).$$

For a general $f \in \mathcal{O}_K(\mathfrak{B}^{\text{perf}}(r_1, r_2))$ which is the limit of a sequence $(f_k)_{k\in\mathbb{N}}$ in $\breve{\mathcal{O}}_K(\mathfrak{B}^{\text{perf}}(r_1, r_2))$ the sequence $(\|f_k\|_t)_{k\in\mathbb{N}}$ converges, and we have $\|f\|_t = \lim_{k\to\infty}\|f_k\|_t$. Note that we can make a similar definition for the rings $\mathcal{O}_K(\mathfrak{B}^{\text{perf}}(r))$. For an element $f \in \mathcal{O}_K(\mathfrak{B}^{\text{perf}}(t))$ we have $\|f\|_{\mathfrak{B}^{\text{perf}}(t)} = \|f\|_t$

Now we show that the restriction is injective, so $\|\cdot\|_t$ actually is a norm. In the following, write $R_r := p^{-qr/e(q-1)}$ for $r \in \mathbb{Q}_{\geq 0}$. Note that $R_r \to 1$ from below if $r \to 0$ from above. Also note that every $s \in (0,1)$ can be written as R_r (with $r = -\log_p(s) \cdot e(q-1)/q$). The following two lemmas are a variant of [27, Lemma 4.2.3] and [27, Lemma 5.2.5]. Note that if $0 < r < 1$, then R_r lies in $p^{\mathbb{Q}} \cap [p^{-q/e(q-1)}, 1)$.

Lemma 3.32. *Let $f \in \mathcal{O}_K(\mathfrak{B}^{\text{perf}}(R_{r_2}, R_{r_1}))$ for $0 < r_1 \leq r_2 < 1$. The function*

$$\begin{aligned}[r_1, r_2] &\mapsto \mathbb{R} \cup \{-\infty\},\\ t &\mapsto \log(\|f\|_{R_t})\end{aligned}$$

for $t \in [r_1, r_2]$ is continuous and convex.

Proof. If $f \in \breve{\mathcal{O}}_K(\mathfrak{B}^{\text{perf}}(R_{r_2}, R_{r_1}))$ with preimage of the form

$$f_{i_0} = a_j T^j \in \mathcal{O}_K(\mathfrak{B}(R_{r_2}^{1/q^{i_0}}, R_{r_1}^{1/q^{i_0}}))$$

for a $j \in \mathbb{Z}$, then the function

$$t \mapsto \log(|a_j| R_t^{j/q^{i_0}}) = \log(|a_j|) + t \cdot \log(p^{-(q/e(q-1)) \cdot j/q^{i_0}})$$

is affine and hence convex. If f has a preimage of the form $f_{i_0} = \sum_{j=-m}^{n} a_j T^j \in \mathcal{O}_K(\mathfrak{B}(R_{r_2}^{1/q^{i_0}} R_{r_1}^{1/q^{i_0}}))$, $n, m \in \mathbb{N}$, we have

$$\|f\|_{R_t} = \max_{-m \leq j \leq n} (\|a_j T^j\|_{R_t^{1/q^{i_0}}}),$$

i.e. the function $t \mapsto \log(\|f\|_{R_t})$ is the maximum of finitely many affine functions, and hence it is convex. Such finite sums are dense in $\mathcal{O}_K(\mathfrak{B}^{\text{perf}}(R_{r_2}, R_{r_1}))$, so for a general $f \in \mathcal{O}_K(\mathfrak{B}^{\text{perf}}(R_{r_2}, R_{r_1}))$ which is the limit of a sequence of such finite sums $(f_k)_{k \in \mathbb{N}}$ in $\breve{\mathcal{O}}_K(\mathfrak{B}^{\text{perf}}(R_{r_2}, R_{r_1}))$, the function $t \mapsto \log(\|f\|_{R_t})$ is the pointwise limit of a sequence $(t \mapsto \log(\|f_k\|_{R_t})_{k \in \mathbb{N}}$ of convex functions and therefore convex.

To show continuity, we see that by a similar argument that the function

$$[r_1, r_2] \to \mathbb{R}_{\geq 0}, t \mapsto \|f\|_{R_t}$$

is continuous if f has a preimage of the form $f_{i_0} = \sum_{j=-m}^{n} a_j T^j \in \mathcal{O}_K(\mathfrak{B}(R_{r_2}^{1/q^{i_0}} R_{r_1}^{1/q^{i_0}}))$ for $n, m \in \mathbb{N}$. For a general $f = \varinjlim_k f_k$ which is the limit of a sequence of such finite sums in $\breve{\mathcal{O}}_K(\mathfrak{B}^{\text{perf}}(R_{r_2}, R_{r_1}))$, note that the sequence $(t \mapsto \|f_k\|_{R_t})_{k \in \mathbb{N}}$ converges uniformly to the function $[r_1, r_2] \to \mathbb{R}_{\geq 0}, t \mapsto \|f\|_{R_t}$, because for every $\varepsilon > 0$ we find an k_0 such that $\max(\|f - f_k\|_{R_{r_2}}, \|f - f_k\|_{R_{r_1}}) < \varepsilon$ for all $k \geq k_0$, so that $\|f - f_k\|_{R_t} < \varepsilon$ and hence $|\|f\|_{R_t} - \|f_k\|_{R_t}| < \varepsilon$ for all $k \geq k_0$ and $t \in [r_1, r_2]$. Then the function $[r_1, r_2] \to \mathbb{R} \cup \{-\infty\}, t \mapsto \log(\|f\|_{R_t})$ is the composition of continuous functions and therefore continuous. □

Lemma 3.33. *For $0 < r_1 \leq s_1 \leq s_2 \leq r_2 < 1$, the restriction map*

$$res : \mathcal{O}_K(\mathfrak{B}^{\text{perf}}(R_{r_2}, R_{r_1})) \to \mathcal{O}_K(\mathfrak{B}^{\text{perf}}(R_{s_2}, R_{s_1}))$$

is injective.

Proof. If $\|f\|_{R_{t_0}} = 0$ for some $t_0 \in [r_1, r_2]$, then we have $\lim_{t \to t_0} \log(\|f\|_{R_t}) = -\infty$. We conclude with the previous lemma that $\|f\|_{R_t} = 0$ for all $t \in [r_1, r_2]$ and hence $f = 0$. □

Lemma 3.34. *We have*

$$\|f\|_{\mathfrak{B}^{\text{perf}}(r_1, r_2)} = \max\{\|f\|_{r_1}, \|f\|_{r_2}\} = \sup_{t \in [r_1, r_2]} \{\|f\|_t\}.$$

In particular, $\| \cdot \|_{\mathfrak{B}^{\text{perf}}(r_1, r_2)}$ is power-multiplicative.

Proof. For $f \in \breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$ the statement is true by the maximum modulus principle. For general f which is the limit of of a sequence $(f_k)_{k\in\mathbb{N}}$ in $\breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$, we have

$$\begin{aligned}\|f\|_{\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)} &= \lim_{k\to\infty} \|f_k\|_{\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)} \\ &= \lim_{k\to\infty} \max\{\|f_k\|_{r_1}, \|f_k\|_{r_2}\} \\ &= \max\{\lim_{k\to\infty} \|f_k\|_{r_1}, \lim_{k\to\infty} \|f_k\|_{r_2}\} \\ &= \max\{\|f\|_{r_1}, \|f\|_{r_2}\}.\end{aligned}$$

In addition, we have $\|f\|_{\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)} \geq \|f\|_t$ for all $t \in [r_1, r_2]$ which proves the second equality. □

Remark 3.35. The $\mathbb{C}_p$-points of $\mathfrak{B}_K^{\mathrm{perf}}(r_1, r_2)$ are

$$\begin{aligned}\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)(\mathbb{C}_p) &= \mathrm{Hom}((\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)), \mathcal{O}_K^+(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)), (\mathbb{C}_p, o_{\mathbb{C}_p})) \\ &= \mathrm{Hom}(\varinjlim_{\varphi,i}(\mathcal{O}_K(\mathfrak{B}(r_1^{1/p^i}, r_2^{1/p^i})), \mathcal{O}_K^+(\mathfrak{B}(r_1^{1/p^i}, r_2^{1/p^i}))), (\mathbb{C}_p, o_{\mathbb{C}_p})) \\ &= \varprojlim_{\varphi,i} \mathrm{Hom}(\mathcal{O}_K(\mathfrak{B}(r_1^{1/p^i}, r_2^{1/p^i})), \mathcal{O}_K^+(\mathfrak{B}(r_1^{1/p^i}, r_2^{1/p^i})), (\mathbb{C}_p, o_{\mathbb{C}_p})) \\ &\cong \varprojlim_{\varphi,i} \mathfrak{B}(r_1^{1/p^i}, r_2^{1/p^i})(\mathbb{C}_p) \\ &\cong \{x \in \mathbb{C}_p^\flat \mid |x|_\flat \in [r_1, r_2]\}.\end{aligned}$$

The last bijection sends an element $(z_0, z_1, ...) \in \varprojlim_{\varphi,i} \mathfrak{B}(r_1^{1/p^i}, r_2^{1/p^i})(\mathbb{C}_p)$ to $(z_0 \mod (\pi), z_1 \mod (\pi), ...) \in o_{\mathbb{C}_p^\flat}$. Note that $|(z_0 \mod (\pi), z_1 \mod (\pi), ...)|_\flat \in [r_1, r_2]$. To see that it is a bijection, note that if $z \equiv z' \mod (\pi)$ for elements $z, z' \in o_{\mathbb{C}_p}$, then $\varphi^i(z) \equiv \varphi^i(z') \mod (\pi^{i+1})$ which follows from [34, Lemma 1.1.1] and the defining properties of φ. If $(z_0, z_1, ...)$ and $(z_0', z_1', ...)$ are elements in $\varprojlim_{\varphi,i} \mathfrak{B}(r_1^{1/p^i}, r_2^{1/p^i})(\mathbb{C}_p)$ such that $z_i \equiv z_i' \mod (\pi)$ for all i, then we have $z_i = \varphi^j(z_{i+j}) \equiv \varphi^j(z_{i+j}') = z_i' \mod (\pi^{j+1})$ for all j, hence $z_i = z_i'$. Therefore the map in question is injective. On the other hand, we define a right inverse similarly to the proof of [34, Lemma 1.4.5]. Let $(\alpha_0, \alpha_1, ...) \in o_{\mathbb{C}_p^\flat}$ and choose representatives $a_i \in o_{\mathbb{C}_p}$ for the α_i. Then the sequence $(\varphi^i(a_i))_{i\in\mathbb{N}}$ converges because we have $a_i \equiv a_{i+1}^q \equiv \varphi(a_{i+1}) \mod (\pi)$ and hence $\varphi^i(a_i) \equiv \varphi^{i+1}(a_{i+1}) \mod (\pi^{i+1})$ for all i. We have $\lim_{i\to\infty} \varphi^i(a_i) \equiv a_0 \mod (\pi)$. The limit is independent of the choices of the representatives a_i since we have $a_i \equiv a_i' \mod (\pi)$ for another choices of representatives a_i', and hence $\varphi^i(a_i) \equiv \varphi^i(a_i') \mod (\pi^{i+1})$. The right inverse is then given by $(\alpha_0, \alpha_1, ...) \mapsto (\lim_{i\to\infty} \varphi^i(a_i), \lim_{i\to\infty} \varphi^i(a_{i+1}), ...)$. Note that after [34, Lemma 1.4.5] we find representatives a_i for the α_i such that $a_{i+1}^q = a_i$ and $|a_0| = |\alpha|_\flat \in [r_1, r_2]$, so that $|a_{i+j}| \in [r_1^{q^{i+j}}, r_2^{q^{i+j}}]$ which implies $|\varphi^i(a_{i+j})| = |a_{i+j}^{q^i}|$ and $\lim_{i\to\infty} \varphi^i(a_{i+j}) \in \mathfrak{B}(r_1^{1/p^j}, r_2^{1/p^j})(\mathbb{C}_p)$.

Remark 3.36. Let K be perfectoid. Consider the group $\widehat{G}_0(K)$ of K-valued locally-analytic characters $\mathbb{Z}_p^d \to K^\times$ with the action of p, $p^* : \widehat{G}_0(K) \to \widehat{G}_0(K), \chi \mapsto \chi(p\cdot)$. Then we have a bijection between the projective limit $\varprojlim_{p^*} \widehat{G}_0(K)$ and the K-valued locally analytic characters $\mathbb{Q}_p^d \to K^\times$ where $(\chi_0, \chi_1, ...) \in \varprojlim_{p^*} \widehat{G}_0(K)$ is sent to the character $z = 1/p^i \cdot a \mapsto \chi_i(a)$, $a \in \mathbb{Z}_p^d$. The inverse map sends a character $\chi : \mathbb{Q}_p^d \to K^\times$ to the element $(\chi_0, \chi_1, ...) \in \varprojlim_{p^*} \widehat{G}_0(K)$ where $\chi_i : \mathbb{Z}_p^d \to K^\times, z \mapsto \chi(z/p^i)$. Moreover, since $\widehat{G}_0(K)$ is isomorphic to a d-dimensional open unit disk around $1 \in \mathbb{Q}_p$, the projective limit $\varprojlim_{p^*} \widehat{G}_0(K) \cong \varprojlim_{p^*} \mathfrak{B}_1^d(K)$ is equal to the open unit disk $\mathfrak{B}_1^d(K^\flat) = \{(z_1, ..., z_d) \in (K^\flat)^d \mid |1 - z_i|_\flat < 1\}$ by the same argument as in the previous remark. Note that if $z = (z_0, z_1, ...) \in \varprojlim_{z \mapsto z^q} K \cong K^\flat$, then $|z - 1|_\flat = |\lim_{i\to\infty}(z_i - 1)^{q^i}| < 1$ if and only if $|z_i - 1| < 1$ for all i.

Lemma 3.37. *We have an isometric isomorphism*

$$\varinjlim_{[p],i} \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^{ei}}, r_2^{1/q^{ei}})) \cong \check{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)) = \varinjlim_{\varphi,i} \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i})).$$

Proof. We have $p = u\pi^e$ for a unit $u \in o_L$. Write $A_i = \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i}))$, then we have commutative diagrams[3]

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{[\pi]} & A_i & \xrightarrow{[u]} & A_i & \xrightarrow{[\pi]} & A_{i+1} & \xrightarrow{[\pi]} & \cdots & \xrightarrow{[\pi]} & A_{i+e} & \xrightarrow{[u]} & A_{i+e} & \xrightarrow{[\pi]} & \cdots \\
 & & \downarrow{\scriptstyle [u]^{-\lfloor i/e \rfloor}} & & \downarrow{\scriptstyle [u]^{-\lfloor (i+1)/e \rfloor}} & & \downarrow{\scriptstyle [u]^{-\lfloor (i+1)/e \rfloor}} & & & & \downarrow{\scriptstyle [u]^{-\lfloor (i+e)/e \rfloor}} & & \downarrow{\scriptstyle [u]^{-\lfloor (i+e+1)/e \rfloor}} & & \\
\cdots & \xrightarrow{[\pi]} & A_i & \xrightarrow[id]{} & A_i & \xrightarrow{[\pi]} & A_{i+1} & \xrightarrow{[\pi]} & \cdots & \xrightarrow{[\pi]} & A_{i+e} & \xrightarrow{id} & A_{i+e} & \xrightarrow{[\pi]} & \cdots
\end{array}$$

where $i = ne$ is a multiple of e. The inductive limit of the upper row is equal to $\varinjlim_{[p],i} \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^{ei}}, r_2^{1/q^{ei}}))$ and the inductive limit of the lower row is equal to $\varinjlim_{\varphi,i} \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i}))$. All vertical maps are isomorphisms. This gives the desired isomorphism which is an isometry since all the maps $[u]^{-[i/e]}$ are isometric. $\square$

Proposition 3.38. *Let K'/K be complete. The Tate-Huber pair*

$$(\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))\widehat{\otimes}_K K', (\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))\widehat{\otimes}_K K')^+)$$

is uniform.

Proof. The arguments in the proof of Lemma 3.19 work for negative powers of T as well, so we can write $f = \sum_{k\in\mathbb{Z}} a_k T^k = \sum_{k\geq 0} a_k T^k + \sum_{k<0} a_k T^k$ and repeat the proof. $\square$

[3] Here, $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{Q}$.

Corollary 3.39. *We have an isomorphism of Tate-Huber pairs*

$$\begin{aligned}&(\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2))\widehat{\otimes}_K K', (\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2))\widehat{\otimes}_K K')^+)\\ &\cong (\mathcal{O}_{K'}(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)), \mathcal{O}^+_{K'}(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)).\end{aligned}$$

Proof. This follows in the same way as in the proof of Corollary 3.20. ☐

Corollary 3.40. *Let K' be a field over K which is perfectoid. The algebra $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2))\widehat{\otimes}_K K'$ is perfectoid.*

Proof. This follows in the same way as in the proof of Lemma 3.21, namely because $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2))\widehat{\otimes}_K K' \cong \mathcal{O}_{K'}(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2))$ is a rational localization of the perfectoid K'-algebra $\widehat{\varinjlim_{\varphi}} K'\langle T\rangle$ if $\varphi(T) = \pi T + T^q$, and by using an isomorphism $\mathcal{O}_{K'}(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2))_\varphi \cong \mathcal{O}_{K'}(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2))_\psi$ for a general Frobenius power series ψ. ☐

It follows that $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2))$ is stably uniform, and $\mathfrak{B}^{\mathrm{perf}}_K(r_1,r_2)$ is an adic space.

Corollary 3.41. *The adic space $\mathfrak{B}^{\mathrm{perf}}_L(r_1,r_2) = \mathrm{Spa}(\mathcal{O}_L(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)), \mathcal{O}^+_L(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2))$ is L-preperfectoid.*

Definition 3.42. *Fix a radius $r_0 \in p^{\mathbb{Q}} \cap [p^{-q/e(q-1)}, 1)$. Set*

$$\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{r_0} := \varprojlim_{r_0<r_1\leq r_2<1} \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2))$$

where the maps in the projective limit are the restrictions. The ring

$$\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}}) := \varinjlim_{0<r_0<1} \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{r_0}$$

is the Robba ring for $\mathfrak{B}^{\mathrm{perf}}$ (over K).

It follows from Lemma 3.33 that the transition maps in the inductive limit are injective. Both $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2))$ and $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{r_0}$ are topological K-algebras: The algebras $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2))$ are Banach algebras. The algebras $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{r_0}$ are projective limits of Banach algebras and hence Fréchet algebras. Then $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})$ is an locally inductive limit of Fréchet spaces, and we endow it with the locally convex inductive limit topology.

We have an isometric (for the supremum norm) embedding

$$\mathcal{O}_K(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i})) \hookrightarrow \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i})) = \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i}))\widehat{\otimes}_K \mathbb{C}_p$$

for any $i \in \mathbb{N}$. This extends to an isometric embedding

$$\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)) \hookrightarrow \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)) = \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1,r_2))\widehat{\otimes}_K \mathbb{C}_p$$

(see Corollary 3.39 for the equality).

Lemma 3.43. *Let* $\ldots \to V_n \to \ldots \to V_0$ *be a sequence of locally convex* K*-vector spaces and* $V := \varprojlim_n V_n$ *its projective limit. Assume that* V_n *is Hausdorff and that the transition maps* $V_{n+1} \to V_n$ *have dense image for every* n*. Then we have*

$$V \widehat{\otimes}_K \mathbb{C}_p = \varprojlim_{n \in \mathbb{N}} (V_n \widehat{\otimes}_K \mathbb{C}_p).$$

Proof. See [6, Lemma 2.4] and the comment directly after the proof. □

Lemma 3.44. *For any* $r \in p^{\mathbb{Q}} \cap [p^{-q/e(q-1)}, 1)$*, we have an isomorphism of topological rings*

$$\mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})^r \cong \mathcal{R}_L(\mathfrak{B}^{\mathrm{perf}})^r \widehat{\otimes}_L \mathbb{C}_p.$$

(Here, $\mathcal{R}_L(\mathfrak{B}^{\mathrm{perf}})^r \widehat{\otimes}_L \mathbb{C}_p$ *has the projective tensor product topology.)*

Proof. We show that the restrictions maps res : $\mathcal{O}_L(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)) \to \mathcal{O}_L(\mathfrak{B}^{\mathrm{perf}}(s_1, s_2))$ for $r_1 \leq s_1 \leq s_2 \leq r_2$ have dense image. Then we can use the previous lemma. As in the proof of [6, Proposition 2.1] one shows that the restrictions

$$\mathrm{res} : \mathcal{O}_L(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i})) \to \mathcal{O}_L(\mathfrak{B}(s_1^{1/q^i}, s_2^{1/q^i}))$$

have dense image for all $i \geq 0$. Then the restrictions

$$\mathrm{res} : \varinjlim_{\varphi, i} \mathcal{O}_L(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i})) \to \varinjlim_{\varphi, i} \mathcal{O}_L(\mathfrak{B}(s_1^{1/q^i}, s_2^{1/q^i}))$$

have dense image, and the same stays true after passing to the completions. □

Remark 3.45. In the setting of Remark 3.18, the isometric isomorphisms

$$\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))_\varphi \to \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))_\psi,$$

extend to isomorphisms

$$\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r_\varphi \to \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r_\psi$$

and

$$\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})_\varphi \to \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})_\psi.$$

3.2.2 The Robba ring as certain completions of Witt vectors

In this section, we explain another approach to define the Robba ring over $\mathfrak{B}^{\mathrm{perf}}$. The main sources are [4], based on [15] and [3], and [34] as well as [27]. See also [39, Kapitel 3].

Let $\mathcal{F}_n$ be the set of π^n-torsion points of $\mathfrak{B}(L^{alg})$ and $L_n := L(\mathcal{F}_n)$. Then L_n/L is a totally ramified extension of L of degree $(q-1)q^{n-1}$ ([34, Proposition 1.3.12]). We let L_∞ be the field

$$L_\infty := \bigcup_{n\geq 1} L_n$$

and let $\hat{L}_\infty$ be its completion.

Proposition 3.46. *$\hat{L}_\infty$ is a perfectoid field.*

Proof. This is Proposition 1.4.12 in [34]. □

Consider the o_L-module

$$T := \varprojlim_{[\pi],n} (\mathcal{F}_n) = \varprojlim_{n} (\mathcal{F}_1 \stackrel{[\pi]}{\rightarrow} \mathcal{F}_2 \stackrel{[\pi]}{\rightarrow} ...).$$

It is a free o_L-module of rank one. In [34, Chapter 1.4] a well-defined map

$$\begin{aligned} \iota : T &\to o_{\hat{L}^\flat_\infty}, \\ (y_n)_{n\geq 1} &\mapsto (0, y_1 \mod (\pi), ..., y_n \mod (\pi), ...) \end{aligned}$$

is constructed. Fix a generator t of T and write $\omega := \iota(t) \in o_{\hat{L}^\flat_\infty}$. We have $|\omega|_\flat = |\pi|^{q/(q-1)}$ ([34, Lemma 1.4.14]). There is a well defined k-algebra map

$$\begin{aligned} k[|T|] &\to o_{\hat{L}^\flat_\infty}, \\ f(T) &\mapsto f(\omega), \end{aligned}$$

which extends to an embedding of fields

$$k((T)) \to \hat{L}^\flat_\infty.$$

Denote the image of this map by $\mathbf{E}_L$, by $o_{\mathbf{E}_L}$ the ring of integers of $\mathbf{E}_L$, and by $\mathfrak{m}_{\mathbf{E}_L} = \omega o_{\mathbf{E}_L}$ the maximal ideal of $o_{\mathbf{E}_L}$. Set

$$\mathbb{M}_{\mathbf{E}_L} := \Phi_0^{-1}(\mathfrak{m}_{\mathbf{E}_L}) \subseteq W(o_{\mathbf{E}_L})_L,$$

where $\Phi_0 : W(\mathbf{E}_L)_L \to \mathbf{E}_L, \sum_{i\geq 0} \pi^i \tau(x_i) \mapsto x_0$. In [34, Chapter 2.1], a map

$$\begin{aligned} \{\cdot\} : \mathbb{M}_{\mathbf{E}_L} &\to \mathbb{M}_{\mathbf{E}_L} \\ \alpha &\mapsto \{\alpha\} := \lim_{i\to\infty} ([\pi] \circ Fr^{-1})^i(\alpha) \end{aligned}$$

is defined which has the following properties:

Lemma 3.47. *For any $\alpha \in \mathbb{M}_{\mathbf{E}_L}$ we have*

1. *$\{\alpha\}$ is the unique element in $\mathbb{M}_{\mathbf{E}_L}$ which satisfies $\{\alpha\} \equiv \alpha \mod V_1(o_{\mathbf{E}_L})_L$ and $[\pi](\alpha) = Fr(\{\alpha\})$.*
2. *$\{\alpha\} = \{\tau(\Phi_0(\alpha))\}$*
3. *$[b](\{\alpha\}) = \{[b](\alpha)\}$ for any $b \in o_L$.*

Proof. Lemma 2.1.11 in [34]. □

We then consider the composition

$$\tau_\varphi : \mathfrak{m}_{\mathbf{E}_L} \xrightarrow{\tau} \mathbb{M}_{\mathbf{E}_L} \xrightarrow{\{\cdot\}} \mathbb{M}_{\mathbf{E}_L}.$$

This leads to an element $\omega_\varphi := \tau_\varphi(\omega) \in W(\mathbb{C}_p^\flat)_L$ which fulfils

$$\tau(\omega) \equiv \omega_\varphi \mod (\pi)$$

and which is invertible in $W(\mathbb{C}_p^\flat)_L$.

Remark 3.48. The construction of $\{\cdot\}$ and the previous lemma work for any $\mathbf{E}_L^{1/q^i}$ as well. Therefore we can consider the elements $\omega_i := \{\tau(\omega^{1/q^i})\} \in \mathbb{M}_{\mathbf{E}_L^{1/q^i}}$ for $i \geq 0$ (then $\omega_0 = \omega_\varphi$). We have

$$Fr(\omega_i) = [\pi](\omega_i) = \omega_{i-1}.$$

On $W(\hat{L}_\infty^\flat)_L$ we have the G_L-action

$$G_L \times W(\hat{L}_\infty^\flat)_L \to W(\hat{L}_\infty^\flat)_L,$$
$$(\sigma, \sum_i \pi^i \tau(x_i)) \mapsto \sum_i \pi^i \tau(\sigma^\flat(x_i))$$

where $\sigma^\flat : \hat{L}_\infty^\flat \to \hat{L}_\infty^\flat, (a_0 \mod (\pi), a_1 \mod (\pi), ...) \mapsto (\sigma(a_0) \mod (\pi), \sigma(a_1) \mod (\pi), ...)$. This action is continuous for the weak topology (see Remark 2.1.14 in [34]). It induces an action of $\Gamma = \Gamma_L := \mathrm{Gal}(L_\infty/L)$. We have an isomorphism of topological groups $\chi_L : \Gamma \to o_L^\times$.

Lemma 3.49. *For any $\gamma \in \Gamma$ we have*

$$\gamma(\omega_i) = [\chi_L(\gamma)](\omega_i).$$

Proof. For $\omega_0 = \omega_\varphi$ this is [34, Lemma 2.1.15]. Then we have

$$\begin{aligned} Fr(\gamma(\omega_1)) &= \gamma(Fr(\omega_1)) \\ &= [\chi_L(\gamma)](Fr(\omega_1)) \\ &= [\chi_L(\gamma)]([\pi](\omega_1)) \\ &= [\pi]([\chi_L(\gamma)](\omega_1)) \\ &= [\pi](\{[\chi_L(\gamma)](\tau(\omega^{1/q}))\}) \\ &= Fr(\{[\chi_L(\gamma)](\tau(\omega^{1/q}))\}) \\ &= Fr([\chi_L(\gamma)](\omega_1)). \end{aligned}$$

Then we use that Fr is injective and see that $\gamma(\omega_1) = [\chi_L(\gamma)](\omega_1)$. Repeating this computation shows that $\gamma(\omega_i) = [\chi_L(\gamma)](\omega_i)$ for any $i \geq 0$. $\square$

If $L \subseteq K \subseteq \mathbb{C}_p$ is a perfectoid field and $r > 0$, then we define

$$W^r(K^\flat)_L := \{\sum_{i\geq 0} \pi^i \tau(x_i) \in W(K^\flat)_L \mid |\pi^i||x_i|_\flat^r \to 0\}.$$

Lemma 3.50. *$W^r(K^\flat)_L$ is a ring on which we have a complete multiplicative norm*

$$|x|_r := \sup_{i\in\mathbb{N}}\{|\pi^i||x_i|_\flat^r\} \quad \text{for } x = \sum_{i\geq 0} \pi^i \tau(x_i) \in W^r(K^\flat)_L$$

which extends multiplicatively to $W^r(K^\flat)_L[\frac{1}{\pi}]$.

Proof. As in [27, Proposition 5.1.2]. $\square$

We set

$$W^\dagger(K^\flat)_L := \bigcup_{r>0} W^r(K^\flat)_L.$$

Remark 3.51. The π-adic completion of $W^\dagger(K^\flat)_L$ is equal to $W(K^\flat)_L$ since they are equal modulo π^i for all $i \in \mathbb{N}$.

Definition 3.52. *We write $\tilde{\mathbf{A}} := W(\mathbb{C}_p^\flat)_L$, $\tilde{\mathbf{A}}_L = W(\hat{L}_\infty^\flat)_L$, and $\tilde{\mathbf{A}}^\dagger = W^\dagger(\mathbb{C}_p^\flat)_L$ as well as $\tilde{\mathbf{A}}_L^\dagger = W^\dagger(\hat{L}_\infty^\flat)_L$. Moreover, we define $\tilde{\mathbf{B}} = \tilde{\mathbf{A}}[\frac{1}{\pi}]$,$\tilde{\mathbf{B}}^\dagger = \tilde{\mathbf{A}}^\dagger[\frac{1}{\pi}]$,$\tilde{\mathbf{B}}_L = \tilde{\mathbf{A}}_L[\frac{1}{\pi}]$, and $\tilde{\mathbf{B}}_L^\dagger = \tilde{\mathbf{A}}_L^\dagger[\frac{1}{\pi}]$.*

We also have the rings $\tilde{\mathbf{A}}^{\dagger,r} := W_{\leq 1}^r(\mathbb{C}_p^\flat)_L$, and $\tilde{\mathbf{A}}_K^{\dagger,r} := W_{\leq 1}^r(K^\flat)_L$ of elements $x = \sum_{i\geq 0} \pi^i \tau(x_i)$ in $W^r(\mathbb{C}_p^\flat)_L$ resp. $W^r(K^\flat)_L$ such that $|\pi^i||x_i|_\flat^r \leq 1$ for all i. We write $\tilde{\mathbf{A}}_L^{\dagger,r} := W_{\leq 1}^r(\hat{L}_\infty^\flat)_L$.

Lemma 3.53. *The rings $\tilde{\mathbf{A}}^{\dagger,r}$ and $\tilde{\mathbf{A}}_K^{\dagger,r}$ are complete for the topology induced by $|\cdot|_r$.*

Proof. As in [4, Lemma 21.5], see also [39, Lemma 3.8]. $\square$

Definition 3.54. *For $0 < s \leq r$, we define $\tilde{\mathcal{R}}^{[s,r]}(K)$ as the completion of $W^r(K^\flat)_L[\frac{1}{\pi}]$ with respect to the norm $\max\{|\cdot|_r, |\cdot|_s\}$, and*

$$\tilde{\mathcal{R}}^r(K) := \varprojlim_{0<s\leq r} \tilde{\mathcal{R}}^{[s,r]}(K),$$

which we give the Fréchet topology. For $K = \hat{L}_\infty$ we write $\tilde{\mathcal{R}}_L^r = \tilde{\mathcal{R}}^r(\hat{L}_\infty^\flat)$.

Remark 3.55. On $\tilde{\mathcal{R}}^r(K)$ we have the multiplicative norm $|\cdot|_r, x = \sum_{i\geq 0} \pi^i\tau(x_i) \mapsto \max_{i\in\mathbb{N}}\{|\pi^i||x_i|_\flat^r\}$.

Definition 3.56. *We define $\tilde{\mathcal{R}}(K) := \varinjlim_{r>0} \tilde{\mathcal{R}}^r(K)$ with the locally convex inductive limit topology. Write $\tilde{\mathcal{R}} := \tilde{\mathcal{R}}(\mathbb{C}_p)$ and $\tilde{\mathcal{R}}_L := \tilde{\mathcal{R}}(\hat{L}_\infty^\flat)$.*

3.2.3 The weak topology on $\tilde{\mathbf{A}}_L$

The ring $\tilde{\mathbf{A}}_L$ carries the *weak topology* which has a basis of neighbourhoods consisting of the sets

$$\pi^n W(\hat{L}_\infty^\flat)_L + \omega_\varphi^n W(o_{\hat{L}_\infty^\flat}).$$

The weak topology is Hausdorff and complete (see [34, Remark 1.5.2]. We define the weak topology on $\tilde{\mathbf{A}}_L^\dagger$ and $\tilde{\mathbf{A}}_L^{\dagger,r}$ as the subspace topology inherited from the weak topology on $\tilde{\mathbf{A}}_L$.

3.2.4 Connection between the two constructions

Lemma 3.57. *We have an isomorphism of o_L-algebras*

$$\widehat{R}_L \cong W(o_{\hat{L}_\infty^\flat})_L.$$

Proof. Set

$$\omega_i := \{\tau(\omega^{1/q^i})\} \in \mathbb{M}_{\mathbb{E}_L^{1/q^i}} \quad \text{for } i \geq 0.$$

We have

$$\mathrm{Fr}(\omega_{i+1}) = [\pi](\omega_{i+1}) = \omega_i,$$

where Fr is the Witt vector Frobenius. We get a compatible system of embeddings

$$\begin{array}{ccc}
o_L[|T|] & \xrightarrow{T \mapsto \omega_{i+1}} & W(o_{\mathbf{E}_L^{1/q^{i+1}}})_L \\
{\scriptstyle T\mapsto[\pi](T)} \uparrow & & \uparrow {\scriptstyle \subseteq} \\
o_L[|T|] & \xrightarrow[T \mapsto \omega_i]{} & W(o_{\mathbf{E}_L^{1/q^i}})_L.
\end{array}$$

Passing to the inductive limit we get an embedding

$$R_L \hookrightarrow W(o_{\mathbf{E}_L^{\mathrm{perf}}})_L \subseteq W(o_{\hat{L}_\infty^\flat})_L.$$

The weak topology on $W(o_{\hat{L}_\infty^\flat})_L$ is the (π, w_0)-adic one. Moreover, $W(o_{\hat{L}_\infty^\flat})$ is complete for the weak topology (see [34, Remark 1.5.2]). Therefore this embedding extends to a ring homomorphism

$$\widehat{R}_L \to W(o_{\hat{L}_\infty^\flat})_L,$$

which is the identity modulo π (see Proposition 1.4.17 in [34]).
Both sides are π-adically complete (see Lemma 3.14 for $\widehat{R}_L$), hence, using [34, Corollary 1.1.24], we get an isomorphism

$$\widehat{R}_L \cong W(o_{\hat{L}_\infty^\flat})_L.$$

□

The map in Lemma 3.57 is a homeomorphism with respect to the weak topology on $W(o_{\hat{L}^\flat_\infty})_L$ and the (π, T)-adic topology on $\widehat{R}_L$ since it identifies $(\pi^n, \omega_\varphi^n)$ and (π^n, T^n).

Lemma 3.58. *Let $r > 0$. If $x = \sum_{i\geq 0} \pi^i \tau(x_i) \in \tilde{\mathbf{A}}_K^{\dagger,r}$ is an element with $|x_0|_\flat = 1$ and $|\pi^i||x_i|_\flat^r < 1$ for $i \geq 1$, then x is a unit in $\tilde{\mathbf{A}}_K^{\dagger,r}$.*

Proof. By substituting x with $x/\tau(x_0)$ we may assume that $x_0 = 1$. Then we have $x = 1 + z$ with $|z|_r < 1$. We then have $x^{-1} = \sum_{n\geq 0}(-z)^n$ which converges in $\tilde{\mathbf{A}}_K^{\dagger,r}$ because $\tilde{\mathbf{A}}_K^{\dagger,r}$ is complete (Lemma 3.53). □

Proposition 3.59. *If $r \leq (q-1)/q$ then $\omega_\varphi/\tau(\omega)$ is a unit in $\tilde{\mathbf{A}}_K^{\dagger,r}$.*

Proof. The proof is from [39, Lemma 3.10]. We have

$$\omega_\varphi \equiv [\pi^i](Fr^{-i}(\tau(\omega))) \mod \pi^{i+1} W(o_{\hat{L}^\flat_\infty})_L.$$

If we write $\omega_\varphi = \sum_{i\geq 0} \pi^i \tau(\alpha_i)$ then the α_i are given by a power series in $\omega^{q^{-i}}$ without constant term and with coefficients in $o_{\hat{L}^\flat_\infty}$. We have $|\alpha_0|_\flat/|\omega|_\flat = 1$. For $i \geq 1$ we have $|\alpha_i|_\flat \leq |\omega^{q^{-i}}|_\flat = p^{-q^{-i+1}/e(q-1)}$ and

$$|\alpha_i|_\flat/|\omega|_\flat \leq p^{-q^{-i+1/e(q-1)}}/p^{-q/e(q-1)} = p^{-(q^{-i+1}-q)/e(q-1)} < p^{iq/e(q-1)}.$$

This implies $|\pi^i||\alpha_i|_\flat^r/|\omega|_\flat^r < 1$ for $i \geq 1$ and hence $\omega_\varphi/\tau(\omega) \in \tilde{\mathbf{A}}_K^{\dagger,r}$. Lemma 3.58 shows that it is a unit in $\tilde{\mathbf{A}}_K^{\dagger,r}$. □

Remark 3.60 (cf. Lemma 2.18 in [4])**.** Let $0 < r < 1$ be of the form $(q-1)/qn$ for an $n \geq 1$. Every element $y \in \tilde{\mathbf{A}}_L^{\dagger,r}$ can be written as $y = \sum_{k\geq 0} y_k(\pi/\omega_\varphi^{(q-1)/qr})^k$ where $y_k \in W(o_{\hat{L}^\flat_\infty})_L$ and $y_k \to 0$ if $k \to \infty$ in the weak topology.

Proof. Let $\sum_{i\geq 0} \pi^i \tau(x_i) \in \tilde{\mathbf{A}}_L^{\dagger,r}$. Then $|\pi^i||x_i|_\flat^r \leq 1$ and $|\pi^i||x_i|_\flat^r \to 0$ for $i \to \infty$. Then we write

$$\sum_{i\geq 0} \pi^i \tau(x_i) = \sum_{i\geq 0} \frac{\pi^i}{\tau(\omega^{(q-1)/qr})^i} \cdot \tau(\omega^{(q-1)/qr})^i \cdot \tau(x_i).$$

The $\tau(\omega^{(q-1)/qr})^i \cdot \tau(x_i)$ lie in $W(o_{\hat{L}^\flat_\infty})_L$ and go to 0 for $i \to \infty$. Therefore it is enough to show that $\pi/\tau(\omega^{(q-1)/qr})$ can be written in this form. For this, write $\tau(\omega^{(q-1)/qr}) = \omega_\varphi^{(q-1)/qr} - z$ with $z = \sum_{i\geq 1} \pi^i \tau(\beta_i)$ for $\beta_i \in o_{\hat{L}^\flat_\infty}$. Then

$$|\beta_i|_\flat \leq |\omega^{1/q^i}|_\flat < 1,$$

and in $W(\hat{L}^\flat_\infty)_L$ we have

$$\begin{aligned}\frac{\pi}{\tau(\omega^{(q-1)/qr})} &= \frac{\pi}{\omega_\varphi^{(q-1)/qr} - z}\\ &= \frac{\omega_\varphi^{(q-1)/qr}}{\omega_\varphi^{(q-1)/qr}} \cdot \frac{\pi}{\omega_\varphi^{(q-1)/qr} - z}\\ &= \frac{\pi}{\omega_\varphi^{(q-1)/qr}} \cdot \frac{1}{1 - z/\omega_\varphi^{(q-1)/qr}}\\ &= \frac{\pi}{\omega_\varphi^{(q-1)/qr}} \cdot \frac{1}{1 - \pi/\omega_\varphi^{(q-1)/qr} \cdot z'}\end{aligned}$$

where $z' = z/\pi \in (\pi, \tau(\omega^{1/q})) \subseteq W(o_{\hat{L}^\flat_\infty})_L$. Then

$$\frac{1}{1 - \pi/\omega_\varphi^{(q-1)/qr} \cdot z'} = \sum_{k \geq 0} (\pi/\omega_\varphi^{(q-1)/qr})^k \cdot (z')^k$$

and $(z')^k \to 0$ if $k \to \infty$ in the weak topology. Then set $y_k := (z')^{k-1}$ and write

$$\frac{\pi}{\tau(\omega^{(q-1)/qr})} = \sum_{k \geq 1} y_k \cdot (\pi/\omega_\varphi^{(q-1)/qr})^k.$$

□

Lemma 3.61. *Let $0 < s \leq r$ be elements of the form $(q-1)/qn$. Let*

$$\tilde{\mathcal{R}}_L^{[s,r],int} = \{x \in \tilde{\mathcal{R}}_L^{[s,r]} \mid \max\{|x|_r, |x|_s\} \leq 1\}$$

be the ring of integers of $\tilde{\mathcal{R}}_L^{[s,r]}$ for max$\{|\cdot|_r, |\cdot|_s\}$. Then $\tilde{\mathcal{R}}_L^{[s,r],int}$ is the π-adic completion of

$$W(o_{\hat{L}^\flat_\infty})_L[\frac{\tau(\omega^{(q-1)/qs})}{\pi}, \frac{\pi}{\tau(\omega^{(q-1)/qr})}].$$

Proof. We have $W(o_{\hat{L}^\flat_\infty})_L[\frac{\tau(\omega^{(q-1)/qs})}{\pi}, \frac{\pi}{\tau(\omega^{(q-1)/qr})}] \subseteq \tilde{\mathcal{R}}_L^{[s,r],int}$ since

$$\begin{aligned}\max\{|\frac{\pi}{\tau(\omega^{(q-1)/qr})}|_r, |\frac{\pi}{\tau(\omega^{(q-1)/qr})}|_s\} &= |\frac{\pi}{\tau(\omega^{(q-1)/qr})}|_r = 1 \quad \text{and}\\ \max\{|\frac{\tau(\omega^{(q-1)/qs})}{\pi}|_r, |\frac{\tau(\omega^{(q-1)/qs})}{\pi}|_s\} &= |\frac{\tau(\omega^{(q-1)/qs})}{\pi}|_s = 1.\end{aligned}$$

On the other hand, if $x = \sum_{i \gg -\infty}^k \pi^i \tau(x_i) \in W^r(\hat{L}^\flat_\infty)_L[1/\pi]$ is a finite sum such that $\max(|x|_s, |x|_r) \leq 1$, then we find elements $u_i \in o_{\hat{L}^\flat_\infty}$ such that $x_i = 1/\omega^{(q-1)i/qr} \cdot u_i$ if $i \geq 0$, and $x_i = \omega^{(q-1)i/qs} \cdot u_i$ if $i < 0$. Here we use $|\omega|_\flat = |\pi|^{q/q-1}$. Then we write

$$\pi^i \tau(x_i) = \begin{cases} \frac{\pi^i}{\tau(\omega^{(q-1)/qr})^i} \cdot \tau(u_i) & \text{for } i \geq 0\\ \frac{\tau(\omega^{(q-1)/qs})^i}{\pi^i} \cdot \tau(u_i) & \text{for } i < 0.\end{cases}$$

Therefore $\pi^i\tau(x_i)$ lies in $W(o_{\hat{L}^\flat_\infty})_L[\frac{\pi}{\tau(\omega^{(q-1)/qr})}, \frac{\tau(\omega^{(q-1)/qs})}{\pi}]$ for every i. These sums are dense in $\tilde{\mathcal{R}}_L^{[s,r],int}$. Note that the norm topology on $\tilde{\mathcal{R}}_L^{[s,r]}$ has a basis of fundamental neighbourhoods of 0 consisting of $\pi^n \cdot \tilde{\mathcal{R}}_L^{[s,r],int}$. So passing to the π-adic completion gives the result.

□

Note that we have $\tilde{\mathcal{R}}_L^{[s,r]} = \tilde{\mathcal{R}}_L^{[s,r],int}[1/\pi]$. Remember that $R_r = p^{-qr/e(q-1)}$ for $0 < r < 1$ and that $R_r \to 1$ from below if $r \to 0$ from above.

Remark 3.62. Write $r_1 = |\pi|^{1/m}$ and $r_2 = |\pi|^{1/n}$ for $m, n \geq 1$. Set

$$R_{L,m,n} = R_L[\frac{T^n}{\pi}, \frac{\pi}{T^m}] = R_L[X,Y]/(\pi X - T^n, T^m Y - \pi, XY - T^{n-m})$$

and consider the Huber pair

$$(R_{L,m,n}[1/\pi], R^+_{L,n,m})$$

where $R^+_{L,n,m}$ is the integral closure of $R_{L,m,n}$ in $R_{L,m,n}[1/\pi]$. Let $(\widehat{R}_{L,m,n}[1/\pi], \widehat{R}^+_{L,m,n})$ be the completion[4]. With similar arguments as in Lemma 3.17 we see that $(\widehat{R}_{L,m,n}[1/\pi], \widehat{R}^+_{L,m,n})$ is uniform. We have an isomorphism

$$(\widehat{R}_{L,m,n}[1/\pi], \widehat{R}^+_{L,n,m}) \cong (\mathcal{O}_L(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)), \mathcal{O}_L^+(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$$

of Tate-Huber pairs (with similar arguments as in the case of $\widehat{R}_{L,n}$).

Lemma 3.63. *Let $s = (q-1)/qn \leq r = (q-1)/qm$ where $m, n \geq 1$ are integers. Then we have a topological isomorphism $\tilde{\mathcal{R}}_L^{[s,r]} \cong \mathcal{O}_L(\mathfrak{B}^{\mathrm{perf}}(R_r, R_s))$ which comes from the isomorphism $W(o_{\hat{L}^\flat_\infty})_L \cong \widehat{R}_L$.*

Proof. By Lemma 3.61 we have

$$\tilde{\mathcal{R}}_L^{[s,r],int} = \pi\text{-adic completion of } W(o_{\hat{L}^\flat_\infty})_L[\frac{\tau(\omega^{(q-1)/qs})}{\pi}, \frac{\pi}{\tau(\omega^{(q-1)/qr})}].$$

It follows from Remark 3.60 that $\tilde{\mathbf{A}}_L^{\dagger,r}$ is contained in the π-adic completion of $W(o_{\hat{L}^\flat_\infty})_L[\frac{\omega_\varphi^{(q-1)/qs}}{\pi}, \frac{\pi}{\omega_\varphi^{(q-1)/qr}}]$.We also have $\tilde{\mathbf{A}}_L^{\dagger,r} \subseteq \tilde{\mathcal{R}}_L^{[s,r,int]}$. The element $\omega_\varphi/\tau(\omega)$ is a unit in $\tilde{\mathbf{A}}_L^{\dagger,r}$ (Remark 3.59) and hence is a unit in the aforementioned rings. This implies that $\tilde{\mathcal{R}}_L^{[r,s,int]}$ coincide with the π-adic completion of $W(o_{\hat{L}^\flat_\infty})_L[\frac{\omega_\varphi^{(q-1)/qs}}{\pi}, \frac{\pi}{\omega_\varphi^{(q-1)/qr}}]$. The ring $\mathcal{O}_L(\mathfrak{B}^{\mathrm{perf}}(R_r, R_s))$ is isomorphic to the completion of $\widehat{R}_L[\frac{T^n}{\pi}, \frac{\pi}{T^m}][1/\pi]$ (Remark 3.62). Since the isomorphism $W(o_{\hat{L}^\flat_\infty})_L \cong \widehat{R}_L$ sends T to ω_φ, we get the desired isomorphism by inverting π, and it is continuous and open. □

[4]This is just the rational localization of $(\widehat{R}_L, \widehat{R}_L)$ for $U = \{\pi^2, T^m\pi, T^{m+n}, \pi T^n\}$ and $s = \pi T^m$.

Corollary 3.64. *We have a topological isomorphism $\tilde{\mathcal{R}}_L \cong \tilde{\mathcal{R}}_L(\mathfrak{B}^{\mathrm{perf}})$.*

Note that this only applies over L. The ring $\tilde{\mathcal{R}}_K(\mathfrak{B}^{\mathrm{perf}})$ is not $\tilde{\mathcal{R}}(K)$ but the base change of $\tilde{\mathcal{R}}_L(\mathfrak{B}^{\mathrm{perf}})$ to K.

Proposition 3.65 (cf. Proposition 4.2 in [18])**.** *Let $x \in W(o_{\hat{L}^\flat_\infty})_L$ correspond to $f \in \widehat{R}_L$ under the isomorphism of Lemma 3.57. Then we have $|x|_r = \|f\|_{R_r}$.*

Proof. We adapt the proof of Proposition 4.2 in [18]. Let $f = \sum_{i\geq 0} a_i T^i \in \varinjlim_\varphi o_L[|T|]$ with $a_i \in o_L = W(\mathbb{F}_q)_L$. Write $a_i = \sum_{n\geq 0} \pi^n \tau(\alpha_{i,n})$ for certain $\alpha_{i,n} \in \mathbb{F}_q$. We have $\|f\|_{R_r} = \sup_i\{a_i R_r^{i/q^j}\}$ for some j depending on the equivalence class of f in the inductive limit.
On the other hand, write $x = \sum_{i\geq 0} \pi^i \tau(x_i) \in W(o_{\hat{L}^\flat_\infty})_L$. Reducing modulo π shows that $x_0 = \sum_i \alpha_{i,0}\omega_j^i$ and

$$|\tau(x_0)|_r = |x_0|_\flat^r = \sup_{i,\text{ with } \alpha_{i,0}\neq 0}\{R_r^{i/q^j}\}.$$

If we set $f_0 = \sum_i \tau(\alpha_{i,0})T^i$, then we have $|\tau(x_0)|_r = ||f_0||_r$. Let $x' := x - \tau(x_0)$ and $f'' := f - f_0$. We have

$$|x|_r = \sup\{|\tau(x_0)|_r, |x'|_r\}, \quad \|f\|_{R_r} = \sup\{\|f_0\|_{R_r}, \|f''\|_r\}.$$

The element corresponding to x' is $f' = f - \tau(x_0)$, but we have

$$\|f_0 - \tau(x_0)\|_{R_r} \leq \|f_0\|_{R_r},$$

therefore $\|f\|_{R_r} = \sup(\|f_0\|_{R_r}, \|f'\|_{R_r})$. Now we divide x' and f' by π and continue recursively to get $|x|_r = \|f\|_{R_r}$.
A general $f \neq 0$ which is the limit of a sequence $(f_i)_{i\in\mathbb{N}}, f_i \in \varinjlim_\varphi o_L[|T|]$, for the (π, T)-adic topology converges π-adically in $\mathcal{O}_L(\mathfrak{B}^{\mathrm{perf}}(s_1, s_2))$ for any $s_1 \leq R_r \leq s_2 < 1$, and we have $\|f\|_{R_r} = \varinjlim_i \|f_i\|_{R_r}$. The corresponding sequence $(x_i)_i \in W(o_{\hat{L}^\flat_\infty})_L$ converges to an element x which corresponds to f, and we have $|x|_r = \varinjlim_i |x_i|_r = \varinjlim_i \|f_i\|_{R_r} = \|f\|_{R_r}$. □

Lemma 3.66. *Let $0 < s \leq r$ be elements of the form $(q-1)/qn$. Let $x \in \tilde{\mathcal{R}}_L^{[s,r]}$ correspond to $f \in \mathcal{O}_L(\mathfrak{B}^{\mathrm{perf}}(R_r, R_s))$. Then we have max$(|x|_r, |x|_s) = \max\{\|f\|_{R_r}, \|f\|_{R_s}\}$.*

Proof. This follows from the previous lemma and the fact that the norms $|\cdot|_r$, $|\cdot|_s$, $\|\cdot\|_{R_r}$, and $\|\cdot\|_{R_s}$ are multiplicative. □

Lemma 3.67 (see Lemma 4.2.3 in [27])**.** *For $x \in \tilde{\mathcal{R}}^r(K)$, the function $t \mapsto \log(|x|_t)$ is continuous and convex on $(0, r]$.*

Proof. For $x = \pi^i \tau(x_i)$ for some $i \in \mathbb{Z}$ and $x_i \in \hat{L}_\infty^\flat$ we have

$$t \mapsto \log(\pi^i |x_i|_\flat^t) = \log(\pi^i) + t \cdot \log(|x_i|_\flat)$$

which is an affine function and therefore continuous and convex. If x is a finite sum of such terms, the function is the maximum of finitely many affine functions and therefore convex and continuous. Now such finite sums are dense in $\tilde{\mathcal{R}}^r(K)$ and the lemma follows. □

Remark 3.68. By copying the proof of the previous lemma, one can deduce that for $x \in \tilde{\mathcal{R}}^{[r,s]}(K)$, the function $t \mapsto \log(|x|_t)$ is continuous and convex on $[r, s]$.

Lemma 3.69 (see Lemma 5.2.5 in [27])**.** *For $0 < r_1 \leq s_1 \leq s_2 \leq r_2 < 1$, the restriction map $\tilde{\mathcal{R}}^{[r_1,r_2]}(K) \to \tilde{\mathcal{R}}^{[s_1,s_2]}(K)$ is injective.*

Proof. The previous Remark 3.68 implies that if $|x|_t = 0$ for some $t \in [r_1, r_2]$, then $|x|_t = 0$ for all $t \in [r_1, r_2]$, and therefore $x = 0$. □

3.3 The monoid action

We define an action of $o_L \setminus \{0\}$ on $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$. Remember hat we assume that all radii like r_1, r_2 lie in $p^{\mathbb{Q}} \cap (p^{-q/e(q-1)}, 1)$.

Let $r_1 \leq r_2$. The $o_L^\times$-action on $\mathcal{O}_K(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i}))$ for $i \in \mathbb{N}$ induces an $o_L^\times$-action on $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$ in the following way: Let $f \in \breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$ an element with preimage $f_{i_0} \in \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^{i_0}}, r_2^{1/q^{i_0}})$ under the canonical map

$$\alpha_{i_0} : \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^{i_0}}, r_2^{1/q^{i_0}})) \to \breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)).$$

Then we define $[u](f) := \alpha_{i_0}([u](f_{i_0})) \in \breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$ for $u \in o_L^\times$. This is well defined since the transition maps φ in the inductive limit commute with $[u]$. We pass to the completion $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$ to get an $o_L^\times$-action on $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$ which is isometric for the norm $\| \cdot \|_{\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)}$. It extends to the rings $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{r_0}$ and $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})$.

To get an action of the full multiplicative monoid $o_L \setminus \{0\}$, we need to define the action of π which we denote by φ. We define it by first defining the map

$$\varphi : \breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)) \to \breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1^{1/q}, r_2^{1/q}))$$

on the dense subsets $\breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)) \subseteq \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$ and $\breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1^{1/q}, r_2^{1/q})) \subseteq \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1^{1/q}, r_2^{1/q}))$ to be the map coming from the maps

$$\varphi : \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i})) \to \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^{i+1}}, r_2^{1/q^{i+1}})).$$

Then we extend φ continuously to the map

$$\varphi : \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)) \to \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1^{1/q}, r_2^{1/q})).$$

Remark 3.70. The map

$$\varphi : \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)) \to \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1^{1/q}, r_2^{1/q}))$$

is isometric.

Furthermore, we can pass to the projective limit since φ commutes with the restriction maps, and get a continuous map

$$\varphi : \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r \to \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{r^{1/q}},$$

and $\varphi : \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}}) \to \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})$.

Lemma 3.71. *The $o_L^\times$-action on the algebras $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$, $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{r_0}$, and $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})$ is continuous.*

Proof. The maps $[u]$ and φ are continuous on the algebras in question. Thanks to the Banach-Steinhaus theorem we only have to show that the orbit maps $o_L^\times \to \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$ respective $o_L^\times \to \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{r_0}$ respective $o_L^\times \to \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})$ are continuous. The $o_L^\times$-action on $\mathcal{O}_K(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i}))$ is continuous (Proposition 2.17 in [6]). Let $f \in \check{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$ with preimage $f_{i_0} \in \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^{i_0}}, r_2^{1/q^{i_0}}))$ under the canonical map $\alpha_{i_0} : \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^{i_0}}, r_2^{1/q^{i_0}})) \to \check{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$. Then the orbit map ρ_f corresponding to f is the composition of the continuous maps

$$o_L^\times \to \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^{i_0}}, r_2^{1/q^{i_0}})) \to \check{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)) \to \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)).$$

Now let $f \in \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$ be a general element. Let $U_\varepsilon(x) \subseteq \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$ be the open ball around $x \in \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$ with radius ε. We find an element $f_{i_0} \in \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^{i_0}}, r_2^{1/q^{i_0}}))$ such that $\|f - \alpha_{i_0}(f_{i_0})\|_{\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)} \leq \varepsilon/2$. Note that every $u \in o_L^\times$ acts isometrically on $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$. Then the preimage of $U_\varepsilon(x)$ under the orbit map ρ_f is equal to the preimage of $U_\varepsilon(x)$ under $\rho_{\alpha_{i_0}(f_{i_0})}$ since we have

$$\|x - [u](\alpha_{i_0}(f_{i_0}))\|_{\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)} \leq \varepsilon \Leftrightarrow \|x - [u](f)\|_{\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)} \leq \varepsilon$$

for all $u \in o_L^\times$. This implies that the orbit map $\rho_f : o_L^\times \to \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$ is continuous.
Now consider $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{r_0} = \varprojlim_{r_0 < r_1 \leq r_2 < 1} \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$. Let $f \in \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{r_0}$ and denote by f_{r_1, r_2} the image of f under the map $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{r_0} \to \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$ We have a commutative diagram

$$\begin{array}{ccc} o_L^\times & \xrightarrow{\rho_f} & \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{r_0} \\ & \searrow^{\rho_{f_{r_1, r_2}}} \quad \swarrow & \\ & \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)). & \end{array}$$

This shows that $\rho_f : o_L^\times \to \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{r_0}$ is continuous.
Let $f \in \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})$. There is an r such that $f \in \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r$. The map

$$\rho_f : o_L^\times \to \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})$$

then factors through the continuous map

$$\begin{aligned} o_L^\times &\to \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r, \\ u &\mapsto [u](f) \end{aligned}$$

and the inclusion $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r \to \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})$, and is thus continuous. □

Remark 3.72. We see with Lemma 3.49 that the so-defined $o_L^\times$-action on $\widehat{R}_L$ correspondents to the Γ-action in Witt vectors under the isomorphism in Lemma 3.57. In particular it is continuous for the (π, T)-adic topology.

The restricted map $\varphi : \widehat{R}_L \to \widehat{R}_L$ correspondents to $Fr : W(o_{\hat{L}_\infty^\flat})_L \to W(o_{\hat{L}_\infty^\flat})_L$.

3.4 Rings of bounded functions

Remember that we assume that all occurring radii lie in $p^{\mathbb{Q}} \cap (p^{-q/e(q-1)}, 1)$. Let r_0 be fixed and let $f \in \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{r_0}$. For $r_0 < r_1 \le s_2 \le s_2 \le r_2 < 1$, the norms $\|\cdot\|_{\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)}$ and $\|\cdot\|_{\mathfrak{B}^{\mathrm{perf}}(s_1,s_2)}$ for f are defined to be the resp. norms of the projection of f to the resp. rings, similarly we define $\|f\|_t$ to be $\|\mathrm{res}(f)\|_{\mathfrak{B}^{\mathrm{perf}}(t,t)}$ for $r_0 < t$. We have

$$\|f\|_{\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)} \ge \|f\|_{\mathfrak{B}^{\mathrm{perf}}(s_1,s_2)}.$$

We say that f is *bounded* if there is a constant C such that

$$\|f\|_{\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)} \le C \quad \text{for all } r_0 < r_1 \le r_2 < 1.$$

Definition 3.73. *We define*

$$\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{r_0,bd} := \{f \in \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{r_0} \mid f \text{ is bounded}\}.$$

and

$$\mathcal{E}_K^\dagger(\mathfrak{B}^{\mathrm{perf}}) := \varinjlim_{r\to 1} \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{r,bd} \subseteq \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}}).$$

Lemma 3.74. *An element $f \in \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{r_0} \subseteq \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})$ lies in $\mathcal{E}_K^\dagger(\mathfrak{B}^{\mathrm{perf}})$ if and only if there is an $r_0 \le s < 1$ such that $\{\|f\|_r \mid s \le r < 1\}$ is bounded.*

Proof. If $\{\|f\|_r \mid s \le r < 1\}$ is bounded for such an s, then there is a constant C such that $\|f\|_r \le C$ for all $s \le r < 1$. Therefore we have

$$\|f\|_{\mathfrak{B}^{\mathrm{perf}}(s,r)} = \max\{\|f\|_s, \|f\|_r\} \le C$$

for all $s \leq r < 1$. This means $f \in \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{s,bd}$. If, on the other hand, $f \in \mathcal{E}_K^\dagger(\mathfrak{B}^{\mathrm{perf}})$, then there is an r_0 such that $f \in \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{r_0,bd}$, and we have a constant C such that

$$\|f\|_{\mathfrak{B}^{\mathrm{perf}}(r_1,r_2)} = \max(\|f\|_{r_1}, \|f\|_{r_2}) \leq C$$

for $r_1, r_2 \in (r_0, 1)$. Then $\|f\|_r \leq C$ for $r_0 < s \leq r < 1$ for any $s > r_0$. □

Fix $r > r_0$. We have $\|f\|_{r,s_1} \leq \|f\|_{r,s_2}$ if $r \leq s_1 \leq s_2$. Therefore we can define a norm

$$\|f\|_{r,1} := \lim_{s\to 1} \|f\|_{\mathfrak{B}^{\mathrm{perf}}(r,s)}$$

for $f \in \mathcal{E}_K^\dagger(\mathfrak{B}^{\mathrm{perf}})$. Moreover, since $\|f\|_{s,1} \leq \|f\|_{r,1}$ for $r \leq s$, we can define a seminorm

$$\|f\|_1 := \lim_{r\to 1} \|f\|_{r,1}$$

for $f \in \mathcal{E}_K^\dagger(\mathfrak{B}^{\mathrm{perf}})$.

Lemma 3.75. *The function $\|\cdot\|_1$ is a multiplicative seminorm on $\mathcal{E}_K^\dagger(\mathfrak{B}^{\mathrm{perf}})$.*

Proof. We have

$$\begin{aligned}\|f\|_1 &= \lim_{r\to 1}(\lim_{s\to 1} \|f\|_{\mathfrak{B}^{\mathrm{perf}}(r,s)}) \\ &= \lim_{r\to 1}(\lim_{s\to 1}(\sup_{t\in[r,s]}\{\|f\|_t\})) \\ &= \lim_{r\to 1}(\lim_{s\to 1}(\max\{\|f\|_r, \|f\|_s\}) \\ &= \lim_{s\to 1} \|f\|_s\end{aligned}$$

where we use Lemma 3.34. The norms $\|f\|_s$ are multiplicative, hence its limit for $s \to 1$ is also multiplicative. □

Set

$$\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\mathrm{perf}}) := \{f \in \mathcal{E}_K^\dagger(\mathfrak{B}^{\mathrm{perf}}) \mid \|f\|_1 \leq 1\}.$$

We have $\mathcal{E}_K^\dagger(\mathfrak{B}^{\mathrm{perf}}) = \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\mathrm{perf}})[1/\pi]$. Note that the action of $o_L^\times$ and the map φ on $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})$ restrict to an action of $o_L^\times$ and a map φ on $\mathcal{E}_K^\dagger(\mathfrak{B}^{\mathrm{perf}})$ and on $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\mathrm{perf}})$. We finally define $\mathcal{E}_K(\mathfrak{B}^{\mathrm{perf}})$ to be the completion of $\mathcal{E}_K^\dagger$ with respect to $\|\cdot\|_1$.

We have an isometric injection $\mathcal{O}_K(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i})) \hookrightarrow \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$ for every $i \in \mathbb{N}$ which sends an element $f \in \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i}))$ to its equivalence class in $\breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)) \subseteq \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$. This induces an injection

$$\mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r^{1/q^i})) = \varprojlim_{r^{1/q^i} < r_1 \leq r_2 < 1} \mathcal{O}_K(\mathfrak{B}(r_1, r_2)) \hookrightarrow \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r = \varprojlim_{r < r_1 \leq r_2 < 1} \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)).$$

To simplify notation we often identify $\mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r^{1/q^i}))$ with its image in $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r$, and similarly for $\mathcal{O}_K(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i})) \hookrightarrow \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$.

Lemma 3.76. *Suppose that $\varphi(T) = \pi T + T^q$. Every $f \in \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r$ can be written uniquely as $f = f^+ + f^-$ where f^+ is the limit of a sequence $(f_i^+)_{i \in \mathbb{N}}, f_i^+ \in \mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r^{1/q^i})) \subseteq \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r$ such that each f_i^+, regarded as an element of $\mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r^{1/q^i}))$, can be written as*

$$f_i^+ = \sum_{n \geq 0} a_n T^n \in \mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r^{1/q^i})),$$

and f^- is the limit of a sequence $(f_i^-)_{i \in \mathbb{N}}, f_i^- \in \mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r^{1/q^i})) \subseteq \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r$ such that each f_i^-, regarded as an element of $\mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r^{1/q^i}))$, can be written as

$$f_i^- = \sum_{n < 0} a_n T^n \in \mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r^{1/q^i})).$$

Moreover, $\|f\|_s = \max\{\|f^+\|_s, \|f^-\|_s\}$ for $r < s$.

Proof. The subset

$$\varprojlim_{r < r_1 \leq r_2 < 1} \breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$$

is dense in $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r$. This is because for every $f \in \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r$ and for every $r < r_1 \leq r_2$, and $\varepsilon > 0$, we find an $i_0 \in \mathbb{N}$ and an element

$$f_{i_0} \in \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^{i_0}}, r_2^{1/q^{i_0}})) \subseteq \mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$$

such that $\|f_{i_0} - f\|_{\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)} \leq \varepsilon$. We can moreover assume that $f_{i_0} = \sum_{n \gg -\infty}^{k} a_n T^n$ is a finite sum since such finite sums are dense in $\mathcal{O}_K(\mathfrak{B}(r_1^{1/q^{i_0}}, r_2^{1/q^{i_0}}))$. Then

$$\begin{aligned} f_{i_0} \in \mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r^{1/q^{i_0}})) &= \varprojlim_{r < r_1 \leq r_2 < 1} \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^{i_0}}, r_2^{1/q^{i_0}})) \\ &\subseteq \varprojlim_{r < r_1 \leq r_2 < 1} \breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)). \end{aligned}$$

Note that $\|f_{i_0} - f\|_{\mathfrak{B}^{\mathrm{perf}}(s_1, s_2)} \leq \|f_{i_0} - f\|_{\mathfrak{B}^{\mathrm{perf}}(r_1, r_2)}$ if $r_1 \leq s_1 \leq s_2 \leq r_2$. Therefore we find a sequence $(f_i)_{i \in \mathbb{N}}$ in $\varprojlim_{r < r_1 \leq r_2 < 1} \breve{\mathcal{O}}_K(\mathfrak{B}^{\mathrm{perf}}(r_1, r_2))$ which converges to f in the Fréchet topology on $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r$. We may assume that

$$f_i \in \varprojlim_{r < r' \leq s < 1} \mathcal{O}_K(\mathfrak{B}(r_1^{1/q^i}, r_2^{1/q^i})) = \mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r^{1/q^i}))$$

for every i. Every f_i can be written as $f_i = f_i^+ + f_i^-$ as described in the statement of the lemma. If $f_j = f_j^+ + f_j^-$ and $f_k = f_k^+ + f_k^-$ for $k \geq j$, then $f_k^- - f_j^- \in \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r$ is the image of the element

$$f_k^- - \varphi^{k-j}(f_j^-) \in \mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r^{1/q^k}))$$

in $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r$. The element $f_k^- - \varphi^{k-j}(f_j^-) \in \mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r^{1/q^k}))$ is of the form $\sum_{n<0} a_n T^n$ since we have[5]

$$\varphi(T^{-1}) = \frac{1}{\pi T + T^q} = \sum_{k \geq 0} \frac{(-\pi)^k}{T^{(k+1)q-k}},$$

i.e. φ sends negative powers of T to a series with only negative powers of T. Similarly, $f_k^+ - f_j^+$ has a preimage of the form $\sum_{n \geq 0} a_n T^n \in \mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r^{1/q^k}))$. Let $r < s$. We compute $\|f_k - f_j\|_s = \|f_k - \varphi^{k-j}(f_j)\|_{\mathfrak{B}(s^{1/q^k}, s^{1/q^k})}$ in $\mathcal{O}_K(\mathfrak{B}(s^{1/q^k}, s^{1/q^k}))$ as follows

$$\begin{aligned}
\|f_k - f_j\|_s &= \|f_k^+ + f_k^- - f_j^+ - f_j^-\|_s \\
&= \|f_k^+ + \varphi^{k-j}(f_j^+) - f_k^- - \varphi^{k-j}(f_j^-)\|_{\mathfrak{B}(s^{1/q^k}, s^{1/q^k})} \\
&= \max\{\|f_k^+ + \varphi^{k-j}(f_j^+)\|_{\mathfrak{B}(s^{1/q^k}, s^{1/q^k})}, \|f_k^- + \varphi^{k-j}(f_j^-)\|_{\mathfrak{B}(s^{1/q^k}, s^{1/q^k})}\} \\
&= \max\{\|f_k^+ - f_j^+\|_s, \|f_k^- - f_j^-\|_s\}.
\end{aligned}$$

This shows that $(f_i^+)_{i \in \mathbb{N}}$ and $(f_i^-)_{i \in \mathbb{N}}$ are Cauchy sequences for the norms $\|\cdot\|_s, r < s < 1$ and hence (using Lemma 3.34) converge in the Fréchet topology of $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r$. We set

$$f^+ := \lim_{i \to \infty} f_i^+, \text{ and } f^- := \lim_{i \to \infty} f_i^-.$$

Then $f = f^+ + f^-$. Moreover, we have

$$\begin{aligned}
\|f\|_s &= \lim_{i \to \infty} \|f_i^+ + f_i^-\|_s \\
&= \lim_{i \to \infty} \max\{\|f_i^+\|_s, \|f_i^-\|_s\} \\
&= \max\{\lim_{i \to \infty} \|f_i^+\|_s, \|f_i^-\|_s\} \\
&= \max\{\|f^+\|_s, \|f^-\|_s\}.
\end{aligned}$$

[5] We have

$$a^{-((k+1)q-k)} \cdot |\pi|^k = (a^{q-1} \cdot |\pi|^{-1})^{-k} \cdot a^{-q}.$$

Thus the sum converges for $a > p^{-1/e(q-1)}$. Moreover,

$$(T^q + \pi T) \sum_{k \geq 0} \frac{(-\pi)^k}{T^{(k+1)q-k}} = 1 + \sum_{k>0} \frac{(-\pi)^k}{T^{kq-k}} + \sum_{k \geq 0} \frac{(-1)^k \pi^{k+1}}{T^{(k+1)q-k-1}} = 1.$$

If $f = g^+ + g^-$ is another decomposition where g^+ is the limit of a sequence $(g_i^+)_{i\in\mathbb{N}}$ and g^- is the limit of a sequence $(g_i^-)_{i\in\mathbb{N}}$ of the described form, then the sequence $(f_i^+ + f_i^- - (g_i^+ + g_i^-))_{i\in\mathbb{N}}$ converges to zero in the norms $\|\cdot\|_s$ for $r < s < 1$. But since we have

$$\|f_i^+ + f_i^- - (g_i^+ + g_i^-)\|_s = \max\{\|f_i^+ - g_i^+\|_s, \|f_i^- - g_i^-\|_s\},$$

the sequences $(f_i^+ - g_i^+)_{i\in\mathbb{N}}$ and $(f_i^- - g_i^-)_{i\in\mathbb{N}}$ converge to zero as well. Hence we have $f^+ = g^+$ and $f^- = g^-$. $\square$

Note that in the situation of Lemma 3.76, we have $\|f^-\|_{s_2} \leq \|f^-\|_{s_1}$ and $\|f^+\|_{s_1} \leq \|f^+\|_{s_2}$ for $r < s_1 \leq s_2 < 1$. Moreover, every element of the form $f = f^-$ lies in $\mathcal{E}_K^\dagger(\mathfrak{B}^{\mathrm{perf}})$ and the norm $\|f^-\|_1$ is defined. The sequence $(f_i^-)_i$ converges to f^- in the $\|\cdot\|_1$-norm. On the other hand, every f^+ can be regarded as an element in $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(r))$ for any $r \in p^{\mathbb{Q}} \cap (p^{-q/e(q-1)}, 1)$.

Remark 3.77. If we choose another Frobenius power series ψ, then we cannot expect part 1 of the lemma to still hold true. But we can use the isomorphism $[1]_{\psi,\varphi}$ to obtain two Cauchy sequences $(h_i)_{i\in\mathbb{N}}$ and $(g_i)_{i\in\mathbb{N}}$ with limits h and g such that $(h_i + g_i)_{i\in\mathbb{N}}$ converges to f and such that $\|g_i\|_r$ decreases if $r \to 1$ and $\|h_i\|_r$ increases if $r \to 1$ for every r, and $\|f\|_r = \max\{\|g\|_r, \|h\|_r\}$.

Corollary 3.78. *If $f \in \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\mathrm{perf}})^r := \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\mathrm{perf}}) \cap \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r$, then there is a sequence $(f_i)_{i\in\mathbb{N}}, f_i \in \mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r^{1/q^i}))$ such that each $f_i \in \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\mathrm{perf}}))$ and $(f_i)_{i\in\mathbb{N}}$ converges to f in $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r$.*

Proof. Suppose that $\varphi(T) = \pi T + T^q$. Then we can use Lemma 3.76 and obtain a decomposition $f = f^+ + f^-$ as described there.
First assume that $f = f^- = \lim_{i\to\infty} f_i^-$. We have $\|f^-\|_{s_2} \leq \|f^-\|_{s_1}$ if $s_2 \geq s_1$. We also note that $\|f^-\|_1 \leq \|f\|_s$ for all $r < s < 1$.
If $\|f^-\|_1 = 0$, then for every $\varepsilon > 0$ we find an s_0 such that $\|f^-\|_s = \lim_{i\in\mathbb{N}} \|f_i^-\|_s \leq \varepsilon$ if $s_0 \leq s < 1$. This implies $\|f_i^-\|_s \leq \varepsilon$ for large i. Hence $\|f_i^-\|_1 \leq 1$ for large i.
Assume now that $\|f^-\|_1 \neq 0$. Then $\|f^-\|_s > 0$ for $r < s < 1$. Fix an $r < s_1 < 1$. We find an i_0 such that

$$\|f_i^- - f^-\|_{s_1} < \|f^-\|_1 \leq \|f^-\|_{s_1},$$

hence $\|f_i^-\|_{s_1} = \|f^-\|_{s_1}$ for $i \geq i_0$. Then also $\|f_i^-\|_{s_2} = \|f^-\|_{s_2}$ for $s_2 \geq s_1$ and $i \geq i_0$ by the same argument. We see that $\|f^-\|_s = \|f_i^-\|_s$ for all $s_1 \leq s < 1$ and $i \geq i_0$ and hence $\|f^-\|_1 = \|f_i^-\|_1$ for $i \geq i_0$. Therefore $\|f^-\|_1 \leq 1$ implies $\|f_i^-\|_1 \leq 1$ for $i \geq i_0$.

If $f = f^+$, we note that f^+ can be regarded as an element in $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(s))$ for $r < s < 1$. The rings $\mathcal{O}_K(\mathfrak{B}^{\mathrm{perf}}(s))$ have compatible Schauder bases described in Lemma 3.26. We write $f^+ = \sum_{k,j_k} a_{k,j_k} T_k^{j_k}$ as in Lemma 3.26. Then set

$$f_i^+ := \sum_{k\leq i, j_k} a_{k,j_k} T_k^{j_k}.$$

We have

$$\|f^+\|_s = \max_{k,j_k}\{|a_{k,j_k}|s^{j_k/q^k}\}.$$

There is a constant C such that $\|f^+\|_s \leq C$ for $r < s < 1$ (Lemma 3.74), this implies that $|a_{k,j_k}| \leq C$ as well for all k, j_k. We compute

$$\begin{aligned}\|f^+\|_1 &= \lim_{s\to 1}\|f\|_s \\ &= \lim_{s\to 1}\sup_{k,j_k}\{|a_{k,j_k}|s^{j_k/q^k}\} \\ &= \sup_{k,j_k}\{|a_{k,j_k}|\} \leq 1.\end{aligned}$$

Therefore $\|f_i^+\|_1 = \sup_{k\leq i,j_k}\{|a_{k,j_k}|\} \leq 1$ and hence $f_i^+ \in \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\mathrm{perf}})$ for all $i \in \mathbb{N}$.

If now $f = f^+ + f^-$, then we have

$$\|f\|_s = \max\{\|f^+\|_s, \|f^-\|_s\}$$

for all $r < s < 1$ after Lemma 3.76. Hence

$$\|f\|_1 = \lim_{s\to 1}(\max\{\|f^+\|_s, \|f^-\|_s\}) \leq 1.$$

This and the fact that $\|f^+\|_s$ increase for $s \to 1$ imply $\|f^+\|_s \leq 1$ for all $r < s < 1$, hence $\|f^+\|_1 \leq 1$. Then also $\|f^-\|_1 \leq 1$ since otherwise $\|f^-\|_s > 1$ for all $r < s < 1$ because $\|f^-\|_s$ decreases for $s \to 1$, which contradicts $\|f\|_1 \leq 1$ because of the above equality.

If ψ is another Frobenius power series, then $[1]_{\psi,\varphi}$ gives an isomorphism

$$\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})_\varphi \cong \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})_\psi$$

which restricts to an isomorphism

$$\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\mathrm{perf}})_\varphi \cong \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\mathrm{perf}})_\psi.$$

Noting that $\mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r^{1/q^i}))_\varphi$ is isomorphic to $\mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(r^{1/q^i}))_\psi$ for each i via $[1]_{\varphi,\psi}$ as well, the lemma follows from the previous discussion. □

Lemma 3.79. *The seminorm* $\|\cdot\|_1$ *is a norm.*

Proof. Again, we only need to consider the case $\varphi(T) = T^q + \pi T$. Let $0 < r < 1$. If $f = f^+ \in \mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^{R_r,bd}$, then $\|f\|_{R_r}$ is increasing as $R_r \to 1$. Therefore $\|f\|_1 = 0$ implies $\|f\|_{R_r} = 0$ and hence $f = 0$. On the other hand, if $f = f^-$, then f is the limit of a sequence of elements $f_i = \sum_{n<0} a_n T^n \in \mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(R_r^{1/q^i}))$ which converge for $T = 1$. Fix $r' > r$. We get a function

$$[0, r'] \to \mathbb{R} \cup \infty,$$
$$t \mapsto \log(\|f^-\|_{R_t}).$$

As in the proof of Lemma 3.32 one shows that this is a continuous and convex function. Thus $\|f^-\|_1 = 0$ implies $\|f^-\|_s = 0$ for $R_{r'} < s < 1$ and hence $f^- = 0$. If $f = f^+ + f^-$, then $\|f\|_1 = \lim_{s\to 1}(\max\{\|f^+\|_s, \|f^-\|_s\}) = 0$ implies $\|f^+\|_s \to 0$ and $\|f^-\|_s \to 0$ for $s \to 1$, so $f^+ = 0$ and $f^- = 0$. □

Lemma 3.80. *We have $\mathcal{E}_L^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}}) \subseteq \tilde{\mathbf{A}}_L^\dagger$ under the isomorphism in Corollary 3.64.*

Proof. Let $0 < r < 1$. Let

$$f \in \mathcal{E}_L^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}})^{R_r} = \mathcal{E}_L^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}}) \cap \mathcal{R}_L(\mathfrak{B}^{\text{perf}})^{R_r}.$$

We find a sequence $(f_i)_{i\in\mathbb{N}}$ such that each f_i lies in the image of $\mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(R_r^{1/q^i}))$ in $\mathcal{R}_L(\mathfrak{B}^{\text{perf}})^{R_r}$, is bounded, and fulfils $\|f_i\|_1 \leq 1$, and which converges to f for the Fréchet topology on $\mathcal{R}_K(\mathfrak{B}^{\text{perf}})^{R_r}$ (Corollary 3.78). We write $f_i = \sum_{n\in\mathbb{Z}} a_n T^n \in \mathcal{O}_K(\mathfrak{B} \setminus \mathfrak{B}(R_r^{1/q^i}))^{bd}$. Then we have $\sup_n\{|a_n|\} \leq 1$. We see that the image of f_i under the isomorphism in Corollary 3.64 lies in $W^s(\hat{L}_\infty^\flat)_L$ for any $s < r$. Moreover, the image of the sequence $(f_i)_{i\in\mathbb{N}}$ converges in $W^s(\hat{L}_\infty^\flat)_L$ in the $|\cdot|_s$-norm (because the f_i converge for $\|\cdot\|_{R_s}$ and Lemma 3.66), and its limit is the image of f which then lies in $\tilde{\mathbf{A}}_L^\dagger$. □

The ring $\mathcal{E}_L^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}})$ therefore carries a weak topology which is defined as the subspace topology coming from $\tilde{\mathbf{A}}_L^\dagger \subseteq \tilde{\mathbf{A}}_L$.

Consider the ring $o_{\mathbb{C}_p} \otimes_{o_L} W(\hat{L}_\infty^\flat)_L$. We define a topology on $o_{\mathbb{C}_p} \otimes_{o_L} W(\hat{L}_\infty^\flat)_L$ by taking the sets

$$U_n := o_{\mathbb{C}_p} \otimes_{o_L} (\pi^n W(\hat{L}_\infty^\flat)_L + \omega_\varphi^n W(o_{\hat{L}_\infty^\flat})_L)$$

as a neighbourhood basis of 0. This coincides with the usual topology on the tensor product as in [1, Tag 0AMU] since we have

$$\pi^n o_{\mathbb{C}_p} \otimes_{o_L} W(\hat{L}_\infty^\flat)_L \subseteq U_n,$$

and hence

$$U_n = (\pi^n o_{\mathbb{C}_p} \otimes_{o_L} W(\hat{L}_\infty^\flat)_L + (o_{\mathbb{C}_p} \otimes_{o_L} (\pi^n W(\hat{L}_\infty^\flat)_L + \omega_\varphi^n W(o_{\hat{L}_\infty^\flat})_L)).$$

Then we form the completion

$$o_{\mathbb{C}_p} \widehat{\otimes}_{o_L} W(\hat{L}_\infty^\flat)_L = \varprojlim_n o_{\mathbb{C}_p}/\pi^n o_{\mathbb{C}_p} \otimes_{o_L} W(\hat{L}_\infty^\flat)_L/(\pi^n W(\hat{L}_\infty^\flat)_L + \omega_\varphi^n W(o_{\hat{L}_\infty^\flat})_L).$$

Since $o_{\mathbb{C}_p}$ is a flat o_L-module, we have an inclusion

$$o_{\mathbb{C}_p} \otimes_{o_L} W(o_{\hat{L}_\infty^\flat})_L \hookrightarrow o_{\mathbb{C}_p} \otimes_{o_L} W(\hat{L}_\infty^\flat)_L. \tag{3.1}$$

Remember that $k = o_L/(\pi)$ is the residue field of L. We have an injective map between k-modules

$$(o_{\mathbb{C}_p} \otimes_{o_L} W(o_{\hat{L}_\infty^\flat})_L)/(\pi) = o_{\mathbb{C}_p}/(\pi) \otimes_k o_{\hat{L}_\infty^\flat} \hookrightarrow o_{\mathbb{C}_p}/(\pi) \otimes_k \hat{L}_\infty^\flat = (o_{\mathbb{C}_p} \otimes_{o_L} W(\hat{L}_\infty^\flat)_L)/(\pi).$$

This implies that the preimage of $\pi(o_{\mathbb{C}_p} \otimes_{o_L} W(\hat{L}_\infty^\flat)_L)$ under 3.1 is given by

$$\pi(o_{\mathbb{C}_p} \otimes_{o_L} W(o_{\hat{L}_\infty^\flat})_L).$$

Since $W(\hat{L}_\infty^\flat)_L$ and by flat base change also $o_{\mathbb{C}_p} \otimes_{o_L} W(\hat{L}_\infty^\flat)_L$ are π-torsion free, the same is true with π replaced by π^n for every n. Let

$$\sum c_i \otimes (\pi^n f_i + \omega_\varphi^n g_i) \in o_{\mathbb{C}_p} \otimes (\pi^n W(\hat{L}_\infty^\flat)_L + \omega_\varphi^n W(o_{\hat{L}_\infty^\flat})_L)$$

for elements $c_i \in o_{\mathbb{C}_p}, f_i \in W(\hat{L}_\infty^\flat)_L, g_i \in W(o_{\hat{L}_\infty^\flat})_L$ such that

$$\sum c_i \otimes (\pi^n f_i + \omega_\varphi^n g_i) = \sum c_j' \otimes g_j' \in o_{\mathbb{C}_p} \otimes_{o_L} W(o_{\hat{L}_\infty^\flat})_L$$

for elements $c_j' \in o_{\mathbb{C}_p}, g_j' \in W(o_{\hat{L}_\infty^\flat})_L$. Then $\sum c_i \otimes \omega_\varphi^n g_i \in o_{\mathbb{C}_p} \otimes_{o_L} W(o_{\hat{L}_\infty^\flat})_L$, so

$$\sum c_i \otimes \pi^n f_i = \sum c_j' \otimes g_j' - \sum c_i \otimes \omega_\varphi^n g_i \in o_{\mathbb{C}_p} \otimes_{o_L} W(o_{\hat{L}_\infty^\flat})_L.$$

Together with the previous observation we see that

$$\sum c_i \otimes \pi^n f_i \in \pi^n(o_{\mathbb{C}_p} \otimes_{o_L} W(o_{\hat{L}_\infty^\flat})_L).$$

This implies

$$U_n \cap (o_{\mathbb{C}_p} \otimes_{o_L} W(o_{\hat{L}_\infty^\flat})_L) = o_{\mathbb{C}_p} \otimes (\pi^n W(o_{\hat{L}_\infty^\flat})_L + \omega_\varphi^n W(o_{\hat{L}_\infty^\flat})_L) =: V_n$$

for every n which we may take as a neighbourhood basis of 0 of $o_{\mathbb{C}_p} \otimes_{o_L} W(o_{\hat{L}_\infty^\flat})_L$ with the same argument as above. Then the maps

$$(o_{\mathbb{C}_p} \otimes_{o_L} W(o_{\hat{L}_\infty^\flat})_L)/V_n \to (o_{\mathbb{C}_p} \otimes_{o_L} W(\hat{L}_\infty^\flat)_L)/U_n.$$

are injective for every n. Completion then yields an injection

$$o_{\mathbb{C}_p} \widehat{\otimes}_{o_L} W(o_{\hat{L}_\infty^\flat})_L \hookrightarrow o_{\mathbb{C}_p} \widehat{\otimes}_{o_L} W(\hat{L}_\infty^\flat)_L,$$

where both sides have the projective limit topology.

Now consider $\widehat{R}_{\mathbb{C}_p}[1/T] = \bigcup_m T^{-m} \widehat{R}_{\mathbb{C}_p}$ with the inductive limit topology. Note that T is not a zero divisor in $\widehat{R}_{\mathbb{C}_p}$ since it is not a zero divisor in $\mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(r))$

for any r and we have an embedding $\widehat{R}_{\mathbb{C}_p} \to \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(r))$ (Remark 3.27). We use the notation $T = T_0$ and T_i and φ_i as in Remark 3.27[6]. Note that with $1/T$ we also have the elements $1/T_i$. This can be seen by writing

$$\varphi_i(T_i) = T_i(\pi + T_i^{q-1} + \text{ terms of higher order}),$$

i.e. T_i divides $\varphi_i(T_i)$. Then the inverse of T_i is given by the inverse of $T_{i-1} = \varphi_i(T_i)$ times an element in $\widehat{R}_{\mathbb{C}_p}$.

Let $\widehat{R}_{\mathbb{C}_p}[1/T]^\wedge$ be the π-adic completion of $\widehat{R}_{\mathbb{C}_p}[1/T]$. Note that $\widehat{R}_{\mathbb{C}_p}[1/T]^\wedge$ is Hausdorff for the topology which has $\pi^n\widehat{R}_{\mathbb{C}_p}[1/T]^\wedge + T^n\widehat{R}_{\mathbb{C}_p}$ as basic open subsets. We claim that we have maps

$$\mathcal{E}^{\dagger,\leq 1}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}) \hookrightarrow \widehat{R}_{\mathbb{C}_p}[1/T]^\wedge \hookrightarrow o_{\mathbb{C}_p}\widehat{\otimes}_{o_L} W(\hat{L}^\flat_\infty)_L.$$

Again we may assume $\varphi(T) = T^q + \pi T$. Let $f \in \mathcal{E}^{\dagger,\leq 1}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})^r = \mathcal{E}^{\dagger,\leq 1}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}) \cap \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})^r$. Let $(f_i^+)_{i\in\mathbb{N}}, (f_i^-)_{i\in\mathbb{N}}$ two sequences as in Corollary 3.78 such that $(f_i^+ + f_i^-)_{i\in\mathbb{N}}$ converges to f in $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})^r$ and such that $f_i^+, f_i^- \in \mathcal{E}^{\dagger,\leq 1}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})$ for every i. Then $f_i^+ \in \widehat{R}_{\mathbb{C}_p}$ for every i, and the sequence converges for the (π, T)-adic topology in $\widehat{R}_{\mathbb{C}_p}$. On the other hand, we may assume that the $f_i^- = \sum_{n=1}^m a_n T^{-n} \in \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B} \setminus \mathfrak{B}(r^{1/q^i}))$ are finite sums. Then the f_i^- lie in $\widehat{R}_{\mathbb{C}_p}[1/T]$ and converge π-adically in $\widehat{R}_{\mathbb{C}_p}[1/T]^\wedge$. This gives a well-defined injection $\mathcal{E}^{\dagger,\leq 1}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}) \to \widehat{R}_{\mathbb{C}_p}[1/T]^\wedge$.

We have an isomorphism

$$o_{\mathbb{C}_p}\widehat{\otimes}_{o_L}\widehat{R}_L = \widehat{R}_{\mathbb{C}_p}.$$

The isomorphism from Lemma 3.57 shows that $o_{\mathbb{C}_p}\widehat{\otimes}_{o_L} W(o_{\hat{L}^\flat_\infty})_L \cong \widehat{R}_{\mathbb{C}_p}$. Then we have a map

$$\widehat{R}_{\mathbb{C}_p}[1/T] \cong (o_{\mathbb{C}_p}\widehat{\otimes}_{o_L} W(o_{\hat{L}^\flat_\infty})_L)[1/\omega_\varphi] \hookrightarrow o_{\mathbb{C}_p}\widehat{\otimes}_{o_L} W(\hat{L}^\flat_\infty)_L.$$

Passing to the π-adic completion (note that $o_{\mathbb{C}_p}\widehat{\otimes}_{o_L} W(\hat{L}^\flat_\infty)_L$ is π-adically complete by the same argument as in Lemma 3.14) then gives a map

$$\widehat{R}_{\mathbb{C}_p}[1/T]^\wedge \hookrightarrow o_{\mathbb{C}_p}\widehat{\otimes}_{o_L} W(\hat{L}^\flat_\infty)_L.$$

We then have maps

$$\mathcal{E}^{\dagger,\leq 1}_K(\mathfrak{B}^{\mathrm{perf}}) \hookrightarrow \mathcal{E}^{\dagger,\leq 1}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}) \hookrightarrow o_{\mathbb{C}_p}\widehat{\otimes}_{o_L} W(\hat{L}^\flat_\infty)_L.$$

Note that image of the sequence $(f_i^+ + f_i^-)_{i\in\mathbb{N}} \in \mathcal{E}^{\dagger,\leq 1}_K(\mathfrak{B}^{\mathrm{perf}})$ converges in $o_{\mathbb{C}_p}\widehat{\otimes}_{o_L} W(\hat{L}^\flat_\infty)_L$ for the weak topology.

[6]This notation is, of course, also possible if the given Frobenius power series is not the special one.

Definition 3.81. *The weak topology on* $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}})$ *is the initial topology with respect to this map.*

Remark 3.82. If we write $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}})^+$ for the subring of $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}})$ which consists of the elements of the form f^+, then the weak topology has a neighbourhood basis of 0 consisting of the sets

$$\pi^n \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}}) + T^n \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}})^+.$$

The $o_L^\times$-action as well as the map φ restrict to $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}})$. The map φ is continuous for the weak topology. This follows from the continuity of the corresponding map

$$id \otimes Fr : o_{\mathbb{C}_p} \widehat{\otimes}_{o_L} W(\hat{L}_\infty^\flat)_L \to o_{\mathbb{C}_p} \widehat{\otimes}_{o_L} W(\hat{L}_\infty^\flat)_L.$$

Lemma 3.83. *Let R be a ring and M, N two linearly topologized topological R-modules. Let G be a profinite group which acts continuously on M and N. Then the diagonal action of G on $M \otimes_R N$ is continuous.*

Proof. Exercise 3.1.9 in [34]. □

Lemma 3.84. *The $o_L^\times$-action on $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}})$ is continuous for the weak topology.*

Proof. The previous lemma shows that the $o_L^\times$-action is continuous on $o_{\mathbb{C}_p} \widehat{\otimes}_{o_L} W(\hat{L}_\infty^\flat)_L$. This implies the continuity of the $o_L^\times$-action on $\mathcal{E}_K^{\dagger}(\mathfrak{B}^{\text{perf}})^{\leq 1}$ because the map $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}}) \to o_{\mathbb{C}_p} \widehat{\otimes}_{o_L} W(\hat{L}_\infty^\flat)_L$ is $o_L^\times$-equivariant. □

We define the weak topology on $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}})^r$ as the subspace topology from $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}})$. We endow $\mathcal{E}_K^{\dagger}(\mathfrak{B}^{\text{perf}}) = \bigcup_{n \in \mathbb{N}} \pi^{-n} \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}})$ with the inductive limit topology.

Lemma 3.85. *The inclusions $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}})^r \hookrightarrow \mathcal{R}_K(\mathfrak{B}^{\text{perf}})^r$ and $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}}) \hookrightarrow \mathcal{R}_K(\mathfrak{B}^{\text{perf}})$ are* not *continuous (where $\mathcal{R}_K(\mathfrak{B}^{\text{perf}})^r$ resp. $\mathcal{R}_K(\mathfrak{B}^{\text{perf}})$ have the Fréchet resp. the locally convex inductive limit topology and $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}})^r$ resp. $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}})$ carry the weak topology).*

Proof. Firstly, let $\varepsilon > 0$, $r' > r$, and $U = \{f \in \mathcal{R}_K(\mathfrak{B}^{\text{perf}})^r \mid \|f\|_{r'} \leq \varepsilon\}$ which is an open subset of $\mathcal{R}_K(\mathfrak{B}^{\text{perf}})^r$. If $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}})^r \hookrightarrow \mathcal{R}_K(\mathfrak{B}^{\text{perf}})^r$ was continuous, then the preimage of U in $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}})^r$ would contain a set of the form $(\pi^n \mathcal{E}^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}}) + T^n \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}})^+) \cap \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}})^r$. But since we always find an element $f \in \pi^n \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}})^r$ such that $\|f\|_{r'} > \varepsilon$ (take for example $f = \pi^n \cdot T^{-m}$ for large m), this is not possible.
We show similarly that the inclusion $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\text{perf}}) \to \mathcal{R}_K(\mathfrak{B}^{\text{perf}})$ is not continuous. Choose a decreasing sequence $(\varepsilon_n)_n$ of real numbers $\varepsilon_n > 0$ which converges to zero, and let $(r_n)_n$ and $(s_n)_n$ be two increasing sequences such that $0 < r_n < s_n < 1$. Set $U_n := \{f \in \mathcal{R}_K(\mathfrak{B}^{\text{perf}})^{r_n} \mid \|f\|_{s_n} \leq \varepsilon_n\}$. Then $U := \sum_n U_n \subseteq \mathcal{R}_K(\mathfrak{B}^{\text{perf}})$ is an open

set. For any k, there is an m such that the element $\pi^k T^{-m}$ does not lie in the preimage of U in $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\mathrm{perf}})$. This is because we find an n_0 such that $\varepsilon_{n_0} \leq |\pi^k|$ and we can choose m large enough such that we have $|\pi^k T^{-m}\|_{r_n} > \varepsilon_n$ for $n = 1, ..., n_0$. We have $\|\pi^k T^{-m}\|_r \geq \|\pi^k T^{-m}\|_1 = |\pi^k|$ for any r, hence $\|\pi^k T^{-m}\|_{s_n} > \varepsilon_n$ for $n > n_0$ as well. Hence the preimage of U in $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\mathrm{perf}})$ does not contain a set of the form $\pi^n \mathcal{E}^{\dagger,\leq 1}(\mathfrak{B}^{\mathrm{perf}}) + T^n \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{B}^{\mathrm{perf}})^+$.

□

4 Construction of $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$

Let $L \subseteq K \subset \mathbb{C}_p$ be a complete nonarchimedean field. We construct a preperfectoid version of the varieties $\mathfrak{X}(r)$ which we denote by $\mathfrak{X}^{\mathrm{perf}}(r)$. Based on that we will construct a "preperfectoid" version of the Robba ring over $\mathfrak{X}$.

4.1 Construction of $\mathfrak{X}^{\mathrm{perf}}(r)$

Let $r \in S_n$ for some n (Definition 2.7). The affinoid subdomains $\mathfrak{X}(r) \subseteq \mathfrak{X}$ from the first chapter form an open covering of $\mathfrak{X}$. The K-algebra $\mathcal{O}_K(\mathfrak{X}(r))$ is a reduced affinoid Tate algebra. We have a Tate-Huber pair $(\mathcal{O}_K(\mathfrak{X}(r)), \mathcal{O}_K(\mathfrak{X}(r))^{\leq 1})$ where

$$\mathcal{O}_K(\mathfrak{X}(r))^{\leq 1} = \mathcal{O}_K(\mathfrak{X}(r))^{\circ} = \{f \in \mathcal{O}_K(\mathfrak{X}(r)) \mid \|f\|_{\mathfrak{X}(r)} \leq 1\}$$

are the power-bounded elements of $\mathcal{O}_K(\mathfrak{X}(r))$, and $(\mathcal{O}_K(\mathfrak{X}(r)), \mathcal{O}_K(\mathfrak{X}(r))^{\leq 1})$ is stably uniform (Remark 1.30). For $i \in \mathbb{N}$, the action of p,

$$p_* : \mathcal{O}_K(\mathfrak{X}(r^{1/p^i})) \to \mathcal{O}_K(\mathfrak{X}(r^{1/p^{i+1}})),$$

is isometric for the supremum norms $\|\cdot\|_{\mathfrak{X}(r^{1/p^i})}$ respective $\|\cdot\|_{\mathfrak{X}(r^{1/p^{i+1}})}$ (Remark 2.11). In particular p_* is continuous and satisfies

$$p_*(\mathcal{O}_K(\mathfrak{X}(r^{1/p^i}))^{\leq 1}) \subseteq \mathcal{O}_K(\mathfrak{X}(r^{1/p^{i+1}}))^{\leq 1}.$$

We have inductive systems of rings

$$\mathcal{O}_K(\mathfrak{X}(r)) \overset{p_*}{\to} \mathcal{O}_K(\mathfrak{X}(r^{1/p})) \overset{p_*}{\to} \dots \overset{p_*}{\to} \mathcal{O}_K(\mathfrak{X}(r^{1/p^i})) \overset{p_*}{\to} \mathcal{O}_K(\mathfrak{B}(r^{1/p^{i+1}})) \overset{p_*}{\to} \dots$$

and

$$\mathcal{O}_K(\mathfrak{X}(r))^{\leq 1} \overset{p_*}{\to} \mathcal{O}_K(\mathfrak{X}(r^{1/p}))^{\leq 1} \overset{p_*}{\to} \dots \overset{p_*}{\to} \mathcal{O}_K(\mathfrak{X}(r^{1/p^i}))^{\leq 1} \overset{p_*}{\to} \mathcal{O}_K(\mathfrak{X}(r^{1/p^{i+1}}))^{\leq 1} \overset{p_*}{\to} \dots$$

Set

$$\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r)) := \varinjlim_{p_*, i} \mathcal{O}_K(\mathfrak{X}(r^{1/p^i})), \quad \text{and}$$
$$\breve{\mathcal{O}}_K^{+}(\mathfrak{X}^{\mathrm{perf}}(r)) := \varinjlim_{p_*, i} \mathcal{O}_K(\mathfrak{X}(r^{1/p^i}))^{\leq 1}.$$

Then $(\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r)), \breve{\mathcal{O}}_K^{+}(\mathfrak{X}^{\mathrm{perf}}(r))$ is the inductive limit of the inductive system consisting of the maps

$$p_* : (\mathcal{O}_K(\mathfrak{X}(r^{1/p^i}), \mathcal{O}_K(\mathfrak{X}(r^{1/p^i})^{\leq 1}) \to (\mathcal{O}_K(\mathfrak{X}(r^{1/p^{i+1}}), \mathcal{O}_K(\mathfrak{X}(r^{1/p^{i+1}})^{\leq 1})$$

for every i in the category of uniform Tate-Huber pairs as in Proposition 1.55. Its pair of definition is $(\breve{\mathcal{O}}_K^{+}(\mathfrak{X}^{\mathrm{perf}}(r)), (\pi))$.

Remark 4.1. $\breve{\mathcal{O}}^+_K(\mathfrak{X}^{\mathrm{perf}}(r)) = \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r))^\circ$, i.e. $\breve{\mathcal{O}}^+_K(\mathfrak{X}^{\mathrm{perf}}(r))$ consists of the power-bounded elements of $\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r))$.

Proof. Let $f \in \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r))^\circ$ be power-bounded. Then any preimage $f_{i_0} \in \mathcal{O}_K(\mathfrak{X}(r^{1/p^{i_0}}))$ of f under the canonical map $\mathcal{O}_K(\mathfrak{X}(r^{1/p^{i_0}})) \to \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r))$ is power-bounded since the transitions maps p_* in the inductive limit are isometric. Therefore $f_{i_0} \in \mathcal{O}_K(\mathfrak{X}(r^{1/p^{i_0}}))^{\leq 1}$ and hence $f \in \breve{\mathcal{O}}^+_K(\mathfrak{X}^{\mathrm{perf}}(r))$. On the other hand, every element of $\breve{\mathcal{O}}^+_K(\mathfrak{X}^{\mathrm{perf}}(r))$ clearly is power-bounded. □

We denote by $\mathcal{O}^+_K(\mathfrak{X}^{\mathrm{perf}}(r))$ the π-adic completion of $\breve{\mathcal{O}}^+_K(\mathfrak{X}^{\mathrm{perf}}(r)) = \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r))^\circ$ and set

$$\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r)) := \mathcal{O}^+_K(\mathfrak{X}^{\mathrm{perf}}(r)) \otimes_{\breve{\mathcal{O}}^+_K(\mathfrak{X}^{\mathrm{perf}}(r))} \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r))$$

which is the completion of $\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r))$. Then $(\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r)), \mathcal{O}^+_K(\mathfrak{X}^{\mathrm{perf}}(r)))$ is the completion of $(\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r)), \breve{\mathcal{O}}^+_K(\mathfrak{X}^{\mathrm{perf}}(r)))$. It is again a uniform Tate-Huber pair.

Remark 4.2. We have $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r)) = \mathcal{O}^+_K(\mathfrak{X}^{\mathrm{perf}}(r))[1/\pi]$.

Proof. $\mathcal{O}^+_K(\mathfrak{X}^{\mathrm{perf}}(r))$ is a ring of definition. The element $\pi \in \mathcal{O}^+_K(\mathfrak{X}^{\mathrm{perf}}(r))$ is a topologically nilpotent unit in $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r))$, hence we have $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r)) = \mathcal{O}^+_K(\mathfrak{X}^{\mathrm{perf}}(r))[1/\pi]$ because of Remark 1.7. □

We consider the inductive limit seminorm on $\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r)) = \varinjlim_{p_*,i} \mathcal{O}_K(\mathfrak{X}(r^{1/p^i}))$ coming from the supremum norms $\|\cdot\|_{\mathfrak{X}(r^{1/p^i})}$ on each $\mathcal{O}_K(\mathfrak{X}(r^{1/p^i}))$. It is a norm on $\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r))$. This is because the transition maps $p_* : \mathcal{O}_K(\mathfrak{X}(r^{1/p^i})) \to \mathcal{O}_K(\mathfrak{X}(r^{1/p^{i+1}}))$ are isometric with respect to the supremum norms for every i (Remark 2.11). Note that this norm induces the topology on $\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r))$ as a Huber ring. Passing to the completion $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r))$ gives us a norm on $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r))$ which induces the topology on $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r))$ as a Huber ring and which we denote by $\|\cdot\|_{\mathfrak{X}^{\mathrm{perf}}(r)}$. It is multiplicative because the norms $\|\cdot\|_{\mathfrak{X}(r^{1/p^i})}$ are multiplicative (see the explanation after Proposition 2.9). For $f \in \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r))$, we have

$$\|f\|_{\mathfrak{X}^{\mathrm{perf}}(r)} = \|f_{i_0}\|_{\mathfrak{X}(r^{1/p^{i_0}})}$$

where $f_{i_0} \in \mathcal{O}_K(\mathfrak{X}(r^{1/p^{i_0}}))$ is a preimage of f under the canonical map $\mathcal{O}_K(\mathfrak{X}(r^{1/p^{i_0}})) \to \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r))$.

Remark 4.3. If we regard the Tate rings $\mathcal{O}_K(\mathfrak{X}(r^{1/p^i}))$ as normed rings using the norm defined in Remark 1.6 and form the (completion of the) inductive limit seminorm coming from these norms, then the induced topology on $\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r))$ resp. $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r))$ coincides with the topology on $\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r))$ resp. $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r))$ as Huber rings as well.

Definition 4.4. *We define the preadic space* $\mathfrak{X}^{\mathrm{perf}}_K(r) := \mathrm{Spa}(\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r)), \mathcal{O}^+_K(\mathfrak{X}^{\mathrm{perf}}(r)))$.

Lemma 4.5. *We have an isometric isomorphism of Banach algebras*

$$\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}(r)) \cong \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(a))$$

for a certain radius a (if $r \in S_n$, then $a \in R_n$).

Proof. If $a \geq p^{-q/e(q-1)}$, then

$$[p]^{-1}(\mathfrak{B}(a^{1/q^{e(i+1)}})) = \mathfrak{B}(a^{1/q^{ei}})$$

for every $i \in \mathbb{N}$ (Lemma 2.8). We have an isometric isomorphism $\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}(r^{1/p^i})) \cong \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(a^{1/q^{ei}}))$ (Proposition 2.9) for every $i \in \mathbb{N}$. Moreover, for every $i \in \mathbb{N}$ the diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}(r^{1/p^i})) & \xrightarrow{\kappa} & \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(a^{1/q^{ei}})) \\ \downarrow p_* & & \downarrow [p] \\ \mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}(r^{1/p^{i+1}})) & \xrightarrow{\kappa} & \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(a^{1/q^{e(i+1)}})) \end{array}$$

commutes. This gives an isometric isomorphism

$$\breve{\mathcal{O}}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}(r)) = \varinjlim_{p_*,i} \mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}(r^{1/p^i})) \cong \breve{\mathcal{O}}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(a)) = \varinjlim_{p_*,i} \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(a^{1/q^{ei}})).$$

The latter is isometrically isomorphic to $\varinjlim_{\varphi,i} \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(a^{1/q^i}))$ (which can be seen as in the proof of Lemma 3.37). Passing to the completion gives an isometric isomorphism

$$\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}(r)) \cong \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(a)).$$

□

Proposition 4.6. *$\mathfrak{X}_L^{\mathrm{perf}}(r)$ is an L-preperfectoid space.*

Proof. We show that the uniform completion of $\mathcal{O}_L(\mathfrak{X}^{\mathrm{perf}}(r))\widehat{\otimes}_L\mathbb{C}_p$ is perfectoid. Firstly consider the algebra $\mathcal{O}_L(\mathfrak{X}^{\mathrm{perf}}(r))\widehat{\otimes}_L\mathbb{C}_p$ with the tensor product norm. Since the tensor product norm is compatible with the inductive limit norm (Lemma 6.13), we have

$$\mathcal{O}_L(\mathfrak{X}^{\mathrm{perf}}(r))\widehat{\otimes}_L\mathbb{C}_p \cong \widehat{\varinjlim}_{p_*,i}(\mathcal{O}_L(\mathfrak{X}(r^{1/p^i}))\widehat{\otimes}_L\mathbb{C}_p)$$

where on the right hand side we take the completion with respect to the inductive limit norm coming from the tensor product norms on the $\mathcal{O}_L(\mathfrak{X}(r^{1/p^i}))\widehat{\otimes}_L\mathbb{C}_p$. The induced spectral norm coming from the tensor product norm on $\mathcal{O}_L(\mathfrak{X}(r^{1/p^i}))\widehat{\otimes}_K\mathbb{C}_p \cong \mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}(r^{1/p^i}))$ coincides with the supremum norm on $\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}(r^{1/p^i}))$ for each i because both are complete power-multiplicative norms which then coincide ([31, Lemma 3.8.3/3 and Theorem 6.2.4/1]). It follows that the spectral seminorm coming from the seminorm on $\varinjlim_{p*,i}(\mathcal{O}_L(\mathfrak{X}(r^{1/p^i}))\widehat{\otimes}_L\mathbb{C}_p)$ coincides with the norm on $\breve{\mathcal{O}}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}(r))$. This implies that the uniform completion of $\mathcal{O}_L(\mathfrak{X}^{\mathrm{perf}}(r))\widehat{\otimes}_L\mathbb{C}_p$ is isomorphic to $\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}(r))$ which is isomorphic to $\mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(a))$ as in Lemma 4.5. But the latter is perfectoid (Lemma 3.21 and Corollary 3.20).

□

Remark 4.7. We have

$$|\mathfrak{X}_K^{\mathrm{perf}}(r)| \cong \varprojlim_{p^*,i} |\mathrm{Spa}(\mathcal{O}_K(\mathfrak{X}(r^{1/p^i}), \mathcal{O}_K^+(\mathfrak{X}(r^{1/p^i}))|$$

for the underlying topological spaces (Proposition 1.55 and Proposition 1.38).

Proposition 4.8. *The K-points of $\mathfrak{X}_K^{\mathrm{perf}}(r)$ are the K-valued locally analytic characters χ of L such that $|\chi(g)-1| \leq r$ for all $g \in o_L$.*

Proof. We have $\mathfrak{X}(r) = \mathfrak{X}_0(r) \cap \mathfrak{X}$, and an isomorphism of rigid varieties

$$\mathfrak{B}_1(r) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o_L, \mathbb{Z}_p) \to \mathfrak{X}_0(r),$$
$$z \otimes \beta \mapsto (g \mapsto z^{\beta(g)}).$$

Therefore the K-points of $\mathfrak{X}(r)$ are the locally analytic characters $\chi : o_L \to K^\times$ such that

$$|\chi(g) - 1| \leq r, \quad g \in o_L.$$

Let $(\chi_0, \chi_1, ...) \in \varprojlim_{p^*,i} \mathfrak{X}(r^{1/p^i})(K)$. We define a character

$$\chi : L \to K^\times,$$
$$1/p^i \cdot o_L \ni x \mapsto \chi_i(p^i \cdot x).$$

Now remember that $r \geq p^{-p/p-1}$. If $\chi : L \to K^\times$ is a character such that $\chi_{|o_L}$ takes values in $\mathfrak{B}_1(r)$, i.e. $\chi_{|o_L} \in \mathfrak{X}(r)(K)$, then we get a character

$$\chi_{|o_L}(1/p\,\cdot) : o_L \to K^\times,$$
$$x \mapsto \chi(x/p)$$

which fulfils $\chi_{|o_L}(1/p\,\cdot)^p = \chi_{|o_L}$ and hence takes values in $\mathfrak{B}_1(r^{1/p})$ (Lemma 2.8). By repeating this argument we get an element

$$(\chi_{|o_L}, \chi_{|o_L}(1/p\,\cdot), \chi_{|o_L}(1/p^2\,\cdot), ...) \in \varprojlim_{p^*,i} \mathfrak{X}(r^{1/p^i})(K).$$

This gives a bijection between $\varprojlim_{p^*,i} \mathfrak{X}(r^{1/p^i})(K)$ and the K-valued locally analytic characters χ on L with $|\chi(g) - 1| \leq r$ for $g \in o_L$.

□

4.1.1 The monoid action on $\mathfrak{X}_K^{\text{perf}}(r)$

To define an $o_L \setminus \{0\}$-action on $\mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r))$, we start with defining an $o_L \setminus \{0\}$-action on the dense subset $\breve{\mathcal{O}}_K(\mathfrak{X}^{\text{perf}}(r)) = \varinjlim_{p_*,i} \mathcal{O}_K(\mathfrak{X}(r^{1/p^i}))$ of $\mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r))$. Let $a \in o_L \setminus \{0\}$ and $f \in \breve{\mathcal{O}}_K(\mathfrak{X}^{\text{perf}}(r))$. Then set

$$a_*(f) := \alpha_{i_0}(a_*(f_{i_0})),$$

where f_{i_0} is a preimage of f under the canonical map

$$\alpha_{i_0} : \mathcal{O}_K(\mathfrak{X}(r^{1/p^{i_0}})) \to \breve{\mathcal{O}}_K(\mathfrak{X}^{\text{perf}}(r)).$$

This is well defined because the action of an element $a \in o_L \setminus \{0\}$ on $\mathcal{O}(\mathfrak{X}(r^{1/p^i}))$ commutes with the transition maps in the inductive limit $p_* : \mathcal{O}_K(\mathfrak{X}(r^{1/p^i})) \to \mathcal{O}_K(\mathfrak{X}(r^{1/p^{i+1}}))$.
The resulting map

$$a_* : \breve{\mathcal{O}}_K(\mathfrak{X}^{\text{perf}}(r)) \to \breve{\mathcal{O}}_K(\mathfrak{X}^{\text{perf}}(r))$$

is continuous and by passing to the completion $\mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r))$ we get a continuous map

$$a_* : \mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r)) \to \mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r)).$$

We have $a_*(\mathcal{O}_K^+(\mathfrak{X}^{\text{perf}}(r))) \subseteq \mathcal{O}_K^+(\mathfrak{X}^{\text{perf}}(r))$, hence a_* defines a morphism of Tate-Huber pairs. We get a corresponding map $a^* : \mathfrak{X}_K^{\text{perf}}(r) \to \mathfrak{X}_K^{\text{perf}}(r)$ between preadic spaces.

4.2 A preperfectoid version of $\mathcal{R}_K(\mathfrak{X})$

We start with constructing rings $\mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2))$ analogue to the rings $\mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r))$, but with the rings $\mathcal{O}_K(\mathfrak{X}(r_1, r_2))$ instead of $\mathcal{O}_K(\mathfrak{X}(r))$.
Throughout the rest of this chapter, we assume that all radii which occur in relation to $\mathfrak{X}$ (as in $\mathcal{O}_K(\mathfrak{X}(r_1, r_2))$ or $\mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r))$) like r_1, r_2 (or s_1, s_2 etc.) fulfil $p^{-(1+e/(p-1))/ep^n} < r_1 \leq r_2 < 1$, with $r_1 \in S_n$ and $r_2 \in \bigcup_{m \geq n} S_m$ (see Definition 2.7), and a single radius (like r) is assumed to lie in some S_n, unless stated otherwise.

The $\mathcal{O}_K(\mathfrak{X}(r_1, r_2)))$ are reduced affinoid Tate algebras (over K). Every $\mathcal{O}_K(\mathfrak{X}(r_1, r_2))$ is stably uniform (Remark 1.30). Hence we have stably uniform Tate-Huber pairs $(\mathcal{O}_K(\mathfrak{X}(r_1, r_2)), \mathcal{O}_K(\mathfrak{X}(r_1, r_2)^{\leq 1})$.

Lemma 4.9. *If $z \in \mathfrak{B}_1$ with $|z^p - 1| > p^{-p/(p-1)}$, then $|(z-1)^p| = |z^p - 1|$.*

Proof. Let $z = x + 1$, then

$$\begin{aligned} |z^p - 1| &= |(x+1)^p - 1| \\ &= |\sum_{k=1}^{p} \binom{p}{k} x^k| \\ &\leq \max_{0<k<p} (p^{-1}|x|^k, |x^p|), \end{aligned}$$

since $|\binom{p}{k}x^k| = p^{-1}|x|^k$ for $0 < k < p$. If $|x| \le p^{-1/(p-1)}$, then

$$\begin{aligned}\max\nolimits_{0<k<p}(p^{-1}|x|^k, |x^p|) &\le \max\nolimits_{0<k<p}(p^{-1}p^{-k/(p-1)}, p^{-p/(p-1)})\\ &\le \max(p^{-1}p^{-1/(p-1)}, p^{-p/(p-1)})\\ &= p^{-p/(p-1)},\end{aligned}$$

since $p^{-1}p^{-1/(p-1)} = p^{-(p-1)/(p-1)}p^{-1/(p-1)} = p^{-p/(p-1)}$.
But this contradicts $|z^p - 1| > p^{-p/(p-1)}$, so $|x| > p^{-1/(p-1)}$. Then

$$p^{-1}\cdot|x|^k \le p^{-1}|x| < |x|^p$$

for $0 < k < p$ since $p^{-1} = p^{-(p-1)/(p-1)} < |x|^{p-1}$. Therefore $|z^p - 1| = |x^p| = |(z-1)^p|$. □

Lemma 4.10. *The preimage of $\mathfrak{X}(r_1, r_2)$ under $p^* : \mathfrak{X} \to \mathfrak{X}$ is $\mathfrak{X}(r_1^{1/p}, r_2^{1/p})$.*

Proof. If $z \in \mathfrak{B}_1$ with $|z^p - 1| > p^{-p/(p-1)}$, then $|(z-1)^p| = |z^p - 1|$ (Lemma 4.9). Write

$$\mathfrak{B}_1^{d,(i)}(r_1, r_2) = \{(x_1, ..., x_d) \in \mathfrak{B}_1^d(r_2) \,|\, |x_i - 1| \ge r_1\},$$

then

$$\begin{aligned}(p^*)^{-1}(\mathfrak{B}_1^{d,(i)}(r_1, r_2)) &= \{(x_1, ..., x_d) \in \mathfrak{B}_1^d \,|\, |p^*(x_i) - 1| \ge r_1 \,\text{and}\, p^*(x) \in \mathfrak{B}_1^d(r_2)\}\\ &= \{(x_1, ..., x_d) \in \mathfrak{B}_1^d \,|\, |x_i^p - 1| \ge r_1 \,\text{and}\, p^*(x) \in \mathfrak{B}_1^d(r_2)\}\\ &= \mathfrak{B}_1^{d,(i)}(r_1^{1/p}, r_2^{1/p}).\end{aligned}$$

Identifying $\mathfrak{B}_1^{d,(i)}$ with $\mathfrak{X}_0^{(i)}$, we see that

$$(p^*)^{-1}(\mathfrak{X}_0^{(i)}(r_1, r_2)) = \mathfrak{X}_0^{(i)}(r_1^{1/p}, r_2^{1/p}).$$

We conclude that the preimage of $\mathfrak{X}_0(r_1, r_2) = \bigcup \mathfrak{X}_0^{(i)}(r_1, r_2)$ under p^* is $\mathfrak{X}_0(r_1^{1/p}, r_2^{1/p})$ because we have $(p^*)^{-1}(\mathfrak{X}_0^{(i)}(r_1, r_2)) = \mathfrak{X}_0^{(i)}(r_1^{1/p}, r_2^{1/p})$ for every i and therefore

$$(p^*)^{-1}(\mathfrak{X}_0(r_1, r_2)) = \mathfrak{X}_0(r_1^{1/p}, r_2^{1/p}).$$

Then we have $(p^*)^{-1}(\mathfrak{X}(r_1, r_2)) = (p^*)^{-1}(\mathfrak{X}_0(r_1, r_2) \cap \mathfrak{X}) = \mathfrak{X}(r_1^{1/p}, r_2^{1/p})$. □

Alternatively, one can compute

$$\begin{aligned}(p^*)^{-1}(\mathfrak{X}(r_1, r_2)) &= (p^*)^{-1}(\mathfrak{X}(r_2) \setminus \mathfrak{X}^-(r_1)) = (p^*)^{-1}(\mathfrak{X}(r_2) \setminus \bigcup_{r<r_1} \mathfrak{X}(r))\\ &= (p^*)^{-1}(\mathfrak{X}(r_2)) \setminus \bigcup_{r<r_1} (p^*)^{-1}(\mathfrak{X}(r))\\ &= \mathfrak{X}(r_2^{1/p}) \setminus \bigcup_{r<r_1^{1/p}} \mathfrak{X}(r)\\ &= \mathfrak{X}(r_1^{1/p}, r_2^{1/p}).\end{aligned}$$

We get a map

$$p_* : \mathcal{O}_K(\mathfrak{X}(r_1, r_2)) \to \mathcal{O}_K(\mathfrak{X}(r_1^{1/p}, r_2^{1/p})).$$

Remember that after base change to $\mathbb{C}_p$, $\mathfrak{X}(r_1, r_2)_{/\mathbb{C}_p}$ is isomorphic to some annulus $\mathfrak{B}(a_1, a_2)_{/\mathbb{C}_p}$ (Remark 2.12).

Lemma 4.11. *For every $i \in \mathbb{N}$, the action of p,*

$$p_* : \mathcal{O}_K(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i})) \to \mathcal{O}_K(\mathfrak{X}(r_1^{1/p^{i+1}}, r_2^{1/p^{i+1}}))$$

is isometric for the supremum norm and therefore injective.

Proof. After base change to $\mathbb{C}_p$, $\mathfrak{X}(r_1, r_2)$ and $\mathfrak{X}(r_1^{1/p}, r_2^{1/p})$ are isomorphic to affinoid annuli $\mathfrak{B}_{\mathbb{C}_p}(a_1, a_2)$ and $\mathfrak{B}_{\mathbb{C}_p}(a_1^{1/q^e}, a_2^{1/q^e})$. The map $p^* : \mathfrak{B}(\mathbb{C}_p) \to \mathfrak{B}(\mathbb{C}_p)$ on $\mathbb{C}_p$-points is surjective. Since the action of p commutes with the isomorphism $\kappa : \mathfrak{B}_{\mathbb{C}_p} \to \mathfrak{X}_{\mathbb{C}_p}$, the map $p^* : \mathfrak{X}(\mathbb{C}_p) \to \mathfrak{X}(\mathbb{C}_p)$ on $\mathbb{C}_p$-points is surjective as well. With Remark 4.10, we see that

$$p^* : \mathfrak{X}(r_1^{1/p}, r_2^{1/p})(\mathbb{C}_p) \to \mathfrak{X}(r_1, r_2)(\mathbb{C}_p)$$

is surjective. Therefore $p_* : \mathcal{O}_K(\mathfrak{X}(r_1, r_2)) \to \mathcal{O}_K(\mathfrak{X}(r_1^{1/p}, r_2^{1/p}))$ is isometric for the supremum norm. □

The pairs $(\mathcal{O}_K(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i})), \mathcal{O}_K(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i}))^{\leq 1})$ for $i \in \mathbb{N}$ together with the maps p_* form an inductive system in the category of uniform Tate algebras. We form its inductive limit by setting

$$\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)) := \varinjlim_{p_*, i} \mathcal{O}_K(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i})),$$

and

$$\breve{\mathcal{O}}_K^+(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)) := \varinjlim_{p_*, i} \mathcal{O}(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i}))^{\leq 1},$$

We get a uniform Tate-Huber pair $(\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)), \breve{\mathcal{O}}_K^+(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)))$. We define $\mathcal{O}_K^+(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2))$ to be the π-adic completion of $\breve{\mathcal{O}}_K^+(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2))$. Then

$$\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)) := \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)) \otimes_{\breve{\mathcal{O}}_K^+(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2))} \mathcal{O}_K^+(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)).$$

is the completion of $\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2))$. Again, we have

$$\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)) = \mathcal{O}_K^+(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2))[1/\pi]$$

because of Remark 1.7.

Next, we define a norm on $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2))$. Consider the inductive limit seminorm on $\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2))$ coming from the supremum norms $\|\cdot\|_{\mathfrak{X}(r_1^{1/p^i},r_2^{1/p^i})}$ on the $\mathcal{O}_K(\mathfrak{X}(r_1^{1/p^i},r_2^{1/p^i}))$. Since the transition maps p_* in the inductive limit are isometric for the supremum norms $\|\cdot\|_{\mathfrak{X}(r_1^{1/p^i},r_2^{1/p^i})}$ (Lemma 4.11), the seminorm is a norm. It induces the topology on $\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2))$ as a Huber ring. We pass to the completion $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2))$ and denote the resulting norm by $\|\cdot\|_{\mathfrak{X}^{\mathrm{perf}}(r_1,r_2)}$. For an element $f\in\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2))$ we have

$$\|f\|_{\mathfrak{X}^{\mathrm{perf}}(r_1,r_2)} := \|f_{i_0}\|_{\mathfrak{X}(r_1^{1/p^{i_0}},r_2^{1/p^{i_0}})}$$

for a preimage f_{i_0} of $f\in\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2))$ under the canonical map

$$\alpha_{i_0}:\mathcal{O}_K(\mathfrak{X}(r_1^{1/p^{i_0}},r_2^{1/p^{i_0}}))\to\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2)).$$

The norm $\|\cdot\|_{\mathfrak{X}^{\mathrm{perf}}(r_1,r_2)}$ is power-multiplicative because the norms $\|\cdot\|_{\mathfrak{X}(r_1^{1/p^i},r_2^{1/p^i})}$ are power-multiplicative for every i. Note that we have an isometric embedding $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2))\to\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2))$ which comes from the isometric embeddings $\mathcal{O}_K(\mathfrak{X}(r_1^{1/p^i},r_2^{1/p^i}))\to\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}(r_1^{1/p^i},r_2^{1/p^i}))$ for every i.

Lemma 4.12. *We have an isometric isomorphism of Banach algebras*

$$\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2))\cong\mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(a_1,a_2))$$

for certain radii a_1,a_2. If $r_1\in S_n$ and $r_2\in\bigcup_{m\geq n}S_m$, then $a_1\in R_n$ and $a_2\in\bigcup_{m\geq n}R_m$.

Proof. We have an isomorphism $\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}(r_1^{1/p^i},r_2^{1/p^i}))\cong\mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(a_1^{1/q^{ei}},a_2^{1/q^{ei}}))$ for every $i\in\mathbb{N}$ and radii a_1,a_2 as in Remark 2.12. We have

$$[p]^{-1}(\mathfrak{B}(a_1^{1/q^{ei}},a_2^{1/q^{ei}}))=\mathfrak{B}(a_1^{1/q^{e(i+1)}},a_2^{1/q^{e(i+1)}})$$

for every $i\in\mathbb{N}$ which follows from Lemma 3.28. Moreover, the diagram

$$\begin{array}{ccc}
\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}(r_1^{1/p^i},r_2^{1/p^i})) & \longrightarrow & \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(a_1^{1/q^{ei}},a_2^{1/q^{ei}})) \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle [p]} \\
\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}(r_1^{1/p^{i+1}},r_2^{1/p^{i+1}})) & \longrightarrow & \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(a_1^{1/q^{e(i+1)}},a_2^{1/q^{e(i+1)}}))
\end{array}$$

commutes and the horizontal maps are isometric isomorphisms. This gives an isometric isomorphism

$$\breve{\mathcal{O}}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2))=\varinjlim_{p_*,i}\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}(r_1^{1/p^i},r_2^{1/p^i}))\cong\mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(a_1,a_2))=\varinjlim_{p_*,i}\mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(a_1^{1/q^{ei}},a_2^{1/q^{ei}})).$$

The latter is isometrically isomorphic to $\varinjlim_{\varphi,i} \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(a_1^{1/q^i}, a_2^{1/q^i}))$ (Lemma 3.37). Passing to the completion gives an isometric isomorphism

$$\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)) \cong \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(a_1, a_2)).$$

□

Proposition 4.13. *The preadic space* $\mathrm{Spa}(\mathcal{O}_L(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)), \mathcal{O}_L^+(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2))$ *is* L*-preperfectoid.*

Proof. This follows by the same arguments as in Proposition 4.6. □

Let $r_1 \leq s_1 \leq s_2 \leq r_2$ with $r_1 \in S_n, r_2, s_1, s_2 \in \bigcup_{m \geq n} S_m$ (after the general assumption on the radii). Then we have restriction maps

$$\mathrm{res} : \mathcal{O}_K(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i})) \to \mathcal{O}_K(\mathfrak{X}(s_1^{1/p^i}, s_2^{1/p^i}))$$

which are injective. This is because there are radii $a_1 \leq b_1 \leq b_2 \leq a_2$ such that the diagram

$$\begin{array}{ccccc}
\mathcal{O}_K(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i})) & \longrightarrow & \mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i})) & \xrightarrow{\cong} & \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(a_1^{1/q^{ei}}, a_2^{1/q^{ei}})) \\
\downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\
\mathcal{O}_K(\mathfrak{X}(s_1^{1/p^i}, s_2^{1/p^i})) & \longrightarrow & \mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}(s_1^{1/p^i}, s_2^{1/p^i})) & \xrightarrow{\cong} & \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(b_1^{1/q^{ei}}, b_2^{1/q^{ei}}))
\end{array}$$

commutes, and the horizontal maps are isometric inclusions resp. isomorphisms, and the map $\mathrm{res} : \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(a_1^{1/q^{ei}}, a_2^{1/q^{ei}})) \to \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(b_1^{1/q^{ei}}, b_2^{1/q^{ei}}))$ is injective. Since the restriction commutes with p_*, we can pass to the inductive limit and get a continuous restriction map

$$\mathrm{res} : \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)) \to \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(s_1, s_2)).$$

Passing to the completions gives a continuous restriction map

$$\mathrm{res} : \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)) \to \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(s_1, s_2)).$$

(Since the restriction $\mathcal{O}_K(\mathfrak{X}(r_1, r_2)) \to \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(t_1, t_2))$ coincides with the composition of the restrictions $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)) \to \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(s_1, s_2)) \to \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(t_1, t_2))$ if $r_1 \leq s_1 \leq t_1 \leq t_2 \leq s_2 \leq r_2$ we just write res without reference to the radii.)

Fix a radius $r_0 \in S_n$. The rings $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2))$ for $r_0 < r_1 \leq r_2$ together with the restriction maps form a projective system. We form their projective limit

$$\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r_0} := \varprojlim_{r_0 < r_1 \leq r_2 < 1} \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)).$$

If $s_0 \leq r_0$, we have continuous restriction maps

$$\mathrm{res} : \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{s_0} \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r_0}.$$

Lemma 4.14. *For every $r_0 \in S_n$ there is an $a_0 \in R_n$ such that we have an isomorphism*

$$\mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}})^{r_0} \cong \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})^{a_0}.$$

Proof. Let $r_1 \le s_1 \le s_2 \le r_2$. There are radii $a_1 \le b_1 \le b_2 \le a_2$ such that the diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i})) & \longrightarrow & \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(a_1^{1/q^{ei}}, a_2^{1/q^{ei}})) \\ \Big\downarrow{\scriptstyle \mathrm{res}} & & \Big\downarrow{\scriptstyle \mathrm{res}} \\ \mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}(s_1^{1/p^i}, s_2^{1/p^i})) & \longrightarrow & \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(b_1^{1/q^{ei}}, b_2^{1/q^{ei}})) \end{array}$$

commutes and the horizontal arrows are isomorphisms for every $i \in \mathbb{N}$. This follows from Remark 2.12. This gives rise to a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)) & \longrightarrow & \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(a_1, a_2)) \\ \Big\downarrow{\scriptstyle \mathrm{res}} & & \Big\downarrow{\scriptstyle \mathrm{res}} \\ \mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}(s_1, s_2)) & \longrightarrow & \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(b_1, b_2)) \end{array}$$

where the horizontal arrows are the isomorphisms from Lemma 4.12, and the lemma follows. □

Definition 4.15. *For $n \in \mathbb{N}$ set*

$$\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})_n := \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{p^{-(1+e/(p-1))/ep^n}}.$$

We define the "preperfectoid" Robba ring over K as

$$\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) := \varinjlim_n \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})_n.$$

If we set

$$\mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})_n := \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})^{p^{-1/e(q-1)q^{en-1}}},$$

then we have an isomorphism $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}})_n \cong \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})_n$. Over $\mathbb{C}_p$, the Robba ring over $\mathfrak{X}^{\mathrm{perf}}$ is isomorphic to the Robba ring over $\mathfrak{B}^{\mathrm{perf}}$, i.e. we have

$$\mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}) \cong \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}).$$

Proposition 4.16. *Let $r_1 \le s_1 \le s_2 \le r_2$. The restriction map*

$$res : \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)) \to \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(s_1, s_2))$$

is injective.

Proof. There are radii $a_1 \leq b_1 \leq b_2 \leq a_2$ such that the diagram

$$\begin{array}{ccccc} \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2)) & \longrightarrow & \mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2)) & \xrightarrow{\cong} & \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(a_1,a_2)) \\ \downarrow \mathrm{res} & & \downarrow \mathrm{res} & & \downarrow \mathrm{res} \\ \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(s_1,s_2)) & \longrightarrow & \mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}(s_1,s_2)) & \xrightarrow{\cong} & \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(b_1,b_2)) \end{array}$$

commutes, and the horizontal maps are isometric inclusions resp. isomorphisms. Therefore it is enough to show that

$$\mathrm{res} : \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(a_1,a_2)) \to \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(b_1,b_2))$$

is injective. But this is Lemma 3.33. □

Corollary 4.17. *Let $s_0 \leq r_0$. The restriction map*

$$res : \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{s_0} \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r_0}$$

is injective.

Proof. This follows from the previous proposition and the fact that projective limits are left exact. □

The K-algebras $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2))$ are Banach algebras. Therefore the rings $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r_0} = \varprojlim_{r_0<r_1\leq r_2<1} \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2))$ are the projective limits of Banach algebras and hence Fréchet algebras. The ring $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ is an inductive limit of Fréchet algebras. We endow $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ with the locally convex inductive limit topology. Over $\mathbb{C}_p$, the isomorphisms $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}})^r \cong \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})^a$ and $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}) \cong \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})$ are topological.

Lemma 4.18. *Let $r_1 \leq s_1 \leq s_2 \leq r_2$. The restriction maps*

$$res : \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2)) \to \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(s_1,s_2))$$

have dense image.

Proof. According to the proof of Proposition 2.1 in [6] the restriction maps

$$\mathrm{res} : \mathcal{O}_K(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i})) \to \mathcal{O}_K(\mathfrak{X}(s_1^{1/p^i}, s_2^{1/p^i}))$$

have dense image for every i. If f lies in $\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(s_1,s_2))$, then a preimage f_{i_0} under the canonical map

$$\alpha_{i_0} : \mathcal{O}_K(\mathfrak{X}(s_1^{1/p^{i_0}}, s_2^{1/p^{i_0}})) \to \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(s_1,s_2))$$

can be approximated by elements in the image of the restriction map

$$\mathrm{res} : \mathcal{O}_K(\mathfrak{X}(r_1^{1/p^{i_0}}, r_2^{1/p^{i_0}})) \to \mathcal{O}_K(\mathfrak{X}(s_1^{1/p^{i_0}}, s_2^{1/p^{i_0}})).$$

The images of these elements under the map α_{i_0} approximate f. We see that the restriction maps

$$\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)) \to \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(s_1, s_2))$$

have dense image. Passing to the completions finishes the proof. □

Proposition 4.19. *$\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ is Hausdorff.*

Proof. We have a continuous inclusion

$$\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \hookrightarrow \mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}) \cong \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}).$$

Therefore it is enough to show that $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})$ is Hausdorff. Let $n_0 \in \mathbb{N}$ and $f, g \in \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})_{n_0} \subseteq \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})$ be two distinct elements. We may assume that $\varphi(T) = T^q + \pi T$. Write $f = f^- + f^+$ and $g = g^- + g^+$ as in Lemma 3.76. Choose an $r_0 > p^{-1/e(q-1)q^{en_0}}$. Then set

$$c := \max\{\|f^+ - g^+\|_{r_0}, \|f^- - g^-\|_1\}.$$

Choose elements s_n such that $p^{-1/e(q-1)q^{e(n+1)}} > s_n > p^{-1/e(q-1)q^{en}}$ for every $n \geq n_0$ and define

$$U_n := \{f \in \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})_n \mid \|f\|_{s_n} \leq c/2\}, \quad \text{and}$$
$$V_n := \{g \in \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})_n \mid \|g\|_{s_n} \leq c/2\}.$$

These are open subsets of $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})_n$. Then in $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})$ we have the open subsets

$$U := f + \sum_{n \geq n_0} U_n, \text{ resp. } V := g + \sum_{n \geq n_0} V_n.$$

Assume that $U \cap V$ is not empty, i.e. there are elements $f_{n_i} \in U_{n_i}$ and $g_{n'_j} \in U_{n'_j}$ such that

$$f + f_{n_1} + \ldots + f_{n_l} = g + g_{s'_1} + \ldots + g_{n'_k}.$$

Then we apply Lemma 3.76 to the f_{n_i} and $g_{n'_i}$ to obtain a decomposition $f_{n_i} = f^+_{n_i} + f^-_{n_i}$ and $g_{n'_i} = g^+_{n'_i} + g^-_{n'_i}$. The uniqueness in Lemma 3.76 implies

$$f^+ - g^+ = f^+_{n_1} + \ldots + f^+_{n_l} - g^+_{n'_1} + \ldots + g^+_{n'_k}, \quad \text{and}$$
$$f^- - g^- = f^-_{n_1} + \ldots + f^-_{n_l} - g^-_{n'_1} + \ldots + g^-_{n'_k}.$$

This implies either

$$\|f^+_{n_1} + \ldots + f^+_{n_l} - g^+_{n'_1} + \ldots + g^+_{n'_k}\|_{r_0} = c, \quad \text{or}$$
$$\|f^-_{n_1} + \ldots + f^-_{n_l} - g^-_{n'_1} + \ldots + g^-_{n'_k}\|_1 = c.$$

We show that both cases are not possible. We have

$$\|f_{n_i}\|_{s_{n_i}} = \max\{\|f_{n_i}^+\|_{s_{n_i}}, \|f_{n_i}^-\|_{s_{n_i}}\} \le c/2$$

after Lemma 3.76 and hence $\|f_{n_i}^+\|_{s_{n_i}} \le c/2$ as well as $\|f_{n_i}^-\|_{s_{n_i}} \le c/2$. Note that the $f_{n_i}^+$ can be regarded as elements in $\mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(r_0))$ so that the norm $\|\cdot\|_{r_0}$ is defined for every $f_{n_i}^+$. Since $r_0 \le s_{n_i}$ for every i we have

$$\|f_{n_i}^+\|_{r_0} \le \|f_{n_i}^+\|_{s_i} \le c/2$$

for all $i = 1, ..., l$. Of course we also have $\|g_{n_i'}^+\|_{r_0} \le c/2$ for $i = 1, ..., k$ with the same arguments. Therefore the first case is not possible.
The second case is impossible for a similar reason. Note that the norm $\|\cdot\|_1$ is defined for every $f_{n_i}^-$ and that we have

$$\|f_{n_i}^-\|_1 \le \|f_{n_i}^-\|_{s_{n_i}} \le c/2$$

for $i = 1, ..., l$. Analogously we have $\|g_{n_i'}^-\|_1 \le c/2$ for all $i = 1, ..., k$, and hence the second case is impossible. Therefore V and U are disjunct open sets in $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})$ which separate f and g. □

Lemma 4.20. *(i) The inclusion*

$$\mathcal{O}_K(\mathfrak{X}(r_1, r_2)) \to \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2))$$

is continuous.

(ii) The inclusion $\mathcal{O}_K(\mathfrak{X} \setminus \mathfrak{X}_n) \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})_n$ is continuous.

(iii) The inclusion $\mathcal{R}_K(\mathfrak{X}) \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ is continuous.

Proof. Part (i) follows from the fact that the inclusion is isometric, (ii) follows from (i). For (iii), we have to show that the composition

$$\mathcal{O}_K(\mathfrak{X} \setminus \mathfrak{X}_n) \to \mathcal{R}_K(\mathfrak{X}) \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$$

is continuous in $\mathcal{O}_K(\mathfrak{X} \setminus \mathfrak{X}_n)$ (see [11, II, 4, Prop. 5]). This follows from (ii) since $\mathcal{O}_K(\mathfrak{X} \setminus \mathfrak{X}_n) \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ factors through $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})_n \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$. □

4.3 Rings of bounded functions

Let $r_0 < r_1 \le s_1 \le s_2 \le r_2$ with $r_0 \in S_n, r_1, r_2, s_1, s_2 \in \bigcup_{m \ge n} S_m$ (after the general assumption on the radii) and $f \in \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r_0}$. The norms $\|f\|_{\mathfrak{X}^{\mathrm{perf}}(r_1,r_2)}$ and $\|f\|_{\mathfrak{X}^{\mathrm{perf}}(s_1,s_2)}$ are defined using the projections $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r_0} \to \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2))$ and $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r_0} \to \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(s_1, s_2))$. We have

$$\|f\|_{\mathfrak{X}^{\mathrm{perf}}(r_1,r_2)} \ge \|f\|_{\mathfrak{X}^{\mathrm{perf}}(s_1,s_2)}.$$

We say that f is *bounded* if there is a constant C such that

$$\|f\|_{\mathfrak{X}^{\mathrm{perf}}(r_1,r_2)} \le C \quad \text{for all } r_0 < r_1 \le r_2.$$

Definition 4.21. *We define*

$$\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r_0,bd} := \{f \in \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r_0} \mid f \text{ is bounded}\}.$$

and

$$\mathcal{E}_K^{\dagger}(\mathfrak{X}^{\mathrm{perf}}) := \varinjlim_{r\to 1} \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r,bd} \subseteq \mathcal{R}(\mathfrak{X}^{\mathrm{perf}}).$$

Let $r_1 > r_0$ and $f \in \mathcal{E}_K^{\dagger}(\mathfrak{X}^{\mathrm{perf}})$. We set $\|f\|_{r_1,1} := \varinjlim_{r\to 1} \|f\|_{\mathfrak{X}^{\mathrm{perf}}(r_1,r)}$. Since $\|\cdot\|_{r_2,1} \leq \|\cdot\|_{r_1,1}$ for $r_1 \leq r_2$, we can define the seminorm

$$\|f\|_1 := \varinjlim_{r\to 1} \|f\|_{r,1}$$

for $f \in \mathcal{E}_K^{\dagger}(\mathfrak{X}^{\mathrm{perf}})$.

Remark 4.22. We have

$$\mathcal{E}_K^{\dagger}(\mathfrak{X}^{\mathrm{perf}}) = \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}})[1/\pi]$$

with

$$\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}}) := \{f \in \mathcal{E}_K^{\dagger}(\mathfrak{X}^{\mathrm{perf}}) \mid \|f\|_1 \leq 1\}.$$

Remark 4.23. The seminorm $\|\cdot\|_1$ is a norm. The isomorphism $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}) \cong \mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}})$ restricts to an isomorphism

$$\mathcal{E}_{\mathbb{C}_p}^{\dagger}(\mathfrak{B}^{\mathrm{perf}}) \cong \mathcal{E}_{\mathbb{C}_p}^{\dagger}(\mathfrak{X}^{\mathrm{perf}}).$$

It is isometric for the $\|\cdot\|_1$-norm on both sides and therefore restricts to an isomorphism $\mathcal{E}_{\mathbb{C}_p}^{\dagger,\leq 1}(\mathfrak{B}^{\mathrm{perf}}) \cong \mathcal{E}_{\mathbb{C}_p}^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}})$.

Proof. The isomorphism $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})^a \cong \mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}})^r$ from Lemma 4.14 restricts to an isomorphism $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})^{a,bd} \cong \mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}})^{r,bd}$. This follows from Lemma 4.12 which moreover implies that the isomorphism is isometric for the $\|\cdot\|_1$-(semi)norm on both sides. Since $\|\cdot\|_1$ is a norm on $\mathcal{E}_{\mathbb{C}_p}^{\dagger}(\mathfrak{B}^{\mathrm{perf}})$ (Lemma 3.79), the same is true for $\|\cdot\|_1$ on $\mathcal{E}_{\mathbb{C}_p}^{\dagger}(\mathfrak{X}^{\mathrm{perf}})$. □

Let $f \in \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2))$ and $t \in \bigcup_{m\geq n} S_m$ with $r_1 \leq t \leq r_2$. We define a norm $\|f\|_t := \|\mathrm{res}(f)\|_{\mathfrak{X}^{\mathrm{perf}}(t,t)}$ analogue to Remark 3.31. Then, using the isometric embedding

$$\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2)) \hookrightarrow \mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2)) \cong \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(a_1,a_2))$$

for certain radii a_1, a_2 (see Lemma 4.12) and that the isomorphism commutes with restrictions on both sides, we note that the isomorphism is isometric for $\|\cdot\|_t$ on $\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2))$ and $\|\cdot\|_a$ on $\mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(a_1,a_2))$ for a certain $a \in [a_1,a_2]$. We see that

$$\|\cdot\|_{\mathfrak{X}^{\mathrm{perf}}(r_1,r_2)} = \max\{\|\cdot\|_{r_1}, \|\cdot\|_{r_2}\}$$

as in Lemma 3.34.

Lemma 4.24. *Let $f \in \mathcal{E}^\dagger_K(\mathfrak{X}^{\mathrm{perf}})$. Then we have*

$$\|f\|_1 = \varinjlim_{r \to 1} \|f\|_r.$$

The norm $\|\cdot\|_1$ is multiplicative.

Proof. We have an isometric embedding $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_0,r)) \to \mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}(r_0,r))$. Therefore we have an isomteric (for $\|\cdot\|_{r_0,1}$) embedding $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r_0,bd} \to \mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}})^{r_0,bd}$, and an isometric embedding (for $\|\cdot\|_1$) $\mathcal{E}^\dagger_K(\mathfrak{X}^{\mathrm{perf}}) \to \mathcal{E}^\dagger_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}})$. Then we can use the isometric isomorphism $\mathcal{E}^\dagger_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}) \cong \mathcal{E}^\dagger_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})$, and the lemma follows from the above observation and Lemma 3.75. □

We finally define $\mathcal{E}_K(\mathfrak{X}^{\mathrm{perf}})$ as the completion of $\mathcal{E}^\dagger_K(\mathfrak{X}^{\mathrm{perf}})$ with respect to $\|\cdot\|_1$.

Lemma 4.25. *An element $f \in \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r_0} \subseteq \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ lies in $\mathcal{E}^\dagger_K(\mathfrak{X}^{\mathrm{perf}})$ if and only if there is an $r_0 \leq s < 1$ such that $\{\|f\|_r \mid s \leq r < 1\}$ is bounded.*

Proof. This follows in same way as in the proof of Lemma 3.74. □

The rings $\mathcal{E}^\dagger_K(\mathfrak{X}^{\mathrm{perf}})$ and $\mathcal{E}^\dagger_K(\mathfrak{X}^{\mathrm{perf}})^r := \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r \cap \mathcal{E}^\dagger_K(\mathfrak{X}^{\mathrm{perf}})$ carry the respective subspace topologies inherited from $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ respective $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r$ (cf. Definition 5.1.3 in [27]).

4.3.1 The weak topology on $\mathcal{E}^{\dagger,\leq 1}_K(\mathfrak{X}^{\mathrm{perf}})$

We have an embedding of rings

$$\mathcal{E}^{\dagger,\leq 1}_K(\mathfrak{X}^{\mathrm{perf}}) \hookrightarrow \mathcal{E}^{\dagger,\leq 1}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}) \cong \mathcal{E}^{\dagger,\leq 1}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}) \hookrightarrow o_{\mathbb{C}_p} \widehat{\otimes}_{o_L} W(\hat{L}^\flat_\infty)_L.$$

The last ring in the row has a weak topology defined in chapter 3. We define the weak topology on $\mathcal{E}^{\dagger,\leq 1}_K(\mathfrak{X}^{\mathrm{perf}})$ to be the subspace topology w.r.t to this embedding.

4.4 The monoid action

4.4.1 The monoid action on $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$

We define an action of $o_L \setminus \{0\}$ on $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$. The action of $a \in o_L^\times$ on the rings $\mathcal{O}_K(\mathfrak{X}(r_1,r_2))$ is isometric for the supremum norm $\|\cdot\|_{\mathfrak{X}(r_1,r_2)}$. Let $f \in \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2)) = \varinjlim_{p_*,i} \mathcal{O}_K(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i}))$ and $a \in o_L^\times$. We define an $o_L^\times$-action on $\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2))$ by setting

$$(a,f) \mapsto \alpha_{i_0}(a_*(f_{i_0})),$$

for a preimage f_{i_0} of f under the canonical map

$$\alpha_{i_0} : \mathcal{O}_K(\mathfrak{X}(r_1^{1/p^{i_0}}, r_2^{1/p^{i_0}})) \to \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1,r_2)).$$

This is well defined because the action of an element $a \in o_L^\times$ on $\mathcal{O}_K(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i}))$ commutes with the transition maps in the inductive limit

$$p_* : \mathcal{O}_K(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i})) \to \mathcal{O}_K(\mathfrak{X}(r_1^{1/p^{i+1}}, r_2^{1/p^{i+1}})).$$

It gives a continuous endomorphism of $\breve{\mathcal{O}}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2))$ for every $a \in o_L^\times$. Passing to the completion then gives a continuous endomorphism a_* of $\mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2))$ which is isometric for $\|\cdot\|_{\mathfrak{X}^{\text{perf}}(r_1,r_2)}$. We get an $o_L^\times$-action on $\mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2))$. The maps a_* commute with the restrictions

$$\text{res} : \mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2)) \to \mathcal{O}_K(\mathfrak{X}^{\text{perf}}(s_1, s_2))$$

for $r_1 \leq s_1 \leq s_2 \leq r_2$, and hence extend to the rings $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^{r_0}$ and then to $\mathcal{R}_K(\mathfrak{X}^{\text{perf}}) = \varinjlim_n \mathcal{R}_K(\mathfrak{X}^{\text{perf}})_n$.

Lemma 4.26. *The* $o_L^\times$*-action on* $\mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2))$ *is continuous.*

Proof. Each element $a \in o_L^\times$ acts by a continuous ring homomorphism on $\mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2))$. It follows from the discussion under Lemma 2.18 in [6] that the orbit maps

$$\rho_f : o_L^\times \to \mathcal{O}_K(\mathfrak{X}(r_1, r_2)),$$
$$a \mapsto a_*(f)$$

for $f \in \mathcal{O}_K(\mathfrak{X}(r_1, r_2))$ are continuous. Let $f \in \breve{\mathcal{O}}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2))$ with preimage f_{i_0} under the map $\alpha_{i_0} : \mathcal{O}_K(\mathfrak{X}(r_1^{1/p^{i_0}}, r_2^{1/p^{i_0}})) \to \breve{\mathcal{O}}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2))$ for some i_0. Then the orbit map

$$\rho_f : o_L^\times \to \mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2)),$$
$$a \mapsto a_*(f)$$

is continuous since it is the composition of $a \mapsto a_* f_{i_0}$, α_{i_0}, and the completion map

$$\breve{\mathcal{O}}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2)) \to \mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2)),$$

which are continuous. If f is the limit of a Cauchy sequence in $\breve{\mathcal{O}}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2))$, then consider the open ball

$$U_\varepsilon(x) = \{g \in \mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2)) \mid \|x - g\|_{\mathfrak{X}^{\text{perf}}(r_1,r_2)} < \varepsilon\}$$

around some $x \in \mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2))$. Let $f_{i_0} \in \breve{\mathcal{O}}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2))$ such that

$$\|f - f_{i_0}\|_{\mathfrak{X}^{\text{perf}}(r_1,r_2)} < \varepsilon/2.$$

Then we have

$$\|x - a_*(f_{i_0})\|_{\mathfrak{X}^{\text{perf}}(r_1,r_2)} < \varepsilon \Leftrightarrow \|x - a_*(f)\|_{\mathfrak{X}^{\text{perf}}(r_1,r_2)} < \varepsilon$$

(since $\|a_*(f) - a_*(f_i)\|_{\mathfrak{X}^{\mathrm{perf}}(r_1,r_2)} = \|f - f_{i_0}\|_{\mathfrak{X}^{\mathrm{perf}}(r_1,r_2)} < \varepsilon/2$), and hence

$$\rho_f^{-1}(U_\varepsilon(x)) = \rho_{f_{i_0}}^{-1}(U_\varepsilon(x))$$

which is open. According to the nonarchimedean Banach-Steinhaus theorem (see [32, Proposition 6.15]) this shows the continuity of the $o_L^\times$-action on $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2))$. □

Lemma 4.27. *The $o_L^\times$-action on $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r$ is continuous.*

Proof. Again, it is enough to show that the orbit maps

$$\rho_f : o_L^\times \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r$$

for $f \in \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r$ are continuous since $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r$ is barrelled. But this follows from the above lemma. □

Lemma 4.28. *The $o_L^\times$-action on $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ is continuous.*

Proof. As an inductive limit of Fréchet spaces, $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ is barrelled, so we can use the nonarchimedean Banach-Steinhaus theorem and show that the orbit maps

$$\rho_f : o_L^\times \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$$

for $f \in \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ are continuous. But this follows from the previous lemma because the map ρ_f factors through $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r$ for some r and the canonical map $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ is continuous. □

Remark 4.29. It follows from the construction of the $o_L^\times$-action and the $o_L^\times$-equivariance of the isomorphism $\mathfrak{B}_{/\mathbb{C}_p} \cong \mathfrak{X}_{/\mathbb{C}_p}$ that the isomorphism $\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)) \cong \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(a_1, a_2))$ from Lemma 4.12 is $o_L^\times$-equivariant. The same is true for the isomorphisms $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}})^r \cong \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})^a$ and $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}) \cong \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})$.

To get an action of the full monoid $o_L \setminus \{0\}$ on $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$, we need the following lemma:

Lemma 4.30. *(i) For any $r \in [p^{-p/(p-1)}, 1) \cap p^{\mathbb{Q}}$ we have $(\pi^*)^{-1}(\mathfrak{X} \setminus \mathfrak{X}(r)) \supseteq \mathfrak{X} \setminus \mathfrak{X}(r^{1/p})$.*

(ii) Let $r_1, r_2 \in [p^{-p/(p-1)}, 1) \cap p^{\mathbb{Q}}$ such that $r_2 \geq r_1^{1/p}$. We have $(\pi^*)^{-1}(\mathfrak{X}(r_1, r_2)) \supseteq \mathfrak{X}(r_1^{1/p}, r_2)$, and hence $(\pi^*)^{-1}(\bigcup_r \mathfrak{X}(r_1, r)) \supseteq \bigcup_r \mathfrak{X}(r_1^{1/p}, r))$.

Proof. (i) This is Lemma 2.11 in [6].

(ii) Let $x \in \mathfrak{X}(r_1^{1/p}, r_2)$. Then $x \in \bigcup_{r \geq r_1^{1/p}} \mathfrak{X}(r_1^{1/p}, r) = \bigcap_{r < r_1^{1/p}} \mathfrak{X} \setminus \mathfrak{X}(r)$ which means $\pi^*(x) \in \bigcap_{r<r_1} \mathfrak{X} \setminus \mathfrak{X}(r) = \bigcup_{r \geq r_1} \mathfrak{X}(r_1, r)$ after (i). On the other hand, we have $x \in \mathfrak{X}(r_2)$ and therefore also $\pi^*(x) \in \mathfrak{X}(r_2)$. Hence $\pi^*(x) \in \mathfrak{X}(r_2) \cap \bigcup_{r \geq r_1} \mathfrak{X}(r_1, r) = \mathfrak{X}(r_1, r_2)$.

□

Let $r_2 > r_1^{1/p}$, $i \in \mathbb{N}$, and $f \in \mathcal{O}_K(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i}))$. Then

$$\pi_*(f) \in \mathcal{O}_K((\pi^*)^{-1}(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i}))),$$

and we have the restriction $\mathrm{res}(\pi_*(f)) \in \mathcal{O}_K(\mathfrak{X}(r_1^{1/p^{i+1}}, r_2^{1/p^i}))$ because of Lemma 4.30. We have commutative diagrams

$$\begin{array}{ccccc}
\mathcal{O}_K(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i})) & \xrightarrow{\pi_*} & \mathcal{O}_K((\pi^*)^{-1}(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i}))) & \xrightarrow{\mathrm{res}} & \mathcal{O}_K(\mathfrak{X}(r_1^{1/p^{i+1}}, r_2^{1/p^i})) \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\
\mathcal{O}_K(\mathfrak{X}(r_1^{1/p^{i+1}}, r_2^{1/p^{i+1}})) & \xrightarrow{\pi_*} & \mathcal{O}_K((\pi^*)^{-1}(\mathfrak{X}(r_1^{1/p^{i+1}}, r_2^{1/p^{i+1}}))) & \xrightarrow{\mathrm{res}} & \mathcal{O}_K(\mathfrak{X}(r_1^{1/p^{i+2}}, r_2^{1/p^{i+1}}))
\end{array}$$

Therefore we can pass to the inductive limit and define a continuous map

$$\varphi : \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)) \to \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1^{1/p}, r_2)),$$

and by passing to the completions

$$\varphi : \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)) \to \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1^{1/p}, r_2)).$$

Since this map φ commutes with the restriction maps, we get a continuous map

$$\varphi : \varprojlim_{r_0<r_1<r_2<1,\, r_1^{1/p}<r_2} \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)) \to \varprojlim_{r_0<r_1<r_2<1,\, r_1^{1/p}<r_2} \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1^{1/p}, r_2)).$$

Since we have topological isomorphisms (continuous bijective linear maps which are then open because of the Open Mapping Theorem)

$$\varprojlim_{r_0<r_1<r_2<1,\, r_1^{1/p}<r_2} \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)) \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r_0} \quad \text{and}$$

$$\varprojlim_{r_0<r_1<r_2<1,\, r_1^{1/p}<r_2} \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1^{1/p}, r_2)) \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r_0^{1/p}},$$

this leads to continuous maps

$$\varphi : \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r^{1/p}}.$$

Especially we have a map

$$\varphi : \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})_n \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})_{n+1}.$$

This leads to a well-defined map

$$\varphi : \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}).$$

To see that the last map is continuous, note that the composition

$$\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})_n \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \xrightarrow{\varphi} \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$$

is continuous since it is equal to the composition

$$\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})_n \xrightarrow{\varphi} \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})_{n+1} \hookrightarrow \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}).$$

Then use [11, II, 4, Prop. 5].

Note that we have $u_*(\varphi^n(f)) = \varphi^n(u_*(f))$ for all $u \in o_L^\times, n \in \mathbb{N}$, and $f \in \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$. We get a continuous action of $o_L \setminus \{0\}$ on $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$.

Remark 4.31. The isomorphism $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}) \cong \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})$ is equivariant for the action of $o_L \setminus \{0\}$ on both sides. This follows from the $o_L \setminus \{0\}$-equivariance of the isomorphism $\mathfrak{X}_{/\mathbb{C}_p} \cong \mathfrak{B}_{/\mathbb{C}_p}$ and the construction of the $o_L \setminus \{0\}$-action on the rings. We already noted this for the $o_L^\times$-action. We look in detail at the action of π : Let $r < r_1^{1/p} < r_2$ and let a, a_1, a_2 the corresponding radii as in Remark 2.12. Then we have commutative diagrams

$$\begin{array}{ccccc}
\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i})) & \xrightarrow{\pi_*} & \mathcal{O}_{\mathbb{C}_p}((\pi^*)^{-1}(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i}))) & \xrightarrow{\mathrm{res}} & \mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}(r_1^{1/p^{i+1}}, r_2^{1/p^i})) \\
\cong\downarrow & & \downarrow\cong & & \downarrow\cong \\
\mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(a_1^{1/q^{ei}}, a_2^{1/q^{ei}})) & \xrightarrow{\pi_*} & \mathcal{O}_{\mathbb{C}_p}((\pi^*)^{-1}(\mathfrak{B}(a_1^{1/q^{ei}}, a_2^{1/q^{ei}}))) & \xrightarrow{\mathrm{res}} & \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}(a_1^{1/q^{e(i+1)}}, a_2^{1/q^{ei}}))
\end{array}$$

which lead to a commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)) & \xrightarrow{\varphi} & \mathcal{O}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}(r_1^{1/p}, r_2)) \\
\downarrow\cong & & \downarrow\cong \\
\mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(a_1, a_2)) & \xrightarrow{\mathrm{res}\circ\varphi} & \mathcal{O}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}(a_1^{1/qe}, a_2))
\end{array}$$

and then to

$$\begin{array}{ccc}
\mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}})^r & \xrightarrow{\varphi} & \mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}})^{r^{1/p}} \\
\downarrow\cong & & \downarrow\cong \\
\mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})^a & \xrightarrow{\mathrm{res}\circ\varphi} & \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})^{a^{1/qe}}.
\end{array}$$

But for an element $f \in \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})^a \subseteq \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})$, the elements $\varphi(f) \in \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})^{a^{1/q}}$ and $\mathrm{res}(\varphi(f)) \in \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})^{a^{1/qe}}$ coincide in the inductive limit $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})$, so that the diagram

$$\begin{array}{ccc}
\mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}) & \xrightarrow{\varphi} & \mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}) \\
\downarrow\cong & & \downarrow\cong \\
\mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}) & \xrightarrow{\varphi} & \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})
\end{array}$$

commutes.

Proposition 4.32. *The action of π on $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})$ is bijective.*

Proof. Firstly, we show that the map

$$p_* : \breve{\mathcal{O}}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2)) \to \breve{\mathcal{O}}_K(\mathfrak{X}^{\text{perf}}(r_1^{1/p}, r_2^{1/p}))$$

which comes from the maps $p_* : \mathcal{O}_K(\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i})) \to \mathcal{O}_K(\mathfrak{X}((r_1^{1/p})^{1/p^i}, (r_2^{1/p})^{1/p^i})$ is bijective. It is injective since it is an isometry (this follows from Lemma 4.11). To show surjectivity, let $f \in \breve{\mathcal{O}}_K(\mathfrak{X}^{\text{perf}}(r_1^{1/p}, r_2^{1/p}))$ be an element with preimage $f_{i_0} \in \mathcal{O}_K(\mathfrak{X}((r_1^{1/p})^{1/p^{i_0}}, (r_2^{1/p})^{1/p^{i_0}}))$ under the canonical map

$$\alpha_{i_0} : \mathcal{O}_K(\mathfrak{X}((r_1^{1/p})^{1/p^{i_0}}, (r_2^{1/p})^{1/p^{i_0}})) \to \breve{\mathcal{O}}_K(\mathfrak{X}^{\text{perf}}(r_1^{1/p}, r_2^{1/p}))$$

for some $i_0 \in \mathbb{N}$. Then the image of f_{i_0} in $\breve{\mathcal{O}}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2))$ under the map

$$\alpha'_{i_0+1} : \mathcal{O}_K(\mathfrak{X}(r_1^{1/p^{i_0+1}}, r_2^{1/p^{i_0+1}})) \to \breve{\mathcal{O}}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2))$$

is a preimage of f under p_* since we have $p_*(\alpha'_{i_0+1}(f_{i_0})) = \alpha_{i_0+1}(p_*(f_{i_0})) = f$. Passing to completions, we see that

$$p_* : \mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2)) \to \mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r_1^{1/p}, r_2^{1/p}))$$

is bijective.
The induced map on the projective limits

$$\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^{r_0} = \varprojlim_{r_0 < r_1 \le r_2 < 1} \mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2)) \to \mathcal{R}_K(\mathfrak{X}^{\text{perf}})^{r_0^{1/p}} = \varprojlim_{r_0^{1/p} < r_1 \le r_2 < 1} \mathcal{O}_K(\mathfrak{X}^{\text{perf}}(r_1, r_2))$$

is bijective. It follows that

$$p_* : \mathcal{R}_K(\mathfrak{X}^{\text{perf}}) \to \mathcal{R}_K(\mathfrak{X}^{\text{perf}})$$

is bijective. This map $p_* : \mathcal{R}_K(\mathfrak{X}^{\text{perf}}) \to \mathcal{R}_K(\mathfrak{X}^{\text{perf}})$ is equal to a power of φ times an automorphism. This is because we have $\pi^e u = p$ for a unit $u \in o_L^\times$. Then the maps

$$u_* \circ \varphi^e : \mathcal{R}_K(\mathfrak{X}^{\text{perf}})^r \to \mathcal{R}_K(\mathfrak{X}^{\text{perf}})^{r^{1/p^e}}$$

and

$$\text{res} \circ p_* : \mathcal{R}_K(\mathfrak{X}^{\text{perf}})^r \to \mathcal{R}_K(\mathfrak{X}^{\text{perf}})^{r^{1/p^e}}$$

agree which implies that p_* and $u_* \circ \varphi^e$ agree on the inductive limit $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})$ (but only there, otherwise the radii are not compatible).
We conclude that

$$\varphi : \mathcal{R}_K(\mathfrak{X}^{\text{perf}}) \to \mathcal{R}_K(\mathfrak{X}^{\text{perf}})$$

is bijective. □

Remark 4.33. The inverse map $\varphi^{-e} : \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ is equal to $u_* \circ p_*^{-1}$ and thus we may regard it as a map

$$\varphi^{-e} : \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r^p}$$

for all $r \in S_n, n \geq 1$.

4.4.2 The monoid action on $\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})$ and $\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^{\leq 1}$

By construction, the map $\varphi : \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ restricts to a map

$$\varphi : \mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}}) \to \mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}}).$$

Furthermore, we have $\|\varphi(f)\|_{r^{1/p},1} \leq \|f\|_{r,1}$ for any $f \in \mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})$. This implies $\|\varphi(f)\|_1 \leq \|f\|_1$ so that φ restricts to a map

$$\varphi : \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}}) \to \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}}).$$

The action of $o_L^\times$ restricts to an action on the rings $\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})$, $\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^r := \mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}}) \cap \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r$, and $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}})$. If we endow these rings with the subspace topologies inherited from the larger rings $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ and $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r$, the action is continuous.

Lemma 4.34 (Lemma 3.1.1 in [19])**.** *Let V be a topological K-vector space and G a topological group which acts on V via linear endomorphisms. Then the action is continuous if and only if it satisfies the following:*

1. *for each $v \in V$, the orbit map $G \to V, g \mapsto gv$ is continuous,*
2. *for each $g \in G$, the map $V \to V, v \mapsto gv$ is continuous,*
3. *the map $G \times V \to V, (g, v) \mapsto gv$ is continuous at $(e, 0) \in G \times V$ (where $e \in G$ denotes the identity element).*

Proposition 4.35. *Endow $\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^r \subseteq \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r$ with the subspace topology, and $\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}}) = \varinjlim_r \mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^r$ with the locally convex inductive limit topology. Then the $o_L^\times$-action on $\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})$ is continuous for this topology.*

Proof. The first and second condition of the above lemma are fulfilled. We have to show that that action is continuous at $(1, 0)$. Let $U \subseteq \mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})$ be a neighbourhood of 0. We may assume that $U = \sum_r U_r$ for neighbourhoods of zero $U_r \subseteq \mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^r$ which are of the form

$$U_r = \{f \in \mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^r \mid \|f\|_s \leq \varepsilon_r \text{ for finitely many } s \in (r, 1)\}$$

for some $\varepsilon_r > 0$ since sets of this form form a defining family of lattices for the topology on $\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^r$, see [32, §5 A, and Proposition 4.3]. Then $o_L^\times \times U$ is in the preimage of U since we have $u_*(U_r) = U_r$ for every r (as the action is isometric). $\square$

Remark 4.36. The $o_L^\times$-action on $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}})$ is continuous for the weak topology.

Proof. The $o_L^\times$-action on $\mathcal{E}_{\mathbb{C}_p}^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}}) \cong \mathcal{E}_{\mathbb{C}_p}^{\dagger,\leq 1}(\mathfrak{B}^{\mathrm{perf}})$ is continuous for the weak topology, and the weak topology on $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}})$ is the subspace topology coming from $\mathcal{E}_{\mathbb{C}_p}^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}})$. □

It is also possible to endow the rings $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}})^r = \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}}) \cap \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r$ with the subspace topology from $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r$. Then the union $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}}) = \varinjlim_r \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}})^r$ has the inductive limit topology. Since $o_L^\times$ is locally compact, we have a homeomorphism

$$o_L^\times \times \varinjlim_r \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}})^r \cong \varinjlim_r (o_L^\times \times \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}})^r).$$

By passing to the limit, this shows that the $o_L^\times$-action on $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}})$ is continuous for this topology as well since

$$o_L^\times \times \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}})^r \to \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}})^r$$

is continuous.

4.5 Towards a preperfectoid character variety

In the rigid analytic world, the $\mathfrak{X}(r)$ form an open covering of the character variety $\mathfrak{X}$. It would be desirable to form an adic *preperfectoid character variety* by glueing together the preadic spaces $\mathfrak{X}^{\mathrm{perf}}(r)$. But we do not know wether

$$\mathcal{O}_{\mathfrak{X}_K^{\mathrm{perf}}(r)}(V) \to \mathcal{O}_{\mathfrak{X}_K^{\mathrm{perf}}(s)}(V)$$

for $s < r$ and a rational subset $V \subseteq \mathfrak{X}_K^{\mathrm{perf}}(r)$ is an isomorphism of topological rings. If one could show that the ring $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r))$ are stably uniform, then this would be the case. But this turns out to be a quite difficult problem, as it is often the case in the adic world. In their preprint [21], Hansen and Kedlaya developed several notions for Huber rings which imply stable uniformity and behave (or are expected to behave) better under several constructions such as étale extensions or profinite étale extensions. In this section, we explain a conjecture from [21] and discuss how it implies the existence of a preperfectoid character variety.

Let A be a complete Tate ring with ring of definition A_0 and ideal of definition I. We say that an A_0-module M is *torsion* if every element of M is killed by a power of I. We say that M is *uniformly torsion* if M is killed by a power of I. If M is chosen from a set $\{M_j\}_{j\in J}$ and M is killed by some power of I which can be chosen independently of j, we will say that M is j-uniformly torsion. In the following, all Huber rings and Huber pairs are assumed to be complete and Tate.

Definition 4.37. *Let (A, A^+) be a Huber pair. The* v-topology *on* $X = \mathrm{Spa}(A, A^+)$ *is the Grothendieck topology whose objects are preadic spaces (in the sense of [21]) over* $\mathrm{Spa}(A, A^+)$*, and where a family of morphisms* $\{f_i : U_i \to X\}_{i\in I}$ *is a covering if every quasicompact open subset of* X *is contained in the image of some quasicompact open subset of* $\coprod_{i\in I} U_i$*. We denote by* $\mathrm{Spa}(A, A^+)_v$ *the resulting site which carries natural presheafs* $\mathcal{O}$ *and* $\mathcal{O}^+$*.*

Definition 4.38.
1. *A Tate-Huber pair (A, A^+) with topologically nilpotent unit $t \in A^+$ is* plus-sheafy *if the A^+-modules* $\mathrm{coker}(A^+/t^nA^+ \to H^0(\mathrm{Spa}(A, A^+), \mathcal{O}^+/t^n))$ *and* $\ker(A^+/t^nA^+ \to H^0(\mathrm{Spa}(A, A^+), \mathcal{O}^+/t^n))$ *are n-uniformly torsion, and for each positive integer i, the A^+-modules* $H^i(\mathrm{Spa}(A, A^+), \mathcal{O}^+/t^n)$ *are n-uniformly torsion.*
2. *Let (A, A^+) be a Huber pair in which p is topologically nilpotent. We define the* v-completion $(\check{A}, \check{A}^+)$ *to be the Huber pair with* $\check{A} = H^0(\mathrm{Spa}(A, A^+)_v, \mathcal{O})$ *and* $\check{A}^+ = H^0(\mathrm{Spa}(A, A^+)_v, \mathcal{O}^+)$*. A Huber pair (A, A^+) is said to be* v-complete *if the natural map $(A, A^+) \to (\check{A}, \check{A}^+)$ is an isomorphism.*
3. *A Huber ring A over $\mathbb{Q}_p$ is* diamantine *if A is plus-sheafy and v-complete.*

According to [2, Theorem, Folgerung 3] every smooth affinoid algebra over a nonarchimedean field is plus-sheafy (cf. Remark 6.16 in [21]).

Lemma 4.39 (Lemma 11.9 in [21])**.** *Diamantine Huber rings are stably uniform.*

Theorem 4.40. *Let A be a reduced (hence uniform) affinoid algebra over a nonarchimedean field of mixed characteristics. Then the following conditions are equivalent:*

1. *The ring A is seminormal (i.e. A is reduced and if $x, y \in A$ fulfil $x^3 = y^2$, then there is an $s \in A$ with $s^2 = x$ and $s^3 = y$).*
2. *The ring A is v-complete.*
3. *The A°-module $H^1(\mathrm{Spa}(A, A^\circ), \mathcal{O}^+)$ is uniformly torsion.*

Proof. Theorem 10.3 in [21]. □

Lemma 4.41. *A reduced smooth affinoid algebra over a nonarchimedean field of mixed characteristic is diamantine. In particular, $\mathcal{O}_K(\mathfrak{X}(r))$ and $\mathcal{O}_K(\mathfrak{X}(r_1, r_2))$ are diamantine.*

Proof. According to [21, Theorem 11.18], any smooth affinoid algebra over a nonarchimedean field is diamantine. The varieties $\mathfrak{X}$ and hence $\mathfrak{X}(r)$ and $\mathfrak{X}(r_1, r_2)$ are smooth ([35, text before Lemma 2.4]). □

Definition 4.42. *A map $A \to B$ of uniform Huber rings is* profinite étale *of B is the completion of an inductive limit of subalgebras which are finite étale over A.*

Conjecture 4.43 (Hansen, Kedlaya)**.** *Let A be a diamantine Huber ring. Then for any profinite étale map $A \to B$, B is also diamantine.*

The maps $\mathcal{O}_K(\mathfrak{X}(r)) \to \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r))$ and $\mathcal{O}_K(\mathfrak{X}(r_1, r_2)) \to \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2))$ are profinite étale. Together with Lemma 4.41 Conjecture 4.43 implies the following

Conjecture 4.44. *The Tate-Huber pairs $(\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r)), \mathcal{O}_K^+(\mathfrak{X}^{\mathrm{perf}}(r)))$ and $(\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)), \mathcal{O}_K^+(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)))$ are stably uniform.*

We now discuss how the conjecture implies the existence of the preperfectoid character variety. Firstly, we need the following lemma:

Lemma 4.45. *Let $(X_i, \alpha_{i,j})_{i,j \in \mathbb{N}}$ and $(Y_i, \beta_{i,j})_{i,j \in \mathbb{N}}$ two projective systems of topological spaces with projective limits X and Y, and assume that we have open maps $f_i : X_i \to Y_i$ for all i such that the diagrams*

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_i \\ \uparrow{\scriptstyle \alpha_{i+1,i}} & & \uparrow{\scriptstyle \beta_{i+1,i}} \\ X_{i+1} & \xrightarrow[f_{i+1}]{} & Y_{i+1} \end{array}$$

are commutative for all i. Assume moreover that if $f_i(x_i) = \beta_{i+1,i}(y_{i+1})$ for $x_i \in X_i$ and $y_{i+1} \in Y_{i+1}$, then there is an $x_{i+1} \in X_{i+1}$ with $\alpha_{i+1,i}(x_{i+1}) = x_i$ and $f_{i+1}(x_{i+1}) = y_{i+1}$. Then the induced map $f : X \to Y$ is open.

Proof. For $i_0 \in \mathbb{N}$ denote by $\mathrm{pr}_{X,i_0} : X \to X_{i_0}$ resp. $\mathrm{pr}_{X,i_0} : Y \to Y_{i_0}$ the projection on X_{i_0} resp. Y_{i_0}. Let $\mathrm{pr}_{X,i_0}^{-1}(U) \subseteq X$ be an open subset where $U \subseteq X_{i_0}$ is open, then we claim that

$$f(\mathrm{pr}_{X,i_0}^{-1}(U)) = \mathrm{pr}_{Y,i_0}^{-1}(f_{i_0}(U)) \subseteq Y$$

which is open: The inclusion $f(\mathrm{pr}_{X,i_0}^{-1}(U)) \subseteq \mathrm{pr}_{Y,i_0}^{-1}(f_{i_0}(U))$ is clear. On the other hand, if $y = (y_0, ..., y_{i_0}, ...) \in \mathrm{pr}_{Y,i_0}^{-1}(f_{i_0}(U))$ then

$$\beta_{i_0+1,i_0}(y_{i_0+1}) = y_{i_0} = f_{i_0}(x_{i_0})$$

for some $x_{i_0} \in U$ and we find an $x_{i_0+1} \in X_{i_0+1}$ with $\alpha_{i_0+1}(x_{i_0+1}) = x_{i_0}$ and $f_{i_0+1}(x_{i_0+1}) = y_{i_0+1}$. Inductively we find elements $x_j \in X_j$ such that $x = (..., x_j, ...) \in \mathrm{pr}_{i_0}^{-1}(U)$ and $f(x) = y$. □

Remark 4.46. The diagram in Lemma 4.45 is called *exact* in [14, (2.1)]. Lemma 4.45 is (a weaker version of) [14, Theorem 3.29].

Proposition 4.47. *If Conjecture 4.44 holds true, then the map $\mathfrak{X}_K^{\mathrm{perf}}(r) \to \mathfrak{X}_K^{\mathrm{perf}}(s)$ is an open immersion of adic spaces for $r \leq s$,.*

Proof. The rigid-analytic space $\mathfrak{X}(r^{1/p^i})$ is an affinoid subdomain of $\mathfrak{X}(s^{1/p^i})$ for all $i \in \mathbb{N}$. Therefore we have open immersions $\mathfrak{X}(r^{1/p^i}) \hookrightarrow \mathfrak{X}(s^{1/p^i})$ for $i \in \mathbb{N}$, which commute with the transition maps p_*. Since open immersions are preserved under the functor r_L from rigid-analytic spaces to adic spaces (see 1.2.2), we have open immersions $\iota_i : \mathfrak{X}(r^{1/p^i})^{\mathrm{ad}} \hookrightarrow \mathfrak{X}(s^{1/p^i})^{\mathrm{ad}}$ of adic spaces for all $i \in \mathbb{N}$. We identify the image of ι_i with $\mathfrak{X}(r^{1/p^i})^{\mathrm{ad}}$.
The induced map $|\mathfrak{X}_K^{\mathrm{perf}}(r)| \cong \varprojlim_i |\mathfrak{X}(r^{1/p^i})^{\mathrm{ad}}| \to \varprojlim_i |\mathfrak{X}(s^{1/p^i})^{\mathrm{ad}}| \cong |\mathfrak{X}_K^{\mathrm{perf}}(s)|$ coincides with the map induced by the map $\mathfrak{X}_K^{\mathrm{perf}}(r) \to \mathfrak{X}_K^{\mathrm{perf}}(s)$ coming from the restriction $\mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(s)) \to \mathcal{O}_K(\mathfrak{X}^{\mathrm{perf}}(r))$.

We get an injective map of topological spaces

$$\iota : |\mathfrak{X}_K^{\mathrm{perf}}(r)| = \varprojlim_i |\mathfrak{X}(r^{1/p^i})^{\mathrm{ad}}| \hookrightarrow \varprojlim_i |\mathfrak{X}(s^{1/p^i})^{\mathrm{ad}}| = |\mathfrak{X}_K^{\mathrm{perf}}(s)|.$$

We consider the commutative diagram

$$\begin{array}{ccc} |\mathfrak{X}(r^{1/p^i})^{\mathrm{ad}}| & \longrightarrow & |\mathfrak{X}(s^{1/p^i})^{\mathrm{ad}}| \\ \uparrow{\scriptstyle p^*} & & \uparrow{\scriptstyle p^*} \\ |\mathfrak{X}(r^{1/p^{i+1}})^{\mathrm{ad}}| & \longrightarrow & |\mathfrak{X}(s^{1/p^{i+1}})^{\mathrm{ad}}|. \end{array}$$

Let $x \in |\mathfrak{X}(r^{1/p^i})^{\mathrm{ad}}|$ with $x = p^*(y)$ for some $y \in |\mathfrak{X}(s^{1/p^{i+1}})^{\mathrm{ad}}|$. By Lemma 2.8 and Remark 1.52 we have $|\mathfrak{X}(r^{1/p^{i+1}})^{\mathrm{ad}}| = (p^*)^{-1}(|\mathfrak{X}(r^{1/p^i})^{\mathrm{ad}}|)$ and hence $y \in |\mathfrak{X}(r^{1/p^{i+1}})^{\mathrm{ad}}| = (p^*)^{-1}(|\mathfrak{X}(r^{1/p^i})^{\mathrm{ad}}|) \subseteq |\mathfrak{X}(s^{1/p^{i+1}})^{\mathrm{ad}}|$. We can apply Lemma 4.45 and see that $|\mathfrak{X}_K^{\mathrm{perf}}(r)| \hookrightarrow |\mathfrak{X}_K^{\mathrm{perf}}(s)|$ is open.

For every i we have an isomorphism $\mathcal{O}_{\mathfrak{X}(r^{1/p^i})} \to \mathcal{O}_{\mathfrak{X}(s^{1/p^i})|\mathfrak{X}(r^{1/p^i})}$. Let

$$V \subseteq |\mathfrak{X}_K^{\mathrm{perf}}(r)| \subseteq |\mathfrak{X}_K^{\mathrm{perf}}(s)|$$

be a rational subset. By identifying the rational subsets of $|\mathfrak{X}_K^{\mathrm{perf}}(r)|$ and $\varprojlim_i |\mathfrak{X}(r^{1/p^i})^{\mathrm{ad}}|$ as in Proposition 1.38 we can assume that V is pulled back from a rational subset $V_{i_0} \subseteq |\mathfrak{X}(r^{1/p^{i_0}})^{\mathrm{ad}}|$ for some i_0. We get induced isomorphisms of topological rings

$$\mathcal{O}_{\mathfrak{X}_K^{\mathrm{perf}}(r)}(V) \to \mathcal{O}_{\mathfrak{X}_K^{\mathrm{perf}}(s)|\mathfrak{X}_K^{\mathrm{perf}}(r)}(V)$$

as both can be computed as the completed inductive limits of the $\mathcal{O}_{\mathfrak{X}(r^{1/p^i})}(V_i)$ respective $\mathcal{O}_{\mathfrak{X}(s^{1/p^i})|\mathfrak{X}(r^{1/p^i})}(V_i)$ with V_i being the preimage of V_{i_0} under the map $|\mathfrak{X}(r^{1/p^i})^{\mathrm{ad}}| \to |\mathfrak{X}(r^{1/p^{i_0}})^{\mathrm{ad}}|$ (see Remark 1.58). Here we use the conjecture. Note that this isomorphism identifies $\mathcal{O}^+_{\mathfrak{X}_K^{\mathrm{perf}}(r)}(V)$ and $\mathcal{O}^+_{\mathfrak{X}_K^{\mathrm{perf}}(s)|\mathfrak{X}_K^{\mathrm{perf}}(r)}(V)$.

If $U \subseteq V$ is another rational subset, the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{X}_K^{\mathrm{perf}}(r)}(V) & \longrightarrow & \mathcal{O}_{\mathfrak{X}_K^{\mathrm{perf}} f(s)|\mathfrak{X}_K^{\mathrm{perf}}(r)}(V) \\ {\scriptstyle \mathrm{res}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ \mathcal{O}_{\mathfrak{X}_K^{\mathrm{perf}}(r)}(U) & \longrightarrow & \mathcal{O}_{\mathfrak{X}_K^{\mathrm{perf}}(s)|\mathfrak{X}_K^{\mathrm{perf}}(r)}(U) \end{array}$$

since the corresponding diagrams

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{X}(r^{1/p^i})}(V_i) & \longrightarrow & \mathcal{O}_{\mathfrak{X}(s)|\mathfrak{X}(r^{1/p^i})}(V_i) \\ {\scriptstyle \mathrm{res}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ \mathcal{O}_{\mathfrak{X}(r^{1/p^i})}(U_i) & \longrightarrow & \mathcal{O}_{\mathfrak{X}(s)|\mathfrak{X}(r^{1/p^i})}(U_i) \end{array}$$

commute. We get an isomorphism of sheaves of topological rings $\mathcal{O}_{\mathfrak{X}_K^{\mathrm{perf}}(r)} \to \mathcal{O}_{\mathfrak{X}_K^{\mathrm{perf}}(s)|\mathfrak{X}_K^{\mathrm{perf}}(r)}$ which induces an isomorphism $\mathcal{O}^+_{\mathfrak{X}_K^{\mathrm{perf}}(r)} \cong \mathcal{O}^+_{\mathfrak{X}_K^{\mathrm{perf}}(s)|\mathfrak{X}_K^{\mathrm{perf}}(r)}$. Then Lemma 1.47 shows that we get an isomorphism of adic spaces $(\mathfrak{X}_K^{\mathrm{perf}}(r), \mathcal{O}_{\mathfrak{X}_K^{\mathrm{perf}}(r)}) \cong (\mathfrak{X}_K^{\mathrm{perf}}(r), \mathcal{O}_{\mathfrak{X}_K^{\mathrm{perf}}(s)|\mathfrak{X}_K^{\mathrm{perf}}(r)})$.

□

If Conjecture 4.44 holds true, we can glue together the spaces $\mathfrak{X}_L^{\mathrm{perf}}(r)$ and define the adic space

$$\mathfrak{X}_L^{\mathrm{perf}} := \varinjlim_r \mathfrak{X}_L^{\mathrm{perf}}(r).$$

This would be a preperfectoid space in the sense of Definition 1.74.

In the following, let $s_1 \le r_1 \le r_2 \le s_2 < 1$. In the same way as desribed before Proposition 4.47, the maps $\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i}) \to \mathfrak{X}(r_2^{1/p^i})$ and $\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i}) \to \mathfrak{X}(s_1^{1/p^i}, s_2^{1/p^i})$ induce maps $\mathfrak{X}_K^{\mathrm{perf}}(r_1, r_2) \to \mathfrak{X}_K^{\mathrm{perf}}(r_2)$ and $\mathfrak{X}_K^{\mathrm{perf}}(r_1, r_2) \to \mathfrak{X}_K^{\mathrm{perf}}(s_1, s_2)$.

Proposition 4.48. *If Conjecture 4.44 holds true, then the* $\mathfrak{X}_K^{\mathrm{perf}}(r_1, r_2) \to \mathfrak{X}_K^{\mathrm{perf}}(r_2)$ *and* $\mathfrak{X}_K^{\mathrm{perf}}(r_1, r_2) \to \mathfrak{X}_K^{\mathrm{perf}}(s_1, s_2)$ *are open immersions.*

Proof. Ad $\mathfrak{X}_K^{\mathrm{perf}}(r_1, r_2) \to \mathfrak{X}_K^{\mathrm{perf}}(r_2)$: We have open immersions $\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i}) \subseteq \mathfrak{X}(r_2^{1/p^i})$ of rigid-analytic spaces for all i since $\mathfrak{X}(r_1, r_2)$ is an affinoid subdomain of $\mathfrak{X}(r_1)$. As in the proof of Proposition 4.47 we get an open immersion of adic spaces

$$\mathfrak{X}^{\mathrm{ad}}(r_1^{1/p^i}, r_2^{1/p^i}) \hookrightarrow \mathfrak{X}^{\mathrm{ad}}(r_2^{1/p^i}),$$

and therefore, by passing to the limits, an injective map of the topological spaces

$$|\mathfrak{X}_K^{\mathrm{perf}}(r_1, r_2)| \hookrightarrow |\mathfrak{X}_K^{\mathrm{perf}}(r_2)|,$$

which coincides with the map induced by the map $\mathfrak{X}_K^{\mathrm{perf}}(r_1, r_2)_K \to \mathfrak{X}_K^{\mathrm{perf}}(r_2)$ defined above. Again, we have commutative diagrams

$$\begin{array}{ccc} |\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i})^{\mathrm{ad}}| & \longrightarrow & |\mathfrak{X}(r_2^{1/p^i})^{\mathrm{ad}}| \\ \uparrow{\scriptstyle p_*} & & \uparrow{\scriptstyle p_*} \\ |\mathfrak{X}(r_1^{1/p^{i+1}}, r_2^{1/p^{i+1}})^{\mathrm{ad}}| & \longrightarrow & |\mathfrak{X}(r_2^{1/p^{i+1}})^{\mathrm{ad}}| \end{array} .$$

If $x \in |\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i})^{\mathrm{ad}}|$ such that $p^*(y) = x$ for some $y \in |\mathfrak{X}(r_2^{1/p^{i+1}})^{\mathrm{ad}}|$, then we have $y \in |\mathfrak{X}(r_1^{1/p^{i+1}}, r_2^{1/p^{i+1}})^{\mathrm{ad}}| = (p^*)^{-1}(|\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i})^{\mathrm{ad}}|)$ (by Lemma 4.10 and Remark 1.52). Again, we can apply Lemma 4.45 and see that the we have an open immersion $|\mathfrak{X}_K^{\mathrm{perf}}(r_1, r_2)| \hookrightarrow |\mathfrak{X}_K^{\mathrm{perf}}(r_2)|$ of topological spaces.

Ad $\mathfrak{X}_K^{\mathrm{perf}}(r_1, r_2) \to \mathfrak{X}_K^{\mathrm{perf}}(s_1, s_2)$: We note that

$$\mathfrak{X}(r_1, r_2) \subseteq \mathfrak{X}(s_1, s_2),$$

and that we have open immersions open immersions $\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i}) \hookrightarrow \mathfrak{X}(s_1^{1/p^i}, s_2^{1/p^i})$ of rigid-analytic spaces for all i since $\mathfrak{X}(r_1, r_2)$ is an affinoid subdomain of $\mathfrak{X}(s_1, s_2)$. As in the proof of Proposition 4.47 we get an open immersion of adic spaces

$$\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i})^{\mathrm{ad}} \hookrightarrow \mathfrak{X}(s_1^{1/p^i}, s_2^{1/p^i})^{\mathrm{ad}},$$

and therefore, by passing to the limits, an injective map of topological spaces

$$|\mathfrak{X}^{\mathrm{perf}}(r_1^{1/p^i}, r_2^{1/p^i})| \hookrightarrow |\mathfrak{X}^{\mathrm{perf}}(s_1^{1/p^i}, s_2^{1/p^i})|,$$

which coincides with the map induced by the map $\mathfrak{X}_K^{\mathrm{perf}}(r_1, r_2) \to \mathfrak{X}_K^{\mathrm{perf}}(s_1, s_2)$ defined above. Again, we have commutative diagrams

$$\begin{array}{ccc} |\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i})^{\mathrm{ad}}| & \longrightarrow & |\mathfrak{X}(s_1^{1/p^i}, s_2^{1/p^i})^{\mathrm{ad}}| \\ {\scriptstyle p^*}\uparrow & & {\scriptstyle p^*}\uparrow \\ |\mathfrak{X}(r_1^{1/p^{i+1}}, r_2^{1/p^{i+1}})^{\mathrm{ad}}| & \longrightarrow & |\mathfrak{X}(s_1^{1/p^{i+1}}, s_2^{1/p^{i+1}})^{\mathrm{ad}}| \end{array}$$

If $x \in |\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i})^{\mathrm{ad}}|$ such that $p^*(y) = x$ for some $y \in |\mathfrak{X}(s_1^{1/p^{i+1}}, s_2^{1/p^{i+1}})^{\mathrm{ad}}|$, then we have $y \in |\mathfrak{X}^{\mathrm{ad}}(r_1^{1/p^{i+1}}, r_2^{1/p^{i+1}})| = (p^*)^{-1}(|\mathfrak{X}(r_1^{1/p^i}, r_2^{1/p^i})^{\mathrm{ad}}|)$ (Lemma 4.10 and Remark 1.52). Again, we can apply Lemma 4.45 and see that the we have an open immersion $|\mathfrak{X}_K^{\mathrm{perf}}(r_1, r_2)| \hookrightarrow |\mathfrak{X}_K^{\mathrm{perf}}(s_1, s_2)|$ of topological spaces.

The proof of the statement about the structure sheaves is basically the same as in the proof of Proposition 4.47. $\square$

5 (φ, Γ)-Modules

5.1 Generalities about φ-modules

Let R be a ring with an endomorphism φ. In the following, the tensor product $R \otimes_{R,\varphi} M$ denotes the usual tensor product, but with R regarded as a right R-module via φ.

Definition 5.1. *A φ-module over R is a finitely generated projective R-module M which carries a semilinear action of φ denoted by φ_M such that the R-linear map*

$$\begin{aligned} \varphi_M^{lin} : R \otimes_{R,\varphi} M &\to M, \\ f \otimes m &\mapsto f\varphi_M(m) \end{aligned}$$

is bijective.

We define the category of φ-modules over R whose objects are φ-modules over R and the morphisms are R-module homomorphisms $\alpha : M \to N$ between φ-modules M and N such that

$$\varphi_N \circ \alpha = \alpha \circ \varphi_M.$$

We denote by $\mathrm{Hom}_\varphi(M, N)$ the set of all such morphisms between M and N.

Lemma 5.2 (see 1.1.4 in [20])**.** *Let M be a φ-module. If $\varphi : R \to R$ is bijective, then $\varphi_M : M \to M$ is bijective if and only if $\varphi_M^{lin} : R \otimes_{R,\varphi} M \to M$ is bijective.*

Proof. If $\varphi : R \to R$ is bijective, then we have $r \otimes m = 1 \otimes \varphi^{-1}(r)m$ for $r \in R$ and $m \in M$, hence every element in $R \otimes_{R,\varphi} M$ has a unique representation $1 \otimes m$ for some $m \in M$. This gives a bijection between M and $R \otimes_{R,\varphi} M$.
Assume φ_M^{lin} is bijective. Then φ_M is surjective since for every $m \in M$ we find elements $m_i \in M$ and $r_i \in R$ such that

$$\begin{aligned} m &= \varphi_M^{lin}(\sum_i r_i \otimes m_i) = \sum_i r_i \varphi_M(m_i) \\ &= \sum_i \varphi_M(\varphi^{-1}(r_i)m_i) = \varphi_M(\sum_i \varphi^{-1}(r_i)m_i). \end{aligned}$$

It is injective since $\varphi_M(m) = 0$ implies $\varphi_M^{lin}(1 \otimes m) = 0$ and then $1 \otimes m = 0$ which in turn implies $m = 0$ (using the bijection described above). On the other hand, bijectivity of φ_M clearly implies surjectivity, and, using the above bijection, also injectivity of φ_M^{lin}. □

Lemma 5.3 (see Lemma 1.5.2 in [27])**.** *Let M be a φ-module over R. Then there is a finite free R-module F equipped with a semilinear φ-action φ_F such that*

$$R \otimes_{R,\varphi} F \to F,$$
$$r \otimes m \mapsto r\varphi_F(m)$$

is an isomorphism (i.e. F is a φ-module), and a φ-equivariant surjection $F \to M$.

Proof. Let $v_1, ..., v_n$ be generators of M and $R^n \to M$ the corresponding surjection. Let $\varphi_M^{lin} : R \otimes_{R,\varphi} M \to M$ be the given isomorphism and let $a_{ij}, b_{ij} \in R$ be elements such that

$$\varphi_M^{lin}(1 \otimes v_j) = \sum_{i=1}^{n} a_{ij} v_i, \quad (\varphi_M^{lin})^{-1}(v_j) = \sum_{i=1}^{n} b_{ij}(1 \otimes v_i)$$

for $j = 1, ..., n$. Set $A = (a_{ij}), B = (b_{ij})$, and write

$$v_k = \varphi_M^{lin}((\varphi_M^{lin})^{-1}(v_k)) = \varphi_M^{lin}(\sum_{j=1}^{n} b_{jk}(1 \otimes v_j)) = \sum_{i,j} a_{ij} b_{jk} v_i.$$

We see that the columns of $C := AB - 1$ lie in the kernel $N = \ker(R^n \to M)$. Let D be the block matrix $\begin{pmatrix} A & C \\ 1 & B \end{pmatrix}$. We compute $\det(D) = \det(AB - C) = 1$ (see [38, Theorem 3]). Therefore D is invertible over R and we define an isomorphism

$$D : R \otimes_{R,\varphi} F \to F$$

for $F := R^n \oplus R^n$. This isomorphism carries $R \otimes_{R,\varphi} (N \oplus R^n)$ into $N \oplus R^n$, and hence we obtain a φ-equivariant surjection $F \to M$ which factors through the chosen surjection $R^n \to M$. □

Corollary 5.4 (see Corollary 1.5.3 in [27])**.** *Let M be a φ-module over R. Then there exists another φ-module N over R such that $M \oplus N$ is a free module over R.*

Proof. Let F be a φ-module such that the underlying R-module is free and such that there is a φ-equivariant surjection $F \to M$ as in the previous lemma. Put $N = \ker(F \to M)$. We have $F = M \oplus N$ since M is projective, and N is the kernel of the projector $F \to F$ which factors through M. Denote by $\iota : N \to M \oplus N$ the canonical inclusion and by $\mathrm{pr} : M \oplus N \to N$ the projection onto N. We define

$$\varphi_N(n) = \mathrm{pr}(\varphi_F(\iota(n)))$$

for $n \in N$. Then we have the following commutative diagram

$$\begin{array}{ccc} R \otimes_{R,\varphi} N & \xrightarrow{\varphi_N^{lin}} & N \\ \downarrow{\scriptstyle id \otimes \iota} & & \uparrow{\scriptstyle \mathrm{pr}} \\ R \otimes_{R,\varphi} F & \xrightarrow[\varphi_F^{lin}]{} & F \end{array}$$

Note that $id \otimes \iota : R \otimes_{R,\varphi} N \to R \otimes_{R,\varphi} F$ is injective and $R \otimes_{R,\varphi} N$ is a direct summand of $R \otimes_{R,\varphi} F$ ([9, Chapter II, §3.7, Corollary 5 to Proposition 7]). Since φ_F^{lin} carries $R \otimes_{R,\varphi} N$ to N, this implies the injectivity of φ_N^{lin}. For surjectivity, let $n \in N$ and $x = \sum_i r_i \otimes (m_i, n_i) \in R \otimes_{R,\varphi} (M \oplus N) = R \otimes_{R,\varphi} F$ such that $\varphi_F^{lin}(x) = \iota(n) = (0, n)$. Then we have

$$\begin{aligned}(0, n) &= \sum_i r_i \varphi_F(m_i, n_i) \\ &= \sum_i r_i(\varphi_M(m_i), n_i'),\end{aligned}$$

for some $n_i' \in N$, and hence

$$(0, n - \sum_i r_i n_i') = (\sum_i r_i \varphi_M(m_i), 0) \in M \cap N \subseteq F$$

which implies $0 = \sum_i r_i \varphi_M(m_i) = \varphi_M^{lin}(\sum_i r_i \otimes m_i)$. Then $\sum_i r_i \otimes m_i = 0$ and $\sum_i r_i \otimes n_i$ is a preimage of n under φ_N^{lin}. We conclude that $\varphi_N^{lin} : R \otimes_{R,\varphi} N \to N$ is an isomorphism. □

Remark 5.5. If N and M are φ-modules over R, then the direct sum $M \oplus N$ is a φ-module over R via

$$\begin{aligned}\varphi_{M\oplus N} : M \oplus N &\to M \oplus N, \\ (m, n) &\mapsto (\varphi_M(m), \varphi_N(n)).\end{aligned}$$

This clearly defines a φ-linear map. We have

$$R \otimes_{R,\varphi} (M \oplus N) \cong (R \otimes_{R,\varphi} M) \oplus (R \otimes_{R,\varphi} N),$$

and conclude that the linearized map $\varphi_{M\oplus N}^{lin}$ is an isomorphism.

We say that a φ-module is free if its underlying R-module is free.

Lemma 5.6. *The category of φ-modules over R has tensor products, duals, and internal homs.*

Proof. Let M and N be two φ-modules over R. The tensor product $M \otimes_R N$ is finitely generated projective over R. We define

$$\begin{aligned}\varphi_{M\otimes N} : M \otimes_R N &\to M \otimes_R N \\ m \otimes n &\mapsto \varphi(m) \otimes \varphi(n).\end{aligned}$$

We have an isomorphism of R-modules

$$\begin{aligned}R \otimes_{R,\varphi} (M \otimes_R N) &\to (R \otimes_{R,\varphi} M) \otimes_R (R \otimes_{R,\varphi} N) \\ r \otimes (m \otimes n) &\mapsto r(1 \otimes m) \otimes (1 \otimes n),\end{aligned}$$

and the linearized map

$$\varphi^{lin}_{M\otimes N} : R \otimes_{R,\varphi} (M \otimes_R N) = (R \otimes_{R,\varphi} M) \otimes_R (R \otimes_{R,\varphi} N) \to M \otimes_R N$$

is an isomorphism.
The dual module $M^* := \mathrm{Hom}_R(M, R)$ is finitely generated projective over R. Define

$$\begin{aligned} \varphi_{M^*} : M^* &\to M^* \\ \alpha &\mapsto \varphi^{lin} \circ (\mathrm{id}_R \otimes \alpha) \circ (\varphi^{lin}_M)^{-1}. \end{aligned}$$

We identify $\mathrm{Hom}_R(M, R)$ with

$$\mathrm{Hom}_R(R \otimes_{R,\varphi} M, R \otimes_{R,\varphi} R) = \mathrm{Hom}_R(M, R \otimes_{R,\varphi} R)$$

via φ^{lin} and φ^{lin}_M. Then we have

$$\begin{aligned} \varphi^{lin}_{M^*} : R \otimes_{R,\varphi} \mathrm{Hom}_R(M, R) &\to \mathrm{Hom}_R(M, R \otimes_{R,\varphi} R) \\ f \otimes \alpha &\mapsto [m \mapsto f\alpha(m)]. \end{aligned}$$

Since M is finitely generated projective and therefore a direct summand of a finite free R-module, we can reduce the bijectivity of the above map first to the case of a finite free module and then to the case $M = R$ (as R-modules) where it is clear. Furthermore, we have an isomorphism of R-modules

$$\begin{aligned} \mathrm{Hom}_R(M, R) \otimes_R N &\to \mathrm{Hom}_R(M, N) \\ \alpha \otimes n &\mapsto n \cdot \alpha \end{aligned}$$

(see [1, Tag 0DVB]), so we may regard $\mathrm{Hom}_R(M, N)$ as a φ-module over R, too. The map φ is given by

$$\varphi_{\mathrm{Hom}_R(M,N)}(\alpha) = \varphi^{lin}_N \circ (\mathrm{id} \otimes \alpha) \circ (\varphi^{lin}_M)^{-1}.$$

This is because for $n \in N, \alpha \in \mathrm{Hom}_R(M, R)$, and $m \in M$ with preimage $\sum_i f_i \otimes m_i \in R \otimes_{R,\varphi} M$ under φ^{lin}_M we have

$$\begin{aligned} \varphi_{\mathrm{Hom}_R(M,R)}(\alpha)(m) \cdot \varphi_N(n) &= \varphi^{lin}(\mathrm{id} \otimes \alpha((\varphi^{lin}_M)^{-1}(m)) \cdot \varphi_N(n) \\ &= \varphi^{lin}(\sum_i f_i \otimes \alpha(m_i)) \cdot \varphi^{lin}_N(1 \otimes n) \\ &= \sum_i f_i \varphi(\alpha(m_i)) \cdot \varphi^{lin}_N(1 \otimes n) \\ &= \sum_i \varphi^{lin}_N(f_i \varphi(\alpha(m_i)) \otimes n) \\ &= \varphi^{lin}_N(\sum_i f_i \varphi(\alpha(m_i)) \otimes n) \\ &= \varphi^{lin}_N(\sum_i f_i \otimes n \cdot \alpha(m_i)) \\ &= \varphi^{lin}_N(\mathrm{id} \otimes n \cdot \alpha(\varphi^{lin}_M)^{-1}(m)) \\ &= \varphi^{lin}_N \circ (\mathrm{id} \otimes n \cdot \alpha) \circ (\varphi^{lin}_M)^{-1}(m). \end{aligned}$$

□

If M is a φ-module over R, we denote by M^φ the elements of M which are fixed by φ_M.

Remark 5.7. We have

$$\mathrm{Hom}_\varphi(M,N) = \mathrm{Hom}_R(M,N)^\varphi,$$

i.e. the R-module homomorphisms $\alpha : M \to N$ which fulfil $\alpha \circ \varphi_M = \varphi_N \circ \alpha$ are exactly the R-module homomorphisms $M \to N$ which are fixed by the action of φ on $\mathrm{Hom}_R(M,N)$.

Proof. Let $\alpha \in \mathrm{Hom}_R(M,N)$. We have $\varphi_N^{lin} \circ (\mathrm{id} \otimes \alpha) \circ (\varphi_M^{lin})^{-1} = \alpha$ if and only if $\varphi_N^{lin} \circ (\mathrm{id} \otimes \alpha) = \alpha \circ \varphi_M^{lin}$ which is equivalent to α being a morphism of φ-modules. $\square$

Lemma 5.8. *Let S be another ring with an endomorphism $\varphi_S : S \to S$ and a φ-equivariant ring map $R \to S$. Let M be a φ-module over R. Then the base change $S \otimes_R M$ is a φ-module over S with*

$$\begin{aligned} \varphi_{S\otimes M} : S \otimes_R M &\to S \otimes_R M \\ s \otimes m &\mapsto \varphi(s) \otimes \varphi(m). \end{aligned}$$

Proof. Clearly $S \otimes_R M$ is a finitely generated projective S-module. We have an isomorphism of S-modules

$$\begin{aligned} S \otimes_{S,\varphi} (S \otimes_R M) &\to S \otimes_R (R \otimes_{R,\varphi} M), \\ s_1 \otimes s_2 \otimes m = s_1\varphi(s_2) \otimes (1 \otimes m) &\mapsto s_1\varphi(s_2) \otimes 1 \otimes m. \end{aligned}$$

The linearized map

$$\varphi_{S\otimes M}^{lin} : S \otimes_{S,\varphi} (S \otimes_R M) = S \otimes_R (R \otimes_{R,\varphi} M) \to S \otimes_R M$$

is an isomorphism. $\square$

Remark 5.9. We have

$$S \otimes_R \mathrm{Hom}_R(M,N) = \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N)$$

as φ-modules.

Proof. Since M is finitely generated projective, the canonical map

$$\begin{aligned} S \otimes_R \mathrm{Hom}_R(M,N) &\to \mathrm{Hom}_R(M, S \otimes_R N) \\ s \otimes \alpha &\mapsto [m \mapsto s \otimes \alpha(m)] \end{aligned}$$

is an isomorphism ([1, Tag 0DVB]). Moreover, $\mathrm{Hom}_R(M, S \otimes_R N)$ is canonically isomorphic to $\mathrm{Hom}_S(S \otimes_R M, S \otimes_R N)$ via $[m \mapsto s \otimes \alpha(m)] \mapsto [1 \otimes m \mapsto s \otimes \alpha(m)]$. We have to check that the isomorphism is compatible with the φ on both sides. Let

$s \otimes \alpha \in S \otimes_R \mathrm{Hom}_R(M, N)$ and $m \in M$ with preimage $\sum_i f_i \otimes m_i$ under φ_M^{lin}, then we compute

$$\begin{aligned}
\varphi_S(s) \otimes \varphi_{\mathrm{Hom}_R(M,N)}(\alpha)(m) &= \varphi_S(s) \otimes \varphi_N^{lin} \circ (\mathrm{id} \otimes \alpha) \circ (\varphi_M^{lin})^{-1}(m) \\
&= \varphi_S(s) \otimes \varphi_N^{lin} \circ (\mathrm{id} \otimes \alpha)(\sum_i f_i \otimes m_i) \\
&= \varphi_S(s) \otimes \varphi_N^{lin}(\sum_i f_i \otimes \alpha(m_i)) \\
&= \varphi_S(s) \otimes \sum_i f_i \varphi_N(\alpha(m_i)) \\
&= \sum_i f_i \varphi_{S \otimes_R N}(s \otimes \alpha(m_i)) \\
&= \varphi_{S \otimes_R N}^{lin} \circ (\mathrm{id} \otimes (s \otimes \alpha)) \circ (\varphi_M^{lin})^{-1}(m) \\
&= \varphi_{\mathrm{Hom}_S(S \otimes_R M, S \otimes_R N)}([1 \otimes x \mapsto s \otimes \alpha(x)])(1 \otimes m).
\end{aligned}$$

□

We usually consider φ-modules M over a topological ring R. In this case, M has a canonical topology, namely the quotient topology with respect to a surjection $R^n \to M$. Note that the resulting topology is independent of the choice of the surjection. If R is a locally convex K-algebra which is barrelled, then M is barrelled as well. If $\varphi : R \to R$ is continuous, then the semilinear map $\varphi_M : M \to M$ is automatically continuous:

Lemma 5.10. *Let $\psi : R \to S$ be a continuous map between topological rings, and let M and N be finitely generated R resp. S-modules. Let $\alpha : M \to N$ be any ψ-linear map. Then α is continuous for the canonical topologies on M and N.*

Proof. This is [6, Remark 2.20]. □

5.2 φ-modules over the Robba rings

We look at φ-modules over the previously constructed rings $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}), \mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})$, and $\mathcal{E}_K^{\dagger,\le 1}(\mathfrak{X}^{\mathrm{perf}})$. The φ is given by the action of π. We assume that every radius which occurs in relation to $\mathfrak{B}^{\mathrm{perf}}$ (i.e. as in $\mathcal{R}_K(\mathfrak{B}^{\mathrm{perf}})^r$) lies in R_n for some n, and that every radius which occurs in relation to $\mathfrak{X}^{\mathrm{perf}}$ (i.e. as in $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r$) lies in S_n for some n.

Definition 5.11. *Let R be one of the rings $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ or $\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})$. A φ-module over R is* étale *if it arises via base change from a φ-module over $\mathcal{E}_K^{\dagger,\le 1}(\mathfrak{X}^{\mathrm{perf}})$.*

Let $r \in S_n$. We write $\mathcal{R}_K(\mathfrak{X})^r = \mathcal{O}_K(\mathfrak{X} \setminus \mathfrak{X}(r))$, $\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^r = \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r \cap \mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})$ and likewise $\mathcal{E}_K^\dagger(\mathfrak{X})^r = \mathcal{R}_K(\mathfrak{X})^r \cap \mathcal{E}_K^\dagger(\mathfrak{X})$.

Definition 5.12. *Let R be one of the rings $\mathcal{R}_K(\mathfrak{X}^{\text{perf}}), \mathcal{E}_K^\dagger(\mathfrak{X}^{\text{perf}}), \mathcal{R}_K(\mathfrak{X})$, or $\mathcal{E}_K^\dagger(\mathfrak{X})$. A φ-module M over R^r is a finitely generated projective R^r-module with a φ-semilinear continuous map $\varphi_M : M \to R^{r^{1/p}} \otimes_{R^r} M$ such that*

$$R^{r^{1/p}} \otimes_{R^r,\varphi} M \to R^{r^{1/p}} \otimes_{R^r} M$$

is an isomorphism.

Remark 5.13. If M is a φ-module over $\mathcal{E}_K^\dagger(\mathfrak{X}^{\text{perf}})^r$, then the base change

$$\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^r \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\text{perf}})^r} M$$

is a φ-module over $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^r$. This is because we have

$$\begin{aligned}&\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^{r^{1/p}} \otimes_{\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^r,\varphi} (\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^r \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\text{perf}})^r} M) = \\ &(\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^{r^{1/p}} \otimes_{\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^r,\varphi} \mathcal{R}_K(\mathfrak{X}^{\text{perf}})^r) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\text{perf}})^r} M,\end{aligned}$$

and the linearized map $\varphi^{lin} : \mathcal{R}_K(\mathfrak{X}^{\text{perf}})^{r^{1/p}} \otimes_{\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^r,\varphi} \mathcal{R}_K(\mathfrak{X}^{\text{perf}})^r \to \mathcal{R}_K(\mathfrak{X}^{\text{perf}})^{r^{1/p}}$ is an isomorphism.

Now we consider the base change of étale φ-modules over $\mathcal{E}_K^\dagger(\mathfrak{X}^{\text{perf}})$ to étale φ-modules over $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})$. We have an injection $M = \mathcal{E}_K^\dagger(\mathfrak{X}^{\text{perf}}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\text{perf}})} M \hookrightarrow \mathcal{R}_K(\mathfrak{X}^{\text{perf}}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\text{perf}})} M$ via flatness.

Proposition 5.14. *Let M be an étale φ-module over $\mathcal{E}_K^\dagger(\mathfrak{X}^{\text{perf}})$. Then we have*

$$(\mathcal{R}_K(\mathfrak{X}^{\text{perf}}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\text{perf}})} M)^\varphi = M^\varphi$$

Proof. Let M be an étale φ-module over $\mathcal{E}_K^\dagger(\mathfrak{X}^{\text{perf}})$ which arises from a φ-module M_0 over $\mathcal{E}_K^{\dagger,\le 1}(\mathfrak{X}^{\text{perf}})$. We have

$$M = \mathcal{E}_K^\dagger(\mathfrak{X}^{\text{perf}}) \otimes_{\mathcal{E}_K^{\dagger,\le 1}(\mathfrak{X}^{\text{perf}})} M_0.$$

Firstly, we reduce to the case that M_0 and hence M is a free module. Let N_0 be a φ-module over $\mathcal{E}_K^{\dagger,\le 1}(\mathfrak{X}^{\text{perf}})$ such that $P_0 = M_0 \oplus N_0$ is a free $\mathcal{E}_K^{\dagger,\le 1}(\mathfrak{X}^{\text{perf}})$-module (Lemma 5.4). Then we define a φ-module structure on P_0 by setting

$$\begin{aligned}\varphi_{P_0} : P_0 = M_0 \oplus N_0 &\to M_0 \oplus N_0 \\ (m, n) &\mapsto (\varphi_{M_0}(m), \varphi_{N_0}(n)).\end{aligned}$$

This defines a φ-module over $\mathcal{E}_K^{\dagger,\le 1}(\mathfrak{X}^{\text{perf}})$ (Remark 5.5). Write

$$N := \mathcal{E}_K^\dagger(\mathfrak{X}^{\text{perf}}) \otimes_{\mathcal{E}^{\dagger,\le 1}(\mathfrak{X}^{\text{perf}})} N_0.$$

Then M is a direct summand of the free $\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})$-module

$$\begin{aligned} P &:= \mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{E}^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}})} P_0 \\ &= (\underbrace{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{E}^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}})} M_0}_{=M}) \oplus (\underbrace{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{E}^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}})} N_0}_{=N}), \end{aligned}$$

and the φ-action on P is given by

$$\varphi_P(f \otimes m, g \otimes n) \mapsto (\varphi(f) \otimes \varphi_{M_0}(m), \varphi(g) \otimes \varphi_{N_0}(n))$$

for $(f \otimes m, g \otimes n) \in M \oplus N$. We have $P^\varphi = M^\varphi \oplus N^\varphi$ and similarly

$$(\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})} P)^\varphi = (\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})} M)^\varphi \oplus (\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})} N)^\varphi.$$

Assume that we have shown the lemma for P. Then

$$\begin{aligned} (\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})} P)^\varphi &= (\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{E}_K^\dagger} (M \oplus N))^\varphi \\ &= (\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})} M)^\varphi \oplus (\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})} N)^\varphi \\ &= P^\varphi \\ &= M^\varphi \oplus N^\varphi, \end{aligned}$$

hence $(\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})} M)^\varphi = M^\varphi$. Therefore we may assume that M is free and comes from a free φ-module M_0 over $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}})$.

Choose a basis $e_1, ..., e_n$ for M_0 and let $A = (g_{ij})_{i,j} \in \mathrm{Mat}_{n \times n}(\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}}))$ be the matrix in this basis corresponding to φ_{M_0}, i.e. we have

$$\varphi_{M_0}(v) = A\varphi(v)$$

for $v \in M_0$. Now consider $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})} M$ which is isomorphic to $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^n$ as $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$-modules. We want to show that if $v = (v_1, ..., v_n)^t \in \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^n$ fulfils

$$v = A\varphi(v),$$

then $v \in \mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^n$.

We reduce to the case $K = \mathbb{C}_p$. Assume that we have shown the lemma for $K = \mathbb{C}_p$. Then we can use the inclusion $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^n \subseteq \mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}})^n$ and note that $\mathcal{E}_{\mathbb{C}_p}^\dagger(\mathfrak{X}^{\mathrm{perf}}) \cap \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) = \mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})$ to obtain the lemma for general K.
We have an isomorphism $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{X}^{\mathrm{perf}}) \cong \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})$ which restricts to $\mathcal{E}_{\mathbb{C}_p}^\dagger(\mathfrak{X}^{\mathrm{perf}}) \cong \mathcal{E}_{\mathbb{C}_p}^\dagger(\mathfrak{B}^{\mathrm{perf}})$ and $\mathcal{E}_{\mathbb{C}_p}^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}}) \cong \mathcal{E}_{\mathbb{C}_p}^{\dagger,\leq 1}(\mathfrak{B}^{\mathrm{perf}})$, and which is φ-equivariant (Remarks 4.23 and 4.31). Therefore we may compute everything over $\mathfrak{B}^{\mathrm{perf}}$, and we regard

the matrix A as well as the vector v as elements in $\mathrm{Mat}_{n\times n}(\mathcal{E}^{\dagger,\leq 1}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}}))$ resp. $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})$.

The entries g_{ij} of the matrix A fulfil $\|g_{ij}\|_1 \leq 1$. Let s_0 be a radius such that $v_i \in \mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})^{s_0}$ and $g_{ij} \in \mathcal{E}^{\dagger}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})^{s_0}$ for all i,j and $v = A\varphi(v)$ in $\mathcal{R}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})^{s_0}$. We reduce to the case that there is an s' such that $\|g_{ij}\|_s \leq 1$ for all $s_0 < s' \leq s < 1$. Choose an s' such that $s_0 < s' < 1$. The element

$$z = \varphi(T)\cdot T^{-1} \in \mathcal{E}^{\dagger}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})$$

fulfils $\|z\|_{s'} < 1$, so multiplying A with a suitable power z^m of z gives a matrix $z^m A$ with entries $\|z^m g_{ij}\|_{s'} \leq 1$. Write

$$z^m g_{ij} = (z^m g_{ij})^+ + (z^m g_{ij})^-$$

as in Lemma 3.76. Then we have $\|z^m g_{ij}\|_s = \max\{\|(z^m g_{ij})^+\|_s, \|(z^m g_{ij})^-\|_s\}$ for all $s_0 < s < 1$. The sequence $\|(z^m g_{ij})^+\|_s$ is monotonously increasing if $s \to 1$ while the sequence $\|(z^m g_{ij})^-\|_s$ is monotonously falling if $s \to 1$. Since $\|z^m g_{ij}\|_s \to 1$ for $s \to 1$, we see that $\|z^m g_{ij}\|_{s'} \leq 1$ implies $\|z^m g_{ij}\|_s \leq 1$ for all $s' \leq s < 1$.
We have

$$v = A\varphi(v) \Leftrightarrow v\cdot T^{-m} = z^m A\varphi(v\cdot T^{-m}),$$

i.e. v is fixed by φ_M if and only if $v\cdot T^{-m}$ is fixed by the semilinear map given by $z^m A$. If the element $v\cdot T^{-m}$ lies in $\mathcal{E}^{\dagger}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})^n$, then there is an s_0' such that the set $\{\max_i\{\|v_i\cdot T^{-m}\|_s\} \mid s_0' \leq s < 1\}$ is bounded (Lemma 3.74). But since we have

$$\|v_i\cdot T^{-m}\|_s = \frac{\|v_i\|_s}{\|T^m\|_s},$$

this implies that $\{\max_i\{\|v_i\|_s\} \mid s_0' \leq s < 1\}$ is bounded as well and hence $v \in \mathcal{E}^{\dagger}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})$(Lemma 3.74). We see that v lies in $\mathcal{E}^{\dagger}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})$ if and only if $v\cdot T^{-m}$ lies in $\mathcal{E}^{\dagger}_{\mathbb{C}_p}(\mathfrak{B}^{\mathrm{perf}})$. Hence we can always change to a matrix $z^m A$ with the desired property without changing the claim.

Now assume that $\|g_{ij}\|_s \leq 1$ for $s' \leq s < 1$ and all g_{ij}. The equality $v = A\varphi(v)$ implies

$$\begin{aligned}\max_{1\leq i\leq n}\{\|v_i\|_{s^{1/q}}\} &= \max_{1\leq i\leq n}\{\|\sum_{j=1}^{n} g_{ij}\varphi(v_j)\|_{s^{1/q}}\}\\ &\leq \max_{i,j}\{\|g_{ij}\|_{s^{1/q}}\}\cdot\max_i\{\|\varphi(v_i)\|_{s^{1/q}}\}\\ &\leq \max_{1\leq i\leq n}\{\|\varphi(v_i)\|_{s^{1/q}}\} = \max_{1\leq i\leq n}\{\|v_i\|_s\}.\end{aligned}$$

Here we use Remark 3.70 for the last equality. We see that $\|v_i\|_{s^{1/q}} \leq \max_i\{\|v_i\|_s\}$ for $i = 1, ..., n$. Iterating the process shows that

$$\|v_i\|_{s^{1/q^k}} \leq \max_i\{\|v_i\|_s\}$$

for all $k \in \mathbb{N}$. Therefore $\{\max_i \|v_i\|_s \mid s' \leq s < 1\}$ is bounded and $v \in \mathcal{E}^\dagger_{\mathbb{C}_p}(\mathfrak{B}^{\text{perf}})^n$. □

For free φ-modules over $\mathcal{R}_K(\mathfrak{B})$ the analogous result can be found in [25, Proposition 1.2.6].

Theorem 5.15. *The base change functor from étale φ-modules over $\mathcal{E}^\dagger_K(\mathfrak{X}^{\text{perf}})$ to étale φ-modules over $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})$ is an equivalence of categories.*

Proof. The functor is essentially surjective by definition. For fully faithfulness, we compute

$$\begin{aligned}
&\text{Hom}_\varphi(\mathcal{R}_K(\mathfrak{X}^{\text{perf}}) \otimes_{\mathcal{E}^\dagger_K(\mathfrak{X}^{\text{perf}})} M, \mathcal{R}_K(\mathfrak{X}^{\text{perf}}) \otimes_{\mathcal{E}^\dagger_K(\mathfrak{X}^{\text{perf}})} N) \\
&= (\text{Hom}_{\mathcal{R}_K(\mathfrak{X}^{\text{perf}})}(\mathcal{R}_K(\mathfrak{X}^{\text{perf}}) \otimes_{\mathcal{E}^\dagger_K(\mathfrak{X}^{\text{perf}})} M, \mathcal{R}_K(\mathfrak{X}^{\text{perf}}) \otimes_{\mathcal{E}^\dagger_K(\mathfrak{X}^{\text{perf}})} N))^\varphi \\
&= (\mathcal{R}_K(\mathfrak{X}^{\text{perf}}) \otimes_{\mathcal{E}^\dagger_K(\mathfrak{X}^{\text{perf}})} \text{Hom}_{\mathcal{E}^\dagger_K(\mathfrak{X}^{\text{perf}})}(M, N))^\varphi \\
&= \text{Hom}_{\mathcal{E}^\dagger_K(\mathfrak{X}^{\text{perf}})}(M, N)^\varphi \\
&= \text{Hom}_\varphi(M, N).
\end{aligned}$$

Here, the first and the last equality are Remark 5.7, the second equality is Remark 5.9. For the third equality, use the previous proposition. □

Lemma 5.16. *Let M be a φ-module over $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})$. Then there is an $r_0 \in S_m$ for some m and a φ-module M^{r_0} over $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^{r_0}$ such that*

$$\mathcal{R}_K(\mathfrak{X}^{\text{perf}}) \otimes_{\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^{r_0}} M^{r_0} = M$$

as φ-modules.

Proof. We adapt the first part of the proof of [6, Proposition 2.24]. Let M be a φ-module over $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})$. Since M is finitely generated projective, we find an $n \geq 1$ and a projector $\Pi : \mathcal{R}_K(\mathfrak{X}^{\text{perf}})^n \to M$. The matrix Π has entries in some $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^r$, so we may define $M^r := \Pi((\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^r)^n)$ which is a finitely generated projective $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^r$-module. Since M^r is finitely generated, we have $\varphi_M(M^r) \subseteq M^{r'}$ for some $r \leq r'$. This implies

$$\varphi_M(M^{r'}) \subseteq M^{r'^{1/p}}$$

since any set of generators of M^r also generates $M^{r'}$. We then have the linearized map

$$\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^{r'^{1/p}} \otimes_{\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^{r'}, \varphi} M^{r'} \to M^{r'^{1/p}}$$

which is an isomorphism after base change to $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$. The cokernel of this map is finitely generated and hence vanishes after enlarging r'. Then the map is surjective and splits by projectivity of the modules. Therefore its kernel is finitely generated as well and vanishes after further enlarging r'. Therefore we have an r_0 such that

$$\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r^{1/p}} \otimes_{\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r,\varphi} M^r \to M^{r^{1/p}}$$

is an isomorphism for all $r_0 \leq r < 1$.

□

Proposition 5.17. *Let $r \in S_n$ be any radius. For any free φ-module M over $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$, there is a φ-module M^r over $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r$ such that $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r} M^r = M$.*

Proof. Remember that $\varphi : \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ is invertible (Lemma 4.32). Let $v_1, ..., v_n \in M$ be a basis of M, and let $A \in \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{n\times n}$ denote the matrix of φ_M in this basis. Then A is invertible. We may assume that A and A^{-1} have entries in $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r_0}$ for some r_0 and that $v_1, ..., v_n \in M^{r_0}$. For $r \geq r_0$, we define M^r to be the base change of M^{r_0} to $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r$.
On the other hand, to get the smaller radii, we use the matrix $U := \varphi^{-1}(A^{-1})$ as base change matrix to obtain another basis of M. The matrix of φ_M in this basis is given by

$$U^{-1}A\varphi(U) = \varphi^{-1}(A)AA^{-1} = \varphi^{-1}(A).$$

Iterating this process gives a basis $v_1', ..., v_n'$ of M such that the matrix of φ_M in this basis is given by $\varphi^{-e}(A)$ which has entries in $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r_0^p}$ (Remark 4.33) and is invertible over $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r_0^p}$. We define $M^{r_0^p}$ to be the free $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^{r_0^p}$-module with basis $v_1', ..., v_n'$. □

In the following we write

$$\breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}})_n := \varprojlim_{p^{-(1+e/(p-1))/ep^n} < r_1 \leq r_2 < 1} \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)),$$

and

$$\breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}) = \varinjlim_n \breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}})_n.$$

Let $f \in \breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}})$ be any element. Assume $f \in \breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}})_n$ for some $n \in \mathbb{N}$ and write

$$f = (f_0, f_1, ..., f_k, ...) \in \varprojlim_{p^{-(1+e/(p-1))/ep^n} < r_1 \leq r_2 < 1} \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2)).$$

Then we find an i_0 such that each f_k has a preimage in $\mathcal{O}_K(\mathfrak{X}(r_1^{1/p^{i_0}}, r_2^{1/p^{i_0}}))$ for the resp. radii r_1, r_2. This is because if an element $g \in \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(s_1, s_2))$ has a preimage $g_{i_0} \in \mathcal{O}_K(\mathfrak{X}(s_1^{1/p^{i_0}}, s_2^{1/p^{i_0}}))$ under the canonical map

$$\mathcal{O}_K(\mathfrak{X}(s_1^{1/p^{i_0}}, s_2^{1/p^{i_0}})) \to \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(s_1, s_2)),$$

and has a preimage g' in $\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2))$ under the restriction map for $r_1 \leq s_1 \leq s_2 \leq r_2$, then g' has a preimage in $\mathcal{O}_K(\mathfrak{X}(r_1^{1/p^{i_0}}, r_2^{1/p^{i_0}}))$ under the canonical map $\mathcal{O}_K(\mathfrak{X}(r_1^{1/p^{i_0}}, r_2^{1/p^{i_0}})) \to \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2))$.
This can be seen directly for $K = \mathbb{C}_p$ because then the rings in question are isomorphic to affinoid annuli, and for general K one can use Remark 2.15.
Now let $g \in \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1, r_2))$ with preimage $g_{i_0} \in \mathcal{O}_K(\mathfrak{X}(r_1^{1/p^{i_0}}, r_2^{1/p^{i_0}})), i_0 > 0$. The the map p_* sends g to an element $p_*(g) \in \breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1^{1/p}, r_2^{1/p}))$. But this element $p_*(g)$ has a preimage $p_*(g_{i_0}) \in \mathcal{O}_K(\mathfrak{X}((r_1^{1/p})^{1/p^{i_0}}, (r_2^{1/p})^{1/p^{i_0}}))$ which is equal to the image of the element $g_{i_0} \in \mathcal{O}_K(\mathfrak{X}(r_1^{1/p^{i_0}}, r_2^{1/p^{i_0}}))$ in $\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1^{1/p}, r_2^{1/p}))$. This implies that by perpetually applying p_* (namely i_0 times) we eventually arrive at an element in $\breve{\mathcal{O}}_K(\mathfrak{X}^{\mathrm{perf}}(r_1^{1/p^{i_0}}, r_2^{1/p^{i_0}}))$ with preimage in $\mathcal{O}_K(\mathfrak{X}(r_1^{1/p^{i_0}}, r_2^{1/p^{i_0}}))$.
This shows that for any element $f \in \breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}})$ there is an $n \in \mathbb{N}$ such that $p_*^n(f) \in \mathcal{R}_K(\mathfrak{X})$, or similarly, an $n' = en \in \mathbb{N}$ such that $\varphi^{n'}(f) \in \mathcal{R}_K(\mathfrak{X})$.

Proposition 5.18. *The category of φ-modules over $\breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}})$ is equivalent to the category of φ-modules over $\mathcal{R}_K(\mathfrak{X})$ via base change.*

Proof. To show essential surjectivity, let M first be a free φ-module over $\breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}})$, and let $v_1, ..., v_n$ be a basis of M. Denote by A the matrix of φ_M corresponding to this basis.
According to the discussion above we find an r and an $i_0 \in \mathbb{N}$ such that A has entries in $\varprojlim_{r<r_1\leq r_2<1}(\mathcal{O}_K(\mathfrak{X}(r_1^{1/p^{i_0}}, r_2^{1/p^{i_0}}))) = \mathcal{O}_K(\mathfrak{X} \setminus \mathfrak{X}(r^{1/p^{i_0}}))$.
We need to find an invertible matrix

$$U \in \mathrm{Mat}_{n\times n}(\breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}))$$

such that $U^{-1}A\varphi(U) \in \mathrm{Mat}_{n\times n}(\mathcal{R}_K(\mathfrak{X}))$. For this, note that A is invertible and that $A^{-1}A\varphi(A) = \varphi(A)$ has entries in $\mathcal{O}_K(\mathfrak{X} \setminus \mathfrak{X}(r^{1/p^{i_0+1}}))$. In the next step, we take $\varphi(A)$ as base change matrix and obtain $\varphi(A^{-1})\varphi(A)\varphi^2(A) = \varphi^2(A)$. Iterating this process gives a matrix

$$U := \varphi^{i_0 e}(A)$$

with the desired property. We write $U = (u_{ij})_{i,j}$ and then take $v_j' := \sum_i u_{ij}v_i$ as a basis for a φ-module M_0 over $\mathcal{R}_K(\mathfrak{X})$.

For a general φ-module M over $\breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}})$, apply Lemma 5.4 to obtain a φ-module N such that $F = M \oplus N$ is a free $\breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}})$-module, and set

$$\varphi_{M\oplus N} := \varphi_M \oplus \varphi_N$$

as in Remark 5.5. We obtain a free φ-module F_0 over $\mathcal{R}_K(\mathfrak{X})$ such that

$$\breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{R}_K(\mathfrak{X})} F_0 = F$$

and $\varphi_F = \varphi \otimes \varphi_{F_0}$. Let $v_1, ..., v_n$ be a basis of F_0, then $\varphi_{F_0}^k(v_1), ..., \varphi_{F_0}^k(v_n)$ is another basis of F_0 for any k. Then $1 \otimes \varphi_{F_0}^k(v_1), ..., 1 \otimes \varphi_{F_0}^k(v_n)$ is a basis of $\breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{R}_K(\mathfrak{X})} F_0$. We have a φ-equivariant projector $\Pi : F \to F$ with image M and kernel N. Write

$$\Pi(1 \otimes v_i) = \sum_j f_{j_i} \otimes v_j \in \breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{R}_K(\mathfrak{X})} F_0,$$

for certain $f_{j_i} \in \breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}})$, then we have

$$\begin{aligned} \Pi(1 \otimes \varphi_{F_0}^k(v_i)) &= \Pi(\varphi_F^k(1 \otimes v_i)) \\ &= \varphi_F^k(\Pi(1 \otimes v_i)) \\ &= \varphi_F^k(\sum_j f_{j_i} \otimes v_j) \\ &= \sum_j \varphi^k(f_{j_i}) \otimes \varphi_{F_0}^k(v_j). \end{aligned}$$

Thus, by choosing k big enough and then using $1 \otimes \varphi_{F_0}^k(v_i), i = 1, ..., n$ as a basis, we can ensure that all $\varphi^k(f_{j_i})$ lie in $\mathcal{R}_K(\mathfrak{X})$ and that Π restricts to a φ-equivariant projector $\Pi_0 : F_0 \to F_0$. Its image $\Pi(F_0) =: M_0$ is a finite projective $\mathcal{R}_K(\mathfrak{X})$-module because it is a direct summand of F_0. We have $F_0 \cong M_0 \oplus N_0$ where $N_0 := \ker(\Pi_0) \subseteq N$. We have the sections $\iota_{M_0} : M_0 \to F_0$ and $\iota_M : M \to F$ such that $\iota_{M_0} = \iota_{M|M_0}$. We have an inclusion

$$M_0 \hookrightarrow \breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{R}_K(\mathfrak{X})} M_0 \hookrightarrow \breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{R}_K(\mathfrak{X})} F_0 \cong F,$$

where the second arrow is given by $id \otimes \iota_{M_0}$, and is injective because M_0 is a direct summand of F_0 ([9, Chapter II, §3.7, Corollary 5 to Proposition 7]). We have a commutative diagram

$$\begin{array}{ccc} \breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{R}_K(\mathfrak{X})} M_0 & \longrightarrow & M \\ {\scriptstyle id \otimes \iota_{M_0}} \downarrow & & \downarrow {\scriptstyle \iota_M} \\ \breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{R}_K(\mathfrak{X})} F_0 & \xrightarrow{\cong} & F \end{array}$$

where the upper horizontal map is given by $f \otimes m \mapsto fm$. This map is surjective, and, because the other maps in the diagram are injective, also injective, so that we have an isomorphism $M \cong \breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{R}_K(\mathfrak{X})} M_0$. We define

$$\begin{aligned} \varphi_{M_0} : M_0 &\to M_0, \\ x &\mapsto \Pi(\varphi_F(\iota_M(x))) = \Pi_{F_0}(\varphi_{F_0}(\iota_{M_0}(x))) \end{aligned}$$

for $x \in M_0 \subseteq M = \breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{R}_K(\mathfrak{X})} M_0$. Then we have $\varphi_M = \varphi \otimes \varphi_{M_0}$. The linearized map $\varphi_{M_0}^{lin}$ is surjective since $\varphi_{F_0}^{lin}$ is surjective. Note that we have

$$\begin{aligned}\breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}),\varphi} M &= \breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}),\varphi} (\breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{R}_K(\mathfrak{X})} M_0) \\ &= \breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{R}_K(\mathfrak{X}),\varphi} M_0\end{aligned}$$

and an injective map $\mathcal{R}_K(\mathfrak{X}) \otimes_{\mathcal{R}_K(\mathfrak{X}),\varphi} M_0 \to \breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{R}_K(\mathfrak{X}),\varphi} M_0$ because M_0 is flat. Then $\varphi_{M_0}^{lin}$ is injective since it is equal to φ_M^{lin} restricted to $\mathcal{R}_K(\mathfrak{X}) \otimes_{\mathcal{R}_K(\mathfrak{X}),\varphi} M_0$. We see that M_0 is a φ-module over $\mathcal{R}_K(\mathfrak{X})$ and $M \cong \breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{R}_K(\mathfrak{X})} M_0$ as φ-modules.

To show fully faithfulness, we again assume that M is a free φ-module over $\mathcal{R}_K(\mathfrak{X})$ of rank m. We show that $(\breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{R}_K(\mathfrak{X})} M)^{\varphi} = M^{\varphi}$. Denote by A the matrix of $\varphi \otimes \varphi_M$ in a basis of $\breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{R}_K(\mathfrak{X})} M$, we may assume that A has entries in $\mathcal{R}_K(\mathfrak{X})$. If $v \in (\breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{R}_K(\mathfrak{X})} M)^{\varphi}$, then

$$v = A\varphi(v).$$

Iteration leads to

$$v = A\varphi(A)...\varphi^n(A)\varphi^n(v)$$

for any n. This implies $v \in \mathcal{R}_K(\mathfrak{X})^m$. For a general φ-module M, we find a φ-module M such that $M \oplus N$ is a free $\mathcal{R}_K(\mathfrak{X})$-module. Then $M \oplus N$ becomes a φ-module via $\varphi_{M\oplus N}(m,n) = (\varphi_M(m), \varphi_N(n))$. We conclude $(\breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{R}_K(\mathfrak{X})} M)^{\varphi} = M^{\varphi}$ as in the proof of Proposition 5.14. Then we see that the base change functor is fully faithful as in the proof of Theorem 5.15. □

The base extension from φ-modules over $\mathcal{R}_K(\mathfrak{X})$ to φ-modules over $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ is probably not essential surjective since this is likely not the case over $\mathfrak{B}^{\mathrm{perf}}$. See [26, Remark 7.9] for a possible counterexample over $\mathfrak{B}^{\mathrm{perf}}$.

If M is a finite projective module over $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})_n$, we can find a finite projective module M_0 over $\breve{\mathcal{R}}_K(\mathfrak{X}^{\mathrm{perf}})_n$ such that $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})_n \otimes M_0 = M$ (see [27, Lemma 2.2.13]). The problem is the φ-action which does not necessarily restrict to an action over the smaller ring $\mathcal{R}_K(\mathfrak{X})$.

5.3 (φ,Γ)-Modules over $\mathfrak{B}^{\mathrm{perf}}$ and $\mathfrak{X}^{\mathrm{perf}}$

In this section, we define (φ,Γ)-modules over the previous discussed rings. We first impose topologies on the rings as follows:

Definition 5.19. *1. The rings $\mathcal{E}_K^{\dagger}(\mathfrak{X}^{\mathrm{perf}}) \subseteq \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ resp. $\mathcal{E}_K^{\dagger}(\mathfrak{X}) \subseteq \mathcal{R}_K(\mathfrak{X})$ carry the subspace topology coming from the rings $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ resp. $\mathcal{R}_K(\mathfrak{X})$.*

2. *The rings $\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^r \subseteq \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r$ resp. $\mathcal{E}_K^\dagger(\mathfrak{X})^r \subseteq \mathcal{R}_K(\mathfrak{X})^r$ carry the subspace topology coming from the rings $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r$ resp. $\mathcal{R}_K(\mathfrak{X})^r$.*

3. *The rings $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}})$ resp. $\mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X})$ carry the weak topology.*

Definition 5.20. *A (φ, Γ)-module over*

$$R \in \{\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}), \mathcal{R}_K(\mathfrak{X}), \mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}}), \mathcal{E}_K^\dagger(\mathfrak{X}), \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}}), \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X})\}$$

is a φ-module over R which carries a semilinear continuous $o_L^\times$-action which commutes with φ.

A morphism of (φ, Γ)-modules M and N is a morphism of the underlying φ-modules which commutes with the $o_L^\times$-action. We denote the set of the morphisms by $\mathrm{Hom}_{\varphi,\Gamma}(M, N)$.

Definition 5.21. *Let $R \in \{\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}), \mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}}), \mathcal{E}_K^{\dagger,\leq 1}(\mathfrak{X}^{\mathrm{perf}}), \mathcal{R}_K(\mathfrak{X}), \mathcal{E}_K^\dagger(\mathfrak{X})\}$. Let $r \in S_n$. A (φ, Γ)-module M over R^r is a φ-module M over R^r which carries a semilinear continuous (for the canonical topology) $o_L^\times$-action which commutes with φ.*

If a (φ, Γ)-module M over R arises via base change from a (φ, Γ)-module over R^r for a radius r, we say that M has a *model* over R^r.

Remark 5.22. If $R = \mathcal{R}_K(\mathfrak{X})$, then for every (φ, Γ)-module M over R there is an $r < 1$ such that M arises via base change from a (φ, Γ)-module M^r over R^r. This follows from [6, Proposition 2.24] which implies that we can descend every (φ, Γ)-module over $\mathcal{R}_K(\mathfrak{X})$ to a (φ, Γ)-module over $\mathcal{R}_K(\mathfrak{X})^r$ for some r.

Definition 5.23. *A (φ, Γ)-module over $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ resp. $\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})$ is called* étale *if its underlying φ-module is étale.*

Let M be a (φ, Γ)-module over $\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})$. It is unclear whether the base change to $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$ with the induced action of φ and $o_L^\times$ is a (φ, Γ)-module over $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$. Of course, we can define a φ-module $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})} M$ with an action of $o_L^\times$. But we do not know whether this action is continuous (the φ-action is automatically continuous). If M has a model, the situation is better:

Lemma 5.24. *Let M be a (φ, Γ)-module over $\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})$ for which there is an r such that M has a model over $\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^r$. Then base change yields a (φ, Γ)-module $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})} M$ over $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$.*

Proof. Write $M = \mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^r} M^r$ for a model M^r of M. Then we have

$$\begin{aligned}\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})} M &\cong \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})} (\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^r} M^r) \\ &\cong \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r} (\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^r} M^r).\end{aligned}$$

This isomorphism is equivariant under the action of $o_L^\times$. We show that the orbit maps

$$o_L^\times \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^r} M^r$$

are continuous. For this, fix $m \in M^r, f \in \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r$ and note that the map

$$\begin{aligned} o_L^\times &\to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r \times M^r, \\ u &\mapsto (u_*(f), u(m)) \end{aligned}$$

is continuous since it is the composition of the continuous maps

$$\begin{aligned} o_L^\times &\to o_L^\times \times o_L^\times \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r \times M^r, \\ u &\mapsto (u, u), \\ (u_1, u_2) &\mapsto ((u_1)_*(f), u_2(m)). \end{aligned}$$

For $n \in \mathbb{N}$ we have a commutative diagram

$$\begin{array}{ccc} \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r \times (\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r)^n & \longrightarrow & (\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r)^n \\ \big\uparrow & & \big\uparrow = \\ \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r \times (\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^r)^n & \longrightarrow & (\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r)^n \end{array}$$

where the vertical maps and the upper horizontal map are continuous. Therefore the lower horizontal map is continuous as well. We have an open projection $pr : \mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^n \to M^r$. Then consider the commutative diagram

$$\begin{array}{ccc} \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r \times (\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^r)^n & \xrightarrow{\otimes} & (\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r)^n \cong \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^r} (\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^r)^n \\ \big\downarrow \mathrm{id}\times pr & & \big\downarrow \mathrm{id}\otimes pr \\ \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r \times M^r & \xrightarrow{\otimes} & \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^r} M^r \end{array}$$

where the vertical maps are open. Together with the continuity of the upper horizontal map this implies that the canonical map

$$\otimes : \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r \times M^r \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r \otimes M^r$$

is continuous. Then the orbit map of an element $f \otimes m$ is the composition of continuous maps

$$o_L^\times \to o_L^\times \times o_L^\times \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r \times M^r \to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r \otimes M^r,$$

and hence continuous. For a general element $\sum_i f_i \otimes m_i$, the orbit map is a sum of continuous maps and therefore continuous.

Now think about $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^s \otimes_{\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^r} (\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^r \otimes_{\mathcal{E}^\dagger_K(\mathfrak{X}^{\text{perf}})^r} M^r)$ for $s \geq r$. Arguing as above, but with M^r replaced by $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^r \otimes_{\mathcal{E}^\dagger_K(\mathfrak{X}^{\text{perf}})^r} M^r$ and base changing from $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^r$ to $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^s$, we see that the orbit maps

$$o_L^\times \to \mathcal{R}_K(\mathfrak{X}^{\text{perf}})^s \otimes_{\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^r} (\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^r \otimes_{\mathcal{E}^\dagger_K(\mathfrak{X}^{\text{perf}})^r} M^r)$$

are continuous. Then we use the continuous inclusions

$$\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^s \otimes_{\mathcal{E}^\dagger_K(\mathfrak{X}^{\text{perf}})^r} M^r \to \mathcal{R}_K(\mathfrak{X}^{\text{perf}}) \otimes_{\mathcal{E}^\dagger_K(\mathfrak{X}^{\text{perf}})} M = \mathcal{R}_K(\mathfrak{X}^{\text{perf}}) \otimes_{\mathcal{E}^\dagger_K(\mathfrak{X}^{\text{perf}})^r} M^r$$

for any $s \geq r$ to see that the orbit maps

$$o_L^\times \to \mathcal{R}_K(\mathfrak{X}^{\text{perf}}) \otimes_{\mathcal{E}^\dagger_K(\mathfrak{X}^{\text{perf}})} M$$

are continuous. This implies that $o_L^\times$ acts continuously on $\mathcal{R}_K(\mathfrak{X}^{\text{perf}}) \otimes_{\mathcal{E}^\dagger_K(\mathfrak{X}^{\text{perf}})} M$ because of the Banach-Steinhaus theorem and Lemma 5.10.

□

Unfortunately we cannot expect the multiplication on $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})$ to be jointly continuous, only separately continuous, so this proof does not work if we replace $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})^r$ with $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})$.
Every (φ,Γ)-module over $\mathcal{R}_K(\mathfrak{X})$ has a model ([6, Proposition 2.24]). But the proof of this proposition relies on the fact that $\mathcal{R}_K(\mathfrak{X})$ is a compactoid inductive limit. But this does not seem to be the case for $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})$. The problem is again the continuity of the $o_L^\times$-action.

Proposition 5.25. *The base change functor $- \otimes_{\mathcal{E}^\dagger_K(\mathfrak{X}^{\text{perf}})} \mathcal{R}_K(\mathfrak{X}^{\text{perf}})$ from étale (φ,Γ)-modules over $\mathcal{E}^\dagger_K(\mathfrak{X}^{\text{perf}})$ with model to étale (φ,Γ)-modules over $\mathcal{R}_K(\mathfrak{X}^{\text{perf}})$ is fully faithful.*

Proof. Let

$$\tilde{\alpha} \in \mathrm{Hom}_{\varphi,\Gamma}(\mathcal{R}_K(\mathfrak{X}^{\text{perf}}) \otimes_{\mathcal{E}^\dagger_K(\mathfrak{X}^{\text{perf}})} M, \mathcal{R}_K(\mathfrak{X}^{\text{perf}}) \otimes_{\mathcal{E}^\dagger_K(\mathfrak{X}^{\text{perf}})} N)$$

be a morphism of (φ,Γ)-modules. We find a morphism of the underlying φ-modules $\alpha \in \mathrm{Hom}_\varphi(M,N)$ such that $\tilde{\alpha} = \alpha \otimes id$ which follows from Theorem 5.15. Then α is already a morphism of (φ,Γ)-modules since

$$\begin{aligned}\tilde{\alpha}(u(m) \otimes 1) &= (\alpha \otimes id)(u(m) \otimes 1)\\ &= u((\alpha \otimes id)(m \otimes 1))\\ &= u(\alpha(m) \otimes 1),\end{aligned}$$

and hence

$$\alpha(u(m)) \otimes 1 = u(\alpha(m)) \otimes 1$$

for $u \in o_L^\times$ and $m \in M$. This implies $\alpha(u(m)) = u(\alpha(m))$ because N is a flat $\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})$-module. Therefore the map

$$\mathrm{Hom}_{\varphi,\Gamma}(M,N) \to \mathrm{Hom}_{\varphi,\Gamma}(M \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})} \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}), N \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})} \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}))$$

is surjective. It is also injective since this is true for the map between morphisms of the underlying φ-modules. □

Example 5.26. Let $\delta : L^\times \to L^\times$ be a continuous character. We define $M_\delta := \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \cdot e_\delta$ with $\varphi(e_\delta) = \delta(\pi) \cdot e_\delta$ and $u(e_\delta) = \delta(u) \cdot e_\delta$ if $u \in o_L^\times$.

Lemma 5.27. *M_δ is a (φ,Γ)-module over $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$.*

Proof. M_δ is a φ-module over $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$. The continuity of the $o_L^\times$-action follows by writing it as the composition of the continuous maps

$$\begin{aligned} o_L^\times \times \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) &\to L^\times \times \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}), \\ (u,f) &\mapsto (\delta(u), f), \quad \text{and} \\ L^\times \times \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) &\to \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}), \\ (a,f) &\mapsto a \cdot f. \end{aligned}$$

□

Lemma 5.28. *Let $\delta : L^\times \to L^\times$ be a continuous character which takes values in o_L. Then M_δ is an étale (φ,Γ)-module.*

Proof. Define $M'_\delta := \mathcal{E}_K^{\dagger,\le 1}(\mathfrak{X}^{\mathrm{perf}}) \cdot e_\delta$ with $\varphi(e_\delta) = \delta(\pi) \cdot e_\delta$ and $u(e_\delta) = \delta(u) \cdot e_\delta$. The continuity of the $o_L^\times$-action follows similarly as above. Then M'_δ is a (φ,Γ)-module over $\mathcal{E}_K^{\dagger,\le 1}(\mathfrak{X}^{\mathrm{perf}})$ such that $M_\delta \cong \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}}) \otimes_{\mathcal{E}_K^{\dagger,\le 1}(\mathfrak{X}^{\mathrm{perf}})} M'_\delta$ as (φ,Γ)-modules. □

We do not know if there is a base change functor from (φ,Γ)-modules over $\mathcal{E}_K^{\dagger,\le 1}(\mathfrak{X}^{\mathrm{perf}})$ (with the weak topology) to $\mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})$, even if we only regard modules with model. The inclusion $\mathcal{E}_K^{\dagger,\le 1}(\mathfrak{X}^{\mathrm{perf}})^r \hookrightarrow \mathcal{R}_K(\mathfrak{X}^{\mathrm{perf}})^r$ likely is not continuous because this is the case over $\mathfrak{B}^{\mathrm{perf}}$, so the proof of Lemma 5.24 does not work.

Lemma 5.29. *$\mathcal{R}_K(\mathfrak{X}) = \varinjlim_n \mathcal{O}_K(\mathfrak{X} \setminus \mathfrak{X}_n)$ is a regular inductive limit (i.e. each bounded subset $U \subseteq \mathcal{R}_K(\mathfrak{X})$ is contained in some $\mathcal{O}_K(\mathfrak{X} \setminus \mathfrak{X}_n)$ and is bounded there).*

Proof. This is [6][Proposition 2.6 i.]. □

Proposition 5.30. *Let M be a (φ,Γ)-module over $\mathcal{E}_K^\dagger(\mathfrak{X})$ such that the underlying $\mathcal{E}_K^\dagger(\mathfrak{X})$-module is free. Then M has a model over some $\mathcal{E}_K^\dagger(\mathfrak{X})^r$.*

Proof. Let n be the rank of M. Denote by M^{r_0} the φ-module over $\mathcal{E}_K^\dagger(\mathfrak{X})^{r_0}$ which comes from repeating the proof of Lemma 5.16 for $\mathcal{E}_K^\dagger(\mathfrak{X})$ and applying it to the underlying φ-module of M. Note that $M^{r_0} \cong (\mathcal{E}_K^\dagger(\mathfrak{X})^{r_0})^n$. Any orbit map

$$\rho_m : o_L^\times \to M$$

for $m \in M$ is continuous, so, $o_L^\times$ being compact, its image $\rho_m(o_L^\times)$ is compact and in particular bounded in M. Then $\rho_m(o_L^\times) \subseteq M \subseteq \mathcal{R}_K(\mathfrak{X}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X})} M$ is bounded as well and hence contained in $\mathcal{R}_K(\mathfrak{X})^r \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^{r_0}} M^{r_0} \cong (\mathcal{R}_K(\mathfrak{X})^r)^n$ for an $r_0 \leq r < 1$ and bounded there according to the previous lemma. Since $\mathcal{E}_K^\dagger(\mathfrak{X})^n \cap (\mathcal{R}_K(\mathfrak{X})^r)^n$ is contained in $(\mathcal{E}_K^\dagger(\mathfrak{X})^s)^n$, we have

$$\rho_m(o_L^\times) \subseteq \mathcal{E}_K^\dagger(\mathfrak{X})^s \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X}^{\mathrm{perf}})^{r_0}} M^{r_0} = M^s$$

for $s \geq r$. We may apply this to the finitely many generators of M^{r_0} and assume that $o_L^\times$ preserves M^s for $s \geq r$. Note that the continuous inclusion of $\rho_m(o_L^\times) \subseteq M^s$ is a homeomorphism onto its image in M. This is because we have a commutative diagram

$$\begin{array}{ccc} \mathcal{R}_K(\mathfrak{X})^s \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X})^s} M^s & \longrightarrow & \mathcal{R}_K(\mathfrak{X}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X})^s} M^s \\ \uparrow & & \uparrow \\ M^s & \longrightarrow & M \end{array}$$

where the vertical maps are a homeomorphism onto their image (the rings in the lower row carry the subspace topology from the rings in the upper row). The set $\rho_m(o_L^\times) \subseteq \mathcal{R}_K(\mathfrak{X})^s \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X})^s} M^s$ is bounded and hence compactoid ([6, Proposition 2.5]), and the inclusion $\mathcal{R}_K(\mathfrak{X})^s \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X})^s} M^s \to \mathcal{R}_K(\mathfrak{X}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X})^s} M^s$ defines a homeomorphism onto its image in $\mathcal{R}_K(\mathfrak{X}) \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X})^s} M^s$ (see [13, Corollary 3.8.39]). Then the inclusion $M^s \to M$ defines a homeomorphism of $\rho_m(o_L^\times)$ onto its image in M as well.
This shows that the orbit maps $o_L^\times \to M^s$ are continuous. Arguing as in the proof of Lemma 5.24 we see that this implies the continuity of the orbit maps

$$o_L^\times \to \mathcal{R}_K(\mathfrak{X})^s \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X})^s} M^s.$$

But over $\mathcal{R}_K(\mathfrak{X})^s$ we can use the Banach-Steinhaus theorem since $\mathcal{R}_K(\mathfrak{X})^s \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X})^s} M^s$ is barrelled. Together wit Lemma 5.10 this implies the continuity of

$$o_L^\times \times \mathcal{R}_K(\mathfrak{X})^s \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X})^s} M^s \to \mathcal{R}_K(\mathfrak{X})^s \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X})^s} M^s,$$

which then implies the continuity of

$$o_L^\times \times M^s \to M^s$$

because $M^s \subseteq \mathcal{R}_K(\mathfrak{X})^s \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X})^s} M^s$ and hence $o_L^\times \times M^s \subseteq o_L^\times \times \mathcal{R}_K(\mathfrak{X})^s \otimes_{\mathcal{E}_K^\dagger(\mathfrak{X})^s} M^s$ carry the subspace topology. □

Corollary 5.31. *The base change functor from the category of free* (φ,Γ)*-modules over* $\mathcal{E}_K^\dagger(\mathfrak{X})$ *to the category of free* (φ,Γ)*-modules over* $\mathcal{R}_K(\mathfrak{X})$ *is fully faithful.*

6 Appendix

6.1 Seminormed groups, rings and modules

We collect several facts about seminormed groups and rings as in [31, Chapter 1.1, 1.2].

Definition 6.1.
- *A* seminormed group *is a pair* $(G, \|\cdot\|)$ *consisting of an abelian group* G *and a function* $\|\cdot\| : G \to \mathbb{R}_{\geq 0}$ *satisfying* $\|0\| = 0$*, and* $\|g - h\| \leq \max(\|g\|, \|h\|)$ *for all* $g, h \in G$*.*
- *A seminorm on a ring* R *is a seminorm* $\|\cdot\|_R$ *on the underlying group* $(R, +)$ *such that* $\|1\|_R = 1$ *and* $\|xy\|_R \leq \|x\|_R\|y\|_R$*. A ring* $(R, \|\cdot\|_R)$ *with a fixed seminorm* $\|\cdot\|_R$ *is called a* seminormed ring.
- *A* seminormed module *module* M *over a seminormed ring* $(R, \|\cdot\|_R)$ *is an* R*-module with a seminorm* $\|\cdot\|_M$ *on the underlying group* $(M, +)$ *such that* $\|xm\|_M \leq \|x\|_R\|m\|_M$ *for all* $x \in R$ *and* $m \in M$*.*

Definition 6.2. *A homomorphism* $f : M \to N$ *between two seminormed groups is called* bounded *if there is a constant* C *such that* $\|f(m)\|_N \leq C\|m\|_M$ *for all* $m \in M$*. If* $C = 1$*, then* f *is called* non-expansive.

Note that every bounded homomorphism between seminormed groups is continuous for the topology induced by the seminorm.

Completions

Definition 6.3. *A seminormed group* $(G, \|\cdot\|)$ *is called* complete *if every Cauchy sequence in* G *has a limit in* G*.*

Definition 6.4. *Let* $(G, \|\cdot\|)$ *be a seminormed group. A pair* $(\widehat{G}, i)$ *is called completion of* G *if the following holds:*

(i) $\widehat{G}$ *is a complete normed group.*

(ii) $i : G \to \widehat{G}$ *is an isometric homomorphism.*

(iii) $i(G)$ *is dense in* $\widehat{G}$*.*

Proposition 6.5. *Each seminormed group admits a completion.*

Proposition 6.6. *Let G and H be seminormed groups, let $(\widehat{G}, i)$ respective $(\widehat{H}, j)$ be a completion of G respective H, and let $\varphi : G \to H$ be a continuous group homomorphism.*

(i) There is a unique group homomorphism $\widehat{\varphi} : \widehat{G} \to \widehat{H}$ such that the diagram is commutative.

(ii) If φ is bounded, then $\widehat{\varphi}$ is bounded with the same bound. If φ is an isometry, then so is $\widehat{\varphi}$.

(iii) If $G = H, (\widehat{G}, i) = (\widehat{H}, j)$, and if φ is the identity, then also $\widehat{\varphi}$ is the identity. If F is a seminormed group with a completion $(\widehat{F}, l)$ and $\psi : F \to G$ is a continuous group homomorphism, then $\widehat{\varphi \circ \psi} = \widehat{\varphi} \circ \widehat{\psi}$.

It follows that completions are uniquely determined up to isometric isomorphisms. Therefore we speak of *the* completion $\widehat{G}$ of a seminormed group G.

Inductive limits

Definition 6.7. *Let $(G_i, \| \cdot \|_i)_{i \in I}$ be an inductive system of seminormed groups with non-expansive transition maps. Let $G := \varinjlim_{i \in I} G_i$ be the inductive limit with $f_i : G_i \to G$ being the canonical maps. We endow G with the* inductive limit seminorm *given by*

$$\|g\| := \inf_{i \in I, g_i \in f_i^{-1}(g)} \|g_i\|_i.$$

This is the maximal seminorm making the f_i non-expansive. Therefore $(G, \| \cdot \|)$ is the inductive limit of $(G_i, \| \cdot \|_i)_{i \in I}$ in the category of seminormed groups with non-expansive homomorphisms.

Lemma 6.8. *Let $(G_i, \| \cdot \|_i)_{i \in I}$ be an inductive system of seminormed groups with non-expansive transition maps. Let $G := \varinjlim_{i \in I} G_i$ be the inductive limit. Then we have a morphism*

$$\varinjlim_{i \in I} G_i \to \widehat{\varinjlim}_{i \in I} (\widehat{G}_i)$$

which coincides with the completion map $\varinjlim_{i \in I} G_i \to \widehat{\varinjlim}_{i \in I} G_i$. Therefore we have an isometric isomorphism

$$\widehat{\varinjlim}_{i \in I} (\widehat{G}_i) \cong \widehat{\varinjlim}_{i \in I} G_i.$$

Proof. Let $\iota_i : G_i \to \widehat{G}_i$ be the completion map for every i. We get an induced map $\iota : \varinjlim_{i \in I} G_i \to \varinjlim_{i \in I} \widehat{G}_i$ which is an isometry since all ι_i are isometric. By composing with the completion map $\varinjlim_{i \in I} \widehat{G}_i \to \widehat{\varinjlim}_{i \in I} \widehat{G}_i$ we get an isometric map $\varinjlim_{i \in I} G_i \to \widehat{\varinjlim}_{i \in I} \widehat{G}_i$. The image of this map is dense. Therefore it fulfils the conditions of Definition 6.4. $\square$

Tensor products

Definition 6.9. *Let R be a normed ring and M, N two seminormed R-modules. The tensor product $M \otimes_R N$ with the tensor product seminorm*

$$|z|_\otimes := \inf(\max_{i=1,\dots,r}|x_i| \cdot |y_i|), \qquad z \in M \otimes_R N,$$

where the infimum is taken over all possible representations

$$z = \sum_{i=1}^{r} x_i \otimes y_i, \qquad x_i \in M, y_i \in N.$$

Remark 6.10. The tensor product seminorm $|\cdot|_\otimes$ is the maximal seminorm such that the bilinear map $\phi : M \times N \to M \otimes_A N$ is non-expansive, i.e. satisfies

$$|\phi(x, y)|_\otimes \leq |x| \cdot |y|.$$

Definition 6.11. *We define $M \widehat{\otimes}_R N$ as the (separated) completion of $M \otimes_R N$. The seminormed R-module $M \widehat{\otimes}_R N$ is called the* completed tensor product *of M and N over R.*

Lemma 6.12. *Let M, N be two semi-normed R-modules, then we have an isometric isomorphism*

$$\widehat{M} \widehat{\otimes}_R \widehat{N} \cong M \widehat{\otimes}_R N.$$

Proof. As in [31, Proposition 2.1.7/4]. $\square$

We denote by $\widehat{\varinjlim}_{i \in I} M_i$ the separated completion of $\varinjlim_{i \in I} M_i$ with respect to the inductive limit seminorm.

Lemma 6.13. *Filtered inductive limits of seminormed modules are compatible with tensor products of seminormed modules.*

Proof. This is Lemma 2.2.12 in [40]. $\square$

Corollary 6.14. *Let $(M_i, \|\cdot\|_i)_{i \in I}$ be a filtered system of seminormed R-modules, $(\widehat{M}, \|\cdot\|_M)$ its completed colimit, and $(N, \|\cdot\|_N)$ a seminormed R-module. Then we have an induced filtered system $(M_i \widehat{\otimes} N)_i$ of seminormed A-modules and an isometric isomorphism*

$$\widehat{M} \widehat{\otimes}_R N \cong \widehat{\varinjlim}_{i \in I} (M_i \widehat{\otimes}_R N).$$

Especially we have $\widehat{M} = \widehat{\varinjlim}_{i \in I} \widehat{M}_i$.

Proof. The previous lemma, and the fact that tensor products commute with inductive limits in the category of R-modules give us an isometric isomorphism

$$M \otimes_R N \cong \varinjlim_{i \in I} (M_i \otimes_R N)$$

of seminormed R-modules. By passing to the completions we get an isometric isomorphism

$$M \widehat{\otimes}_R N \cong \widehat{\varinjlim}_{i \in I} (M_i \otimes_R N).$$

The map

$$\iota : \varinjlim_{i \in I} (M_i \otimes_R N) \to \varinjlim_{i \in I} (M_i \widehat{\otimes} N)$$

induced by the completion maps

$$\iota_i : M_i \otimes_R N \to M_i \widehat{\otimes} N$$

is an isometry. We show that its image is dense: If $x \in \varinjlim_{i \in I} (M_i \widehat{\otimes} N)$, then let $x_{i_0} \in M_{i_0} \widehat{\otimes} N$ be a preimage under the canonical map $M_{i_0} \widehat{\otimes} N \to \varinjlim_i (M_i \widehat{\otimes} N)$ for some i_0. The image of the completion map $\iota_{i_0} : M_{i_0} \otimes N \to M_{i_0} \widehat{\otimes} N$ is dense, therefore for any $\varepsilon > 0$ we find a $y_{i_0} \in M_{i_0} \otimes N$ with

$$||\iota_{i_0}(y_{i_0}) - x_{i_0}||_{M_{i_0} \widehat{\otimes} N} \leq \varepsilon.$$

Let y be the image of y_{i_0} in $\varinjlim_{i \in I} (M_i \otimes N)$, then $\iota(y)$ is the image of $\iota_{i_0}(y_{i_0})$ in $\varinjlim_{i \in I} (M_i \widehat{\otimes} N)$, and we have

$$||\iota(y) - x||_{\varinjlim_{i \in I} (M_i \widehat{\otimes} N)} \leq \|\iota_{i_0}(y_{i_0}) - x_{i_0}\|_{M_{i_0} \widehat{\otimes} N} \leq \varepsilon.$$

This shows the density of the image of ι. We see that $\widehat{\varinjlim}_i (M_i \otimes_R N) \cong \widehat{\varinjlim}_i (M_i \widehat{\otimes} N)$, and together with Lemma 6.12 we get

$$\widehat{M} \widehat{\otimes}_R \widehat{N} \cong M \widehat{\otimes}_R N \cong \widehat{\varinjlim}_{i \in I} (M_i \otimes_R N) \cong \widehat{\varinjlim}_{i \in I} (M_i \widehat{\otimes}_R N).$$

□

Lemma 6.15. *Let (A_i, f_i) and (B_i, g_i) be two inductive systems of seminormed rings. Assume that we have isomorphisms of rings $h_i : A_i \to B_i$ such that the diagrams*

$$\begin{CD} A_i @>{f_i}>> A_{i+1} \\ @V{h_i}VV @VV{h_{i+1}}V \\ B_i @>>{g_i}> B_{i+1} \end{CD}$$

commute for all i. Then $\varinjlim_{f_i, i \in I} A_i \cong \varinjlim_{g_i, i} B_i$ as rings. If the h_i are isometric, then this isomorphism is isometric for the inductive limit seminorm.

Proof. The maps h_i give rise to an isomorphism $h : \varinjlim_{f_i,i} A_i \cong \varinjlim_{g_i,i} B_i$. If all h_i are isometric, the same is true for h which follows from the definition of the inductive limit seminorm. □

6.2 Locally convex vector spaces

The main source is [32]. Let V be topological K-vector space.

Definition 6.16. *A* lattice L *in* V *is an* o_L*-submodule such that for any* $v \in V$ *there is a nonzero* $a \in K^\times$ *such that* $av \in L$.

Let $(L_j)_j$ be a nonempty family of lattices in V such that

1. for any $j \in J$ and any $a \in K^\times$ there is a $k \in J$ such that $L_k \subseteq aL_j$,
2. for any two $i, j \in J$ there is a $k \in J$ such that $L_k \subseteq L_i \cap L_j$.

Then the convex subsets $v + L_j$ for $v \in V$ form a basis of a topology on V. This topology is then called the *locally convex topology on* V *defined by the family* $(L_j)_j$. For any $v \in V$ the convex subsets $v + L_j$ form a fundamental system of open and closed neighbourhoods of v in this topology.

Definition 6.17. *A locally convex vector space over* K *is a* K*-vector space equipped with a locally convex topology.*

If V is a locally convex K-vector space, then addition and scalar multiplication are continuous.

Let $(V_h)_h$ be a family of locally convex K-vector spaces together with linear maps $f_h : V \to V_h$. The coarsest topology for which all the maps f_h are continuous is called the *initial topology* on V with respect to the family $(f_h)_h$. This is a locally convex topology. It can be defined by all lattices which are finite intersections of lattices $(f_h^{-1}(L_{hj}))_{h,j}$ where $(L_{hj})_j$ is a defining family of lattices for the topology on V_h. Examples include the subspace topology, the projective limit topology.

In contrast, let $(V_h)_h$ be a family of locally convex K-vector spaces with linear maps $f_h : V_h \to V$. There is a unique finest locally convex topology on V for which all f_h are continuous. It is called the *locally convex final topology* on V. In general it is strictly coarser than the finest topology which makes all f_h continuous.

Lemma 6.18. *Assume* V *has the locally convex final topology with respect to a family of linear maps* $f_h : V_h \to V$. *Then:*

1. *A* K*-linear map* $f : V \to W$ *into another locally convex* K*-vextor space* W *is continuous if and only if all the maps*

$$f \circ f_h : V_h \to W$$

are continuous;

2. *Assume that the topology on V_h is defined by the family of lattices $(J_{hj})_{j\in J(h)}$ and that we have $V = \sum_{h\in H} f_h(V_h)$, then the topology on V is defined by the family of lattices $\{\sum_{h\in H} f_h(L_{hj(h)}) \mid j(h) \in J(h)\}$.*

Proof. Lemma 5.1 in [32]. □

Proposition 6.19. *For a Hausdorff locally convex K-vector space V the following assertions are equivalent:*

1. *V is metrizable;*
2. *the topology on V can be defined by a translation invariant metric which fulfils the strict triangle equation;*
3. *the topology on V can be defined by a countable family of lattices;*
4. *the topology on V can be defined by a countable family of seminorms.*

Proof. Proposition 8.1 in [32]. □

Definition 6.20. *A locally convex vector space V over K is called a K-*Fréchet space *if V is metrizable and complete.*

Definition 6.21. *A locally convex vector space V is called* barrelled *if every closed lattice in V is open.*

Example 6.22. 1. Fréchet spaces are barrelled.

2. Let the topology on V be the locally convex final topology with respect to a family of linear maps $f_h : V_h \to V$. If all the V_h are barrelled, then so is V.

Definition 6.23. *A subset $H \subseteq \mathrm{Hom}_K(V, W)$ is* equicontinuous *if for any open lattice $M \subseteq W$ there is an open lattice $L \subseteq V$ such that $f(L) \subseteq M$ for every $f \in H$.*

Proposition 6.24 (Open mapping theorem)**.** *Let V be a Fréchet space. If $f : V \to W$ is a continuous linear surjection onto a Hausdorff and barrelled locally convex K-vector space W, then f is open.*

Proof. Proposition 8.6 in [32]. □

Theorem 6.25 (Banach-Steinhaus)**.** *If V is barrelled then any bounded subset $H \subseteq \mathcal{L}_S(V, W)$ is equicontinuous.*

Corollary 6.26. *Let V be barrelled and G be a locally compact topological group such that G acts on V. Then this action is continuous if and only if it is separately continuous.*

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