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On extension of rigid analytic objects

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Abstract. The identity principle for analytic functions predicts the value of an analytic function on a connected open subset at any point if the germ of the function is known at one given point. Therefore, in higher dimension, it can happen that the domain of definition of analytic functions on a connected open subset G of a polydisc $\mathbb{D}^n$ is larger than the given G. It depends on the geometry of G. For example, if G is the periphery of a 2-dimensional polydisc, then every analytic function on G is actually defined on the whole polydisc. Such a property is true for many other objects which are analytically defined, such as meromorphic functions, closed analytic subsets, vector bundles or coherent sheaves. The property of continuity depends on different parameters. The concavity of G inside a polydisc in relation to the dimension of the surrounding space plays an important role. The extension property of analytic objects depends on the balance between concavity on the one hand and on parameters of the analytic object such as the dimension of the closed analytic subset or the homological dimension of a coherent sheaf on the other hand. In complex analysis, these subjects were studied by Siu and Trautmann; for a systematic account, see [32].

In rigid geometry, John Tate has introduced a topology such that the identity principle holds for rigid analytic functions. Therefore, one can expect that statements on continuity are true in rigid geometry as well. In the first section, § 1, we present all the extension properties precisely and describe the shape of concavity for the different problems. The shape of a domain G inside a polydisc were suggested by Hans Grauert who advised W. Bartenwerfer around 1970 to study the problem for meromorphic functions. In complex analysis, these geometric constellations were well-known; cp. the thesis of Riemenschneider [29]. Then Bartenwerfer published a series of papers concerning such problems. Later on, the author contributed to these questions also. The hardest part is the extension problem for vector bundles which was solved in [27] by the author. In an unpublished paper, the author completed the picture by showing the continuity for coherent sheaves.

The intention of this paper is to present a single organized treatment of the extension properties of analytic objects. Some results are known but spread across the literature and mostly hard to access, especially the results on extension of meromorphic functions and of analytic subsets. In this paper, we provide simplifications and improvements of their proofs. The results in Sections 5 and 7 are due to the author and published many years ago. Since they are so central, they should not be omitted in this treatment. The results in Section 8 are partly new. The appendix is certainly of more general interest since the given proofs bring the real arguments to light.

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It is not surprising that after so many years the proofs can be strengthened and organized in a straight line as it is done in this paper which is self-contained except for general results in rigid geometry; cp. [9]. We should mention that the analysis of the geometry of ball figures and of Hartogs figures was completely developed by Bartenwerfer. The new ingredients of this paper are the use of descent theory, which makes the extension properties for meromorphic functions more transparent and also allows to show the continuity for coherent sheaves in an accessible way. One word concerning étale descent: if one looks at Hartogs figures, one has to reduce the geometric situation to a standard one by finite morphisms. But usually, one meets only quasi-finite maps. So, by étale base change, one can transform the situation into finite maps, and afterwards, one has to descend to the original setting.

CONTENTS

A short advice concerning the notation: in this paper, we denote by K a complete field with respect to a nontrivial nonarchimedean valuation $|\cdot|$, by $|K^\times|$ its value group, by R its valuation ring, by $\mathrm{Spf}(R)$ the formal spectrum of R and by k its residue field. We list some often used notations:

$$\begin{aligned}
T_n &:= K\langle\zeta_1,\dots,\zeta_n\rangle \\
&:= \Big\{f = \sum_{\nu\in\mathbb{N}^n} c_\nu\zeta^n \in K[[\zeta]];\ c_\nu \to 0\Big\} && \text{Tate algebra,}\\
\mathbb{A}^n_K &:= \mathrm{MaxSpec}\, K[\zeta_1,\dots,\zeta_n] && \text{affine } n\text{-space,}\\
\mathbb{D}^n_K &:= \{x\in\mathbb{A}^n_K;\ |\zeta_\nu(x)|\le 1 \text{ for } \nu=1,\dots,n\} && n\text{-dimensional unit ball,}\\
\mathbb{D}^n_K(\varepsilon) &:= \{x\in\mathbb{A}^n_K;\ |\zeta_\nu(x)|\le \varepsilon \text{ for } \nu=1,\dots,n\} && n\text{-ball of radius } \varepsilon,\\
\partial\mathbb{D}^n_K &:= \{x\in\mathbb{D}^n;\ |\zeta_\nu(x)| = 1 \text{ for some } \nu\} && \text{periphery of } \mathbb{D}^n_K,\\
\mathbb{D}^n_+ &:= \{x\in\mathbb{D}^n;\ |\zeta_\nu(x)| < 1 \text{ for } \nu=1,\dots,n\} && \text{formal fiber at 0 of } \mathbb{D}^n_K.
\end{aligned}$$

In our notation $\mathbb{D}^n_K$ for the n-dimensional unit ball, we usually drop the subindex K since it is always clear what the base field is; we need this subindex for other purposes. An affinoid K-algebra is a residue algebra of some T_n. We denote by $\mathrm{Sp}(A)$ the affinoid space associated to A whose points are the maximal ideals of A. Especially, we have $\mathrm{Sp}(T_n) = \mathbb{D}^n_K$.

If $X := \mathrm{Sp}(A)$ is an affinoid space, we denote by $|\cdot|$ the spectral norm on A as well, and by $A^\circ \subset A$ the R-algebra of power bounded functions; its canonical reduction is $\tilde{A}$, and usually, $\rho\colon A^\circ \to \tilde{A}$ is the reduction map. Reduced objects are usually denoted by a $\tilde{\ }$ on top. Note that we omit the sign $\tilde{\ }$ in the case of variables, so the reduction of the Tate algebra $T_n = K\langle\zeta_1, \dots, \zeta_n\rangle$ is denoted by $k[\zeta_1, \dots, \zeta_n]$

We assume that the reader is familiar with the basic theory of rigid analytic spaces in the sense of Tate [34]; cp. [9]. In particular, Weierstraß theory is often used; cp. [9, § 2.2]. At some places, we use formal geometry; for a reference see [9] as well.

1. Introduction

Let us first explain the several problems we deal with in this paper.

Extension problems. In the following, let X be a rigid space, mostly being affinoid of pure dimension, let $G \subset X$ be a nonempty open subset of X, and let $n \in \mathbb{N}$ be a natural number. Now we consider the following extension properties with respect to the pair (G, X).

(H_n) Let $S \subset X$ be an irreducible closed analytic subset of X with $\dim S \geq n+1$. Equip S with its reduced structure. Denote by $\mathcal{O}'$ the entire closure of the structure sheaf $\mathcal{O}$ in the sheaf $\mathcal{M}$ of meromorphic functions. Then the restriction map $\mathcal{O}'(S) \xrightarrow{\sim} \mathcal{O}'(S \cap G)$ is bijective.

(M_n) Let $S \subset X$ be an irreducible closed analytic subset of X with $\dim S \geq n+1$. Equip S with its reduced structure. Then the restriction $\mathcal{M}(S) \xrightarrow{\sim} \mathcal{M}(S \cap G)$ is bijective.[1]

(A_n) For any closed analytic subset $S \subset G$ with irreducible components S_i of dimension $\dim S_i \geq n+1$, there exists a closed analytic subset $\underline{S} \subset X$ with $\underline{S} \cap G = S$.
The extension $\underline{S}$ is unique in the sense that any closed analytic subset $\underline{T} \subset X$ with $\underline{T} \cap G = S$ and with irreducible components T_j of dimension $\dim T_j \geq n+1$ equals $\underline{S}$; *i.e.*, $\underline{T} = \underline{S}$.

(U_n) For any coherent sheaf $\underline{\mathcal{G}}$ on X and any coherent subsheaf $\mathcal{F} \subset \mathcal{G} := \underline{\mathcal{G}}|_G$ with $\mathcal{F} = \mathcal{F}_{[n]\mathcal{G}}$, there is a coherent subsheaf $\underline{\mathcal{F}} \subset \underline{\mathcal{G}}$ with $\underline{\mathcal{F}}|_G = \mathcal{F}$ and $\underline{\mathcal{F}} = \underline{\mathcal{F}}_{[n]\underline{\mathcal{G}}}$.[2] For any further coherent subsheaf extension $\underline{\mathcal{H}} \subset \underline{\mathcal{G}}$ of $\mathcal{F}$ with $\underline{\mathcal{H}}_{[n-1]\underline{\mathcal{G}}} = \underline{\mathcal{H}}$, it holds $\underline{\mathcal{H}} = \underline{\mathcal{F}}$.

[1] A meromorphic function on a reduced rigid space is a element which can, locally with respect to the Grothendieck topology, be represented by a fraction of two affinoid functions where the denominator is a nonzero divisor; cp. Definition 2.1.

[2] A coherent subsheaf $\mathcal{F}$ of a coherent sheaf $\mathcal{G}$ satisfies $\mathcal{F} = \mathcal{F}_{[n]\mathcal{G}}$ if, for every open subset U and every closed analytic subset A of U with $\dim A \leq n$, it holds $\Gamma(U, \mathcal{F}) = \Gamma(U, \mathcal{G}) \cap \Gamma(U - A, \mathcal{F})$; cp. Definition 4.1.

(G_n) For any coherent sheaf $\mathcal{G}$ on G with $\mathcal{G} = \mathcal{G}^{[n]}$, there exists a coherent sheaf $\underline{\mathcal{G}}$ on X with $\underline{\mathcal{G}}|_G = \mathcal{G}$ and $\underline{\mathcal{G}} = \underline{\mathcal{G}}^{[n]}$.[3]
If $\underline{\mathcal{H}} = \underline{\mathcal{H}}^{[n-1]}$ is a further extension of $\mathcal{G}$, then the isomorphism $\mathcal{H} \xrightarrow{\sim} \mathcal{G}$ over G extends to an isomorphism $\underline{\mathcal{H}} \xrightarrow{\sim} \underline{\mathcal{G}}$ over X.

($G(n)$) For every $m \geq n$ and for any coherent sheaf $\mathcal{G}$ on G with $\mathcal{G} = \mathcal{G}^{[n]}$ satisfying $0_{[m+1]\mathcal{G}} = 0$ and $0_{[m+2]\mathcal{G}} = \mathcal{G}$, there exist a coherent sheaf $\underline{\mathcal{G}} = \underline{\mathcal{G}}^{[n]}$ on X and an isomorphism $\underline{\mathcal{G}}|_G \xrightarrow{\sim} \mathcal{G}$ over G.

Property ($G(n)$) is stated only for technical reasons. It is a special case used for proving property (G_n). The uniqueness assertion is mostly a consequence of the following property.

(E_n) Any irreducible closed analytic subset S of X with $\dim S \geq n$ meets the subdomain G.

In the next section, we will introduce ball figures and Hartogs figure of dimension n. Then the main goal of this paper is to show that ball figures of dimension $(n-1)$ resp. Hartogs figures of dimension n fulfill all these properties at level n; cp. Proposition 3.3, Theorem 8.4, Theorem 4.6 resp. Theorem 3.6, Theorem 8.11 resp. Theorem 8.12. Moreover, the extension properties are shown for complements of closed analytic subvarieties in Theorem 8.5.

Ball figures. We will study the extension properties for special pairs (G, X) which will be introduced in the following. The standard ball figure is the following configuration.

Definition 1.1. Let $d \geq 1$ and $n \geq 0$ be integers. Let $X := \mathbb{D}^{n+d}$ be the $(n+d)$-dimensional unit polydisc, and set

$$B := \mathbb{D}^n \times \partial\mathbb{D}^d.$$

The pair (B, X) is called *standard ball figure* of dimension n inside the $(n+d)$-dimensional unit polydisc.

The basic fact about extension problems is the following proposition.

Proposition 1.2. *Let (B, X) be a standard ball figure of dimension n in the $(n+2)$-dimensional polydisc $X := \mathbb{D}^{n+2}$. Then the restriction morphism $\mathcal{O}_X(X) \xrightarrow{\sim} \mathcal{O}_X(B)$ is bijective.*

Proof. Due to the identity principle, the restriction map is injective. For showing the surjectivity, denote by ζ_1, ζ_2 the coordinate function on $\mathbb{D}^2$ and by $\xi := (\xi_1, \ldots, \xi_n)$ the coordinate functions on $\mathbb{D}^n$. Then any affinoid function $f \in \mathcal{O}(\mathbb{D}^n \times \mathbb{D}^1 \times \partial\mathbb{D}^1)$ has a Laurent series expansion

$$f = \sum_{\mu \in \mathbb{N}, \nu \in \mathbb{Z}} a_{\mu,\nu}(\xi) \cdot \zeta_1^{\mu} \cdot \zeta_2^{\nu}.$$

[3] A coherent sheaf $\mathcal{G}$ satisfies $\mathcal{G} = \mathcal{G}^{[n]}$ if, for every open subset U and every closed analytic subset A of U with $\dim A \leq n$, the restriction map $\Gamma(U, \mathcal{G}) \to \Gamma(U - A, \mathcal{G})$ is bijective; cp. Definition 6.6.

If $f \in \mathcal{O}(B)$, then the restriction of f to $\mathbb{D}^n \times \partial\mathbb{D}^1 \times \mathbb{D}^1$ has the Laurent expansion

$$f = \sum_{\mu \in \mathbb{Z}, \nu \in \mathbb{N}} a_{\mu,\nu}(\xi) \cdot \zeta_1^{\mu} \cdot \zeta_2^{\nu}.$$

This shows that the coefficients $a_{\mu,\nu}(\xi) = 0$ if $\mu < 0$ or $\nu < 0$. Thus we see that f is induced by a function on $\mathbb{D}^{n+2}$. □

The standard ball figure can be generalized to arbitrary affinoid space by using the notion of reductions of affinoid algebras. We begin by recalling some basic facts.

Reminder 1.3. Let A be an affinoid K-algebra. We denote by

$$A^\circ := \{a \in A;\ |a| \le 1\} \supset A^\vee := \{a \in A;\ |a| < 1\}$$

the R-subalgebra of A consisting of all power bounded elements of A and by $A^\vee \subset A^\circ$ the ideal of all topologically nilpotent elements. Then $\tilde{A} := A^\circ/A^\vee$ is a reduced affine k-algebra; cp. [10, § 1.2.5]. Recall [10, § 6.3] that $A \rightsquigarrow \tilde{A}$ is a functor and that a morphism $\varphi\colon A \to B$ is finite if and only if $\tilde{\varphi}\colon \tilde{A} \to \tilde{B}$ is finite. If $\dim A = n$, then we also have $\dim \tilde{A} = n$, and moreover, if A is of pure dimension, then $\tilde{A}$ is as well.

If we denote by $X = \mathrm{Sp}(A)$ the associated affinoid space, then $\tilde{X} := \mathrm{Spec}(\tilde{A})$ is the associated affine space; this space is called the *(standard) reduction* of X. There is a reduction map $\rho\colon A^\circ \to \tilde{A};\ a \mapsto \tilde{a}$. For affinoid spaces, we also write $\rho_X : X \to \tilde{X}$ which maps a maximal ideal $x \in X$ to the maximal ideal $x \cap A^\circ$ mod $A^\vee$. If no confusion is possible, we will write ρ instead of ρ_X. If $S \subset X$ is a closed analytic subset of dimension m, then $\rho(S) \subset \tilde{X}$ is a Zariski-closed subset of $\tilde{X}$ of the same dimension m. In particular, if S is of pure dimension, then $\tilde{S}$ is too. Indeed, due to [10, Prop. 7.1.5/2], we have a commutative diagram

$$\begin{array}{ccccc} S & & \xrightarrow{\quad\iota\quad} & & X \\ \big\downarrow{\scriptstyle \rho_S} & & & & \big\downarrow{\scriptstyle \rho_X} \\ \tilde{S} & \xrightarrow{\ \tilde{\iota}\ } & \rho(S) & \hookrightarrow & \tilde{X} \end{array}$$

where $\iota\colon S \to X$ is the inclusion map. The map $\tilde{\iota}$ is finite and surjective. Mostly, we will identify $\tilde{S}$ with its image $\rho(S) \subset \tilde{X}$, and hence we write $\tilde{S}$ for $\rho(\tilde{S})$.

Finally, we add a fact which will often be used in this article.

Let $X := \mathrm{Sp}(A)$ be an affinoid space and $Y := \mathrm{Sp}(B) \subset X \times \mathbb{D}^1$ a closed analytic subvariety. If $Y \cap (X \times \partial\mathbb{D}^1) = \varnothing$, then the projection $\varphi\colon X \times \mathbb{D}^1 \to X$ gives rise to a finite morphism $\varphi|_Y : Y \to X$.

Indeed, the reduction map $\tilde{\varphi}\colon \widetilde{A\langle\zeta\rangle} = \tilde{A}[\tilde{\zeta}] \to \tilde{B}$ is finite as $A\langle\zeta\rangle \to B$ is surjective. Since $\tilde{\varphi}(\tilde{\zeta}) = 0$, we see that $\tilde{A} \to \tilde{B}$ is finite, and hence $\varphi|_Y$ is finite. □

Definition 1.4. Let $d \geq 1$ and $n \geq 0$ be integers, and let $X := \operatorname{Sp} A$ be an affinoid space of pure dimension $n + d$ with standard reduction $\tilde{X} := \operatorname{Spec}(\tilde{A})$. A *ball figure* B of dimension n in X is given by functions $f_\sigma \in A$ with spectral norm $|f_\sigma| = 1$ and numbers $\varepsilon_\sigma \in |K^\times|$ with $\varepsilon_\sigma \leq 1$ for $\sigma = 1, \ldots, s$ such that B equals the set

$$B = X_{\underline{f},\underline{\varepsilon}} := \bigcup_{\sigma=1}^{s} \{x \in X;\ |f_\sigma(x)| \geq \varepsilon_\sigma\},$$

where the locus $\tilde{N} := V(\tilde{f}_1, \ldots, \tilde{f}_s)$ on the reduction $\tilde{X}$ of the reduced functions $\tilde{f}_1, \ldots, \tilde{f}_s$ has dimension $\dim \tilde{N} \leq n$.

For a better understanding, it might be reasonable to consider the codimension d of $\tilde{N}$ in $\tilde{X}$. Namely, the larger the codimension, the bigger is the ball figure. These figures were first considered by Bartenwerfer in [3, p. 205].

If the numbers $\varepsilon_1, \ldots, \varepsilon_s$ are equal to 1, then B depends only on $\tilde{N}$; in this case, we will write $B := X_{\tilde{N}}$. If $\tilde{N} = V(\tilde{f})$ for a single function $f \in A^\circ$, we will also write $X_{\tilde{f}}$ instead of $X_{\tilde{N}}$. The affinoid ring associated to $X_{\tilde{f}}$ is $A\langle f^{-1}\rangle$, and its reduction is $\tilde{A}_{\tilde{f}}$; cp. [10, Prop. 7.2.6/3]

The simplest example is given by the standard ball figure. So let X be the unit ball $\mathbb{D}^{n+d}$ with coordinate functions $\xi_1, \ldots, \xi_n; \zeta_1, \ldots, \zeta_d$. The standard ball figure is given by the functions $f_\sigma := \zeta_\sigma$ and $\varepsilon_\sigma := 1$ for $\sigma = 1, \ldots, d$. Then we have $B := \mathbb{D}^n \times \partial\mathbb{D}^d = X_{\underline{\zeta},\underline{1}}$.

The most important example is the case $X_{\tilde{N}}$ where $\tilde{N} \subset \tilde{X}$ is the image of a closed analytic subset $N \subset X$ of an affinoid space under the reduction map $X \to \tilde{X}$.

If $S \subset X_{\tilde{N}}$ is a purely m-dimensional closed analytic subset of a ball figure $X_{\tilde{N}}$, then the image of S under the reduction map $X_{\tilde{N}} \to \tilde{X} - \tilde{N}$ is purely m-dimensional too; cp. Reminder 1.3. Then we denote by $\tilde{S}$ the Zariski closure of this image in $\tilde{X}$.

Lemma 1.5. *Let $d, m, n \in \mathbb{N}$ be natural numbers with $n + d \geq m > n$. Let $X = \operatorname{Sp}(A)$ be an affinoid space of pure dimension $(n + d)$, let $B \subset X$ be a ball figure of dimension n as in Definition 1.4, and let $S \subset B$ be a closed analytic subset with $\dim S = m$. Then there exists a finite morphism $\phi\colon X \to \mathbb{D}^{n+d}$ such that*

$$\phi^{-1}(\mathbb{D}^n \times \partial\mathbb{D}^d) \subset X_{\tilde{N}} \subset B \quad \textit{and} \quad \phi^{-1}(\mathbb{D}^m \times \partial\mathbb{D}^{n+d-m}) \cap S = \varnothing.$$

Let $p := \mathbb{D}^{n+d} \to \mathbb{D}^m$ be the projection onto the first coordinates; then the restriction

$$\psi := p \circ \phi|_{\ldots}\colon S \cap (p \circ \phi)^{-1}(\mathbb{D}^n \times \partial\mathbb{D}^{m-n}) \to \mathbb{D}^n \times \partial\mathbb{D}^{m-n}$$

is finite. Moreover, the induced map $\tilde{p}\colon \tilde{S} \to \mathbb{A}_k^m$ is finite; cp. Reminder 1.3 for the notation.

Proof. Let $\tilde{N} \subset \tilde{X}$ be the algebraic n-dimensional subset of $\tilde{X}$ associated to B, and let $\tilde{S} \subset \tilde{X}$ be the m-dimensional algebraic subset induced by $S \cap X_{\tilde{N}}$;

cp. Reminder 1.3. Due to [31, Thm. III-20], there exists a finite morphism $\tilde{\phi}\colon \tilde{X} \to \mathbb{A}_k^{n+d}$ with the following properties:

$$\widetilde{\phi}(\widetilde{S}) \subset V(\tilde{\zeta}_{m+1}, \dots, \tilde{\zeta}_{n+d}) \quad \text{and} \quad \widetilde{\phi}(\widetilde{N}) \subset V(\tilde{\zeta}_{n+1}, \dots, \tilde{\zeta}_{n+d}),$$

where $(\tilde{\zeta}_1, \dots, \tilde{\zeta}_{n+d})$ are the coordinate functions of $\mathbb{A}_k^{n+d}$. Lifting everything to the affinoid site, we obtain a morphism $\phi\colon X \to \mathbb{D}^{n+d}$ which fulfills all the assertions. □

In particular, we obtain the following proposition.

Proposition 1.6. *Any n-dimensional ball figure satisfies property* (E_{n+1}).

Even more is true.

Lemma 1.7. *If $B' \subset B \subset X = \mathrm{Sp}(A)$ are ball figures of dimension n, then any closed analytic subset S of B with $\dim S \geq n+1$ meets B' as well.*

Proof. Let $B = X_{\underline{f},\underline{\varepsilon}}$ be the n-dimensional ball figure. Let $S \subset B$ of dimension $m > n$, and assume that

$$S \cap \{x \in X;\, \varepsilon_1 \leq |f_1(x)| \leq 1\} \neq \varnothing.$$

Now let $\phi\colon X \to \mathbb{D}^{n+d}$ be a finite morphism as in Lemma 1.5. Then consider the map

$$\psi := (\phi, f_1)\colon S \cap \{x \in X;\, \varepsilon_1 \leq |f_1(x)| \leq 1\} \to \mathbb{D}^{n+d} \times A(\varepsilon_1, 1),$$

where $A(\varepsilon_1, 1)$ is the annulus with radii $\varepsilon_1, 1$. Then ψ is finite. If the image $\psi(S)$ does not meet $\mathbb{D}^n \times \partial\mathbb{D}^d \times A(\varepsilon_1, 1)$, then the projection along $\mathbb{D}^d$ to $\mathbb{D}^n \times A(\varepsilon_1, 1)$, restricted to $\psi(S)$, is finite due to Reminder 1.3 and surjective by reasons of dimensions. So there exists a point $x \in S$ with $|f_1(x)| = 1$. Since $\phi^{-1}(\mathbb{D}^n \times \partial\mathbb{D}^d) \subset X_{\tilde{N}}$, there always exists a point $x \in S \cap X_{\tilde{N}}$, meaning $S \cap X_{\tilde{N}} \neq \varnothing$. Thus the algebraic subset $\tilde{S}$ induced by the subset $S \cap X_{\tilde{N}} \neq \varnothing$ is of dimension m. If now $\tilde{N}' \subset \tilde{X}$ is the algebraic subset associated to B' of dimension n, then by reasons of dimension, there is a point $\tilde{s} \in \tilde{S} - (\tilde{N}' \cup \tilde{N})$. Thus we see that $S \cap B' \neq \varnothing$. □

With these lemmata, we will be able to prove all the extension properties on level $n+1$ for n-dimensional ball figures by reducing the questions to problems of the standard ball figure.

Hartogs figures. We will distinguish two types of Hartogs figures, rectilinear respectively affinoid ones. If we will talk about Hartogs figures in general, then we mean both types.

Definition 1.8. Let Y be an irreducible, reduced rigid space of dimension n, and let $V \subset Y$ be a nonempty open subdomain. Let $d \geq 1$ be an integer. Consider the rigid space

$$H := (V \times \mathbb{D}^d) \cup (Y \times \partial\mathbb{D}^d).$$

The pair (H, X) is called a *rectilinear Hartogs figure* of dimension n inside the rigid space $X := Y \times \mathbb{D}^d$ which is of dimension $(n+d)$.

Obviously, the standard ball figure of dimension n is a rectilinear Hartogs figure of dimension $n+1$. The basic fact about extension problems is the following proposition.

Proposition 1.9. *Let (H, X) be a rectilinear Hartogs figure of dimension n in the $(n+1)$-dimensional space $X := Y \times \mathbb{D}^1$ as in Definition 1.8. Then the restriction map $\mathcal{O}(X) \tilde{\to} \mathcal{O}(H)$ is bijective.*

Proof. Due to the identity principle the restriction map is injective. For showing the surjectivity, denote by ζ the coordinate function on $\mathbb{D}^1$. Then any $f \in \mathcal{O}(Y \times \partial\mathbb{D}^1)$ has a Laurent series expansion

$$f = \sum_{\nu \in \mathbb{Z}} a_\nu(\xi) \cdot \zeta^\nu.$$

If $f \in \mathcal{O}(H)$, the restriction of f to $V \times \mathbb{D}^1$ shows that the coefficients $a_\nu = 0$ for all $\nu < 0$. Thus we see that f is induced by a function on $X = Y \times \mathbb{D}^1$. □

As in the case of ball figures, one can introduce more general Hartogs figures.

Definition 1.10. Let $X = \mathrm{Sp}(A)$ be an affinoid space of pure dimension $(n+d)$. Let $B \subset X$ be a ball figure of dimension n in X. Let $\tilde{N} \subset \tilde{X}$ be the algebraic subset associated to B; cp. Definition 1.4. Let $\tilde{N}_1, \dots, \tilde{N}_r$ be the irreducible components of $\tilde{N}$. For any irreducible component $\tilde{N}_j$ of dimension n, there is a complete intersection

$$\tilde{M}_j = V(\tilde{g}_{j,1}, \dots, \tilde{g}_{j,n}) \quad \text{with } g_{j,\nu} \in A^\circ$$

of codimension n in $\tilde{X}$ such that $\dim \tilde{N} \cap \tilde{M}_j = 0$ for $j = 1, \dots, r$ and such that, for every irreducible component $\tilde{N}_i$ of dimension n, there exists at least one $\tilde{M}_j$ with $\tilde{N}_i \cap \tilde{M}_j \neq \varnothing$. Note $\dim \tilde{M}_j = d$. Let T be the union of the *tubes*

$$T_j := \{x \in X;\, |g_{j,\nu}(x)| \leq \delta_{j,\nu} \text{ for } 1 \leq \nu \leq n\},$$

where $\delta_{j,\nu} \in |K^\times|$ with $0 < \delta_{j,\nu} < 1$. The subset $H := T \cup B$ is called an *affinoid Hartogs figure* of dimension n. The tubes are called *maximal* if the T_j are defined by $|g_{j,\nu}(x)| < 1$ instead of $|g_{j,\nu}(x)| \leq \delta_{j,\nu}$.

Obviously, a Hartogs figure $H \subset X$ of dimension n contains a ball figure of dimension n. A rectilinear Hartogs figure is an affinoid Hartogs figure if the open subset $V \subset Y$ can be described by n functions $g_1, \dots, g_n$; this is not always possible. Of course, in the case $Y = \mathbb{D}^n$, this is satisfied. The situation for these more general affinoid Hartogs figures seems more difficult because we do not have a projection statement like Lemma 1.5 for ball figures. In this paper, we will show that any Hartogs figure of dimension n satisfies all the extension properties of level n. Historically, the notion of an affinoid Hartogs figure was introduced by Bartenwerfer in [5, p. 90] for general d and earlier in [4, p. 157] for $d = 1$.

Lemma 1.11. *Let $H = T \cup B$ be a Hartogs figure of dimension n as defined in Definition 1.10. Then, for every tube T_j of T, there exist the functions*

$(h_{n+1},\dots,h_{n+d})$ such that $\phi_j := (g_{j,1},\dots,g_{j,n},h_{n+1},\dots,h_{n+d})$ gives rise to a finite morphism

$$\phi_j : X \cap \{x \in X;\ |g_{j,\nu}(x)| < 1 \text{ for } 1 \le \nu \le n\} \to \mathbb{D}^n_+ \times \mathbb{D}^d$$

satisfying $\phi_j^{-1}(\mathbb{D}^n_+ \times \partial\mathbb{D}^d) \subset B$, where $\mathbb{D}^n_+$ is the subset of $\mathbb{D}^n$ consisting of all $z \in \mathbb{D}^n$ with coordinates $(z_1,\dots,z_n)$ and $|z_\nu| < 1$ for all $\nu = 1,\dots,n$.

Proof. Denote by $\underline{g}$ the tuple $(g_{j,1},\dots,g_{j,n})$. As $V(\underline{\tilde{g}})$ is of codimension n in $\tilde{X}$ and $\dim \tilde{M}_j \cap \tilde{N} = 0$, by Noether's Normalization Theorem [31, Thm. III-20], there exist functions $h_{n+1},\dots,h_{n+d}$ in $\mathcal{O}_X(X)$ with spectral norm $|h_\delta| \le 1$ such that their reductions belong to the vanishing ideal of $\tilde{N}$ and such that the map $\underline{\tilde{h}}\colon \tilde{X} \cap V(\underline{\tilde{g}}) \to \mathbb{A}^d_k$ is finite. This yields the assertion by Reminder 1.3. □

Proposition 1.12. *Any Hartogs figure of dimension n has property* (E_n).

Proof. Let $H = T \cup B$ be defined in Definition 1.10. Let $S \subset X$ be an irreducible closed analytic subset of dimension $m \ge n$. If $m \ge n+1$, then S meets B due to Proposition 1.6. So we may assume $\dim S = n$. Let $\tilde{N} \subset \tilde{X}$ be the algebraic subset of $\tilde{X}$ associated to B, and let $\tilde{S} \subset \tilde{X}$ be the algebraic subset induced by S via the reduction map. Then $\tilde{S}$ is also of dimension n. If $\tilde{S} \not\subset \tilde{N}$, then S meets B obviously. So we may assume that $\tilde{S} \subset \tilde{N}$. Then there exists an irreducible component $\tilde{S}_0$ of $\tilde{S}$ such that $\tilde{S}_0$ coincides with an irreducible component $\tilde{N}_i$ of $\tilde{N}$. Due to the definition of a Hartogs figure, there is a complete intersection $\tilde{M}_j$ with $\tilde{M}_j \cap \tilde{N}_i \ne \varnothing$. In particular, we also have $\tilde{M}_j \cap \tilde{S}_0 \ne \varnothing$. Now consider the maximal tube T_j^+ associated to $\tilde{M}_j$. Since $\tilde{S} \cap X_{\tilde{N}} = \varnothing$, the morphism of Lemma 1.11 gives rise to a finite morphism

$$(g_{j,1},\dots,g_{j,n})\colon S \cap T_j^+ \to \mathbb{D}^n_+$$

Since $\tilde{S} \cap \tilde{M}_j \ne \varnothing$, the source $S \cap T_j^+$ is not empty. By reasons of dimensions, this map is surjective. Thus we see that $S \cap T_j \ne \varnothing$ and hence that S meets the Hartogs figure H.

The case of a rectilinear Hartogs figure goes similarly. □

We will need a preliminary tool on finite projections which is due to Bartenwerfer; cp. [4, Satz 4.1] and [5, Satz 5.3]. After that, we will lift the result to the affinoid site.

Lemma 1.13. *Assume that k has infinitely many elements. Let X be an affine k-scheme of finite type of pure dimension $n+d$ with $n \ge 1$ and $d \ge 1$. Let $M = V(g_1,\dots,g_n)$ be a complete intersection defined by regular functions $g := (g_1,\dots,g_n)$, and let $N \subset X$ be a closed subset of dimension n such that $\dim(M \cap N_i) = 0$ for every irreducible component N_i of N. Then there exists a finite morphism $\phi\colon X \to \mathbb{A}^{d+n}$ with the following properties, where the coordinate functions on $\mathbb{A}^{d+n}$ are denoted by $\zeta_1,\dots,\zeta_{d+n}$:*

(o) *$\phi^{-1}(V(\zeta_{d+1},\dots,\zeta_{d+n}))$ is a complete intersection on $\tilde{X}$ and on $\tilde{N}$;*

(i) *$\phi^{-1}(V(\zeta_{d+1},\dots,\zeta_{d+n})) = M \,\dot{\cup}\, M'$ is a disjoint union of closed subsets of X;*

(ii) $\phi(N_i) \cap V(\zeta_{d+1}, \ldots, \zeta_{d+n})) = \{0\}$ *for every irreducible component* N_i *of* N*;*
(iii) $\phi(N \cap M')$ *is disjoint from* $\phi(N \cap M)$*.*
In particular, there is a polynomial $h \in k[\zeta_1]$ *such that* $\phi^* h$ *vanishes identically on* $N \cap M'$ *and is invertible over* $N \cap M$*.*

Proof. Due to Noether's Normalization Theorem [31, Thm. III-20], there exists a finite morphism $\psi\colon X \to \mathbb{A}^{d+n}$ such that
(a) $\psi(M) = V(\xi_{d+1}, \ldots, \xi_{d+n})$,
(b) $\psi(M \cap N) = V(\xi_1, \ldots, \xi_d, \xi_{d+1}, \ldots, \xi_{d+n})$,
where $(\xi_1, \ldots, \xi_d, \xi_{d+1}, \ldots, \xi_{d+n})$ are the coordinate functions of $\mathbb{A}^{d+n}$. Then we obtain a finite morphism $X \to \mathbb{A}^d \times \mathbb{A}^n \times \mathbb{A}^n$ with the additional functions $g_1, \ldots, g_n$ for the middle factor. Obviously, we can replace X with its image $\psi(X)$ and M by $V(\xi_{d+1}, \ldots, \xi_{d+n})$ and N by $\psi(N)$ for our problem. Thus we may assume that
(c) $M = X \cap V(\xi_{d+1}, \ldots, \xi_{d+n}, \xi_{d+n+1}, \ldots, \xi_{d+2n}) = X \cap V(\xi_{d+1}, \ldots, \xi_{d+n})$.

Now we proceed by descending induction on $r = d + 2n$ to reduce the number of variables $\xi_1, \ldots, \xi_{d+2n}$ to new variables $\zeta_1, \ldots, \zeta_{d+n}$ via projections for fulfilling the properties of the assertion. At each induction step, we can replace X, M and N for our problem as done above. The beginning of the induction is trivial with $M' = \varnothing$.

Now consider the case $r > d + n$. Let $\mathfrak{a} = I(X) \subset k[\xi_1, \ldots, \xi_r]$ be the vanishing ideal of X. Due to Hilbert's Nullstellensatz, condition (c) implies

$$\text{rad}(\mathfrak{a} + (\xi_{d+1}, \ldots, \xi_{d+n})) = \text{rad}(\mathfrak{a} + (\xi_{d+1}, \ldots, \xi_r)). \tag{1}$$

Now consider transformations ψ of $\mathbb{A}^r$ of the type

$$\psi^* \xi_r = \zeta_r \quad \text{and} \quad \psi^* \xi_\nu = \zeta_\nu - \beta_\nu \cdot \zeta_r^{t_\nu} \quad \text{for } \nu = 1, \ldots, r-1,$$

with $\beta_\nu \in k^\times$ and $t_\nu \in \mathbb{N}$. Due to (1), we have

$$\psi^*(\xi_{d+1}, \ldots, \xi_r) \subset \text{rad}(\psi^* \mathfrak{a} + (\psi^* \xi_{d+1}, \ldots, \psi^* \xi_{d+n})).$$

For sufficiently large $t \in \mathbb{N}$, we obtain an equation

$$\zeta_r^t = \psi^* \xi_r^t = \psi^* g + h_{d+1} \cdot \psi^* \xi_{d+1} + \cdots + h_{n+d} \cdot \psi^* \xi_{d+n}$$

with some $g \in \mathfrak{a}$ and $h_{d+\nu} \in k[\zeta]$ for $\nu = 1, \ldots, n$. Now replace $\psi^* \xi_\nu$ by $\zeta_\nu - \beta_\nu \zeta_r^{t_\nu}$. We choose $t_\nu > t$. So we obtain

$$\zeta_r^t \cdot \left(1 + \sum_{\nu=d+1}^{d+n} \beta_\nu \cdot h_\nu \cdot \zeta_r^{t_\nu - t}\right) \in \psi^* \mathfrak{a} + (\zeta_{d+1}, \ldots, \zeta_{d+n}).$$

Since the expression in the brackets is invertible over M due to assumption (c) and $\zeta_r = \xi_r$, we have that $X \cap V(\zeta_{d+1}, \ldots, \zeta_{d+n}) = M \mathbin{\dot\cup} M'$ is a disjoint union. Moreover, we can choose t large enough such that there is a polynomial in $\psi^* \mathfrak{a}$ which is monic in the variable ζ_r. The induced morphism $\pi := pr \circ \psi\colon X \to \mathbb{A}^{r-1}$ is finite, and

$$\pi(M) = V(\zeta_{d+1}, \ldots, \zeta_{d+n}) \quad \text{and} \quad \pi^{-1}(\pi(M)) = M \mathbin{\dot\cup} M'.$$

Thus we have a finite morphism $\phi\colon X \to \mathbb{A}^{r-1}$ satisfying conditions (i) and (ii).

Till now, we are free to choose the coefficients $\beta_1, \ldots, \beta_{r-1} \in k^\times$. At first, we choose the constants $\beta_{d+1}, \ldots, \beta_{d+n} \in k^\times$ to satisfy condition (o). Since $\xi_r|_M = 0$, we have $\zeta_i|_M = \xi_i|_M$ for all $i = d+1, \ldots, r-1$. Thus we see $M \subset V(\zeta_{d+1}, \ldots, \zeta_{d+n})$ for any choice of the β's. We will stepwise choose $\beta_{d+1}, \ldots, \beta_{d+n} \in k^\times$ such that $\dim X \cap V(\zeta_{d+1}, \ldots, \zeta_{d+j}) = d+n-j$ and $\dim N \cap V(\zeta_{d+1}, \ldots, \zeta_{d+j}) = n-j$. Assume that we have already chosen $\beta_{d+1}, \ldots, \beta_{d+j}$. If there exists an irreducible component

$$M'_j \text{ of } X \cap V(\zeta_{d+1}, \ldots, \zeta_{d+j}) \quad \text{with } \dim M'_j = (d+n-j) \text{ and } \xi_r|_{M'_j} = 0,$$

then for any choice of $\beta_{d+j+1}, \ldots, \beta_{d+n}$, we have that

$$M'_j \cap X \cap V(\zeta_{d+1}, \ldots, \zeta_{d+n}) \subset V(\xi_{d+1}, \ldots, \xi_{d+n})$$

is a complete intersection by hypothesis. If there is an irreducible component N_j of the intersection $N \cap V(\zeta_{d+1}, \ldots, \zeta_{d+j})$ with $\dim N_j = (n-j)$ such that ξ_r vanishes on N_j, then

$$N_j \cap V(\zeta_{d+1}, \ldots, \zeta_{d+n}) \subset N \cap V(\xi_{d+1}, \ldots, \xi_{d+n}),$$

and hence its dimension is 0 for any choice of $\beta_{d+j+1}, \ldots, \beta_{d+n}$ as well. If ξ_r does not vanish on an irreducible component of $V(\zeta_{d+1}, \ldots, \zeta_{d+j})$ resp. of $N \cap V(\zeta_{d+1}, \ldots, \zeta_{d+j})$, there are only finitely many β's for which the dimension of the intersection with the vanishing locus $V(\xi_{d+j+1} - \beta \cdot \xi_{d+j+1}^{t_{d+j+1}})$ does not drop by 1. So, by excluding these finitely many elements β's in k, there exists some $\beta_{d+1}, \ldots, \beta_{d+j+1} \in k$ such that the dimension of $V(\zeta_{d+1}, \ldots, \zeta_{d+j+1})$ and of $N \cap V(\zeta_{d+1}, \ldots, \zeta_{d+j+1})$ drops by 1. Proceeding by induction similarly, we can find constants $\beta_{d+1}, \ldots, \beta_{d+n} \in k^\times$ to satisfy condition (o) and the condition $\dim N \cap V(\zeta_{d+1}, \ldots, \zeta_{d+n}) = 0$.

Since it holds that $M \cap M' = \varnothing$ and $N \cap M = V(\zeta_1, \ldots, \zeta_r)$, there exist constants $\beta_1, \ldots, \beta_d, \beta_{d+n+1}, \ldots, \beta_{r-1}$ such that the projection $\mathbb{A}^r \to \mathbb{A}^d$ with respect to the coordinate function $\zeta_1, \ldots, \zeta_d$ maps $N \cap M$ to the origin and the finitely many points of $N \cap M'$ to points different from the origin. Thus condition (iii) is satisfied as well. □

Proposition 1.14. *Assume that k has infinitely many elements. Let X be an affinoid space of pure dimension $n+d$ with $n \geq 1$ and $d \geq 1$. Let $H \subset X$ be a Hartogs figure of dimension n as defined in Definition 1.10 but with precisely one maximal tube; i.e., the complete intersection $\tilde{M}$ meets all irreducible components of $\tilde{N}$ and $H = \rho^{-1}(\tilde{M}) \cup X_{\tilde{N}}$, where $\rho\colon X \to \tilde{X}$ is the reduction map. Then there exists a finite morphism $\phi\colon X \to \mathbb{D}^{d+n}$ and a polynomial $h \in K[\zeta_1]$ with $|h| = 1$ such that the following properties are satisfied:*

(o) $\phi^{-1}(V(\zeta_{d+1}, \ldots, \zeta_{d+n}))$ *is a complete intersection on $\tilde{X}$ and on $\tilde{N}$;*
(i) $\phi^{-1}(V(\zeta_{d+1}, \ldots, \zeta_{d+n})) = M \mathbin{\dot\cup} M'$ *is a disjoint union of closed subsets;*
(ii) $\phi(N_i) \cap V(\zeta_{d+1}, \ldots, \zeta_{d+n}) = \{0\}$ *for all irreducible components N_i of N;*
(iii) $\tilde{\phi}(\tilde{N} \cap \tilde{M}') \subset V(\tilde{h})$ *and $\tilde{h}$ is invertible over $\tilde{\phi}(\tilde{N} \cap \tilde{M})$.*

In particular,

$$H' := ((\mathbb{D}^d \times \mathbb{D}^n_+) \cup \mathbb{D}^{d+n}_{\tilde{\phi}(\tilde{N})}) \cap \mathbb{D}^{d+n}_{\tilde{h}}$$

is a Hartogs figure of dimension n *in* $\mathbb{D}^{d+n}_{\tilde{h}}$ *with* $\phi^{-1}(H') \subset H$*, and* $\tilde{N} \cap V(\tilde{\phi}^*\tilde{h})$ *gives rise to a ball figure of dimension* $n+1$ *in* X*.*

Proof. This follows by lifting the result of Lemma 1.13. □

Some results on affinoid smoothness. The main tool here is the following result of Kiehl; cp. [23, Satz 1.12 and Thm. 1.18].

Proposition 1.15. *Let* X *be a smooth affinoid space of dimension* $d+n$*, and let* $A \subset X$ *be a smooth closed analytic subset of dimension* n*. Then there exists an open neighborhood of* A *which is a union of finitely many open affinoid subsets* $U_1, \dots, U_r$ *such that there are isomorphisms*

$$\phi_i \colon U_i \xrightarrow{\sim} V_i \times \mathbb{D}^d \quad \text{for } i = 1, \dots, r,$$

with $\phi_i(A \cap U_i) = V_i \times \{0\}$ *for* $i = 1, \dots, r$*.*

For further applications, we have to improve an result of Kiehl [23, Satz 1.12 and 1.14]. In the following, we call a morphism ψ a *perturbation* of a morphism φ between affinoid spaces if $\tilde{\psi} = \tilde{\varphi}$. It is clear that such a ψ is finite if φ is finite; cp. Reminder 1.3.

Proposition 1.16. *Let* $X = \mathrm{Sp}(A)$ *be a* d*-dimensional smooth affinoid space.*

(a) *Fix a finite morphism* $\varphi \colon X = \mathrm{Sp}(A) \to \mathbb{D}^d = \mathrm{Sp}(T)$ *of rank* n*. For any point* $x \in X$*, there is a perturbation* ψ *of* φ *such that* $\psi \colon X \to \mathbb{D}^d$ *is étale at all points of the fiber of* x*.*

(b) *Let* $\varphi \colon X \to Y = \mathrm{Sp}(B)$ *be a finite morphism which is étale over an open subvariety* V *of* Y*. Then there exists a finite covering* $\mathfrak{V} = \{\mathfrak{V}_1, \dots, \mathfrak{V}_n\}$ *of* V *by Zariski-open subsets* V_i *such that* $X \times_Y V_i$ *is isomorphic to* $V(\omega_i) \subset V_i \times \mathbb{D}^1$*, where* $\omega_i \in B[\eta]$ *is a Weierstraß polynomial and its derivative* ω_i' *has no zeros on* $V(\omega_i)$ *for* $i = 1, \dots, n$*.*

(c) *Assume that the module of differential form* $\Omega^1_{X/K}$ *is a free* A*-module of rank* d *and has a basis given by total differentials. If the characteristic of* k *is positive, then there exists a finite étale morphism* $\varphi \colon X \to \mathbb{D}^d$*.*

Proof. (a) This follows similarly to [23, Satz 1.12]. We fix a point $x \in X$. Then there exist total differentials $dy_1, \dots, dy_d$ which generate the module $\Omega^1_{X/K,x_i}$ for all points $x_1, \dots, x_s$ in the fiber $\varphi^{-1}(\varphi(x))$. Then there is small perturbation ψ of φ such that the number of points in $\psi^{-1}(\psi(x))$, where ψ is étale is maximal, since the total number of points in a fiber is bounded by the degree of $\tilde{\varphi}$. Then we claim that ψ is étale at all the points of the fiber $\psi^{-1}(\psi(x))$. Indeed, if ψ is étale at the points $x_1, \dots, x_r$ and not étale at all the other points in the fiber, then ϕ remains étale in a neighborhood of the points $x_1, \dots, x_r$ for any small perturbation ϕ of ψ. Now look at some point x' in the fiber $\psi^{-1}(\psi(x))$, where ψ is not étale if there is any; otherwise, we are done. By adding suitable small $c_1y_1, \dots, c_dy_d$ to each component of ψ with small $c_i \in K^\times$, we obtain a new perturbation ϕ such that $d\phi_1, \dots, d\phi_d$ generate $\Omega^1_{X/K,x'}$. Since this remains true in a small neighborhood of $x', x_1, \dots, x_r$, we get a perturbation such that ϕ is étale at all the points of the neighborhood of

$x', x_1, \dots, x_r$. So ϕ is étale at least $r+1$ points of $\phi^{-1}(\phi(x)$. This contradicts the maximality.

(b) This assertion follows from the local structure of étale morphisms; cp. [30, Chap. V, Thm. 1].

(c) This part is mainly contained in [23, Satz 1.12]. Let $df_1, \dots, df_d$ be total differentials which generate the module $\Omega^1_{X/K}$ of differential forms. We may assume $|f_i| < 1$ for $i = 1, \dots, d$. Now choose a finite morphism given $(h_1, \dots, h_d)\colon X_i \to \mathbb{D}^d$. Then the morphism

$$(f_1 + h_1^{p^t}, \dots, f_d + h_d^{p^t})\colon X \to \mathbb{D}^d$$

with a high power p^t of p yields the assertion. □

Proposition 1.17. *Assume that $X = \mathrm{Sp}(A)$ has smooth reduction $\tilde{X}$ of dimension d.*

(a) *Then, for every point $x \in X$, there exists a finite morphism $\varphi\colon X \to \mathbb{D}^d$ which is formally étale at any point of the fiber $\varphi^{-1}(\varphi(x))$.*

(b) *Assume that $\Omega^1_{X/K}$ is a free A-module of rank d and that it has a basis given by total formal differentials. If the characteristic of k is positive, there exists a finite formally étale morphism $\phi\colon X \to \mathbb{D}^d$.*

Formally, étale at a given point $x \in X$ here means that the induced formal morphism $\varphi\colon \mathrm{Spf}(A^\circ) \to \mathrm{Spf}(R\langle\zeta_1, \dots, \zeta_n\rangle)$ is a formal étale morphism at the specialization $\tilde{x}$ of the point x in question.

Proof. (a) This follows by lifting from Lemma 1.18 below.

(b) This follows in the same manner as Proposition 1.16 (c). □

Lemma 1.18. *Let $X = \mathrm{Spec}(A) \subset \mathbb{A}^n_k$ be a smooth irreducible subvariety of dimension d. Let $x_0 = 0 \in X$ be the origin. Then there exists a finite morphism $\phi\colon X \to \mathbb{A}^d_k$ such that ϕ is étale at all the points of the fiber $\phi^{-1}(\phi(x_0))$.*

Proof. Denote by $\zeta_1, \dots, \zeta_n$ the coordinate functions of $\mathbb{A}^n_k$. We may assume that $d\zeta_1, \dots, d\zeta_d$ generate the module of differential forms $\Omega^1_{X/k}$ at x_0. We will proceed by descending induction on n. For $n \geq d+2$, we claim that there exists a linear transformation

$$\xi_n \mapsto \zeta_n \quad \text{and} \quad \xi_\nu \mapsto \zeta_\nu - a_\nu \zeta_n \quad \text{for } \nu = 1, \dots, n-1$$

such that the following holds for the projection $\phi\colon X \to \mathbb{A}^{n-1}_k$ with respect to the new coordinate functions $\xi_1, \dots, \xi_{n-1}$:

(i) $\phi\colon X \to \mathbb{A}^{n-1}_k$ is finite;

(ii) $d\xi_1, \dots, d\xi_d$ generate $\Omega^1_{X/k}$ at x_0;

(iii) $\phi^{-1}(\phi(x_0)) = \{x_0\}$.

It is well-known that the set of points $a := (a_1, \dots, a_{n-1}) \in k^{n-1}$ which satisfy condition (i) is Zariski-open and dense in k^{n-1}. Indeed, take a nonzero polynomial vanishing on X. Let f_m be the homogenous component of f of highest degree. Then, for any $b := (a, 1)$ with $f_m(b) \neq 0$, the associated map ϕ is finite. Moreover, there is dense open subset of k^{n-1} such the total differentials $d\zeta_1 - a_1 d\zeta_n, \dots, d\zeta_d - a_d d\zeta_n$ generate $\Omega^1_{X/k} \otimes_A k$. Both conditions are fulfilled by a dense open subset of k^{n-1}. After having chosen $(a_1,\dots,a_d)$, there is dense open subset of $(a_{d+1},\dots,a_{n-1})$ such that $(a_\nu\zeta_\nu-\zeta_n)(x)\neq 0$ at least for one $\nu\in\{d+1,\dots,n-1\}$ for the finitely many points with $\xi_1(x)=\cdots=\xi_d(x)=0$ and $x\neq x_0$.

Now assume $n=d+1$. In this case, $X=V(f)$ is a locus of a prime polynomial f. Let f_m be as above. The points $a\in k^n$ with $f_m(a)\neq 0$ are a dense open subset $W\subset k^n$. Let S be the singular locus of X. Since X is smooth at x_0, the dimension of S is less than d. Any $a\in k^n$ gives rise to a line $L_a := k\cdot a\subset k^n$. Then there is a dense open subset $V\subset k^n$ such that $L_a\cap S=\varnothing$ for all $a\in V$. The tangent spaces T_a of X at a point $a\in X$ are defined by the locus of the linear form

$$df(a)\cdot\zeta := \zeta_1\frac{\partial f}{\partial\zeta_1}(a)+\cdots+\zeta_n\frac{\partial f}{\partial\zeta_n}(a).$$

At smooth points a of X, this linear form is not degenerated. So there exists a dense open subset $U\subset k^n$ such that $df(a)\cdot a\neq 0$ for $a\in U$. Combining the three conditions, we see that there exists a dense subset $Z\subset k^n$ such that all three conditions are fulfilled for $a\in Z$. After an eventually renumbering of the coordinates, there is a linear transformation as above such that the projection

$$\psi := (\xi_1,\dots,\xi_d)\colon X=V(f)\to\mathbb{A}_k^d$$

is finite and étale at all points of the fiber $\psi^{-1}(\psi(x_0))$. Composing ψ with the morphism $\varphi\colon X\to\mathbb{A}_k^{d+1}$, we obtain the desired morphism. □

2. Meromorphic functions

Let us first recall the definition of a meromorphic function.

Definition 2.1. We denote by $\mathcal{M}$ the *sheaf (with respect to the Grothendieck topology) of meromorphic functions* on a reduced rigid space X which associates to an open affinoid subdomain $U=\mathrm{Sp}(A)$ of X the total field of fractions $\mathrm{Frac}(A)$ consisting of all fractions f/g with $f,g\in A$, where g is a nonzero divisor of A. The restriction maps are the canonical ones.

A *meromorphic function* on X is a global section of $\mathcal{M}$. Such a section is given by an admissible covering of X by affinoid subdomains $\{U_i,\ i\in I\}$ and fractions $m_i=f_i/g_i$ on $U_i=\mathrm{Sp}(A_i)$ of affinoid functions, where the denominator g_i is a nonzero divisor on A_i such that the functions m_i coincide on the overlaps $U_i\cap U_j$ for all $i,j\in I$.

It follows from Kiehl's Theorem A [22] that any meromorphic function m on an affinoid space $X=\mathrm{Sp}(A)$ is (globally) a fraction f/g of affinoid functions $f,g\in A$, where g is a nonzero divisor on A.

Let us first clarify the relationship of properties (M_n) and (H_n).

Proposition 2.2. *If a couple* (G,X) *satisfies properties* (M_n) *and* (E_n)*, then it satisfies property* (H_n) *as well.*

Proof. Consider a function f on $S\cap G$ which belongs to $\mathcal{O}'(S\cap G)$ and is meromorphic on S. If f does not belong to $\mathcal{O}'(S)$, then its pole divisor on the normalization S' of S is of codimension 1. Due to property (E_n), the pole divisor has to meet $S \cap G$, and hence f is neither holomorphic on S' nor on S. $\square$

I guess that (M_n) implies (H_n) without property (E_n). Indeed, consider a function f as above. Now consider a series $h := \sum_{\nu=1}^{\infty} c^\nu f^\nu$ for some $c \in K^\times$ with $|c| < |f|$. Then h gives rise to an element of $\mathcal{O}'(S \cap G)$, but h is not extendable to S as a meromorphic function. Of course, some details have to be filled in, but we leave it to the reader since, in this paper, our couples (G, X) usually satisfy (E_n); cp. Proposition 1.6 and Proposition 1.12.

Now we turn to showing the extension property for ball and Hartogs figures. The extension property for meromorphic functions is based on an old observation by Levi.

Lemma 2.3. *Let $Y = \mathrm{Sp}(A)$ be an irreducible and reduced rigid space. Let (H, X) be a rectilinear Hartogs figure as defined in Definition 1.8 with $X = Y \times \mathbb{D}^1$ and*

$$H := (V \times \mathbb{D}^1) \cup (Y \times \partial\mathbb{D}^1).$$

Let $f \in \mathcal{M}(H)$ be a meromorphic function such that f is holomorphic on $B := Y \times \partial\mathbb{D}^1$. Then f extends to a meromorphic function on X. Actually, it is sufficient that the function f is holomorphic on $Y \times \partial\mathbb{D}^1$ and that the restriction $f|_{\{y\}\times\partial\mathbb{D}^1}$ extends to $\{y\} \times \mathbb{D}^1$ for all points y of V. In particular, if Y is normal, there is a monic polynomial $p \in A[\eta]$ such that $p \cdot f$ is holomorphic on $Y \times \mathbb{D}^1$ and $V(p) \cap (Y \times \partial\mathbb{D}^1) = \varnothing$.

Proof. Restricted to $Y \times \partial\mathbb{D}^1$, the function f has a Laurent series expansion

$$f = \sum_{\nu\in\mathbb{Z}} a_\nu \cdot \eta^\nu \in \mathcal{O}_Y(Y)\langle \eta, 1/\eta\rangle.$$

Since f is meromorphic on $V \times \mathbb{D}^1$, there is a monic polynomial

$$p := b_0 + \cdots + b_r\eta^r \in \mathcal{O}_Y(V)[\eta]$$

such that $p \cdot f$ is holomorphic on $V \times \mathbb{D}^1$. This means that the system of linear equations

$$\sum_{\nu=0}^{r} a_{-s-\nu} \cdot \beta_\nu = 0 \quad \text{for all } s = -1, -2, -3, \ldots$$

has a nontrivial solution $b := (b_0, \ldots, b_r) \in \mathcal{O}_Y(V)^{r+1}$. This is a system of linear equations over the smaller ring $\mathcal{O}_Y(Y)$. Then it follows by simple linear algebra that there exists also a nontrivial solution $(c_0, \ldots, c_r) \in \mathcal{O}_Y(Y)^{r+1}$ of this system of linear equations. Thus, putting $q := c_0 + \cdots + c_r\eta^r \in \mathcal{O}_Y(V)[\eta]$, we see that $q \cdot f$ extends to a holomorphic function on $Y \times \mathbb{D}^1$, and hence f extends to a meromorphic function on X.

For the additional assertion, consider the determinants d_n given by the determinant of the matrices $(n \times n)$-matrices which are defined by the rows $(a_{-s}, \ldots, a_{-s-n})$ for $s = 1, \ldots, n$. Since $f|_{\{y\}\times\partial\mathbb{D}^1}$ extends to $\{y\} \times \mathbb{D}^1$ for all points y of V, all these determinants $d_n(y) = 0$ have to vanish for large $n \in \mathbb{N}$

and $y \in V$. Then we have that $d_n = 0$ for large n. Now one can proceed as above. The pole set $\mathrm{Pol}(f) \subset Y \times \mathbb{D}^1$ is a divisor and the projection $\mathrm{Pol}(f) \to Y$ is finite. So there is a monic polynomial vanishing on $\mathrm{Pol}(f)$. Since Y is normal, $\mathrm{Pol}(f)$ is the locus of monic polynomial; cp. Proposition A.13. □

Lemma 2.4. *Let $\phi\colon X = \mathrm{Sp}(A) \to Y = \mathrm{Sp}(B)$ be a finite morphism of irreducible and reduced affinoid spaces of dimension n. Let $V \subset Y$ be an open subspace, and set $U := \phi^{-1}(V)$. If any meromorphic function on V extends uniquely to a meromorphic function on Y, then any meromorphic function on U extends uniquely to a meromorphic function on X, too.*

Proof. Denote by $\mathcal{M}$ the sheaf of meromorphic functions. The rings A and B are domains. If the degree of ϕ is r, then there exists a B-linearly independent set $\{e_1, \dots, e_r\}$ in A such that

$$\mathcal{M}(X) = \mathcal{M}(Y) \cdot e_1 \oplus \cdots \oplus \mathcal{M}(Y) \cdot e_r.$$

Moreover, we also have

$$\mathcal{M}(U) = \mathcal{M}(V) \cdot e_1 \oplus \cdots \oplus \mathcal{M}(V) \cdot e_r.$$

Since $\mathcal{M}(Y) \stackrel{\sim}{\to} \mathcal{M}(V)$ is bijective, the restriction map $\mathcal{M}(X) \stackrel{\sim}{\to} \mathcal{M}(U)$ is bijective, too. □

For further applications, we add the following fact which will be used in the proof of the following proposition. There exists a function $b \in B - \{0\}$ such that

$$b \cdot A \subset B \cdot e_1 \oplus \cdots B \cdot e_r.$$

Then we also have

$$b \cdot \mathcal{O}_X(U) \subset \mathcal{O}_Y(V) \cdot e_1 \oplus \cdots \oplus \mathcal{O}_y(V) \cdot e_r.$$

Proposition 2.5. *Let (B, X) be a ball figure in dimension n on an irreducible and reduced affinoid space X of dimension $n+2$. Then the restriction map $\mathcal{M}(X) \stackrel{\sim}{\to} \mathcal{M}(B)$ is bijective; i.e., property (M_{n+1}) is fulfilled by the couple (B, X). If X is normal, then the same is true for the holomorphic functions.*

Proof. Let $\tilde{N} \subset \tilde{X}$ be the n-dimensional algebraic subset induced by B. Let $f \in \mathcal{M}(B)$ be a nonzero meromorphic function on B. Denote by $S \subset X_{\tilde{N}}$ the set of poles of $f|_{X_{\tilde{N}}}$. Then S is at most of dimension $n+1$. Now S induces an algebraic subset $\tilde{S} \subset \tilde{X}$ via the Zariski closure of the image of S under the reduction map. By Lemma 1.5, there exists a finite morphism

$$\phi\colon X \to \mathbb{D}^n \times \mathbb{D}^2 \quad \text{with } \phi^{-1}(\mathbb{D}^n \times \partial\mathbb{D}^2) \subset B.$$

Moreover, we can also assume that $\phi^{-1}(\mathbb{D}^{n+1} \times \partial\mathbb{D}^1) \cap S = \varnothing$. Now we can apply Lemma 2.3 and Lemma 2.4 and its additional remark. Thus we see that any meromorphic function on B extends to meromorphic function on X. The uniqueness follows from Lemma 1.7. The assertion on holomorphic functions follows from Proposition 1.6 since, on a normal space, the set of poles of a meromorphic function is either empty or of pure codimension 1; *i.e.*, of dimension $n+1$ in this case. So it has to meet B if it is not empty. □

Now we turn to Hartogs figures.

Proposition 2.6. *Let $Y = \mathrm{Sp}(A)$ be an irreducible and reduced affinoid space of dimension n. Consider the rectilinear Hartogs figure*

$$H := (V \times \mathbb{D}^1) \cup (Y \times \partial\mathbb{D}^1)$$

on $X := Y \times \mathbb{D}^1$, where $V \subset Y$ is a nonempty open subset of Y. Then any meromorphic function on H extends to a meromorphic function on X.

Proof. Consider a meromorphic function $f \in \mathcal{M}(Y \times \partial\mathbb{D}^1)$. It can be written in the form $f = h/g$ with $h, g \in \mathcal{O}_X(Y \times \partial\mathbb{D}^1)$.

Let us first assume that $V(h, g) = \varnothing$. Then there exist elements $a, b \in \mathcal{O}_X(Y \times \partial\mathbb{D}^1)$ such that $1 = ah + bg$. Now we can approximate a resp. b by A-rational functions α and β such that $\alpha \cdot h + \beta \cdot g$ is still a unit. Next consider the function[4] $m := \alpha \cdot f + \beta \in \mathcal{M}(H)$. Since $g \cdot m = \alpha \cdot h + \beta \cdot g$ is a unit in $\mathcal{O}_X(Y \times \partial\mathbb{D}^1)$, the function m^{-1} has no poles on $Y \times \partial\mathbb{D}^1$. Then it follows by Lemma 2.3 that m^{-1}, and hence f extends to $X := Y \times \mathbb{D}^1$.

Now assume that $\dim V(h, g) = n - 1$. If $n = 1$, then the image $\phi(V(h, g))$ of $V(h, g)$ under the projection $\phi\colon (Y \times \partial\mathbb{D}^1) \to Y$ is a finite set of points in Y. Then one concludes as above that m extends to a meromorphic function on $Y_{\tilde{N}} \times \mathbb{D}^1$, where $\tilde{N} \subset \tilde{Y}$ is an algebraic subset of dimension 0. Thus we obtain the extension of f to the ball figure $(Y_{\tilde{N}} \times \mathbb{D}^1) \cup (Y \times \partial\mathbb{D}^1)$. Finally, we succeed by Proposition 2.5. The case $n = 1$ can be used to show the more general result. Namely, one can restrict the function f on $C \times \partial\mathbb{D}^1$, where $C \subset Y$ is any irreducible curve. Due to the result above, we know that $f|_{C \times \partial\mathbb{D}^1}$ extends to a meromorphic function on $C \times \mathbb{D}^1$. So $f|_{\{y\} \times \partial\mathbb{D}^1}$ extends to $\{y\} \times \mathbb{D}^1$ for all $y \in C$ except finitely many points of C. Then we spread this result by restricting meromorphic functions to irreducible curves in Y; for more details, see the proof of Proposition 3.12. So we obtain that the given function f on $Y \times \partial\mathbb{D}^1$ has the property that, for any point $y \in Y$, the restriction of $m|_{\{y\} \times \partial\mathbb{D}^1}$ extends to a meromorphic functions on $\{y\} \times \mathbb{D}^1$. Then one can conclude by the additional assertion in Lemma 2.3. Finally, we obtain the result by Proposition 2.5.

In the general case, one can proceed as before by restricting f to curves C contained in Y. Except for finitely many points of C, the local rings at points of $C \times \partial\mathbb{D}^1$ are factorial. So one can write f in the form we discussed above, and we can proceed with the restriction of f to $C \times \mathbb{D}^1$. So we can finish the proof of the general case as above. ☐

Corollary 2.7. *Let $Y = \mathrm{Sp}(A)$ be an irreducible and reduced affinoid space, and let $V \subset Y$ be an open nonempty subset. Let $p \in A[\eta]$ be a monic polynomial with $|p| = 1$. Set*

$$H := (V \times \mathbb{D}^1) \cup (Y \times \mathbb{D}^1)_{\tilde{p}}.$$

[4] This trick was introduced by Bartenwerfer in [3, proof of Satz 1.2].

Then every meromorphic function $m \in \mathcal{M}(H)$ extends uniquely to a meromorphic function on $X := Y \times \mathbb{D}^1$. If, in addition, X is normal, the same holds for holomorphic functions.

Proof. The induced map $\phi := (\mathrm{id} \times p)\colon Y \times \mathbb{D}^1 \to Y \times \mathbb{D}^1$ is finite. Then the assertion follows from Proposition 2.6 by Lemma 2.4. The case of holomorphic functions follows in the same way as Proposition 2.5. □

Next we will treat a special case of a Hartogs figure which settles the crucial point of the proof in the case of a general Hartogs figure. We start with a lemma on descent.

Lemma 2.8. *Let $Y = \mathrm{Spf}(A)$ be a smooth affine formal R-space with irreducible reduction $\tilde{Y}$ of dimension n. Let $p \in A[\eta]$ be a polynomial with $|p| = 1$ such that the coefficients of $\tilde{p}$ generate the unit ideal of $\tilde{A}$. Let $Y' = \mathrm{Spf}(B) \to Y$ be a formally étale surjective map such that its reduction $\tilde{Y'}$ is irreducible as well. Denote by $\phi\colon Y' \times \mathbb{D}^1 \to Y \times \mathbb{D}^1$ the induced morphism. Let $Z \subset Y' \times \mathbb{D}^1$ be a formal open subscheme such that $\phi\colon Z \to Y \times \mathbb{D}^1$ is surjective. Consider a holomorphic function $m \in \mathcal{M}((Y \times \mathbb{D}^1)_{\tilde{p}})$. If $\phi^* m$ extends to a meromorphic function on Z, then m extends to a meromorphic function on $Y \times \mathbb{D}^1$.*

Proof. By a meromorphic function on a formal scheme, we mean a function that is locally a fraction of two formal functions where the denominator is a nonzero divisor on the special fiber. The symbol $\mathbb{D}^1$ here denotes the formal affine line over $\mathrm{Spf}(R)$. In the following, we have to distinguish between the formal scheme Y and its associated rigid analytic space which we denote by Y_K. Let $\pi \in R$ be nonzero with $|\pi| < 1$. We denote the reduction mod $\pi^{\nu+1}$ by the subindex ν at the symbols for the formal schemes; *i.e.*, $Y_\nu := Y \times_R (R/R\pi^{\nu+1})$ for $\nu \in \mathbb{N}$. The same applies to Y', and Z, etc.

At first, we assume in addition that $\phi^* m$ extends to a holomorphic function on Z. We may assume $|\phi^* m| = 1$ and hence that $\phi^* m \in \mathcal{O}_Z(Z)$. Since the coefficients of $\tilde{p}$ generate the unit ideal of $\tilde{A}$, the function m gives rise to a Y-rational map

$$m_\nu\colon (Y \times \mathbb{D}^1)_\nu \dashrightarrow \mathbb{D}^1_\nu = \mathbb{A}^1_{R/R\pi^{\nu+1}},$$

where $\mathbb{A}^1$ is the formal affine line over $\mathrm{Spf}(R)$. Indeed, m_ν is defined on the subset $(Y \times \mathbb{D}^1_\nu)_{\tilde{p}}$. Its pullback $\phi^* m_\nu$ is defined on Z_ν due to the additional assumption. Due to the descent of the domain of definition of a rational morphism [12, Prop. 2.5.6], the map m_ν is defined on $(Y \times \mathbb{D}^1)_\nu$ for all $\nu \in \mathbb{N}$. Thus we have that m extends to a holomorphic function on $Y_K \times \mathbb{D}^1$.

Now we turn to the meromorphic case. Since Z_K is locally regular and hence factorial, then $\phi^* m$ gives rise to a well-defined pole divisor S and hence to a divisor ideal $\mathcal{J} \subset \mathcal{O}_{Z_K}$. In the following, let

$$p_i\colon Z'_K := Z_K \times_{Y \times \mathbb{D}^1} Z_K \to Z_K$$

be the i-th projection for $i = 1, 2$. Then we can consider $p_i^* \phi^* m$ for $i = 1, 2$. Since m is defined on $(Y_K \times \mathbb{D}^1)_{\tilde{p}}$, we have $p_1^* \phi^* m = p_2^* \phi^* m$. Note that $p_i^* \phi^* m$

has a well-defined pole divisor $S_i \subset Z'_K$ for $i = 1, 2$ since the rigid spaces under consideration are locally regular and hence factorial. Because of $p_1^*\phi^* m = p_2^*\phi^* m$, the associated ideals $\mathcal{J}_i$ of S_i inherit a canonical descent datum on $\mathcal{J}$. This descent is effective due to [11, Thm. 3.1]. So there exists a divisor ideal $\mathcal{I} \subset \mathcal{O}_{Y\times\mathbb{D}^1}$ which induces $\mathcal{J} = \phi^*\mathcal{I}$. Now put $I_K := \mathcal{I}(Y_K \times \mathbb{D}^1)$, and set $I := I_K \cap A\langle\eta\rangle$. Then $A\langle\eta\rangle/I$ is free of R-torsion since $Y \times \mathbb{D}^1$ is formally smooth over R. Thus $A\langle\eta\rangle/I$ is flat over R. Then it follows from [24, Lem. 6.2.3] that I_K extends to a formal Cartier divisor on $Y \times \mathbb{D}^1$; actually, the reflexive closure I^{**} is the extension. Now we want to show that I_K is locally trivial over Y_K in the formal sense; *i.e.*, there is a formal open covering $\{Y_1, \dots, Y_r\}$ of Y such that $I_K|_{Y_i\times\mathbb{D}^1}$ is free for $i = 1, \dots, r$. For doing this, it suffices to show that the restriction of $I^{**} \otimes_R k$ to $\tilde{Y} \times \mathbb{A}^1_k$ is locally free over $\tilde{Y}$. The latter is clear since the local rings of $\tilde{Y}$ are factorial. Then we divide out the pole of m. So we can assume that $\phi^* m$ extends to a holomorphic function on $Z_K \times_Y Y_i$ for $i = 1, \dots, r$. Then we obtain the assertion by the first case we discussed before. □

Lemma 2.9. *Let $Y = \mathrm{Spf}(A)$ be a smooth affine formal R-space with irreducible reduction $\tilde{Y}$. Let $p \in A[\eta]$ be a polynomial with $|p| = 1$ such that the coefficients of $\tilde{p}$ generate the unit ideal of $\tilde{A}$. Let $\tilde{y} \in \tilde{Y}$ be a closed point, and assume that every irreducible component of $V(\tilde{p})$ intersects $\{\tilde{y}\} \times \mathbb{A}^1_k$ in a nonempty set of finitely many points. Let $V \subset Y_K$ be a nonempty open subset which specializes into $\tilde{y}$ under the reduction map. Then every holomorphic function on $(Y_K \times \mathbb{D}^1)_{\tilde{p}}$ extends to a meromorphic function on $Y_K \times \mathbb{D}^1$ if it extends to a meromorphic function on $V \times \mathbb{D}^1$.*

Proof. As in Lemma 2.8, the notation $\mathbb{D}^1$ here means the formal affine line over $\mathrm{Spf}(R)$, and Y_K is the rigid space associated to Y. There exists an étale neighborhood $\tilde{\varphi}\colon (\tilde{Y}', \tilde{y}') \to (\tilde{Y}, \tilde{y})$ of $\tilde{y}$ such that $\tilde{\varphi}^{-1}V(\tilde{p})$ decomposes into two sets $\tilde{\varphi}^{-1}V(\tilde{p}) = V(\tilde{q}_1) \cup V(\tilde{q}_2)$, where $\tilde{q}_1$ is a monic polynomial and $\tilde{q}_2$ is a polynomial with $V(\tilde{q}_2) \cap (\{\tilde{y}'\} \times \mathbb{A}^1) = \varnothing$; cp. [12, Prop. 2.3/8]. In addition, we may assume that $V(\tilde{q}_1)$ is disjoint from $V(\tilde{q}_2)$. Now we lift the étale extension and obtain a formal étale extension $Y' \times \mathbb{D}^1 \to Y \times \mathbb{D}^1$, which covers the tube associated to $\{\tilde{y}\} \times \mathbb{A}^1_k$. Since $V(\tilde{q}_1) \cap V(\tilde{q}_2) = \varnothing$, we can decompose the holomorphic function

$$m|_{(Y'\times\mathbb{D}^1)_{\tilde{p}}} = \sum_{\nu\in\mathbb{N}} a_\nu p^{-\nu} = \sum_{\nu\in\mathbb{N}} a_{\nu,1} q_1^{-\nu} + \sum_{\nu\in\mathbb{N}} a_{\nu,2} q_2^{-\nu}.$$

The second summand is holomorphic on $(Y' \times \mathbb{D}^1)_{\tilde{q}_2}$; the first summand is holomorphic on $(Y' \times \mathbb{D}^1)_{\tilde{q}_1}$ and meromorphic on the tube associated to $\tilde{y}' \times \mathbb{A}^1$ since $V(\tilde{q}_2)$ does not meet the tube $\tilde{y}' \times \mathbb{A}^1$. So, due to Corollary 2.7, we have that $\varphi^* m$ extends to a meromorphic function on $(Y' \times \mathbb{D}^1)_{\tilde{q}_2}$. Since an étale map is open, there exists a function $a \in \mathcal{O}(Y)$ with $|a| = 1$ and $\tilde{a}(\tilde{x}) \neq 0$ such that $Y'_{\tilde{a}} \to Y_{\tilde{a}}$ is surjective. Then the morphism $(Y'_{\tilde{a}} \times \mathbb{D}^1)_{\tilde{q}_2} \to Y_{\tilde{a}} \times \mathbb{D}^1$ is also surjective. Indeed, $V(\tilde{q}_1) \to V(\tilde{p})$ is surjective as follows by our assumption on the irreducible components of $V(\tilde{p})$ and $V(\tilde{q}_1) \cap V(\tilde{q}_2) = \varnothing$. By the descent argument, Lemma 2.8, we have that m extends to a meromorphic function on $(Y_{\tilde{a}} \times \mathbb{D}^1) \cup (Y \times \mathbb{D}^1)_{\tilde{p}}$. The latter is a ball figure of dimension $(n-1)$. Finally, the assertion follows from Proposition 2.5. □

Lemma 2.10. *Let $\xi := (\xi_1, \ldots, \xi_n)$ be the coordinate functions of $\mathbb{D}^n$, and let η be the coordinate function of $\mathbb{D}^1$. Let $p, q \in \mathcal{O}(\mathbb{D}^{n+1})$ be holomorphic functions on the $(n+1)$-dimensional unit polydisc with $|p| = |q| = 1$, and assume that $V(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)$ intersects $V(\tilde{p})$ in finitely many points and does not meet $V(\tilde{q})$. Let $\tilde{p} = \tilde{p}_1 \cdot \ldots \cdot \tilde{p}_r$ be the prime decomposition of $\tilde{p}$, and assume that the locus $V(\tilde{p}_j)$ meets $V(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)$ for every $j = 1, \ldots, r$. Then every meromorphic function m on the Hartogs figure $(\mathbb{D}^n(\varepsilon) \times \mathbb{D}^1) \cup \mathbb{D}^{n+1}_{\tilde{q}\tilde{p}}$ extends uniquely to a meromorphic function on $\mathbb{D}^{n+1}_{\tilde{q}}$.*

Proof. From Corollary 2.7, we deduce that m extends to a meromorphic function on $\mathbb{D}^n_+ \times \mathbb{D}^1$, by using Lemma 1.11. Namely, the restriction of $\tilde{p}$ behaves like a polynomial in one variable. Furthermore, we may assume that $m = f/g$ with f, g in $\mathcal{O}(\mathbb{D}^{n+1}_{\tilde{q}\tilde{p}})$ with $|f| = |g| = 1$. The ring $\mathcal{O}(\mathbb{D}^{n+1}_{\tilde{q}\tilde{p}})$ is factorial, as seen by arguments similar to the ones used in [24, Lem. 6.2.3], because the reduction of $\mathcal{O}(\mathbb{D}^{n+1}_{\tilde{q}\tilde{p}})$ is factorial. So we can choose f and g in such a way that $V(f, g)$ has dimension at most $(n-1)$. So there exists a function $r \in \mathcal{O}(\mathbb{D}^n)$ with $|r| = 1$ such that the reduction of $V(f, g)$ is contained in $V(\tilde{r})$. Unfortunately, it can happen that $\tilde{r}(\tilde{x}) = 0$ for $\tilde{x} \in V(\tilde{\xi}_1, \ldots, \tilde{\xi}_n, \tilde{p})$. Otherwise, we would succeed by Lemma 2.9 with the trick as in the proof of Proposition 2.6.

Due to the maximum principle, there exists some $\varepsilon \in |K^\times|$ with $\varepsilon < 1$ such that $V(f, g)$ is contained in $\{x \in \mathbb{D}^{n+1};\, |r(x)| < \varepsilon\}$. If we stick to the domain $\{|r(x)| \geq \varepsilon\}$, then $V(f, g)$ is empty over that subdomain. So, as in the proof of Proposition 2.6, we may assume that m is holomorphic on the subdomain $\{x \in \mathbb{D}^{n+1};\, |r(x)| \geq \varepsilon \text{ and } |qp(x)| = 1\}$.

Due to [12, Prop. 2.3/8], there is a formal étale neighborhood $(Y = \mathrm{Spf}(B), \tilde{y})$ of $(\mathbb{D}^n, 0)$ such that $\tilde{p}$ splits into a product $\tilde{p} = \tilde{q}_1 \cdot \tilde{q}_2$ over $\tilde{B}$ such that $\tilde{q}_1$ is monic and $V(\tilde{q}_2)$ is disjoint from $V(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)$. In addition, we may assume that the two polynomials generate the unit ideal in $\tilde{B}[\tilde{\eta}]$. Indeed, since $\dim V(\tilde{q}, \tilde{p}) \leq n - 1$, there is a function $s \in \mathcal{O}(\mathbb{D}^n)$ with $|s| = 1$ and $V(\tilde{q}, \tilde{p}) \subset V(\tilde{s})$ and $\tilde{s}(\tilde{x}) \neq 0$. So the polynomials $\tilde{p}$ and $\tilde{q}$ are comaximal on $\mathbb{A}^n_{\tilde{s}}$. Thus, over $\mathbb{D}^{n+1}_{\tilde{s}}$, we can split

$$m|_{\{x;\, |s(x)|=1,\, |r(x)| \geq \varepsilon\}} = \sum_{\mu \in \mathbb{N}} a_\mu \cdot q^{-\mu} + \sum_{\nu \in \mathbb{N}} b_\nu \cdot p^{-\nu}.$$

Leaving the first summand aside, we see that the second sum is defined on $(Y_{\tilde{s}} \times \mathbb{D}^1)_{\tilde{p}}$ and also on $\mathbb{D}^n_+ \cap \{|r(x)| \geq \varepsilon\}$ which is nonempty and open. Now, over $Y_{\tilde{s}} \times \mathbb{D}^1$, we can split the second series

$$\sum_{\nu \in \mathbb{N}} b_\nu \cdot p^{-\nu} = \sum_{\nu=0}^{\infty} b_{\nu,1} \cdot q_1^{-\nu} + \sum_{\nu=1}^{\infty} b_{\nu,2} \cdot q_2^{-\nu}.$$

Denote by m_1 resp. m_2 the first resp. the second summand. So m_1 is a meromorphic function on $(Y_{\tilde{s}} \times \mathbb{D}_1)_{\tilde{p}} \cup \phi^{-1}(\{x \in \mathbb{D}^n; 1 > |r(x)| \geq \varepsilon\} \times \mathbb{D}^1)$. Now it follows from Corollary 2.7 and Lemma 2.4 that m_1 extends to $Y_{\tilde{s}} \times \mathbb{D}^1$. Then set $Z := (Y_{\tilde{s}} \times \mathbb{D}^1)_{\tilde{q}_2}$. Now $\phi^* m$ is meromorphic on Z. The restricted map $\phi\colon Z \to \mathbb{D}^n_{\tilde{s}} \times \mathbb{D}^1$ is formally étale and surjective since $\phi(V(\tilde{q}_2))$ is contained in $V(\tilde{p})$ and $\phi(V\tilde{q}_1)$ equals $V(\tilde{p})$ as every irreducible component of $\tilde{p}$ meets $V(\tilde{\xi}_1, \dots, \tilde{\xi}_n)$. Then it follows from Corollary 2.7 that the second sum is defined on $(Y \times \mathbb{D}^1)_{\tilde{s}\tilde{r}}$ and hence on $Y \times \mathbb{D}^1$ due to Proposition 2.5. Then, by the descent argument, Lemma 2.8, we deduce as in Lemma 2.9 that m can be extended to $\mathbb{D}^{n+1}_{\tilde{q}\tilde{s}\tilde{r}}$. Finally, m is extendable to a meromorphic function $\mathbb{D}^{n+1}_{\tilde{q}}$ due to Proposition 2.5. □

Theorem 2.11. *Let X be a reduced affinoid space of pure dimension $n+1$, and let $H \subset X$ be a Hartogs figure H of dimension n. Then any meromorphic function m on H extends to a meromorphic function on X, and the extension is unique; i.e., the restriction map $\mathcal{M}(X) \xrightarrow{\sim} \mathcal{M}(H)$ is bijective. So (H, X) has property* (M_n). *If, in addition, X is normal, then the restriction map for holomorphic functions $\mathcal{O}(X) \xrightarrow{\sim} \mathcal{O}(H)$ is bijective, too.*

Proof. Consider a Hartogs figure as in Definition 1.10. In the following, we keep the notions introduced in Definition 1.10. At first, we discuss the case where there is only one tube; *i.e.*, there is one complete intersection $\tilde{M}$ which meets any irreducible component of $\tilde{N}$. Due to Proposition 1.14, we have a finite morphism $\phi\colon X \to \mathbb{D}^{n+1}$ and a Hartogs figure $H' \subset \mathbb{D}^{n+1}_{\tilde{h}}$ of dimension n in $\mathbb{D}^{n+1}_{\tilde{h}}$ with $\phi^{-1}(H') \subset H$. Then it follows from Lemma 2.4 that $\mathcal{M}(X_{\tilde{h}}) \to \mathcal{M}(H \cap X_{\tilde{h}})$ is bijective due to Lemma 2.10, where $\tilde{q}$ has to be replaced by $\tilde{h}$. Moreover, $V(\tilde{h})$ intersects any irreducible component of $\tilde{N}$ at most in a closed subset of dimension $n-1$ since $\tilde{h}$ does not meet $\tilde{M}$ and any irreducible component of $\tilde{N}$ meets $\tilde{M}$. So we obtain the extension by Proposition 2.5.

In the general case, where there are more tubes, we make induction on the number of tubes. So pick one of the tubes $\tilde{M}_1$. Let $\tilde{Q}_1$ be the union of all irreducible components of $\tilde{N}$ which meet $\tilde{M}$, and let $\tilde{Q}_2$ the union of the remaining components. Then there exists a function $q \in \mathcal{O}(X)$ with $|q| = 1$ such that $V(\tilde{q})$ does not meet $\tilde{M}_1 \cap \tilde{Q}_1$ and contains $\tilde{Q}_2$. So $H \cap X_{\tilde{q}}$ gives rise to a Hartogs figure on $X_{\tilde{q}}$ with exactly one tube defined by $\tilde{M}_1$. Due to the case discussed above we have that any meromorphic function m on H extends to $X_{\tilde{q}}$. Since the dimension of $\tilde{Q}_1 \cap V(\tilde{q})$ is less than n, it follows from Proposition 2.5 that m extends to $X_{\tilde{Q}_2}$. Now it remains to show that m extends to X. But this follows by the induction hypothesis.

This is the proof for a Hartogs figure as defined in Definition 1.10; for rectilinear Hartogs figures, the proof was done in Proposition 2.6.

The uniqueness follows from Proposition 1.12. In the case where X is normal, the extension of holomorphic functions follows from that in the meromorphic case and Proposition 1.12 since the set of poles is empty or of dimension n.

In the latter case, it has to meet the Hartogs figure due to Proposition 1.12. Thus we see that the set of poles is empty. □

Remark 2.12. In the proof, we made use of Proposition 1.14 which requires that the residue field k has infinitely many elements. This assumption is actually not necessary for this application. Indeed, we can choose a base field extension K'/K in such a way that the residue field k' of K' has infinitely many elements. So we obtain the extension of meromorphic functions after such a field extension. Then an easy descent argument shows the extension property over the given base field.

Concerning the extension of functions through closed analytic subsets, this easily follows from Proposition 1.15. We have the following properties.

Proposition 2.13. *Let X be an irreducible reduced rigid space, and let $A \subset X$ be a closed analytic subset of X.*
(a) *If $\dim A \leq \dim X - 2$, then any meromorphic function on $X - A$ extends to a meromorphic function on X. If, in addition, X is normal, the analog is true for holomorphic functions.*
(b) *If $\dim A = \dim X - 1$ and X is normal, then any bounded holomorphic function on $X - A$ extends to a holomorphic function on X.*

Proof. (a) follows directly from Proposition 2.5.

(b) We may assume that X is affinoid. If $X = Y \times \mathbb{D}^1$ and $A = Y \times \{0\}$, the assertion follows by looking at the Laurent expansion of the given bounded function. Due to Proposition 2.5, it suffices to show that there exists a ball figure B of codimension 2 in X such that a given bounded function extends to a holomorphic function on B. If X is geometrically normal, the set of singular points is of codimension 2. Furthermore, due to Galois decent, we may assume that A is geometrically reduced, so its set of singular points is of codimension 1 in A and hence of codimension 2 in X. The set of singular points of X and A give rise to a ball figure B in X of codimension 2 such that $X \cap B$ and $A \cap B$ are smooth. Then, due to Proposition 1.15, we are reduced to the special case discussed at the very beginning of this proof.

Now we turn to the general case. There exists a finite field extension K'/K such that the reduced space $(X \otimes_K K')_{\mathrm{red}}$ of $X \otimes_K K'$ is geometrically reduced and that its normalization is geometrically normal. Thus we have that our bounded function f extends to a meromorphic function on $(X \otimes_K K')_{\mathrm{red}}$. Since f is defined on $X - A$, one easily shows that f actually extends to a meromorphic function on X. Next we choose a finite map $\varphi\colon X \to \mathbb{D}^n$ for $n = \dim X$. The characteristic polynomial of f with respect to φ has bounded coefficients on $X - \varphi(A)$, and hence they extend to affinoid functions on X, due to what we have proved above. Since X is normal, f belongs to $\mathcal{O}(X)$. □

Theorem 2.11 is essentially due to Bartenwerfer; cp. [3, 4]. His proof is very long and hard to follow. The proof given here runs smoothly because of the use of formal étale extensions and descent theory [11].

3. Analytic subsets

The proof of the extension property (A_n) will be reduced to the extension of holomorphic functions due to the following lemma.

Lemma 3.1. *Let $\phi\colon X = \mathrm{Sp}(A) \to Y = \mathrm{Sp}(B)$ be a morphism of affinoid spaces. Assume that Y is normal and connected and that Y has dimension n. Let $V \subset Y$ be an open nonempty subspace, and set $U := \phi^{-1}(V)$. Assume that any bounded holomorphic function on V extends to a holomorphic function on Y.*

Let $S \subset U$ be an irreducible closed analytic subset of dimension n. Assume that the induced morphism $\phi|_S\colon S \to V$ is finite. Then there exists a closed irreducible analytic subset $\underline{S} \subset X$ with $\underline{S} \cap U = S$. In particular, the induced morphism $\phi\colon \underline{S} \to Y$ is finite.

Endow $\underline{S}$ with its reduced structure. Then any meromorphic function on S extends uniquely to a meromorphic function on $\underline{S}$ if any meromorphic function on V extends to a meromorphic function on Y.

Proof. We endow S with its reduced structure. Let t be the degree of the morphism $\phi|_S$. For a holomorphic function $f \in \mathcal{O}(X)$, we denote by

$$P_f(\eta) = \eta^t + c_{t-1} \cdot \eta^{t-1} + \cdots + c_0 \in \mathcal{O}_Y(V)[\eta]$$

the characteristic polynomial of $f|_S$. The coefficients are meromorphic functions on V. Since Y is normal, the coefficients are holomorphic on V. Moreover, the coefficients are elementary symmetric functions in the values $f(x)$ for $x \in \phi^{-1}(v)$ for all $v \in V$. So they are bounded, and hence, due to our assumption, they are holomorphic on Y. Thus we have $P_f(\eta) \in \mathcal{O}_Y(Y)[\eta]$, and $P_f(f)$ vanishes on S for all $f \in \mathcal{O}_X(X)$. Thus the locus of $(P_f(f);\, f \in \mathcal{O}_X(X))$ is a closed analytic subset $\underline{S} \subset X$ of X with $\underline{S} \cap U = S$. Note that the ideal $(P_f(f);$ $f \in \mathcal{O}_X(X))$ is finitely generated because $\mathcal{O}_X(X)$ is noetherian. The assertion concerning the meromorphic functions on S follows from Lemma 2.4. □

Corollary 3.2. *Let $\phi\colon X = \mathrm{Sp}(A) \to Y = \mathrm{Sp}(B)$ be a morphism of affinoid spaces. Let $V \subset Y$ be an open nonempty subspace, and set $U := \phi^{-1}(V)$. Let $S \subset U$ be an irreducible closed analytic subset of dimension $m \geq n+1$. Assume that $\phi|_S\colon S \to V$ is finite. If the couple (V, Y) satisfies properties (A_n) and (M_n), then S extends to an irreducible closed analytic subset $\underline{S} \subset X$. Moreover, $(S, \underline{S})$ satisfies properties (A_n) and (M_n).*

In particular, if $\phi\colon X \to Y$ is finite, then (U, X) satisfies (A_n) and (M_n) if (V, Y) does.

Proof. Let $T := \phi(S) \subset V$ be the image of S and $\underline{T} \subset Y$ its extension to Y which exists due to (A_n). Let $T' \to T$ be its normalization. The induced morphism $\phi|_S\colon S \to T$ is finite. As in the proof of Lemma 3.1, consider the characteristic polynomial $P_f(\eta) \in \mathcal{O}'_{\underline{T}}(T)$ for any $f \in \mathcal{O}_X(X)$. Due to condition (M_n) and implicitly (H_n), we have that $P_f(\eta) \in \mathcal{O}'_{\underline{T}}(\underline{T})$. Then the locus of $(P_f(f);\, f \in \mathcal{O}_X(X))$ gives rise to a closed analytic subset $\underline{S} \subset X$ of X with $\underline{S} \cap U = S$. Indeed, firstly, we obtain an extension in the fiber product $X \times_Y \underline{T}'$, and hence, by the projection to X, we obtain the extension $\underline{S}$. The assertion concerning (M_n) follows by Lemma 2.4 from property (M_n) of (V, Y). $\square$

Proposition 3.3. *Let (B, X) be a ball figure in dimension n on an affinoid space X of dimension $n+d$ with $d \geq 2$. Then the couple (B, X) has properties (A_{n+1}) and (M_{n+1}).*

Proof. We apply the projection lemma, Lemma 1.5. Then the assertion follows from Proposition 2.5 by using Lemma 3.1 respectively Corollary 3.2. The uniqueness follows from Proposition 1.6. $\square$

The case of Hartogs figure is much harder to prove since we do not know such a nice projection lemma as in the case of ball figures. Let $H \subset X$ be a Hartogs figure of dimension n in an affinoid space of pure dimension $n+d$, and let $S \subset H$ be a closed analytic subset of pure dimension $n+t$ with $t \geq 1$. If one wants to show the extension of S to a closed analytic subset $\underline{S}$ of X, we usually reduce the problem by finite projection to a standard Hartogs figure in $\mathbb{D}^{n+t}$ by using Lemma 3.1 and extension properties for meromorphic functions. In the last proposition, we have seen how it works. Similarly, it works for all the other extension properties as we will see in the sequel. We only have the projection type Proposition 1.14 which yields a map to a standard Hartogs figure in a polydisc, but it does not induce a finite map from a given analytic subset $S \subset H$ of dimension $n+t$ to a Hartogs figure of dimension $n+t$. So we need a new type of projection result which gives additional information for the standard Hartogs figure obtained in Proposition 1.14. Let us start with the case of the standard Hartogs figure.

The following two results are more or less contained in [5, § 3].

Lemma 3.4. *Let $\zeta_1, \ldots, \zeta_n, \eta_1, \ldots, \eta_d$ be the coordinate functions on $\mathbb{D}^{n+d}$ with $d \geq 2$. Consider the Hartogs figure $H := T \cup B \subset X := \mathbb{D}^{n+d}$, where*

$$T := (\mathbb{D}^n(\underline{\varepsilon}) \times \mathbb{D}^d) \quad \text{and} \quad B := \mathbb{D}^n \times \partial\mathbb{D}^d.$$

Then (H, X) has properties (A_n) and (M_n).

Proof. Let $S \subset H$ be a closed analytic subset whose irreducible components have dimension $m = n+t$ with $t \geq 1$. The case $t \geq 2$ is covered by Proposition 3.3. So we may assume that $t = 1$ and $d \geq 2$. For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, we will stepwise increase each ε_ν to 1 for $\nu = 1, \ldots, n$. So we may assume that $\varepsilon_2 = 1, \ldots, \varepsilon_n = 1$. Then we have to extend S to a closed analytic subset of $\mathbb{D}^{n+d}$. Consider the reduction $\tilde{S} \subset \mathbb{A}_k^{n+d}$ in the sense of Reminder 1.3.

If $d \geq 3$, there exists a polynomial $\tilde{g} \in k[\zeta_2, \ldots, \eta_d]$ with $0 \neq \tilde{g}$ such that $\tilde{S} \subset V(\tilde{g})$. After a transformation of type $\zeta_\nu \mapsto \zeta_\nu + \eta_d^{s_\nu}$ for $1 \leq \nu \leq n$ and $\eta_i \mapsto \eta_i + \eta_d^{t_i}$ for $1 \leq i < d$ and $\eta_d \mapsto \eta_d$, we may assume that $\tilde{g}$ is monic in η_d. So we have a finite map $S \to H' := (T' \cup B')$, where $T' := \mathbb{D}^n(\underline{\varepsilon}) \times \mathbb{D}^{d-1}$ and $B' := \mathbb{D}^n \times \partial\mathbb{D}^{d-1}$. Then, by Corollary 3.2, we reduce to $d-1$.

So it remains to deal with the case $d = 2$. Then we arrange the irreducible components of $\tilde{S}$ into subsets $\tilde{S}_1$ and $\tilde{S}_2$ such that $\tilde{S} = \tilde{S}_1 \cup \tilde{S}_2$ satisfying

$\tilde{S}_1 \subset V(\tilde{\zeta_1})$ and $\dim(\tilde{S}_2 \cap V(\tilde{\zeta_1})) \leq n$. Now there exists a nonzero polynomial $\tilde{f} \in k[\zeta_2, \ldots, \eta_d]$ such that $(\tilde{S}_2 \cap V(\tilde{\zeta_1})) \subset V(\tilde{f})$. After a transformation of type $\zeta_\nu \mapsto \zeta_\nu + \eta_2^{s_\nu}$ for $1 \leq \nu \leq n$ and $\eta_1 \mapsto \eta_1 + \eta_2^{t_1}$ and $\eta_2 \mapsto \eta_2$, we may assume that $\tilde{f}$ is monic in η_2. Again by applying Lemma 3.1, we may assume that $(\tilde{S}_2 \cap V(\tilde{\zeta_1})) \subset V(\tilde{\eta}_2)$. Due to the maximum principle, there exist numbers $\delta_1, \delta_2 \in \sqrt{|K^\times|}$ less than 1 such that

$$S \cap \{x \in X,\, |\zeta_1(x)| \geq \delta_1,\, |\eta_2(x)| \geq \delta_2(x)\} = \varnothing.$$

Then the projection by the coordinate functions $\zeta_2, \ldots, \zeta_n, \eta_1, \eta_2$,

$$p|_{\ldots}\colon S \cap p^{-1}(H') \to H' := T' \cap B' \subset \mathbb{D}^{n-1} \times \mathbb{D}^2_{|\eta_2(x)| \geq \delta_2},$$

is finite, where

$$T' := \mathbb{D}^{n-1+2}_{\tilde{\eta}_2} \quad \text{and} \quad B' := \mathbb{D}^{n-1} \times \partial\mathbb{D}^1 \times A(\delta_2, 1).$$

Thus, by Corollary 3.2 and Proposition 2.6, we obtain the extension to the set $\mathbb{D}^n \times A(\delta_2, 1)$.

Now we have more freedom on the geometry of the Hartogs figure. We have $\tilde{S} = V(\tilde{f})$ for some polynomial $f \in K[\zeta, \eta]$ with $|f| = 1$. Then we choose a transformation Φ of type

$$\zeta_\nu \mapsto \zeta_\nu + \eta_2^{s_\nu} \quad \text{for } 1 \leq \nu \leq n; \quad \eta_1 \mapsto \eta_1 + \eta_2^{t_1} \quad \text{and} \quad \eta_2 \mapsto \eta_2$$

such that $\delta_2^{s_1} < \varepsilon_1$ such that $\tilde{f}$ becomes a Weierstraß divisor with respect to η_2. The inverse image of the figure $H'' := T'' \cup B''$ under Φ is contained in the domain of definition of S, where

$$T'' := (\mathbb{D}^n(\underline{\varepsilon}) \times \mathbb{D}^2) \quad \text{and} \quad B'' := \mathbb{D}^n \times \partial\mathbb{D}^2.$$

In particular, the projection by the coordinate functions $\zeta_1, \ldots, \zeta_n, \eta_1$,

$$p|_{\ldots}\colon S \cap p^{-1}(H') \to H' := (\mathbb{D}^n(\underline{\varepsilon}) \times \mathbb{D}^1) \cup (\mathbb{D}^n \times \partial\mathbb{D}^1),$$

is finite. Thus the extension property follows from Corollary 3.2 and Proposition 2.6. ☐

Lemma 3.5. *Let $\zeta_1, \ldots, \zeta_n, \eta_1, \ldots, \eta_d$ be the coordinate functions on $\mathbb{D}^{n+d}$ with $d \geq 2$. Consider the Hartogs figure $H := T \cup B \subset X := \mathbb{D}^{n+d}$ of dimension n, where*

$$T := (\mathbb{D}^n(\underline{\varepsilon}) \times \mathbb{D}^d)_{\tilde{h}} \quad \text{and} \quad B := \mathbb{D}^{n+d}_{\tilde{N}}$$

with $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$, where $\tilde{h} \in k[\eta_1]$ is monic polynomial with $\tilde{h}(0) \neq 0$. Assume that $\tilde{N}$ is of dimension n with $\dim \tilde{N} \cap V(\zeta) = 0$ and that every irreducible component of $\tilde{N}$ contains the origin. Then (H, X) has properties (A_n) and (M_n).

Proof. Let $S \subset H$ be a closed analytic subset whose irreducible components have dimension $m = n + t$ with $t \geq 1$. The case $t \geq 2$ is covered by Proposition 3.3. So we may assume that $t = 1$ and $d \geq 2$. As in the proof of Lemma 3.4, we will stepwise increase the radii $\varepsilon_1, \ldots, \varepsilon_n$ to 1. So we may assume that $\varepsilon_2 = 1, \ldots, \varepsilon_n = 1$. Also by the same procedure, we can reduce to the case $d = 2$ and $\tilde{N} = V(\tilde{g}, \eta_2)$, where $\tilde{g} \in k[\zeta, \eta_1]$ and every irreducible component of $\tilde{g}$ meets the origin and $\tilde{g}(0, \eta_1) \neq 0$ does not vanish identically. Indeed, the projection of $\tilde{N}$ to $\mathbb{A}_k^{n-1} \times \mathbb{A}^2$ is of dimension less than or equal to $n < n - 1 + 2$ and hence contained in a hypersurface $V(\tilde{f})$. After a transformation of coordinates, we may assume that $\tilde{f}$ is monic in η_d. By the usual application of Corollary 3.2, we may assume that $\tilde{f} = \eta_d$. Note that we can arrange the transformations in such a way that the polynomial h can be replaced by some other polynomial with the same property. So we still have $V(\tilde{\zeta}, \tilde{\eta}_2) \cap V(\tilde{g}) = \{0, \tilde{y}_1, \dots, \tilde{y}_r\} \subset \mathbb{A}_k^1$.

Now we proceed as in the proof of Lemma 3.4. For the reduction $\tilde{S} \subset \mathbb{A}_k^{n+2}$ of S, we have a decomposition $\tilde{S} = \tilde{S}_1 \cup \tilde{S}_2$ as a union of Zariski-closed subsets of $\mathbb{A}_k^{n+2}$ such that $\tilde{S}_1 \subset V(\tilde{\zeta}_1)$ and $\dim(\tilde{S}_2 \cap V(\tilde{\zeta}_1)) < n + 1$. In a first step, one shows the extension to the domain $\mathbb{D}^n \times A(\delta_2, 1)$ for some $\delta_2 \in \sqrt{|K^\times|}$ with $\delta_2 < 1$. In a second step, we can start a transformation as at the end of the proof of Lemma 3.4. So we obtain a finite map

$$p|_{\dots}\colon S \cap p^{-1}(H') \to H' := T' \cup \mathbb{D}_{\tilde{g}}^{n+1},$$

where $T' := \mathbb{D}^n(\varepsilon_1)_{\tilde{h}} \times \mathbb{D}^1$. Since H' gives rise to a Hartogs figure in $\mathbb{D}_{\tilde{h}}^{n+1}$ due to the condition on $\tilde{g}$, we obtain the extension to $\mathbb{D}_{\tilde{h}}^{n+d}$ by Corollary 3.2 and Proposition 2.6. Thus we have the extension to the union $\mathbb{D}_{\tilde{N}}^{n+d} \cup \mathbb{D}_{\tilde{h}}^{n+d}$ which is ball figure of dimension $(n-1)$. Finally, we succeed by Proposition 3.3. □

We want to stress the fact that the problem of extending S is reduced to showing it in cases where there is a finite map of some shrinking of S to a Hartogs figure of dimension n in an $(n+1)$-dimensional space.

Theorem 3.6. *Let (H, X) be a Hartogs figure of dimension n in an affinoid space of pure dimension $n + d$. Then the couple (H, X) has properties* (A_n) *and* (M_n).

Proof. We may assume that S is irreducible of dimension m with $m \geq n+1$. If $m \geq n + 2$, the assertion follows from Proposition 3.3. So we assume that $m = n+1$. Let $\tilde{N} \subset \tilde{X}$ be the Zariski-closed subset associated to the ball figure contained in H. The subset $S \subset X_{\tilde{N}}$ gives rise to a Zariski-closed algebraic subset $\tilde{S} \subset \tilde{X} - \tilde{N}$. We denote by $\tilde{S} \subset \tilde{X}$ its closure in $\tilde{X}$ as well since no confusion can happen. Now $\tilde{S}$ is of pure dimension $m = n + 1$.

At first, we apply Lemma 1.11 and Lemma 3.4 to obtain the extension of S to the maximal tubes. As in the proof of Theorem 2.11, we may assume that there is only one tube present. Then we apply the projection lemma, Proposition 1.14. So there is a finite morphism $\phi\colon X \to \mathbb{D}^{n+d}$ and a polynomial $\tilde{h} \in k[\eta_1]$ with the following properties.

(o) Set $H' := T \cup B \subset \mathbb{D}^{n+d}$ with $T := (\mathbb{D}^n(\underline{\varepsilon}) \times \mathbb{D}^d)$ and $B := \mathbb{D}_{\tilde{N}}^{n+d}$.

(i) $H' \subset \mathbb{D}_{\tilde{h}}^{n+d}$ is a Hartogs figure in $\mathbb{D}_{\tilde{h}}^{n+d}$ with $\phi^{-1}(H') \subset H$.

(ii) The induced map $\phi\colon \phi^{-1}(H') \to H'$ is finite.

Then the assertion follows by Corollary 3.2 from Lemma 3.5. □

Remark 3.7. As in the proof of Theorem 2.11, we made use of Proposition 1.14 which requires that the residue field k has infinitely many elements. This assumption is actually not necessary for this application. Indeed, we can choose a base field extension K'/K in such a way the residue field k' of K' has infinitely many elements. So we obtain the extension of closed analytic subsets after such a field extension. Then an easy Galois descent argument shows the extension property over the given base field.

The extension theorem for closed analytic subsets on rectilinear Hartogs figures like $H := (V \times \mathbb{D}^d) \cup (Y \times \partial\mathbb{D}^d)$ is more delicate, because here we do not know such a nice projection lemma as the one we used in the proof of Theorem 3.6. Of course, any closed analytic subset of pure dimension $m \geq 2 + \dim Y$ extends due to Proposition 3.3. So it remains to study the case of pure dimension $\dim S = 1 + \dim Y$. The case $d = 2$ will easily be solved with a completely different method in Remark 5.8 and Corollary 5.9 which is of interest for itself. But that proof does not allow to show property (M_n). Therefore, we introduce a new method.

In the following, we consider a rectilinear Hartogs figure

$$H := (V \times \mathbb{D}^d) \cup (Y \times \partial\mathbb{D}^d) \subset X := (Y \times \mathbb{D}^d),$$

where Y is an irreducible affinoid space and $V \subset Y$ is a nonempty open subset.

Lemma 3.8. *Assume that* $\dim Y = 1$. *If* $S \subset H$ *is an irreducible closed analytic subset of dimension* $\dim Y + 1$, *then* S *uniquely extends to an irreducible closed analytic subset* $\underline{S} \subset X$.

Proof. If $d = 2$, then the assertion follows from Corollary 5.9. In the following, assume $d \geq 3$. We proceed by descending induction on d. Consider the Zariski closure $\tilde{S} \subset (\tilde{Y} \times \mathbb{A}^d_k)$ of the reduction of S. We know that $\tilde{S}$ is of pure dimension 2. Then the projection $q\colon \tilde{Y} \times \mathbb{A}^d_k \to \mathbb{A}^d_k$ maps $\tilde{S}$ into a Zariski-closed subset of dimension of at most $2 < d$. So there exists a nonzero polynomial $\tilde{g} \in k[\eta_1, \ldots, \eta_d]$ such that $q(\tilde{S}) \subset V(\tilde{g})$. After a suitable transformation of coordinates of type $\eta_i \mapsto \eta_i + \eta_d^{t_i}$ for $1 \leq i \leq d-1$ and $\eta_d \mapsto \eta_d$, we may assume that $\tilde{g}$ is monic in η_d. Then consider the projection $p\colon Y \times \mathbb{D}^d \to Y \times \mathbb{D}^{d-1}$. Its restriction $p\colon S \cap p^{-1}(H') \to H'$ is finite, where $H' := (V \times \mathbb{D}^{d-1}) \cup (Y \times \partial\mathbb{D}^{d-1})$. So $p(S) \subset H'$ is a closed analytic subset of dimension 2. Due to the induction hypothesis, $p(S)$ extends to a closed analytic subset $T \subset Y \times \mathbb{D}^{d-1}$. So we see that $S \subset p^{-1}(T)$ is a subset of a closed analytic subset $p^{-1}(T)$ of codimension 1 in $Y \times \mathbb{D}^d$. Obviously, we can find projections p such that a given point $x \in H - S$ is outside $p^{-1}(p(T)$. Then the intersection of the $p_i^{-1}(T)$ for suitable projections $p_i\colon Y \times \mathbb{D}^d \to Y \times \mathbb{D}^{d-1}$ yields an extension of S. □

For generalizing the last result to higher dimension, we start with some preparations.

Definition 3.9. The lines $L \subset \mathbb{D}^n$ through the origin can be parameterized by the points of $\mathbb{P}^{n-1}_K$. A family of such lines is called *dense* if the set of reductions of its members to $\mathbb{A}^n_k$ induce a subset of $\mathbb{P}^{n-1}_k$ which is not contained in a countable union of proper closed subsets of $\mathbb{P}_k^{n-1}$. We remark that the set of lines $L \subset \mathbb{D}^n$ through the origin is dense if the residue field k of K is not countable. Moreover, the complement of a countable union of proper closed subsets of $\mathbb{P}_k^{n-1}$ is a dense family as well.

Lemma 3.10. *Assume that the residue field k of K is not countable. Let $p \in K[[\eta_1, \ldots, \eta_n]]$ be a formal power series which converges on $\mathbb{D}^n(\varepsilon)$ for some $\varepsilon > 0$. If the restriction $p|_L$ converges for a dense family of lines $L \subset \mathbb{D}^n$ through the origin and satisfies $|p|_L| \leq 1$, then p converges on $\mathbb{D}^n$.*

Proof. Let $p := \sum_{m=0}^{\infty} p_m$ be the expansion with respect to the total degree. It means

$$p_m = \sum_{|\mu|=m} c_\mu \cdot \eta_1^{\mu_1} \cdot \ldots \cdot \eta_n^{\mu_n} \in K[\eta_1, \ldots, \eta_n]$$

is a homogenous polynomial of degree m. Let $\pi_m \in K^\times$ with $|\pi_m| := |p_m|$ be the spectral norm of p_m if $p_m \neq 0$. Now look at the reduction $\tilde{p}_m$ of p_m/π_m. Note that any L gives rise to a point $[\tilde{L}]$ in $\mathbb{P}_k^{n-1}$ and $V(\tilde{p}_m)$ gives rise to proper closed subset of $\mathbb{P}_k^{n-1}$. Then $[\tilde{L}] \notin V(\tilde{p}_m)$ is equivalent to $|p_m|_L| = |\pi_m|$.

Due to our assumption, there exists a line L in our family such that $|p_m|_L| = |\pi_m|$. Since $|p|_L \leq 1$, we have that $|p_m| \leq 1$ for all $m \in \mathbb{N}$. Let $I \subset \mathbb{N}$ be the subset consisting of all $m \in \mathbb{N}$ with $\pi_m \neq 0$. Due to our assumption, there exists a dense family of lines L such that $\tilde{L}$ is not contained in $\bigcup_{m \in I} V(\tilde{p}_m)$. Since $p|_L$ converges, we have that π_m converges to 0. Thus we see that $p = \sum_{m \in \mathbb{N}} p_m$ belongs to $K\langle \eta_1, \ldots, \eta_n \rangle$. □

Lemma 3.11. *Assume that the residue field k of K is non-countable. Let $P \in T_n\langle \eta \rangle$ be an irreducible Weierstraß polynomial. Set $X = V(P)$, and let $\phi\colon X \to \mathbb{D}^n$ be the projection. Assume that ϕ is étale over the origin $0 \in \mathbb{D}^n$ and that its fiber consists of rational points. Then there exists a dense family, Definition 3.9, of lines L through 0 in $\mathbb{D}^n$ such that $\phi^{-1}(L) \subset X$ is irreducible.*

Proof. Let $x_1, \ldots, x_s \in X$ be the points with $\phi(x_i) = 0$. Since ϕ is étale above 0, there exists an $\varepsilon > 0$ such that $\phi^{-1}(\mathbb{D}^n(\varepsilon))$ decomposes into sheets $U_1, \ldots, U_s$. So we can write

$$P = (\eta - a_1) \cdot \ldots \cdot (\eta - a_s) \in \mathcal{O}_{\mathbb{D}^n}(\mathbb{D}^n(\varepsilon)).$$

Set $I := \{1, \ldots, s\}$, and consider, for any nonempty subset $\sigma \subset I$, the polynomial

$$P_\sigma := \prod_{i \in \sigma} (\eta - a_i) = \sum_{j=0}^{t_\sigma} b_{j,\sigma} \eta^j \in \mathcal{O}_{\mathbb{D}^n}(\mathbb{D}^n(\varepsilon))[\eta].$$

Any $b_{j,\sigma} \in \mathcal{O}_{\mathbb{D}^n}(\mathbb{D}^n(\varepsilon))$ is a power series. If, for some proper subset $\sigma \subset I$, the restriction of all the coefficients $b_{j,\sigma}|_L$ to a dense family of lines L is holomorphic on L, then $P_\sigma \in T_n[\eta]$ due to Lemma 3.10, and hence the polynomial $P \in T_n[\eta]$ cannot be irreducible. Thus we see that $\phi^{-1}(L)$ is an irreducible curve contained in X for a dense family of lines through 0. □

Proposition 3.12. *Keep the situation introduced above Lemma 3.8. If $S \subset H$ is an irreducible closed analytic subset of dimension* $\dim Y + 1$*, then S uniquely extends to an irreducible closed analytic subset $\underline{S} \subset X$.*

Proof. The case $\dim Y = 1$ was settled in Lemma 3.8 which will be used in the sequel. Now we turn to the general case $n = \dim Y > 1$. Consider the projection $p\colon (\tilde{Y} \times \mathbb{A}_k^d) \to \tilde{Y}$. Look at the generic points $\tilde{y}_1, \dots, \tilde{y}_r$ of $\tilde{Y}$ which are associated to the irreducible components $\tilde{Y}_1, \dots, \tilde{Y}_r$. If p is not dominant over $\tilde{Y}_i$, then there exists an element $\tilde{a}_i \in \tilde{A}$ such that all the geometric fibers over $Y_{\tilde{a}_i}$ are empty. If $p|_{\tilde{S}}$ is dominant over $\tilde{Y}_i$, the fiber of $\tilde{y}_i$ is of dimension 1 by reasons of dimensions. Then there exists a transformation of coordinates as above and a function $\tilde{a}_i \in \tilde{A}$ such that the projection $p_i\colon \tilde{S} \cap (\tilde{Y}_{\tilde{a}_i} \times \mathbb{A}_k^d) \to \tilde{Y}_{\tilde{a}_i} \times \mathbb{A}_k^1$ is finite and $p_i^{-1}(\tilde{Y}_{\tilde{a}_i} \times (\mathbb{A}_k^1 - \{0\})) \subset (\tilde{Y}_{\tilde{a}_i} \times (\mathbb{A}_k^d - \{0\}))$. Then, for any function $f \in \mathcal{O}_X(Y_{\tilde{a}_i} \times \mathbb{D}^d)$, we have the characteristic polynomial

$$P_f(\eta) = \eta^t + c_{t-1} \cdot \eta^{t-1} + \cdots + c_0 \in \mathcal{O}_Y(Y_{\tilde{a}_i} \times \partial\mathbb{D}^1)[\eta]$$

of $f|_S$. Its coefficients c_τ are bounded holomorphic functions on $Y_{\tilde{a}_i} \times \partial\mathbb{D}^1$. Now we perform a base field extension $K \to K'$ such that the residue field of K' is not countable. Due to Lemma 3.11 and Lemma 3.8, there is dense subset of $Y_{\tilde{a}_i}$ such that $S \cap (\{y\} \times \mathbb{D}^d)$ extends to $\{y\} \times \mathbb{D}^d$. Then it follows from Lemma 2.3 that the coefficients c_τ extend to holomorphic functions on $Y_{\tilde{a}_i} \times \mathbb{D}^1$. One easily shows that the coefficients are actually defined over the base field K. Thus S is extended by the locus of the functions $(P_f(f))$, where f runs over all functions $f \in \mathcal{O}_X(Y_{\tilde{a}_i} \times \mathbb{D}^d)$. So S extends to the ball figure $(Y \times \partial\mathbb{D}^d) \cup (Y_{\tilde{a}} \times \mathbb{D}^d))$ of dimension $n-1$. Then the assertion follows from Proposition 3.3. □

Proposition Proposition 3.12 does not show the full property (M_n); the part concerning the extension of meromorphic functions is missing. This lack is due to the fact that we proved Proposition 3.12 by using the method of § 5 instead of the standard technique of Lemma 3.1. Nevertheless, property (M_n) holds for any rectilinear Hartogs figure of dimension n. Although we now know the extension property for closed analytic subsets S in rectilinear Hartogs figures, the extension property for meromorphic functions on S requires more involved methods.

Lemma 3.13. *Let Y be a smooth connected curve, and let $V \subset Y$ be a non-empty open subdomain. Consider the rectilinear Hartogs figure*

$$H := (V \times \mathbb{D}^d) \cup (Y \times \partial\mathbb{D}^d),$$

and let $S \subset Y \times \mathbb{D}^d$ be an irreducible closed analytic subset of dimension 2. Then any meromorphic function on $S \cap H$ extends to a meromorphic function on S.

Proof. Let us first discuss the case where Y has a smooth formal model. Then H contains an affinoid Hartogs figure $H' \subset H$ of dimension n. Indeed, let $y_0 \in V$ be a point. Then it follows from [24, Prop. 4.1.12] that there exists a function $g \in \mathcal{O}_Y(y)$ such that $V(g) \cap Y' = \{y_0\}$, where $Y' \subset Y$ is a formal open neighborhood of $\tilde{y}_0$. So, for a small $\varepsilon \in |K^\times|$, the figure

$$H' := \{x \in Y' \times \mathbb{D}^d;\ |g(x)| \leq \varepsilon\} \cup (Y' \times \partial\mathbb{D}^d) \subset H \cap (Y' \times \partial\mathbb{D}^d)$$

is a Hartogs figure in $Y' \times \partial\mathbb{D}^d$. So property (M_n) for Hartogs figures implies the extension for meromorphic functions on $S \cap (Y' \times \partial\mathbb{D}^d)$. Then the assertion follows by Theorem 3.6.

In the general case, we use the stable reduction theorem for curves [24, Thm. 4.5.3]. So we may assume that Y has a semi-stable reduction where no irreducible component has a self-intersection. Assume firstly that V is contained in the formal fiber of a double point $\tilde{y}_0$. By what we have shown already, we may assume that V is a concentric annulus of height 1 contained in the formal fiber. Let $\tilde{Y}_1$ and $\tilde{Y}_2$ be the components which contain $\tilde{y}_0$. Then it is easy to see that we have a Hartogs figure in $Y' \times \mathbb{D}^d$, where Y' is the formal open part Z which reduces into the smooth part of $\tilde{Y}_1 \cup \tilde{Y}_2$ and $\tilde{y}_0$ and where the tube is given by a function $g \in \mathcal{O}_{Y'}(Z)$ such that $V := \{y \in Z;\ |g(y)| \leq \delta\}$. So we obtain the extension of the meromorphic function to $S \cap (Z \times \mathbb{D}^d)$. If one or both of these components are complete, we blow them down. Then we can continue as above since we can define new Hartogs figures where the tube is given by the blown-down part. □

So, due to Lemma 3.13, we have properties (A_n) and (M_n) in the case where Y is a smooth curve without using Corollary 5.9, but we used the more involved existence of a semi-stable reduction for smooth curves. Then the general case follows as in the proof of Proposition 3.12. Moreover, one can add in that proof the extension property for meromorphic functions once it is known for curves. So we arrive at the full assertion for rectilinear Hartogs figures.

Proposition 3.14. *The rectilinear Hartogs figure* $H := (V \times \mathbb{D}^d) \cup (Y \times \partial\mathbb{D}^d)$ *has properties* (A_n) *and* (M_n).

Next we turn to the case of extension through closed analytic subsets which was studied by Thullen, Remmert and Stein in complex analysis.

Theorem 3.15. *Let X be a rigid analytic space and $A \subset X$ a closed analytic subset of dimension n. Then the following holds.*
(a) *Any closed analytic subset $S \subset X - A$ of pure dimension* $\dim S \geq n+1$ *extends to a closed analytic subset of X.*
(b) *Any closed analytic subset $S \subset X - A$ of pure dimension n extends to a closed analytic subset of X if S extends to $U \cup (X - A)$, where $U \subset X$ is an open subset which meets every irreducible component of A.*

Proof of Theorem 3.15 (a). The case where S is of pure dimension $\dim S \geq n+2$ follows directly from the case where one considers a ball figure of dimension n defined by $\tilde{A}$; cp. Proposition 3.3.

Let us now consider the case where S is of pure dimension $n+1$. Obviously, we may assume that X is affinoid and hence that $X = \mathbb{D}^N$ for some $N \in \mathbb{N}$. By Galois decent, we may assume that A are geometrically reduced. So A

is smooth outside a closed subvariety T of dimension $n-1$. Due to Proposition 3.3, it suffices to show that $S \cap X_{\tilde{T}}$ extends to $X_{\tilde{T}}$. Then we may assume that $X = Y \times \mathbb{D}^d$ and $A = Y \times \{0\}$ due to Proposition 1.15.

Denote by $\tilde{N} \subset \tilde{X}$ the locus $V(\tilde{\zeta}_1, \dots, \tilde{\zeta}_d)$. Let $\tilde{S} \subset \tilde{X} - \tilde{N}$ be the closed algebraic subset induced by $S \cap X_{\tilde{N}}$. Due to Proposition 3.3, it suffices to show that S extends to $V \times \mathbb{D}^d$, where V is a formal open subscheme of X such that $\tilde{V}$ contains all the generic points of $\tilde{Y}$. Thus we may assume that $\tilde{Y}$ is irreducible. Now consider the projection $\tilde{p}\colon \tilde{Y} \times \mathbb{A}^d_k \to \tilde{Y}$. If $\tilde{p}|_{\tilde{S}}$ is not dominant, then there is a formal dense open part V of Y such that $\tilde{p}(\tilde{S}) \subset \tilde{Y} - \tilde{V}$. Moreover, due to Lemma 1.5, we may assume that S does not meet $p^{-1}(V)$, where $p\colon Y \times \mathbb{D}^d \to Y$ is a lifting of $\tilde{p}$. So we are done by Proposition 3.3. Otherwise, $\tilde{p}(\tilde{S})$ contains the generic point $\tilde{y}$ of $\tilde{X}$. In this case, the fiber $\tilde{p}|_{\tilde{S}}$ over $\tilde{y}$ has dimension 1. Then there exists a coordinate transformation of type $\zeta_i \mapsto \xi_i^{t_i} + \xi_d$ for $i = 1, \dots, d-1$ and $\zeta_d \mapsto \xi_d$ such that the projection of $\tilde{S} \cap (\{\tilde{y}\} \times \mathbb{A}^d_k)$ to $\{\tilde{y}\} \times \mathbb{A}^{d-1}_k$ is finite. One can repeat this process until we arrive at a finite projection to $\{\tilde{y}\} \times \mathbb{A}^1_k$. Since only finitely many coefficients are involved, there exists an open neighborhood $\tilde{V}$ of $\tilde{y}$ in $\tilde{Y}$ where everything is defined. Since these projections do not effect the set A, by lifting to the affinoid site, we arrive at a situation of a finite morphism $p\colon S \to V \times (\mathbb{D}^1 - \{0\})$ studied in Lemma 3.1. So the assertion follows in this case by Proposition 2.13 (b). □

For the proof of part (b), which is much harder to show, we provide some preparations. Before we start the proof of Theorem 3.15 (b), we recall two types of the Weierstraß Division Theorem.

Proposition 3.16. *Let B be a reduced affinoid algebra, and set $Y := \mathrm{Sp}(B)$. Let $\mathbb{D}^1(r)$ be the 1-dimensional disc of radius $r \in |K^\times|$. Let $g = \sum_{\nu \in \mathbb{N}} b_\nu \cdot \eta^\nu$ be a power series in $B\langle\eta\rangle_r$ which converges on $X := Y \times \mathbb{D}^1(r)$. Assume that there exists an integer $n \in \mathbb{N}$ such that, for all $y \in Y$,*

$$|b_n(y)| \cdot r^n \geq |b_\nu(y)| \cdot r^\nu \quad \text{for all } \nu \in \mathbb{N},$$
$$|b_n(y)| \cdot r^n > |b_\nu(y)| \cdot r^\nu \quad \text{for all } \nu > n.$$

Such a power series is called a Weierstraß divisor *on $Y \times \mathbb{D}^1(r)$. Then one can uniquely write $g = u \cdot \omega$, where $u \in \mathcal{O}_X(Y \times \mathbb{D}^1(r))$ is a unit and $\omega \in B[\eta]$ is a monic polynomial of degree n which satisfies estimates for the coefficients similar to the ones above.*

Proof. First one reduces to $r = 1$ and $b_n = 1$ by dividing g by b_n which is unit in B. Then one follows the usual method; cp. [9, Thm. 2.2.8]. □

Proposition 3.17. *Let B be a reduced affinoid algebra, and set $Y := \mathrm{Sp}(B)$. Let $A(r_1, r_2)$ be an annulus with radii $r_1 \leq r_2$ for $r_1, r_2 \in |K^\times|$. Then let $g = \sum_{i \in \mathbb{Z}} b_i \cdot \eta^i \in B_{r_1}\langle\eta^{\pm}\rangle_{r_2}$ be a Laurent series which converges on the space $X := Y \times A(r_1, r_2)$. Assume that there are integers $n_1 \leq n_2$ such that, for all $y \in Y$,*

$$|b_{n_1}(y)| \cdot r_1^{n_1} \geq |b_i(y)| \cdot r_1^i \quad \text{for all } i \in \mathbb{Z},$$
$$|b_{n_1}(y)| \cdot r_1^{n_1} > |b_i(y)| \cdot r_1^i \quad \text{for all } i < n_1;$$

$$|b_{n_2}(y)| \cdot r_2^{n_2} \geq |b_i(y)| \cdot r_2^i \quad \textit{for all } i \in \mathbb{Z},$$
$$|b_{n_2}(y)| \cdot r_2^{n_2} > |b_i(y)| \cdot r_2^i \quad \textit{for all } i > n_2.$$

Such a Laurent series is called a Weierstraß divisor *on $Y \times A(r_1, r_2)$. Then one can uniquely write $f = u \cdot \omega$, where $u \in \mathcal{O}_X(Y \times A(r_1, r_2))$ is a unit and $\omega \in B[\eta]$ is a monic polynomial of degree $n = n_2 - n_1$ which satisfies similar estimates for the coefficients as above.*

Proof. For the proof of this type of Weierstraß Theorem, one follows the usual method; cp. [9, Thm. 2.2.8]. At first, one reduces to the case $n_1 = 0$. Then we show the division theorem with estimates for elements $f \in \mathcal{O}_X(Y \times A(r_1, r_2))$ in the following style.

Decompose $f = q \cdot g + r + f_1$, where $q \in \mathcal{O}_X(Y \times A(r_1, r_2))$ satisfying the conditions $|q|_y \leq |f|_y / \min\{|g|_{y,r_1}, |g|_{y,r_2}\}$, where $r \in B[\eta]$ is a polynomial of degree less than n with $|r|_y \leq |f|_y$ and where $f_1 \in \mathcal{O}_X(Y \times A(r_1, r_2))$ with $|f_1|_y \leq \varepsilon \cdot |f|_y$ for all $y \in Y$ with

$$\varepsilon := \max_{y \in Y} \big\{ \max_{j > n} \big\{ |b_j(y)/b_n(y)| \cdot r_2^{j-n}, \max_{j<0} \{ |b_j(y)/b_0(y)| \cdot r_1^j \} \big\} \big\} < 1.$$

Here $|\cdot|_y$ denotes the spectral norm of the function on the fiber $\{y\} \times A(r_1, r_2)$ and $|\cdot|_{y,r_i}$ denotes the spectral norm on $\{y\} \times A(r_i, r_i)$ for $i = 1, 2$. Then one iterates this division $f_i = q_i \cdot g + r_i + f_{i+1}$ as above. The remaining parts f_i converge to 0. Thus one obtains $f = q \cdot g + r$ in the limit. Finally, one applies the division to η^n, and hence one gets $\eta^n = v \cdot g + r$. Now one easily shows that the Laurent series of v satisfies the conditions of our proposition with $n_1 = n_2 = 0$. Then it is clear that v is a unit of the form $v = v_0 \cdot (1 + h)$, where $v_0 \in B^\times$ is a unit and $|h|_y < 1$ for all $y \in Y$. Finally, this yields $g = u \cdot \omega$ with $u = 1/v$ and $\omega = b_n \eta^n - r$. □

The following corollary follows from a well-known fact in commutative algebra; cp. Proposition A.13.

Corollary 3.18. *Keep the situation of Proposition 3.17. Assume that B is normal. Let $S \subset Y \times A(r_1, r_2)$ be a closed analytic subset of pure dimension* $\dim Y$. *If $f \in \mathcal{O}(Y \times A(r_1, r_2))$ vanishes on S and satisfies the conditions of Proposition 3.17, then $S = V(p)$ is the locus of a monic polynomial $p \in B[\eta]$ whose coefficients satisfy conditions similar to the ones of f.*

Proof. The vanishing ideal $\mathfrak{a}$ of S is a divisor ideal which contains a monic polynomial ω. Due to the inequalities, $S = V(\mathfrak{a})$ can be regarded as a closed algebraic subset of $Y \times \mathbb{A}^1_K$. Since the monic polynomial ω is contained in $\mathfrak{a}$, the ideal $\mathfrak{a}$ is generated by a monic polynomial p which divides ω, because B is normal; cp. Proposition A.13. □

We add a general method to construct rational coverings which is often used.

Lemma 3.19. *Let B be a reduced affinoid algebra, and set $Y := \mathrm{Sp}(B)$. Let $r_1 < r_2 \leq 1$ be numbers of $\sqrt{|K^\times|}$. Let $f = \sum_{\nu=-\infty}^{\infty} b_\nu \cdot \eta^\nu \in B[[\eta, 1/\eta]]$ be* *a Laurent series over an affinoid algebra B which converges on the relative annulus $Y \times A(r_1, r_2)$. Assume that the coefficients $(b_\nu \in B;\ \nu \in \mathbb{Z})$ have no common zeros. Then there exists a finite affinoid covering $\mathfrak{V} = \{V_0, \dots, V_n\}$ of $Y = \mathrm{Sp}(B)$ and numbers $\rho_0, \dots, \rho_n \in \sqrt{|K^\times|}$ with $\rho_\nu > r_1$ close to r_1 and numbers $\varrho_0, \dots, \varrho_n \in \sqrt{|K^\times|}$ with $\varrho_\nu < r_2$ close to r_2 such that f has no zeros on $V_\nu \times A(\rho_\nu, \rho_\nu)$ and on $V_\nu \times A(\varrho_\nu, \varrho_\nu)$ for $\nu = 0, \dots, n$. In particular, the projection $p\colon V(f) \cap (V_\nu \times A(\rho_\nu, \varrho_\nu)) \to V_\nu$ is finite for $\nu = 0, \dots, n$.*

Proof. Since the coefficients $(b_\nu;\ \nu \in \mathbb{N})$ have no common zeros, there exist indices n_1, n_2 with $n_1 \le n_2$ such that $b_{n_1}, \dots, b_{n_2}$ have no common zeros. So there exists a positive number b such that

$$b < \max\{|b_{n_1}(y)|, \dots, |b_{n_2}(y)|\} \quad \text{for all } y \in Y.$$

Since the sequence $(|b_\nu| \cdot r_1^\nu)$ converges to 0 for $\nu \to -\infty$ and $(|b_\nu| \cdot r_2^\nu)$ converges to 0 for $\nu \to \infty$, we may assume that, for all $\nu \in \mathbb{Z} - \{n_1, \dots, n_2\}$,

$$|b_\nu(y)| \cdot r_1^\nu < \max\{|b_{n_1}(y)| \cdot r_1^{n_1}, \dots, |b_{n_2}(y)| \cdot r_1^{n_2}\} \quad \text{for all } y \in Y,$$
$$|b_\nu(y)| \cdot r_2^\nu < \max\{|b_{n_1}(y)| \cdot r_2^{n_1}, \dots, |b_{n_2}(y)| \cdot r_2^{n_2}\} \quad \text{for all } y \in Y.$$

Moreover, there exist numbers $\rho, \varrho \in \sqrt{|K^\times|}$ with $\rho > r_1$ close to r_1 and $\varrho < r_2$ and close to r_2 such that, for all $\nu \in \mathbb{Z} - \{n_1, \dots, n_2\}$,

$$|b_\nu(y)| \cdot \rho^\nu < \max\{|b_{n_1}(y)|\rho^{n_1}, \dots, |b_{n_2}(y)|\rho^{n_2}\} \quad \text{and all } y \in Y,$$
$$|b_\nu(y)| \cdot \varrho^\nu < \max\{|b_{n_1}(y)|\varrho^{n_1}, \dots, |b_{n_2}(y)|\varrho^{n_2}\} \quad \text{and all } y \in Y.$$

Of course, we may assume $n_1 = 0$ and put $n := n_2$. For $m = n_1, \dots, n_2$, put

$$Y_m^\varrho := \{y \in Y;\ |b_i(y)| \cdot \varrho^i \le |b_m(y)| \cdot \varrho^m \text{ for } i \le m\}.$$

Then f restricted to $Y_n^\varrho \times \mathbb{D}^1(\varrho)$ is a Weierstraß divisor of degree n. So, if we increase ϱ a little bit, then $b_n\eta^n$ becomes a dominating term of $f|_{Y_n^\varrho \times \mathbb{D}^1(\varrho)}$, and hence f has no zeros on $Y_n^\varrho \times A(\varrho, \varrho)$; cp. [24, Prop. 1.3.4]. Due to the maximum principle, there exists a number $c \in \sqrt{|K^\times|}$ with $c > 1$ such that f has no zeros on $Y_n^c \times A(\varrho, \varrho)$. Now we can look at the union

$$Z := \bigcup_{\nu=1}^{n-1} \{y \in Y;\ |b_\nu(y)| \cdot \varrho^\nu \ge c^{n-\nu}|b_n(y)| \cdot \varrho^n\}.$$

Note that Z is disjoint from Y_n^ϱ. Proceeding by decreasing induction on n, we obtain a rational covering $\mathfrak{V}$ of Y which satisfies the assertion for the higher radius. In the analog way, one deals with the lower radius. A common refinement of the coverings yields the assertion. □

Lemma 3.20. *Let Y be a smooth affinoid space, and set $\mathbb{D}_0^1 := \mathbb{D}^1 - \{0\}$. Let $S \subset Y \times \mathbb{D}_0^1$ be a closed analytic subset of pure dimension $\dim Y$ with $S \cap (Y \times A(1,1)) = \varnothing$. Assume that $S \subset V(f)$ with a holomorphic function $f = \sum_{\nu \in \mathbb{Z}} a_\nu \cdot \eta^\nu \in \mathcal{O}(Y \times \mathbb{D}_0^1)$ such that the coefficients of f have no common zeros on Y.*

Then there exists a covering $\{Y_1, \dots, Y_m\}$ of Y by finitely many connected open affinoid subdomains Y_μ such that $S \cap (Y_\mu \times \mathbb{D}^1_0) = V(f_\mu)$ is a principal divisor of a suitable function $f_\mu = \sum_{i=0}^{\infty} b_{\mu,i} \cdot 1/\eta^i \in \mathcal{O}(Y_\mu \times \mathbb{D}^1_0)$.

Proof. Let $\mathcal{L}$ be the line bundle associated to S. Since $\mathcal{L}|_{Y \times A(1,1)}$ is principal, we can glue it with the trivial line bundle to obtain a line bundle on $Y \times \mathbb{P}^1_0$, where $\mathbb{P}^1_0$ is a projective line punctured at the origin. Obviously, $\mathbb{P}^1_0$ is isomorphic to the affine line $\mathbb{A}^1_K$. Since $\mathcal{L}|_{Y \times \{\infty\}}$ is trivial, it follows from Theorem A.17 that there exists $\{Y_1, \dots, Y_m\}$ of Y by connected affinoid subdomains Y_μ such that $\mathcal{L}|_{Y_\mu \times \mathbb{P}^1_0}$ is trivial. □

Proof of Theorem 3.15 (b). At first, we discuss the special case $X = Y \times \mathbb{D}^1$ and $A := Y \times \{0\}$, where Y is a connected normal affinoid space. Moreover, we choose a nonzero function $f = \sum_{\nu \in \mathbb{Z}} a_\nu \cdot \eta^\nu \in \mathcal{O}(Y \times \mathbb{D}^1_0)$ which vanishes on S. By reason of dimension and in view of Theorem 3.15 (a), we may assume that the coefficients of f have no common zeros on Y and that Y is smooth. Over $U \times \mathbb{D}^1$, the subset S extends a closed subset $\underline{S}$ of $U \times \mathbb{D}^1$. After shrinking U to a nonempty open subset set U_0, we may assume that $\underline{S} \cap (U_0 \times \mathbb{D}^1)$ is the vanishing locus $V(\omega_0)$ for some monic polynomial which is a Weierstraß polynomial. Due to Lemma 3.20, there exists an admissible covering $\{Y_1, \dots, Y_m\}$ of Y by connected affinoid subdomains Y_μ such that $S \cap (Y_\mu \times \mathbb{D}^1_0) = V(f_\mu)$ is a principal divisor for $\mu = 1, \dots, n$.

For a moment, we replace f by f_μ, Y by Y_μ and U by $U \cap Y_\mu$ if $U \cap Y_\mu \neq \varnothing$. Then the restriction of $f = \sum_{\nu \in \mathbb{Z}} a_\nu \cdot \eta^\nu$ onto $U \times \mathbb{D}^1$ can be written in the form $f = u \cdot \omega$, where $\omega \in \mathcal{O}(U)[\eta]$ is a monic polynomial with $|\omega| = 1$ and u is invertible on $U \times \mathbb{D}^1_0$. Now u is of the form $u = \eta^N \cdot \varepsilon$, where $\varepsilon \in \mathcal{O}(U \times \mathbb{D}^1)$ is a unit. So the coefficients $a_\nu = 0$ vanish for all $\nu < N$. Thus we see that f/η^N is holomorphic on $Y \times \mathbb{D}^1$. This also happens to each f_μ if $Y_\mu \cap U \neq \varnothing$. So we see that S extends to $Y_\mu \times \mathbb{D}^1$ for such μ. Since Y is connected, this is passed to all the Y_μ. Finally, this shows the extension of S to the whole $Y \times \mathbb{D}^1$.

It remains to reduce the general case to the special case just discussed. As exercised in the proof of Theorem 3.15 (a), due to Proposition 1.15, we may assume that $X = Y \times \mathbb{D}^d$, where Y is smooth and $A := Y \times \{0\}$. The case $d = 1$ was settled above. So assume that $d \geq 2$ and $S \subset Y \times \mathbb{D}^d_0$ is irreducible of dimension $\dim Y$ with $\mathbb{D}^d_0 := \mathbb{D}^d - \{0\}$. Let $p\colon Y \times \mathbb{D}^d \to Y \times \mathbb{D}^{d-1}$ be the projection. If $S \subset p^{-1}(Y \times \{0\})$, then the assertion follows by the induction hypothesis. So we may assume that $p(S) \cap (Y \times \mathbb{D}^{d-1}_0) \neq \varnothing$ is not empty. It suffices to find a closed irreducible analytic subset $T \subset Y \times \mathbb{D}^{d-1}$ of dimension $\dim Y$ such that $p(S) \subset T$. The dimension of $T \cap (Y \times \{0\})$ is at most $\dim Y - 1$, and hence we are done due to Theorem 3.15 (a) since S is an analytic subset of $p^{-1}((Y \times \{0\}) \cap T) \times \mathbb{D}^d_0$.

Now we want to show that there exists a finite covering $\mathfrak{Y} = \{Y_1, \dots, Y_m\}$ of Y and for each Y_μ numbers $\rho_\mu, \varrho_\mu \in \sqrt{|K^\times|}$ such that

$$p\colon Y_\mu \times \mathbb{D}^{d-1}_0(\rho_\mu) \times \mathbb{D}^1(\varrho_\mu) \to Y_\mu \times \mathbb{D}^{d-1}_0(\rho_\mu)$$

restricts to a finite morphism $p|_S$. Let $r_1, r_2 \in \sqrt{|K^\times|}$ with $r_1 < r_2$. We choose an affinoid function $f \in \mathcal{O}_X(Y \times \mathbb{D}^{d-1} \times A(r_1, r_2))$ with

$$S \cap (Y \times \mathbb{D}^{d-1} \times A(r_1, r_2)) \subset V(f).$$

As above, we may assume that f does not vanish on $\{y\} \times \{0\} \times A(r_1, r_2)$ identically for any $y \in Y$. Then we look at the Laurent series

$$f = \sum_{\nu \in \mathbb{Z}} h_\nu \cdot \eta^\nu \in \mathcal{O}(Y \times \mathbb{D}^{d-1})\langle r_1/\eta, \eta/r_2\rangle.$$

Now we apply Lemma 3.19 to the restriction of f to $Y \times \{0\} \times A(r_1, r_2)$. So there exist a covering $\{Y_1, \ldots, Y_m\}$ and numbers $\varrho_\mu \in \sqrt{|K^\times|}$ with $r_1 < \varrho_\mu < r_2$ for $\mu = 1, \ldots, m$ such that f has no zeros on $Y_\mu \times \{0\} \times A(\varrho_\mu, \varrho_\mu)$. Then there is an index $n \in \mathbb{Z}$ such that

$$|h_\nu(y, 0)| \cdot \varrho_\mu^\nu < |h_n(y, 0)| \cdot \varrho_\mu^n \quad \text{for all } \nu \neq n \text{ and all } y \in Y_\mu.$$

Due to the maximum principle, these inequalities remain true in a small neighborhood of $\{0\}$,

$$|h_\nu(y, x)| \cdot \varrho_\mu^\nu < |h_n(y, x)| \cdot \varrho_\mu^n \quad \text{for all } \nu \neq n, \text{ for all } (y, x) \in Y_\mu \times \mathbb{D}^{d-1}(\rho_\mu),$$

for a small radius $\rho_\mu > 0$. Then the projection

$$p\colon S \cap (Y_\mu \times \mathbb{D}_0^{d-1}(\rho_\mu) \times \mathbb{D}^1(\varrho_\mu)) \to Y_\mu \times \mathbb{D}_0^{d-1}(\rho_\mu)$$

is finite. So, by the argument given above, it suffices to show the extension of $p(S)$. Hence we are finished by the induction hypothesis and finally by the special case $Y \times \mathbb{D}_0^1$. □

Theorem 3.15 was first shown in [25]. The more elegant proof given here was made possible by the recent result of Kerz, Saito and Tamme [20] which is explained in Theorem A.17.

4. Subsheaves

The extension property for subsheaves (U_n) is a formal consequence of properties (M_n) and (A_n). By a precise analysis of the primary decomposition of a coherent subsheaf $\mathcal{F}$ of a given coherent sheaf $\mathcal{G}$, the proof of (U_n) will be reduced to a special extension problem for coherent subsheaves $\mathcal{N} \subset \mathcal{O}_S^p$, where $S \subset G$ is an irreducible analytic subset of G of dimension $\dim S \geq n+1$ and where $\mathcal{O}_S^p/\mathcal{N}$ is a torsion-free $\mathcal{O}_S$-module.

Relative gap-sheaves. In this section, let $X := \mathrm{Sp}(A)$ be an affinoid space, let $\mathcal{M} := \tilde{M}$ be the coherent sheaf associated to a finitely generated A-module M, and let $\mathcal{N} = \tilde{N} \subset \tilde{M}$ be the coherent subsheaf associated to an A-submodule $N \subset M$. If $S \subset X$ is a closed analytic subset, then the *relative gap-subsheaf with respect to* S is the subsheaf $\mathcal{N}[S]_\mathcal{M} \subset \mathcal{M}$ which associates to an open subset $U \subset X$ the submodule of $\Gamma(U, \mathcal{M})$ given by

$$\Gamma(U, \mathcal{N}[S]_\mathcal{M}) := \{s \in \Gamma(U, \mathcal{M});\ s_x \in \mathcal{N}_x \text{ for all } x \in U - S\}.$$

Definition 4.1. For an integer $n \in \mathbb{N}$, the *n-th (relative) gap-subsheaf* $\mathcal{N}_{[n]\mathcal{M}}$ is the subsheaf which associates to an open subset $U \subset X$ the submodule of $\Gamma(U, \mathcal{M})$ given by

$$\Gamma(U, \mathcal{N}_{[n]\mathcal{M}}) := \{s \in \Gamma(U, \mathcal{M});\ s_x \in \mathcal{N}_x \text{ for all } x \in U - T \text{ for a closed analytic subset } T \subset U \text{ with } \dim T \leq n\}.$$

Next we will show that these functors are sheaves and that they are coherent. For an affinoid subdomain $U = \mathrm{Sp}(B)$ of $X = \mathrm{Sp}(A)$ and a finitely generated A-module M, we define $MB := M \otimes_A B = \Gamma(U, \tilde{M})$. For a submodule $P \subset M$, we write $M \cap (PB)$ for the preimage $\Gamma(U, \tilde{P}) \subset \Gamma(U, \tilde{M})$ under the restriction $\Gamma(X, \tilde{M}) \to \Gamma(U, \tilde{M})$.

Lemma 4.2. *Let $Q \subset M$ be a primary submodule; then $Q = M \cap \tilde{Q}_x$ for all $x \in \mathrm{Supp}(\tilde{M}/\tilde{Q})$.*

Proof. Since $Q \subset M$ is primary, the canonical map $M/Q \to (M/Q)_x$ to its localization with respect to the maximal ideal associated to x is injective for $x \in \mathrm{Supp}(\tilde{M}/\tilde{Q})$. Moreover, the map $(M/Q)_x \to (\tilde{M}/\tilde{Q})_x$ is injective, too. □

Lemma 4.3. *Let $N = \bigcap_{i \in I} Q_i$ be a reduced primary decomposition of N in M. Now consider a reduced primary decomposition $Q_i B = \bigcap_{j \in J_i} P_{i,j}$ of $Q_i B$ in MB. Then NB has the primary decomposition $NB = \bigcap_{i \in I'} \bigcap_{j \in J_i} P_{i,j}$ in MB which is reduced as well, where*

$$I' := \{i \in I;\ \mathrm{Supp}(\tilde{M}/\tilde{Q}_i) \cap U \neq \varnothing\}.$$

Let $S \subset X$ be a closed analytic subset. Then it holds
(a) $\dim \mathrm{Supp}(\tilde{M}/\tilde{Q}_i) = \dim \mathrm{Supp}(\tilde{M}/\tilde{P}_{i,j})$ *for all $j \in J_i$ and $i \in I'$.*
(b) *If* $\mathrm{Supp}(\tilde{M}/\tilde{Q}_i) \not\subset S$, *then* $\mathrm{Supp}(\tilde{M}/\tilde{P}_{i,j}) \not\subset S$ *for all $j \in J_i$.*

Proof. Since $A \to B$ is flat, we have $NB = \bigcap_{i \in I'} \bigcap_{j \in J_i} P_{i,j}$. If $\mathfrak{q}_i \subset A$ resp. $\mathfrak{p}_{i,j} \subset B$ are the prime ideals associated to Q_i resp. $P_{i,j}$, then the set of associated primes is given by

$$\mathrm{Ass}_{MB}\, NB = \bigcup_{i \in I} \mathrm{Ass}_B\, \mathfrak{q}_i B = \bigcup_{i \in I'} \{\mathfrak{p}_{i,j};\ j \in J_i\};$$

cp. [31, Prop. 15, p. IV-25]. Since, for $x \in U$, the completions $\hat{A}_x = \hat{B}_x$ are canonically isomorphic and since the localizations A_x resp. B_x are residue rings of regular rings, we have $\dim B/\mathfrak{p}_{i,j} = \dim A/\mathfrak{q}_i$ for $i \in I'$ and $j \in J_i$. The loci $V(\mathfrak{p}_{i,j})$ are the irreducible components of $V(\mathfrak{q}_i) \cap U$ for $i \in I'$. So the primes $(\mathfrak{p}_{i,j};\ j \in J_i,\ i \in I')$ are pairwise different. Thus $NB = \bigcap_{i \in I'} \bigcap_{j \in J'} P_{i,j}$ is a reduced primary decomposition of NB in MB. Assertion (b) follows implicitly. □

Proposition 4.4. *If $\mathcal{N} = \tilde{N} \subset \mathcal{M} = \tilde{M}$ is a coherent subsheaf of a coherent sheaf $\mathcal{M}$, then the gap-subsheaves $\mathcal{N}_{[n]\mathcal{M}}$ and $\mathcal{N}[S]_{\mathcal{M}}$ are coherent.*

More precisely, if $N = \bigcap_{i \in I} Q_i$ is a reduced primary decomposition of N in M, then $\mathcal{N}_{[n]\mathcal{M}}$ resp. $\mathcal{N}[S]_{\mathcal{M}}$ is equal to the coherent sheaf associated to the *submodule*

$$N_{[n]M} := \bigcap_{i \in K} Q_i \quad \textit{with } K := \{i \in I;\, \dim \operatorname{Supp}(\tilde{M}/\tilde{Q}_i) \geq n+1\}$$

respectively to

$$N[S]_M = \bigcap_{i \in L} Q_i \quad \textit{with } L := \{i \in I;\, \dim \operatorname{Supp}(\tilde{M}/\tilde{Q}_i) \not\subset S\}.$$

In particular, it holds $N_{[n]M} = N[T]_M$ *with* $T := \bigcup_{i \in I-K} \operatorname{Supp}(\tilde{M}/\tilde{Q}_i)$.

Proof. By Lemma 4.2, it is clear that $\Gamma(X, \tilde{N}_{[n]M}) = \Gamma(X, \mathcal{N}_{[n]\mathcal{M}})$. Due to Lemma 4.3, we have for any open affinoid subdomain $U = \operatorname{Sp}(B)$ of X that the primary decomposition of $N_{[n]}B$ is of the same type as the one of $N_{[n]}$. Thus we see that $\Gamma(U, \tilde{N}_{[n]M}) = \Gamma(U, \tilde{N}_{[n]\mathcal{M}})$. The assertion for $\mathcal{N}[S]_{\mathcal{M}}$ can be shown in the same way. □

Corollary 4.5. *For any* $x \in X$*, the associated prime ideals* $\mathfrak{p} \subset \mathcal{O}_{X,x}$ *of the* $\mathcal{O}_{X,x}$*-module* $\mathcal{N}_{[n+1]\mathcal{M}}/\mathcal{N}_{[n]\mathcal{M}}$ *are of dimension* $n+1$ *exactly.*

Extension of subsheaves. In this section, let $X = \operatorname{Sp}(A)$ be an affinoid space, $G \subset X$ a nonempty open subset, and let $n \in \mathbb{N}$ be an integer. Moreover, we consider a coherent sheaf $\underline{\mathcal{G}}$ on X and a coherent subsheaf $\mathcal{F} \subset \mathcal{G} := \underline{\mathcal{G}}|_G$. We assume that $\mathcal{F}$ satisfies the condition $\mathcal{F} = \mathcal{F}_{[n]\mathcal{G}}$.

Theorem 4.6. *If the couple* (G, X) *has properties* (M_n), (A_n) *and* (E_n)*, then it also has property* (U_n)*. In particular, ball figures of dimension* $(n-1)$ *and Hartogs figures of dimension* n *have property* (U_n)*.*

For the proof, we have only to show that there is a coherent subsheaf $\underline{\mathcal{F}} \subset \underline{\mathcal{G}}$ with $\underline{\mathcal{F}}|_G = \mathcal{F}$. The assertion of the uniqueness follows from (E_n), explained in § 1. For proving the existence, we firstly concentrate on a special case; this part is due to Siu and Trautmann in the complex case; cp. [33]. For completeness, we discuss it here with full proofs.

So let $S \subset G$ be an irreducible closed analytic subset of dimension $\dim S = m+1 \geq n+1$. Due to property (A_n), the subset S extends to a closed analytic subset $\underline{S} \subset X$. Next we equip S and $\underline{S}$ with their reduced structure. So, for any section $f \in \mathcal{O}_S(U)$ over an open subset $U \subset S$, the following holds.
(a) If $\dim \operatorname{Supp}(f\mathcal{O}_S) \leq m$, then $f = 0$.
(b) If $\dim \operatorname{Supp}(\mathcal{O}_S/f\mathcal{O}_S) \leq m$, then f is a nonzero divisor in $\mathcal{O}_S(U)$.

Lemma 4.7. *Keep the situation introduced above. Let* $\mathcal{S}$ *be a coherent sheaf on* S*, and let* $\varphi\colon \mathcal{O}_S^r \to \mathcal{S}$ *be a morphism. If there is a point* $x \in S$ *such that* ϕ_x *is an isomorphism, then* φ *is injective and there is a unique factorization*

$$\begin{array}{ccc} \mathcal{O}_S^r & \xrightarrow{\ \varphi\ } & \mathcal{S} \\ & \searrow & \big\downarrow{\scriptstyle \psi} \\ & & \mathcal{M}_S^r \end{array}$$

into the r-fold product of meromorphic functions $\mathcal{M}_S$ on S. Especially, the kernel of ψ has dimension $\dim \operatorname{Supp}(\ker(\psi)) \leq m$.

Proof. Since S is irreducible, the injectivity of φ follows from the above remark since φ_x is an isomorphism. For $T := \operatorname{Supp}(\operatorname{coker}(\varphi))$, we have $\dim T \leq m$ as well. Thus any factorization ψ' coincides with ψ and $S - T$. Since $\mathcal{M}_S$ has no sections with support of lower dimension, we have $\psi = \psi'$. This settles the uniqueness.

Concerning the existence, note that, for any open affinoid subset $U \subset S$, there exists a nonzero section $f \in \mathcal{O}_S(U)$ such that $f \cdot \mathcal{S}|_U \subset \operatorname{Im}(\varphi|_U)$ since $\mathcal{S}(U)$ is finitely generated. Then we define $\psi_U(s) := f^{-1}\varphi^{-1}(f \cdot s)$ for sections $s \in \mathcal{S}(U)$. It is clear that ψ_U is well-defined and gives rise to a morphism $\psi\colon \mathcal{S} \to \mathcal{M}_S^r$. The kernel of ψ is contained in T. $\square$

Lemma 4.8. *Keep the situation of Lemma 4.7. Let $\underline{\mathcal{R}}$ be a coherent sheaf on X. If $\mathcal{H} \subset \mathcal{R} := \underline{\mathcal{R}}|_G$ is a coherent subsheaf on G such that $\mathcal{R}/\mathcal{H}$ is an $\mathcal{O}_S$-module which is free of torsion, then there exists a coherent subsheaf $\underline{\mathcal{H}} \subset \underline{\mathcal{R}}$ such that $\underline{\mathcal{R}}/\underline{\mathcal{H}}$ is an $\mathcal{O}_{\underline{S}}$-module which is free of torsion with $\underline{\mathcal{H}}|_G = \mathcal{H}$.*

Proof. Let $(\underline{r}_1, \ldots, \underline{r}_p)$ be a system of generators of $\underline{\mathcal{R}}$. Then $\mathcal{S} := \mathcal{R}/\mathcal{H}$ is generated by the residue classes $s_1, \ldots, s_p$ of $\underline{r}_1, \ldots, \underline{r}_p$. Since the flat locus of a torsion-free $\mathcal{O}_S$-module is open and not empty over a domain, there exists a point $x \in S$ such that $\mathcal{S}_x$ is free. We may assume that $s_1, \ldots, s_q$ is an $\mathcal{O}_{S,x}$-basis of $\mathcal{S}_x$. Due to Lemma 4.7, there exists a commutative diagram

$$\begin{array}{ccccccc}
\mathcal{O}_S^q = \bigoplus_{i=1}^q \mathcal{O}_S \cdot e_i & \xrightarrow{\varphi'} & \mathcal{S} = \mathcal{R}/\mathcal{H} & \longleftarrow & \mathcal{R} & \xleftarrow{\varphi} & \mathcal{O}_X^p = \bigoplus_{i=1}^p \mathcal{O}_S \cdot e_i \\
& \searrow & \downarrow \psi & & & \swarrow \chi & \\
& & \mathcal{M}_S^q = \bigoplus_{i=1}^q \mathcal{M}_S \cdot e_i & & & &
\end{array}$$

where $\varphi'(e_i) := s_i$ for $i = 1, \ldots, q$ and $\varphi(e_i) := r_i$ for $i = 1, \ldots, p$. Since the couple (G, X) satisfies property (M_n), any element $\psi(s_i)$ extends to an element $\underline{t}_i \in \mathcal{M}^q_{\underline{S}}(\underline{S})$ for $i = 1, \ldots, p$. So we obtain a coherent $\mathcal{O}_{\underline{S}}$-module $\underline{\mathcal{S}} := \mathcal{O}_{\underline{S}} \cdot \underline{t}_1 + \cdots + \mathcal{O}_{\underline{S}} \cdot \underline{t}_p$ which is free of torsion. Now we define a morphism $\underline{\chi}\colon \mathcal{O}_X^p \to \underline{\mathcal{S}}$ by setting $\underline{\chi}(\underline{e}_i) := \underline{t}_i$ for $i = 1, \ldots, p$. Then $\underline{\mathcal{H}} := \underline{\varphi}(\ker(\underline{\chi})) \subset \underline{\mathcal{R}}$ is the desired extension of $\mathcal{H}$, where $\underline{\varphi}\colon \mathcal{O}_X^p \to \underline{\mathcal{R}}$ is defined by $\underline{\varphi}(\underline{e}_i) := \underline{r}_i$ for $i = 1, \ldots, p$. Indeed, by Lemma 4.7, we know that ψ is injective and hence that $\mathcal{H} = \varphi(\ker(\chi))$, where $\chi := \underline{\chi}|_S$. $\square$

Proof of Theorem 4.6. If $m \geq \dim X$, then we have $\mathcal{F} = \mathcal{F}_{[m]\mathcal{G}}$. Now let $m \in \mathbb{N}$ be the largest number with $\mathcal{F} = \mathcal{F}_{[m]\mathcal{G}}$. We proceed by descending induction on m, and we may assume $m < n := \dim X$. Due to the induction hypothesis, we may assume that $\mathcal{F}_{[m+1]\mathcal{G}}$ extends to a coherent sheaf on X. So we may assume that $\mathcal{G} = \mathcal{F}_{[m]\mathcal{G}}$. Then, due to Corollary 4.5, we know that $S := \operatorname{Supp}(\mathcal{G}/\mathcal{F})$ has pure dimension $m + 1$. Because of $m \geq n$ and property (A_n), the closed analytic subset S extends to a closed analytic subset $\underline{S}$ of X which is of pure dimension $m+1$. Denote by $\underline{S}_1,\dots,\underline{S}_r$ the irreducible components of $\underline{S}$. Then $S_i := \underline{S}_i \cap G$ for $i=1,\dots,r$ are the irreducible components of S. Now we set

$$\mathcal{F}^i := \mathcal{F}\Big[\bigcup_{j\neq i} S_j\Big]_G, \quad \text{and hence} \quad \mathcal{F} = \bigcap_{i=1}^{r} \mathcal{F}^i$$

by Proposition 4.4. So it suffices to show that the subsheaves $\mathcal{F}^1,\dots,\mathcal{F}^r$ extend to X. Thus we may assume that $S = \mathrm{Supp}(\mathcal{G}/\mathcal{F})$ is irreducible. Now let $\mathcal{I} \subset \mathcal{O}_X$ be the reduced sheaf of ideals associated to $\underline{S}$. There exists an integer $\ell \in \mathbb{N}$ with $\mathcal{I}^\ell \cdot \mathcal{G} \subset \mathcal{F}$. Then we consider the quotients

$$\mathcal{G} = (\mathcal{F} : \mathcal{I}^\ell) \supset \cdots \supset \mathcal{F}_t := (\mathcal{F} : \mathcal{I}^t) \supset \cdots (\mathcal{F} : \mathcal{I}) \supset \cdots \supset \mathcal{F}.$$

Due to [31, Prop. 4, p. I-13], it holds $(\mathcal{F}_t)_{[m]\mathcal{G}} = \mathcal{F}_t$. The successive quotients $\mathcal{F}_{t+1}/\mathcal{F}_t$ are $\mathcal{O}_S$-modules which are free of torsion. By Lemma 4.8 and descending induction, we see that each $\mathcal{F}_t$ extends to a coherent subsheaf $\underline{\mathcal{F}}_t \subset \underline{\mathcal{F}}_{t+1} \subset \underline{\mathcal{F}}_\ell = \underline{\mathcal{G}}$.

The uniqueness follows by property (E_n). Indeed, consider two extensions $\underline{\mathcal{F}}$ and $\underline{H}$ of $\mathcal{F}$. Then consider the subsheaf $\mathcal{R} := (\mathcal{H}+\mathcal{F})/\mathcal{H}$. Since $\underline{\mathcal{H}} = \underline{\mathcal{H}}_{[n-1]}$, the support $\mathrm{Supp}(\underline{\mathcal{R}})$ is at least of dimension n or empty. So, by property (E_n), we have $\underline{\mathcal{F}} \subset \underline{H}$ and, by symmetry, $\underline{\mathcal{H}} \subset \underline{\mathcal{F}}$ as well. The assertion concerning ball figures follows from Proposition 2.5, Proposition 3.3 and Proposition 1.12. The ones for Hartogs figures follow from Theorem 2.11 and Theorem 3.6. □

5. Invertible sheaves

In the following, let A be an affinoid domain, $Y := \mathrm{Sp}(A)$ the associated affinoid space and $V \subset Y$ a nonempty open subset of Y. At first, we recall two well-known facts; cp. [24, Prop. 1.3.4]

Lemma 5.1. *The following statements hold.*
(a) *A function $f = \sum_{i=0}^{\infty} f_i \cdot \eta^i \in A\langle\eta\rangle$ has no zeros if and only if $f_0 \in A^\times$ is unit and $|f_i \cdot f_0^{-1}| < 1$ for all $i \geq 1$.*
(b) *A function $f = \sum_{i=-\infty}^{\infty} f_i \cdot \eta^i \in A\langle\eta^{\pm 1}\rangle$ has no zeros if and only if there exists an $m \in \mathbb{Z}$ such that $f_m \in A^\times$ is a unit and $|f_i \cdot f_m^{-1}| < 1$ for all $i \neq m$.*

Lemma 5.2. *Any invertible function $h \in A\langle\eta^{\pm1}\rangle^\times$ can be written as a product $h = \eta^m \cdot h^+ \cdot h^-$, where $h^+ \in A\langle\eta\rangle^\times$ and $h^- \in A\langle\eta^{-1}\rangle^\times$ are units.*

Lemma 5.3. *If an invertible function*

$$e = \sum_{i\leq 0, j\leq 0} e_{i,j} \cdot \eta_1^i \cdot \eta_2^j \in A\langle\eta_1^{-1}, \eta_2^{-1}\rangle^\times$$

has a representation $e = f_1 \cdot f_2$ in $A\langle\eta_1^{\pm1}, \eta_2^{\pm1}\rangle^\times$ by units $f_1 \in A\langle\eta_1^{\pm1}, \eta_2\rangle^\times$ and $f_2 \in A\langle\eta_1, \eta_2^{\pm1}\rangle^\times$, then e can be presented in the form $e = g_1^- \cdot g_2^-$ with units $g_i^- \in A\langle\eta_i^{-1}\rangle^\times$ for $i = 1,2$. Furthermore, the coefficients $e_{i,j}$ of e satisfy $e_{i,j} = e_{i,0} \cdot e_{0,0}^{-1} \cdot e_{0,j}$ for all i,j.

Proof. Due to Lemma 5.2, the unit $e = f_1 \cdot f_2$ can be decomposed into a product

$$e = \eta_1^m \cdot \eta_2^n \cdot f^{++} \cdot f^{+-} \cdot f^{-+} \quad \text{with units } f^{**} \in A\langle \eta_1^*, \eta_2^* \rangle^\times.$$

Since the coefficient $e_{0,0}$ dominates the other coefficients, we have $m = n = 0$. Then we obtain

$$f^{++} \cdot f^{+-} = e \cdot (f^{-+})^{-1} \in A\langle \eta_1, \eta_2^{\pm 1} \rangle \cap A\langle \eta_1^{-1}, \eta_2^{\pm 1} \rangle^\times = A\langle \eta_2^{\pm 1} \rangle.$$

Again by Lemma 5.2, we have that $f^{++} \cdot f^{+-} = g_2^+ \cdot g_2^-$ with $g_2^* \in A\langle \eta_2^{*1} \rangle$. So it follows

$$g_1^- := f^{-+} \cdot g_2^+ = e \cdot (g_2^-)^{-1} \in A\langle \eta_1^{-1}, \eta_2 \rangle \cap A\langle \eta_1^{-1}, \eta_2^{-1} \rangle = A\langle \eta_1^{-1} \rangle.$$

Thus we obtain $e = g_1^- \cdot g_2^-$. The formula for the coefficients follows by computing the coefficients of $e = g_1^- \cdot g_2^-$. □

Theorem 5.4. *Let $Y = \mathrm{Sp}(A)$ be an irreducible and reduced affinoid space. Let $\mathcal{L}$ be a line bundle on $X := Y \times \partial\mathbb{D}^2$. Assume that $\mathcal{L}|_{X_i}$ is free on $X_i := \{x = (y, z_1, z_2) \in X; |z_i| = 1\}$ for $i = 1, 2$. If there is an open nonempty subset $V \subset Y$ such that $\mathcal{L}|_{V \times \partial\mathbb{D}^2}$ is trivial, then $\mathcal{L}$ is free.*

Proof. We may assume that $V = \mathrm{Sp}(B)$ is irreducible. Since $\mathcal{L}|_{X_i}$ is trivial, $\mathcal{L}$ is presented by a unit $e \in B\langle \eta_1^{\pm 1}, \eta_2^{\pm 1} \rangle$. Due to Lemma 5.2, the unit e can be written in the form

$$e = \eta_1^m \cdot \eta_2^n \cdot e^{++} \cdot e^{+-} \cdot e^{-+} \cdot e^{--}$$

with units $e^{**} \in A\langle \eta_1^*, \eta_2^* \rangle^\times$. Now we transform the basis of $\Gamma(X_1, \mathcal{L})$ by the units $\eta_1^m \cdot e^{++} \cdot e^{-+}$ and the basis of $\Gamma(X_2, \mathcal{L})$ by the unit $\eta_2^n \cdot e^{+-}$. Thus we see that $\mathcal{L}$ can be represented by

$$e = e^{--} = \sum_{i \le 0, j \le 0} e_{i,j} \cdot \eta_1^i \cdot \eta_2^j \in A\langle \eta_1^{-1}, \eta_2^{-1} \rangle^\times.$$

Since $\mathcal{L}$ is trivial over $V \times \mathbb{D}^2$, the formula of Lemma 5.3 can be applied. So we have $e_{i,j} = e_{i,0} \cdot e_{0,0}^{-1} \cdot e_{0,j}$ for all i, j in the ring B. By the identity principle, this equation holds in A already. Therefore, e decomposes into a product $e = g_1^- \cdot g_2^-$ with $g_i^- \in A\langle \eta_i^- \rangle$ for $i = 1, 2$. □

Corollary 5.5. *There are line bundles on $\partial\mathbb{D}^2$ which are not extendable onto $\mathbb{D}^2$ even as a coherent sheaf.*

Proof. Let $e \in K\langle \eta_1^{-1}, \eta_2^{-1} \rangle^\times$ be a unit such that its coefficients do not satisfy the rule of Lemma 5.3. Such a unit defines a line bundle on $\partial\mathbb{D}^2$. If $\mathcal{L}$ would be extendable as a coherent sheaf $\underline{\mathcal{L}}$ on $\mathbb{D}^2$, then its bi-dual $\mathcal{L}^{**}$ would be an extension of $\mathcal{L}$ as well. But in dimension 2, any reflexive coherent sheaf is locally free. Therefore, $\mathcal{L}$ would be extendable as a line bundle and hence as the trivial line bundle. Thus the coefficients $e_{i,j}$ would have to satisfy the rule of Lemma 5.3. Contradiction! There even exist units e such that the associated line bundle has only the trivial global section; for example, set $e := 1 + \sum_{\nu=1}^{\infty} c^\nu \eta_1^{-\nu} \eta_2^{-\nu}$ for some $c \in K^\times$ with $|c| < 1$. □

Remark 5.6. Any line bundle on $\mathbb{D}^2 - \{0\}$ is free.

Proof. Any line bundle on $\mathbb{D}^2 - \{0\}$ is associated to an invertible function $e \in \Gamma(\mathbb{D}^2 - V(\eta_1\eta_2), \mathcal{O}_{\mathbb{D}^2})$. Now it easily follows from Lemma 5.1 that there exist integers $m, n \in \mathbb{N}$ such that $e \cdot \eta_1^m \cdot \eta_2^n$ is an invertible function on $\mathbb{D}^2$. $\square$

Remark 5.7. There exist vector bundles of rank 2 on $\mathbb{D}^2 - \{0\}$ which are not extendable to $\mathbb{D}^2$ as a coherent sheaf.

Proof. Let $c_i \in K^\times$ be constants such that $e := \sum_{i=1}^{-\infty} c_i \cdot \eta_1^i \cdot \eta_2^i$ is holomorphic on $\mathbb{D}^2 - V(\eta_1 \cdot \eta_2)$. Let $\mathcal{F}$ be the vector bundle on $\mathbb{D}^2 - \{0\}$ given by the matrix $\left(\begin{smallmatrix} 1 & e \\ 0 & 1 \end{smallmatrix}\right)$. Now, if $\mathcal{F}$ would be extendable to a coherent sheaf on $\mathbb{D}^2$, passing to the reflexive closure which is locally free over a regular ring of dimension 2, we see that $\mathcal{F}$ would be extendable to a vector bundle on $\mathbb{D}^2$. So there would exist sections $s_i := (a_i, b_i) \in \Gamma(\mathbb{D}^2 - \{0\}, \mathcal{O}_{\mathbb{D}^2})$ for $i = 1, 2$ generating $\mathcal{F}$; *i.e.*, $a_i, b_i \in \Gamma(\mathbb{D}^2 - V(\eta_1), \mathcal{O}_{\mathbb{D}^2})$ such that the sections $a_i - e \cdot b_i$ and b_i belong to $\Gamma(\mathbb{D}^2 - V(\eta_2), \mathcal{O}_{\mathbb{D}^2})$. In particular, the section b_i necessarily belongs to $\Gamma(\mathbb{D}^2, \mathcal{O}_{\mathbb{D}^2})$ and the locus $V(b_1, \ldots, b_n)$ must be contained in $V(\eta_1)$. Then there exist functions h_1, h_2 in $\Gamma(\mathbb{D}^2, \mathcal{O}_{\mathbb{D}^2})$ such that a power η_1 has a representation

$$\eta_1^r = h_1 \cdot b_1 + h_2 \cdot b_2.$$

Then we obtain

$$s = h_1 \cdot s_1 + h_2 \cdot s_2 = (a, \eta_1^r) \in \Gamma(\mathbb{D}^2 - \{0\}, \mathcal{F}).$$

This implies $a - e \cdot \eta_1^r \in \Gamma(\mathbb{D}^2 - V(\eta_2), \mathcal{O}_{\mathbb{D}^2})$. The latter is impossible since a belongs to $\Gamma(\mathbb{D}^2 - V(\eta_1), \mathcal{O}_{\mathbb{D}^2})$ and the term $e \cdot \eta_1^r$ has infinitely many non-vanishing terms $c_i \cdot \eta_1^{i+r} \cdot \eta_2^i$ with $i \leq 0$. $\square$

Remark 5.8. Theorem 5.4 reproves a special case of the already known result Theorem 3.6: any hypersurface $S \subset (\mathbb{D}^n \times \partial\mathbb{D}^2) \cup (U \times \mathbb{D}^2)$ of the Hartogs figure $(\mathbb{D}^n \times \partial\mathbb{D}^2) \cup (U \times \mathbb{D}^2)$ extends to a hypersurface $\underline{S} \subset \mathbb{D}^{n+2}$, and the extension is unique.

Using the result [27, Satz 2] that any line bundle on $X \times \mathbb{D}^1 \times \partial\mathbb{D}^1$ is locally free over X, we obtain the more general result from Theorem 5.4, first shown in [26].

Corollary 5.9. *Let X be an irreducible affinoid space and $U \subset X$ a nonempty open subset. Consider the Hartogs figure $H := (X \times \partial\mathbb{D}^2) \cup (U \times \mathbb{D}^2)$. Then any hypersurface $S \subset H$ extends to a closed hypersurface $\underline{S} \subset X \times \mathbb{D}^2$ uniquely.*

Proof. We may assume that X is reduced. So X is regular outside a closed subset A of codimension 1. Due to Proposition 3.3, it suffices to show that S extends to $(X - A) \times \mathbb{D}^2$. Note that $X - A$ meets U. Since any regular ring is factorial, the local rings of $(X - A) \times \mathbb{D}^2$ are factorial, and hence the sheaf of ideals associated to S is a divisor ideal. Thus we can apply Theorem 5.4, and hence we see that S extends to a closed analytic subset of $(X - A) \times \mathbb{D}^2$. $\square$

6. Cohomology of ball figures

In this section, we will provide some technical tools which are necessary to treat the extension of coherent sheaves. We will study local cohomology of coherent sheaves, absolute gap sheaves and the torsion of the cohomology of ball figures.

Local cohomology. In the following, let X be a rigid space, mostly an affinoid space, and let $Y \subset X$ be a closed analytic subset, and set $U := X - Y$; we denote by $j\colon Y \hookrightarrow X$ its embedding. As in [18], one defines local cohomology groups $H^q_Y(X, \mathcal{G})$ with support in Y for $n \in \mathbb{N}$ on the category of coherent sheaves; *i.e.*, this is the derived functor of $\Gamma_Y(\mathcal{G}) := \ker(\mathcal{G}(X) \to \mathcal{G}(U))$. In the same way, one introduces local cohomology sheaves $\mathcal{H}^q_Y(\mathcal{G})$ resp. the direct images $R^q j_*(\mathcal{G})$ for $q \in \mathbb{N}$. In the complex analytic case, the following result of Frenkel [1, p. 218] or [32, Lem. 3.2] is essential which can be proved in rigid geometry by the same arguments.

Proposition 6.1. *Let $X = \mathrm{Sp}(A)$ be an affinoid space assumed to be connected and $U = \mathrm{Sp}(B) \subset X$ a nonempty open affinoid subdomain. Consider the Hartogs figure $H := (U \times \mathbb{D}^d) \cup (X \times \partial\mathbb{D}^d)$ with $d \geq 1$. Then we have that*
(a) *$H^0(X \times \mathbb{D}^d, \mathcal{O}_{X\times\mathbb{D}^d}) \xrightarrow{\sim} H^0(H, \mathcal{O}_{X\times\mathbb{D}^d})$ is bijective and*
(b) *the cohomology groups $H^q(H, \mathcal{O}_{X\times\mathbb{D}^d}) = 0$ vanish for all $1 \leq q < d$.*
These results hold for locally free $\mathcal{O}_{X\times\mathbb{D}^d}$-modules as well.

Proof. (a) The assertion follows from Proposition 1.9.

(b) In principle, the assertion is shown by splitting the Laurent expansions. Indeed, let $\eta_1, \dots, \eta_d$ be the coordinates on $\mathbb{D}^d$. Let $\mathfrak{V} := \{V_0, \dots, V_d\}$ with

$$V_0 := U \times \mathbb{D}^d \quad \text{and} \quad V_i := \{z \in X \times \partial\mathbb{D}^d;\, |\eta_i(z)| = 1\} \quad \text{for } i = 1, \dots, d.$$

For $q = 1, \dots, d-1$ and $i_0 < \dots < i_q$, set $V_{i_0,\dots,i_q} := V_{i_0} \cap \dots \cap V_{i_q}$. For a function f on $V_{i_0,\dots,i_q}$ with Laurent expansion $f = \sum_{\nu\in\mathbb{Z}} b_\nu \eta_j^\nu$ for $j \geq 1$, set

$$e_j(f) = \sum_{\nu\in\mathbb{N}} b_\nu \eta_j^\nu.$$

For $\xi = (\xi_{i_0,\dots,i_q}) \in C^q(\mathfrak{V}, \mathcal{O})$, we have $(1-e_1)\cdot\ldots\cdot(1-e_d)(\xi) = 0$. Indeed, for $j \notin \{i_0, \dots, i_q\}$, it is obvious for the component $\xi_{i_0,\dots,i_q}$ and hence for ξ if $1 \leq q < d-1$. For $(i_1, \dots, i_q) = (1, \dots, d)$ and $\xi \in Z^{d-1}(\mathfrak{V}, \mathcal{O})$, we have

$$0 = \partial^{d-1}(\xi) = \xi_{1,\dots,d} + \sum_{j=1}^{d} (-1)^j \xi_{0,1,\dots,\hat{j},\dots,d} \quad \text{on } V_0 \cap \dots \cap V_d.$$

Thus we see $(1-e_1)\cdot\ldots\cdot(1-e_d)\xi_{1,\dots,d} = 0$. So we obtain the identity

$$\begin{aligned}(2)\qquad \xi_{i_0,\dots,i_q} &= (1-e_2)\dots(1-e_d)e_1\xi_{i_0,\dots,i_q} \\ &\quad + (1-e_3)\dots(1-e_d)e_2\xi_{i_0,\dots,i_q} + \dots + e_d\xi_{i_0,\dots,i_q}.\end{aligned}$$

Now we define the section $\sigma_q\colon Z^q(\mathfrak{V},\mathcal{O}) \to C^{q-1}(\mathfrak{V},\mathcal{O})$ by mapping ξ to $\sigma_q(\xi) = \zeta := (\zeta_{i_0,\dots,i_{q-1}})$, where

$$\zeta_{i_0,\dots,i_{q-1}} := (1-e_2)\dots(1-e_d)e_1\xi_{1,i_0,\dots,i_{q-1}} \\ + (1-e_3)\dots(1-e_d)e_2\xi_{2,i_0,\dots,i_{q-1}} + \dots + e_d\xi_{d,i_0,\dots,i_{q-1}}$$

for every $\xi \in Z^q(\mathfrak{V},\mathcal{O})$ and every $q = 1,\dots,d-1$.

Since $0 = (\partial^q\xi)_{j,i_0,\dots,i_q} = \xi_{i_0,\dots,i_q} + \sum_{\ell=0}^{q}(-1)^{\ell+1}\xi_{j,i_0,\dots,\hat{i_\ell},\dots,i_q}$, we have

$$e_j\xi_{i_0,\dots,i_q} = \sum_{\ell=0}^{q}(-1)^{\ell} e_j\xi_{j,i_0,\dots,\hat{i_\ell},\dots,i_q}.$$

Then, by using (2), we obtain $\partial^{q-1}\zeta = \xi$ for $\xi \in Z^q(\mathfrak{V},\mathcal{O})$ and $1 \le q \le d-1$.

The additional assertion holds since any finitely generated locally free module is a direct summand of a finitely generated free module. $\square$

For a coherent sheaf $\mathcal{F}$, we define the *depth* or *homological codimension*

$$\operatorname{cdh}(\mathcal{F}) := \min\{\operatorname{cdh}(\mathcal{F}_x);\ x \in X\},$$

where $\operatorname{cdh}(M)$ of a finitely generated module M over a local ring is the maximal length of an M-sequence; *i.e.*, $(a_1,\dots,a_p)$ are elements in the maximal ideal such that each a_i is a nonzero divisor on $M/(a_1,\dots,a_{i-1})M$; cp. [31, p. IV-12]. We remark that $\operatorname{cdh}(M) = \operatorname{cdh}(\hat{M})$ for the completion $\hat{M}$ of M; cp. [31, Prop. 8, p. IV-16]. Therefore, if $X = \operatorname{Sp}(A)$ is an affinoid space and if a coherent $\mathcal{O}_X$-module $\mathcal{F}$ is associated to a finitely generated A-module M, then $\operatorname{cdh}(\mathcal{F}_x) = \operatorname{cdh}(M_x)$ coincide, where M_x is the localization of M with respect to a maximal ideal x of A.

Corollary 6.2. *Keep the situation of Proposition 6.1, and assume that the local rings of X are regular. Let $\mathcal{F}$ be a coherent sheaf on $X \times \mathbb{D}^d$. Then the following canonical maps are*

(a) (a.0) $H^0(X \times \mathbb{D}^d, \mathcal{F}) \xrightarrow{\sim} H^0(H, \mathcal{F})$, *bijective for* $0 < \operatorname{cdh}(\mathcal{F}) - \dim X$,
    (a.1) $H^q(H, \mathcal{F}) = 0$ *for* $1 \le q < \operatorname{cdh}(\mathcal{F}) - \dim X$,
(b) (b.0) $H^0(X \times \mathbb{D}^d, \mathcal{F}) \xrightarrow{\sim} H^0(X \times \partial\mathbb{D}^d, \mathcal{F})$, *bijective for* $1 < \operatorname{cdh}(\mathcal{F}) - \dim X$,
    (b.1) $H^q(X \times \partial\mathbb{D}^d, \mathcal{F}) = 0$ *for* $1 \le q < \operatorname{cdh}(\mathcal{F}) - \dim X - 1$.

Proof. (a) Since the local rings of $X \times \mathbb{D}^d$ are regular, too, we have the following formula for the homological dimension [31, Prop. 21, p. IV-36]:

$$\operatorname{dh}(\mathcal{F}_z) = (\dim X + d) - \operatorname{cdh}(\mathcal{F}_z) \quad \text{for all } z \in X \times \mathbb{D}^d.$$

We choose short exact sequences

$$0 \to \mathcal{Q}_i \to \mathcal{O}_{X\times\mathbb{D}^d}^{t_i} \to \mathcal{Q}_{i-1} \to 0$$

for $i = 0,\dots,r$ with $r := (\dim X + d) - \operatorname{cdh}(\mathcal{F})$ starting with $\mathcal{Q}_0 := \mathcal{F}$. Then the sheaf $\mathcal{Q}_r$ is locally free. Due to Proposition 6.1, we know that the canonical morphism

$$H^q(H,\mathcal{F}) \xrightarrow{\sim} H^{q+r}(H,\mathcal{Q}_r) = 0 \quad \text{for all } 1 \le q < d - r = \operatorname{cdh}(\mathcal{F}) - \dim X$$

is bijective. The case $q = 0$ follows by Proposition 1.9 since $H^1(H, \mathcal{Q}_1) = H^r(H, \mathcal{Q}_r) = 0$. The latter is true due to Proposition 6.1 since $r = \dim X - \operatorname{cdh}(\mathcal{F}) + d < d$.

Part (b) follows from (a) because $X \times \partial\mathbb{D}^d$ is a Hartogs figure of dimension $\dim X + 1$. □

Corollary 6.3. *Let $\mathcal{F}$ be a coherent sheaf on an affinoid space X of pure dimension $n + d$.*
(a) *Let $B \subset X$ be a ball figure of dimension n. Then*
  (a.0) *the restriction $H^0(X, \mathcal{F}) \xrightarrow{\sim} H^0(B, \mathcal{F})$ is bijective for $1 < \operatorname{cdh}(\mathcal{F}) - n$,*
  (a.1) *$H^q(B, \mathcal{F}) = 0$ for $1 \le q < \operatorname{cdh}(\mathcal{F}) - \dim X - 1$.*
(b) *Let $Y \subset X$ be a closed analytic subspace with $\dim Y < \dim X$. Then*
  (b.0) *$H^0(X, \mathcal{F}) \xrightarrow{\sim} H^0(X - Y, \mathcal{F})$ is bijective if $1 < \operatorname{cdh}(\mathcal{F}) - \dim Y$,*
  (b.1) *$H^q(X - Y, \mathcal{F}) = 0$ for $1 \le q \le \operatorname{cdh}(\mathcal{F}) - \dim Y - 1$.*
(c) *The cohomology sheaves $\mathcal{H}^q_Y \mathcal{F} = 0$ vanish for $0 \le q < \operatorname{cdh}(\mathcal{F}) - \dim Y$.*

Proof. (a) Let $B := X_{\underline{f},\underline{\varepsilon}}$ be a ball figure with $\underline{f} = (f_1, \ldots, f_s)$. Due to Lemma 1.5, there is a finite map $\phi \colon X \to \mathbb{D}^{n+d}$ with $\phi^{-1}(\mathbb{D}^n \times \partial\mathbb{D}^d) \subset B$. Then the map

$$\psi := (\phi, \underline{f}) \colon X \to \mathbb{D}^{n+d+s}$$

is finite as well, and we have $\psi^{-1}(B') \subset B$, where

$$B' := (\mathbb{D}^n \times \partial\mathbb{D}^d \times \mathbb{D}^s) \cup (\mathbb{D}^{n+d} \times \partial\mathbb{D}^s(\underline{\varepsilon}, 1)).$$

Then we apply Corollary 6.2 (b) to $\psi_*\mathcal{F}$ and B', where $\partial\mathbb{D}^s(\underline{\varepsilon}, 1)$ denotes $\mathbb{D}^s(1)$ minus the open disc $\mathbb{D}^s_+(\underline{\varepsilon})$. Note that Proposition 6.1 remains true if $\partial\mathbb{D}^s(1)$ is replaced by $\partial\mathbb{D}^s(\underline{\varepsilon}, 1)$.

Assertion (b) follows in a similar way by replacing $\partial\mathbb{D}^s(\underline{\varepsilon}, 1)$ by $\mathbb{D}^s - \{0\}$ if $\underline{f}$ is chosen such that $Y = V(\underline{f})$ and $|f_\sigma| \le 1$.

Part (c) follows from (b) since $H^q(X, \mathcal{F}) = 0$ for all $q \ge 1$ and there are no nonzero sections of $\mathcal{F}$ with support contained in Y for $q = 0$ if we have $\operatorname{cdh}(\mathcal{F}) > \dim Y$. □

For a coherent $\mathcal{O}_X$-module $\mathcal{F}$, we define

$$\mathrm{S}_m(\mathcal{F}) := \{x \in X;\ \operatorname{cdh}(\mathcal{F}_x) \le m\}.$$

If $X = \operatorname{Sp}(A)$ and $\mathcal{F}$ is associated to an A-module M; *i.e.*, $\mathcal{F} = \tilde{M}$, then we have

$$\mathrm{S}_m(\mathcal{F}) = \{x \in X;\ \operatorname{cdh}(M_x) \le m\},$$

where M_x is the localization of M with respect to the maximal ideal $x \subset A$. Thus we know that $\mathrm{S}_m(\mathcal{F})$ is a closed analytic subset of X since the subset, where a sequence of elements is an M-sequence, is open. Moreover, we have that $\dim \mathrm{S}_m(\mathcal{F}) \le m$. Indeed, we may assume that $A = T_n$ is regular. Then consider a resolution with finitely generated free A-modules L_i,

$$0 \to K \to L_{n-m-2} \to \cdots \to L_0 \to M \to 0.$$

Consider now a prime ideal $\mathfrak{p} \in \operatorname{Spec}(S)$ with $\dim A/\mathfrak{p} = m+1$. For the localization of A with respect to $\mathfrak{p}$, we have $\dim A_{\mathfrak{p}} = n-(m+1)$. So the localization $K_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module since $A_{\mathfrak{p}}$ is a regular ring. Then there exists a maximal ideal $x \in X$ with $x \in V(\mathfrak{p})$ such that K_x is a free A_x-module. Then it follows by [31, Prop. 21, p. IV-35] that

$$\operatorname{cdh}(M_x) \geq n-(n-(m+1)) = m+1.$$

In particular, x does not belong to $\mathrm{S}_m(\mathcal{F})$. Thus every irreducible component of $\mathrm{S}_m(\mathcal{F})$ has dimension at most m.

Due to the vanishing result Corollary 6.3, all theorems about local cohomology in algebraic geometry remain true in rigid geometry; cp. [18, Exps. II and III].

Proposition 6.4. *Let $\mathcal{F}$ be a coherent sheaf on an affinoid space X and $Y \subset X$ a closed analytic subset. Denote by $\iota\colon X-Y \to X$ the inclusion map. Then we have the following:*
(a) *$\mathcal{H}^q_Y\mathcal{F}$ is coherent for all q with $0 \leq q < \operatorname{cdh}(\mathcal{F}|_{X-Y}) - \dim Y$;*
(b) *$R^q\iota_*(\mathcal{F})$ is coherent for all q with $0 \leq q < \operatorname{cdh}(\mathcal{F}|_{X-Y}) - \dim Y - 1$.*

Proof. At first, we remark that $\mathcal{H}^0_Y\mathcal{F} = 0[Y]_{\mathcal{F}}$ is always coherent due to Proposition 4.4.

(a) We may assume that $X = \mathbb{D}^n$. Let us first consider the case where $\mathcal{F}$ is locally free on $X-Y$. In this case, we consider the canonical map $\mathcal{F} \to \mathcal{F}^{**}$ from $\mathcal{F}$ to its bi-dual $\mathcal{F}^{**}$; this morphism is bijective on $X-Y$. Then we have an exact sequence

$$0 \to \mathcal{K} \to \mathcal{F} \overset{\alpha}{\to} \mathcal{F}^{**} \to \mathcal{C} \to 0.$$

The sheaves $\mathcal{K} = \mathcal{H}^0_Y\mathcal{F}$ and the cokernel $\mathcal{C}$ of α are coherent as well, and their support is contained in Y. Therefore, we have $\mathcal{H}^q_Y\mathcal{K} = 0$ and $\mathcal{H}^q_Y\mathcal{C} = 0$ for all $q \geq 1$.

Now we consider $q \geq 1$. So we may assume $\dim X - \dim Y \geq 2$. The canonical map $\mathcal{F}^{**} \overset{\sim}{\to} \iota_*\mathcal{F}^{**}$ is bijective. Thus we obtain $\mathcal{H}^q_Y\mathcal{F} = \mathcal{H}^q_Y\mathcal{F}^{**}$ for all $q \geq 1$. So it suffices to show the coherence of $\mathcal{H}^q_Y(\mathcal{F}^{**})$. Now look at an exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}^t_X \to \mathcal{F}^* \to 0.$$

This sequence gives rise to an exact sequence

$$0 \to \mathcal{F}^{**} \to \mathcal{O}^t_X \to \mathcal{Q}_1 \to 0.$$

The sheaf $\mathcal{Q}_1$ is locally free on $X-Y$. Due to Corollary 6.3, the coherence of $\mathcal{H}^q_Y(\mathcal{F}^{**})$ is equivalent to the coherence of $\mathcal{H}^{q-1}_Y\mathcal{Q}_1$ for $1 \leq q < n - \dim Y$. This shows the coherence of $\mathcal{H}^1_Y\mathcal{F}$. Next we apply the same process to $\mathcal{Q}_1$. So we obtain the coherence of $\mathcal{H}^2_Y\mathcal{F}$ and so on.

For the general case, we proceed by descending induction on $\operatorname{cdh}(\mathcal{F}|_{X-Y})$. The beginning at $n = \operatorname{cdh}(\mathcal{F}|_{X-Y})$ was done above. For the induction step, consider an exact sequence

$$0 \to \mathcal{Q} \to \mathcal{L} \to \mathcal{F} \to 0,$$

where $\mathcal{L}$ is locally free on X. Then $\operatorname{cdh}(\mathcal{Q}|_{X-Y}) \geq \operatorname{cdh}(\mathcal{F}|_{X-Y}) + 1$. Due to Corollary 6.3 (c), we have $\mathcal{H}^{q-1}_Y\mathcal{Q} = \mathcal{H}^q_Y\mathcal{F}$ for $1 \leq q < n - \dim Y$. By the induction hypothesis, we obtain the coherence of $\mathcal{H}^{q-1}_Y\mathcal{Q}$ for $1 \leq q < \operatorname{cdh}(\mathcal{Q}|_{X-Y}) - \dim Y$ and hence the coherence of $\mathcal{H}^q_Y\mathcal{F}$ for $0 \leq q < \operatorname{cdh}(\mathcal{F}|_{X-Y}) - \dim Y$.

(b) It remains to show the coherence of $R^q\iota_*\mathcal{F}$. Due to [18, Exp. II, Cor. 2.11], we have the exact sequences

$$0 \to \mathcal{H}^0_Y\mathcal{F} \to \mathcal{F} \to \iota_*\mathcal{F} \to \mathcal{H}^1_Y\mathcal{F} \to 0,$$
$$0 \to R^q\iota_*\mathcal{F} \to \mathcal{H}^{q+1}_Y\mathcal{F} \to 0.$$

The coherence of $\mathcal{H}^1_Y\mathcal{F}$ implies the coherence of $\iota_*\mathcal{F}$. For $q \geq 1$, we have the identification $R^q\iota_*\mathcal{F} = \mathcal{H}^{q+1}_Y\mathcal{F}$, and hence the coherence of $R^q\iota_*\mathcal{F}$ follows for $q + 1 < \operatorname{cdh}(\mathcal{F}|_{X-Y}) - \dim Y$. □

Lemma 6.5. *Let $\mathcal{F}$ be a coherent sheaf on* $\operatorname{Sp}(A)$ *associated to a finitely generated A-module M. Then we have* $\dim \mathrm{S}_{k+j}(\mathcal{F}) \leq k$ *for $r \leq k < m$ if and only if* $\operatorname{cdh}(M_\mathfrak{p}) \geq j$ *for all* $\mathfrak{p} \in \operatorname{Spec}(A)$ *with* $r + 1 \leq \dim(A/\mathfrak{p}) \leq m$.

Proof. We may assume that $A = T_n$ is the n-dimensional Tate algebra. Then we consider a resolution with free T_n-modules L_*,

$$0 \to K \to L_{n-k-j-2} \to \cdots \to L_0 \to M \to 0.$$

Since T_n is regular, we obtain, for the homological dimension of the localization of K at x,

$$\operatorname{dh}(K_x) = n - \operatorname{cdh}(M_x) - (n - k - j - 1).$$

If $\mathfrak{p} \in \operatorname{Spec}(A)$ satisfies $r + 1 \leq \dim(T_n/\mathfrak{p}) = k + 1 \leq m$, then there is a point $x \in V(\mathfrak{p}) - \mathrm{S}_{k+j}(\mathcal{F})$. Since K_x is free module, $K_\mathfrak{p}$ is free, too, and hence $\operatorname{cdh}(M_\mathfrak{p}) = n - k - 1 - \operatorname{dh}(M_\mathfrak{p}) \geq j$.

Conversely, if $K_\mathfrak{p}$ is free for a $\mathfrak{p} \in \operatorname{Spec}(T_n)$ with $r + 1 \leq \dim(T_n/\mathfrak{p}) = k + 1 \leq m$, then there exists an $x \in V(\mathfrak{p})$ such that K_x is free. So we have $\operatorname{cdh}(\mathcal{F}_x) \geq k + j + 1$. Thus we see that $\dim \mathrm{S}_{k+j}(\mathcal{F}) \leq k$ for all $r \leq k < m$. □

Definition 6.6. For a coherent sheaf $\mathcal{F}$ on X, we define the m-th *absolute gap sheaf* $\mathcal{F}^{[m]}$ of $\mathcal{F}$ as the sheafification of the presheaf which associates the direct limit $\lim_{\to} \mathcal{F}(U - S)$ to an open subdomain U, where the limit runs over all closed analytic subsets $S \subset X$ of dimension $\dim S \leq m$.

Proposition 6.7. *Let $\mathcal{F} = \tilde{M}$ be a coherent sheaf on an affinoid space X, and let $m \in \mathbb{N}$ be an integer. Then we have the following.*
(a) $\dim \operatorname{Supp}(0_{[m](\mathcal{F})}) \leq m - 1$ *if and only if* $\dim \mathrm{S}_m(\mathcal{F}) \leq m - 1$.
(b) *If* $\dim \mathrm{S}_{m+1}(\mathcal{F}) \leq m$*, then* $\mathcal{F}^{[m]}$ *is coherent.*
(c) $\mathcal{F} = \mathcal{F}^{[m]}$ *is true if and only if* $\dim \mathrm{S}_{k+2}(\mathcal{F}) \leq k$ *holds for all* $k < m$.
(d) *If* $\dim \operatorname{Supp}(0_{[m+2]\mathcal{F}}) \leq m + 1$*, then* $\mathcal{F}^{[m]}$ *is coherent.*
(e) *If* $\mathcal{F} = \mathcal{F}^{[m]}$*, then* $0_{[m+1]\mathcal{F}} = 0$.

Proof. We may assume that $X = \mathbb{D}^n$.

(a) Due to Corollary 4.5, the module M has no associated prime ideals with $\dim \mathfrak{p} \geq m$ if and only if $\dim \operatorname{Supp}(0_{[m]\mathcal{F}}) \leq m - 1$. So the assertion is equivalent to asking that $\mathrm{cdh}(M_{\mathfrak{p}}) \geq 1$ for all prime ideals of dimension m. By Lemma 6.5, the assertion follows.

(b) The assertion follows from Corollary 6.3 and Proposition 6.4 applied to $Y := \mathrm{S}_{m+1}(\mathcal{F})$. Indeed, by Corollary 6.3, $\mathcal{F}^{[m]} = \iota_*\mathcal{F}$, where $\iota\colon (X - Y) \to X$ is the inclusion. By Proposition 6.4, the sheaf $\iota_*\mathcal{F} = \mathcal{F}^{[m]}$ is coherent.

(c) If $\mathcal{F} = \mathcal{F}^{[m]}$, then for any prime ideal $\mathfrak{p} \subset T_n$ with $\dim(T_n/\mathfrak{p}) \leq m$, the canonical map $\Gamma(\mathrm{Spec}(T_n), M^a) \xrightarrow{\sim} \Gamma(\mathrm{Spec}(T_n) - V(\mathfrak{p}), M^a)$ is bijective as well, where M^a is the algebraic sheaf on $\mathrm{Spec}(T_n)$ associated to M. Due to [18, Exp. III, Cor. 3.5], the latter is equivalent to $\mathrm{cdh}(M_{\mathfrak{p}}) \geq 2$ for all $\mathfrak{p} \in \mathrm{Spec}(T_n)$ with $\dim(T_n/\mathfrak{p}) \leq m$. Then the "only if" is clear by Lemma 6.5.

For the converse implication, we have that $\mathcal{G} := \mathcal{F}^{[m]}$ is coherent due to (b) since $\dim S_{m+1}(\mathcal{F}) \leq m$. Then we will show by decreasing induction that the canonical morphism $\mathcal{F} \to \mathcal{G}$ is an isomorphism over $X - S_{k+1}(\mathcal{F})$ for $k = m, \dots, 1$. For $k = m$, we have $\mathrm{cdh}(\mathcal{F}_x) - \dim Y \geq m + 2 - m = 2$ for $x \in X - S_{m+1}\mathcal{F}$ and $\dim Y \leq m$. Then we have $\mathcal{F}(U) = \mathcal{F}(U - Y)$ for any open subdomain U of X and any closed subvariety Y of U of dimension $\dim Y \leq m$ due to Corollary 6.3 (b). Thus we have that $\mathcal{F} \to \mathcal{G}$ is bijective over $X - S_m(\mathcal{F})$. Now we turn to the induction step. So we assume that $\mathcal{F} \to \mathcal{G}$ is bijective over $X - S_{k+1}(\mathcal{F})$. Since $\mathrm{cdh}(\mathcal{F}_x) \geq k + 1$ for $X - S_k(\mathcal{F})$ and $\dim S_{k+1}(\mathcal{F}) \leq k - 1$, we have $\mathrm{cdh}(\mathcal{F}_x) - \dim S_{k+1}(\mathcal{F}) \geq 2$. Then it follows by Corollary 6.3 (b) that $\mathcal{F} \to \mathcal{G}$ is bijective over $X - S_k(\mathcal{F})$. Because of $S_1(\mathcal{F}) = \varnothing$, the assertion follows.

(d) This follows from (a) and (b).

(e) This follows from (a) and (c). □

Extension of sections in coherent sheaves. For the assertion on the uniqueness in (G_n), we need extension properties for morphism between coherent sheaves of type $\mathcal{G} = \mathcal{G}^{[m]}$. A morphism is a section of $\mathcal{H} := \mathcal{H}om(\mathcal{F}, \mathcal{G})$. If $\mathcal{F} = \mathcal{F}^{[m]}$ and $\mathcal{G} = \mathcal{G}^{[m]}$, then it also holds $\mathcal{H} = \mathcal{H}^{[m]}$. Therefore, it suffices to study extension properties of coherent sheaves $\mathcal{G}$ fulfilling $\mathcal{G} = \mathcal{G}^{[m]}$.

Proposition 6.8. *Let X be an affinoid space of pure dimension $n + d$ with $d \geq 2$ and $B \subset X$ a ball figure of dimension n. If $\mathcal{F} = \mathcal{F}^{[n]}$ is a coherent sheaf on X, then the restriction map $\Gamma(X, \mathcal{F}) \to \Gamma(B, \mathcal{F})$ is bijective.*

Proof. Due to Proposition 6.7 (e), the condition $\mathcal{F} = \mathcal{F}^{[n]}$ implies that the support of any nonzero section of $\mathcal{F}$ has dimension at least $n + 2$. So the restriction map $\Gamma(B, \mathcal{F}) \to \Gamma(B', \mathcal{F})$ is injective for any ball figure B' of dimension n by Lemma 1.7. Thus it suffices to show the assertion for a special ball figure $B' \subset B$. Since $\mathcal{F} = \mathcal{F}^{[n]}$, we have $\dim S_{n+1} \leq n - 1$ due to Proposition 6.7 (c). Then there exists a finite map $\phi\colon X \to \mathbb{D}^{n+d}$ by Lemma 1.5 such that

$$B' := \phi^{-1}(\mathbb{D}^n \times \partial\mathbb{D}^d) \subset B \quad \text{and} \quad \mathrm{S}_{n+1}(\mathcal{F}) \cap \phi^{-1}(\mathbb{D}^{n-1} \times \partial\mathbb{D}^{d+1}) = \varnothing.$$

So it remains to show that the restriction map

$$\Gamma(\mathbb{D}^{n+d}, \phi_*\mathcal{F}) \to \Gamma(\mathbb{D}^{n-1} \times \partial\mathbb{D}^d, \phi_*\mathcal{F})$$

is bijective. If $n = 0$, then $\mathrm{S}_1(\mathcal{F}) = \varnothing$, and hence the assertion follows from Corollary 6.2 (b.0). If $n \geq 1$, then it follows from Corollary 6.2 (b.0) that any section of $\Gamma(\mathbb{D}^n \times \partial\mathbb{D}^d, \phi_*\mathcal{F})$ can be extended to a section on $\mathbb{D}^{n-1} \times \partial\mathbb{D}^1 \times \mathbb{D}^d$; *i.e.*, to a section of $\Gamma(\mathbb{D}^{n-1} \times \partial\mathbb{D}^{d+1}, \phi_*\mathcal{F})$. Thus we see that the assertion follows by induction. □

Moreover, we still need a sharper version of Proposition 6.8 in the following situation:

$$\begin{aligned} X &:= \mathbb{D}^{n-1} \times A(\varepsilon, 1) \supset U = \mathbb{D}^{n-1} \times \partial\mathbb{D}^1, \\ A(\varepsilon, 1) &:= \{z \in \mathbb{D}^1;\, \varepsilon \leq |z| \leq 1\} \quad \text{for some } \varepsilon \in |K^\times|, \\ H &:= (U \times \mathbb{D}^d) \cup (X \times \partial\mathbb{D}^d). \end{aligned}$$

Proposition 6.9. *Keep the situation introduced above. Let $\mathcal{F} = \mathcal{F}^{[n-1]}$ be a coherent sheaf on $X \times \mathbb{D}^d$; then the restriction map $\Gamma(X \times \mathbb{D}^d, \mathcal{F}) \tilde{\to} \Gamma(H, \mathcal{F})$ is bijective.*

Proof. Since $\dim \mathrm{S}_n(\mathcal{F}) \leq n-2$ due to Proposition 6.7 (c), there exists an $h \in T_{n-1}$ with $|h| = 1$ such that the reduction $\widetilde{\mathrm{S}_n(\mathcal{F})}$ of $\mathrm{S}_n(\mathcal{F})$ is contained in the locus of $\tilde{h}$. By Corollary 6.2 (a.0), we have that the restriction map $\Gamma(X_{\tilde{h}} \times \mathbb{D}^d, \mathcal{F}) \tilde{\to} \Gamma(H_{\tilde{h}}, \mathcal{F})$ is bijective. Since $(X_{\tilde{h}} \times \mathbb{D}^d) \cup (X \times \partial\mathbb{D}^d)$ is a ball figure of dimension $(n-1)$, the assertion follows from Proposition 6.8. □

Proposition 6.10. *Let $X = \mathrm{Sp}(A)$ be an affinoid space of dimension n whose local rings are Cohen–Macaulay. Assume that X is irreducible, and let $U \subset X$ be a nonempty open subdomain. Let $\tilde{g} \in \tilde{A}[\eta]$ be a monic polynomial, and let $\mathcal{F} = \mathcal{F}^{[n-1]}$ be a coherent sheaf on $X \times \mathbb{D}^1$. Assume that $\mathcal{F}$ is locally free on $(X \times \mathbb{D}^1)_{\tilde{g}}$. Then the restriction morphism*

$$\Gamma(X \times \mathbb{D}^1, \mathcal{F}) \tilde{\to} \Gamma(H, \mathcal{F}) \quad \textit{for } H := (U \times \mathbb{D}^1) \cup (X \times \mathbb{D}^1)_{\tilde{g}}$$

is bijective.

Proof. Let $g \in A[\eta]$ be a monic lifting of $\tilde{g}$. Then g gives rise to a finite morphism $\phi\colon X \times \mathbb{D}^1 \to X \times \mathbb{D}^1$ such that $\phi^{-1}(X \times \partial\mathbb{D}^1) = (X \times \mathbb{D}^1)_{\tilde{g}}$. Because of $\dim \mathrm{S}_n(\mathcal{F}) \leq n-2$ due to Proposition 6.7 (c), its image $S \subset X$ under the projection to X is a closed subset of dimension $n-2$ in X. Namely, $\mathrm{S}_n(\mathcal{F})$ reduces into $V(\tilde{g})$, and hence the projection is finite when restricted to $\mathrm{S}_n(\mathcal{F})$. So there exists a nonzero element $a \in A$ such that $\mathrm{S}_n(\mathcal{F}) \subset V(a)$. Due to Proposition 6.8, we may assume $a = 1$. Then $\mathcal{F}$ is associated to a locally free module P. Then there exists a direct summand Q such that $P \oplus Q$ is a free $A[\eta]$-module. Now the assertion follows by the extension property of holomorphic functions; cp. Lemma 2.3. □

Proposition 6.11. *Let X be an affinoid space of pure dimension $n + d$ with $d \geq 2$, and let $B' \subset B := X_{\underline{f},\underline{\varepsilon}}$ be ball figures of dimension n. If $\mathcal{F} = \mathcal{F}^{[n]}$ is a coherent sheaf on X, then the restriction map $\Gamma(B, \mathcal{F}) \tilde{\to} \Gamma(B', \mathcal{F})$ is bijective.*

Proof. The restriction map is injective; cp. Lemma 1.7. By Lemma 1.5, there exists a finite morphism $\phi\colon X \to \mathbb{D}^{n+d}$ with $\phi^{-1}(\mathbb{D}^n \times \partial\mathbb{D}^d) \subset B'$. We may assume that $B' = \phi^{-1}(\mathbb{D}^n \times \partial\mathbb{D}^d)$. For any $g \in \mathcal{O}_X(X)$ with $|g| = 1$, the set $B'_{\tilde{g}} \subset X_{\tilde{g}}$ is a ball figure of dimension n. By Proposition 6.8, we have $\Gamma(X_{\tilde{f}_\sigma}, \mathcal{F}) = \Gamma(B_{f_\sigma}, \mathcal{F})$ for any f_σ in $\underline{f}$. So every $z \in \Gamma(B', \mathcal{F})$ has an extension $z'_\sigma \in \Gamma(X_{\tilde{f}_\sigma}, \mathcal{F})$ and hence an extension $z_\sigma \in \Gamma(X_{f_\sigma, \varepsilon_\sigma}, \mathcal{F})$. Since the differences $(z_\sigma - z_\tau)$ vanish on $X_{\tilde{f}_\sigma \tilde{f}_\tau}$, they vanish also on $X_{f_\sigma,\varepsilon_\sigma} \cap X_{f_\tau,\varepsilon_\tau}$. Note that the support of $(z_\sigma - z_\tau)$ is at least of dimension $(n+2)$. Thus these sections fit together to build a section of $\Gamma(B, \mathcal{F})$ extending z. □

Corollary 6.12. *Let $B' \subset B$ be ball figures of dimension n on an affinoid space X. If $\mathcal{F} = \mathcal{F}^{[n]}$ and $\mathcal{G} = \mathcal{G}^{[n]}$ are coherent sheaves on B, then any morphism resp. isomorphism $\psi'\colon \mathcal{F}|_{B'} \to \mathcal{G}|_{B'}$ extends to a morphism resp. isomorphism $\psi\colon \mathcal{F} \to \mathcal{G}$ over B.*

Proof. Put $\mathcal{H} := \mathcal{H}om(\mathcal{F}, \mathcal{G})$. Then $\mathcal{H}$ satisfies $\mathcal{H} = \mathcal{H}^{[n]}$. Thus the assertion follows from Proposition 6.11. □

Corollary 6.13. *Keep the situation of Corollary 6.12. Let $\mathcal{F} = \mathcal{F}^{[n]}$ be a coherent sheaf on B, and let $\underline{\mathcal{G}}$ be a coherent sheaf on X. If $\psi'\colon \mathcal{F}|_{B'} \xrightarrow{\sim} \mathcal{G}|_{B'}$ is an isomorphism, then $\underline{\mathcal{G}}^{[n]}$ is coherent and the isomorphism ψ' extends to an isomorphism $\psi\colon \mathcal{F} \xrightarrow{\sim} \underline{\mathcal{G}}^{[n]}|_B$ over B.*

Proof. Since B' is a ball figure of dimension n, it follows by Lemma 1.7 that $\dim \mathrm{S}_{n+1}(\underline{\mathcal{G}}) \leq n$. Now the assertion follows from Proposition 6.7 (b) and Corollary 6.12. □

Torsion of cohomology groups of ball figures. The study of the torsion of the cohomology groups will be an important tool to switch from property $(\mathrm{G}(m))$ for all $m \geq n$ to property (G_n) in Section 8.

Lemma 6.14. *Let X be a smooth connected affinoid space of dimension $n+d$, and let $B \subset X$ be a ball figure of dimension n. Let $\mathcal{Q}$ be a coherent sheaf on X such that $\mathcal{Q}|_B$ is locally free. Put $S := \mathrm{S}_{n+d-1}(\mathcal{Q})$ and $\mathfrak{a} := \mathrm{Id}(S) \subset \mathcal{O}_X(X)$ the vanishing ideal of S. Then there exists an integer $k \in \mathbb{N}$ such that $\mathfrak{a}^k \cdot H^q(B, \mathcal{Q}) = 0$ for $1 \leq q \leq d-2$.*

Proof. We may assume that $\mathcal{Q} = \mathcal{Q}^{**}$ is reflexive. Then we obtain exact sequences

$$0 \to \mathcal{K} \to \mathcal{O}_X^t \to \mathcal{Q}^* \to 0 \quad \text{and} \quad 0 \to \mathcal{Q} \to \mathcal{O}_X^t \to \mathcal{R} \to 0,$$

where $\mathcal{R} \subset \mathcal{K}^* := \mathcal{H}om(\mathcal{K}, \mathcal{O}_X)$ is coherent with $\mathrm{Supp}(\mathcal{K}^*/\mathcal{R}) \subset S$. So there exists an integer $k \in \mathbb{N}$ with $\mathfrak{a}^k \cdot (\mathcal{K}^*/\mathcal{R}) = 0$. Since $\mathcal{R}|_B = \mathcal{K}^*|_B$ and $\mathcal{K}^* = (\mathcal{K}^*)^{[n]}$, we have the commutative diagram with exact rows

$$\begin{array}{ccccccc}
\Gamma(X, \mathcal{O}_X^t) & \longrightarrow & \Gamma(X, \mathcal{K}^*) & \longrightarrow & \Gamma(X, \mathcal{K}^*/\mathcal{R}) & \longrightarrow & 0 \\
\| & & \| & & \| & & \\
\Gamma(B, \mathcal{O}_X^t) & \longrightarrow & \Gamma(B, \mathcal{R}) & \longrightarrow & H^1(B, \mathcal{Q}) & \longrightarrow & 0
\end{array}$$

Due to Corollary 6.3 (a.1), the group $H^1(B, \mathcal{O}_X) = 0$ vanishes since $d \geq 3$. Due to Proposition 6.8, the first two vertical arrows are bijective, and hence the third vertical map is bijective. So we have

$$0 = \mathfrak{a}^k \cdot \Gamma(X, \mathcal{K}^*/\mathcal{R}) = \mathfrak{a}^k \cdot H^1(B, \mathcal{Q}).$$

The assertion for $i = 2, \ldots, d-2$ follows by induction and the canonical isomorphism $H^{q-1}(B, \mathcal{R}) = H^q(B, \mathcal{Q})$ since $H^q(B, \mathcal{O}_X^t) = 0$ for $q = 1, \ldots, d-2$ by Corollary 6.3. Indeed, we have the exact sequence

$$H^{q-1}(B, \mathcal{K}^*/\mathcal{R}) \to H^q(B, \mathcal{R}) \to H^q(B, \mathcal{K}^*) \to H^q(B, \mathcal{K}^*/\mathcal{R}).$$

For $q \geq 2$, the first and the last term are killed by some power $\mathfrak{a}^k$, and one can apply the induction hypothesis to $H^q(B, \mathcal{K}^*)$ for $q \leq d-3$. □

For proving the extension property (G_n) for Hartogs figures of dimension n, we need a result which is stronger than Lemma 6.14. Let us fix the situation for the following. Let X be a smooth connected affinoid space, and let $U \subset X$ be a nonempty open subdomain. Set $B := X \times \partial\mathbb{D}^d$ and $B_U := U \times \partial\mathbb{D}^d$ for some $d \in \mathbb{N}$ with $d \geq 2$.

Lemma 6.15. *The group $H^{d-1}(B, \mathcal{O}_{X\times\partial\mathbb{D}^d})$ can be computed directly as in Proposition 6.1 by*

$$H^{d-1}(B, \mathcal{O}_{X\times\partial\mathbb{D}^d}) = \mathcal{O}_X(X)\langle\eta_1^{\pm1}, \ldots, \eta_d^{\pm1}\rangle \Big/ \bigoplus_{j=1}^{d} \mathcal{O}_X(X)\langle\eta_1^{\pm1}, .., \eta_j^{+1}, .., \eta_d^{\pm1}\rangle.$$

Therefore, any $\xi \in H^{d-1}(B, \mathcal{O}_{X\times\partial\mathbb{D}^d})$ is uniquely represented by a Laurent series

$$\xi = \sum_{\nu_1<0,\ldots,\nu_d<0} a_{\nu_1,\ldots,\nu_d} \cdot \eta_1^{\nu_1} \cdot \ldots \cdot \eta_d^{\nu_d} \in \mathcal{O}_X(X)\langle\eta_1^{-1}, \ldots, \eta_d^{-1}\rangle.$$

If the restriction of such a cohomology class onto B_U is annihilated by a nonzero function f_j of $\mathcal{O}_X(U)\langle\eta_j\rangle$, then this implies, for any index $\nu(\,\cdot\,) = (\nu_1, \ldots, \hat{\nu}_j, \ldots, \nu_d)$,

$$f_j \cdot \xi_{\nu(\cdot)} = f_j \cdot \sum_{\mu<0} a_{\nu(\mu)}\eta_j^{\mu} \in O_X(U)\langle\eta_j\rangle, \quad \text{where } \xi_{\nu(\cdot)} := \sum_{\mu<0} a_{\nu(\mu)}\eta_j^{\mu}.$$

Thus we have that each $\xi_{\nu(\cdot)}|_{U\times\mathbb{D}^1}$ is meromorphic on $U \times \mathbb{D}^1$. Since $\xi_{\nu(\cdot)}|_{X\times\partial\mathbb{D}^1}$ is holomorphic, there exists a monic polynomial $p_j \in \mathcal{O}_X(X)[\eta_j]$ satisfying $V(p_j) \cap (X \times \partial\mathbb{D}^1) = \varnothing$ such that $p_j \cdot \xi_{\nu(\cdot)}$ extends to a holomorphic function on $X \times \mathbb{D}^1$; cp. Lemma 2.3.

Lemma 6.16. *If the restriction of a class $\xi \in H^{d-1}(B, \mathcal{O}_{X\times\mathbb{D}^d})$ onto B_U is annihilated by a nonzero function $f_j \in \mathcal{O}_X(U)\langle\eta_j\rangle$ in $H^{d-1}(B_U, \mathcal{O}_{X\times\mathbb{D}^d})$, then there exists a monic polynomial $p_j \in \mathcal{O}_X(X)[\eta_j]$ such that $p_j \cdot \xi = 0$ in $H^{d-1}(B, \mathcal{O}_{X\times\mathbb{D}^d})$ and the intersection $V(p_j) \cap (X \times \partial\mathbb{D}^1) = \varnothing$ is empty.*

Proof. Keep the notations from above. We have seen that all the functions $\xi_{\nu(\cdot)}$ are meromorphic. Since there is a single $f_j \in \mathcal{O}_X(U)\langle\eta_j\rangle$ such that $f_j \cdot \xi_{\nu(\cdot)}$ is meromorphic on $X \times \mathbb{D}^1$ for all indices $\nu(\cdot)$, a single polynomial $p_j \in \mathcal{O}_X(X)[\eta_j]$ will do as well. Namely, p_j must be a divisor of f_j, and there are only finitely many divisors. □

Proposition 6.17. *Let X be a smooth connected affinoid space of dimension n, and let $U \subset X$ be a nonempty open subdomain. Set $B := X \times \partial\mathbb{D}^d$ and $B_U := U \times \partial\mathbb{D}^d$ with $d \geq 2$. Let $\mathcal{G} = \mathcal{G}^{[n]}$ be a coherent sheaf on $X \times \mathbb{D}^d$ with $\operatorname{cdh}(\mathcal{G}|_B) \geq n+2$. Let $\mathfrak{a} := \operatorname{Id}(\mathrm{S}_{n+1}(\mathcal{G}))$ be the vanishing ideal of $\mathrm{S}_{n+1}(\mathcal{G})$. Then there exists an integer $k \in \mathbb{N}$ with the following property.*

If the restriction of a cohomology class $\xi \in H^1(B, \mathcal{G})$ onto B_U is annihilated by a nonzero function $f_j \in \mathcal{O}(U)\langle\eta_j\rangle$, then there even exists a polynomial $p_j \in \mathcal{O}(X)[\eta_j]$ such that its locus $V(p_j)$ does not meet $X \times \partial\mathbb{D}^1$ and p_j annihilates $\mathfrak{a}^k\xi$; i.e., $p_j \cdot \mathfrak{a}^k\xi = 0$.

Proof. Starting with $\mathcal{Q}_0 := \mathcal{G}$, there are exact sequences

$$0 \to \mathcal{Q}_{i+1} \to \mathcal{O}^{t_i}_{X\times\mathbb{D}^d} \to \mathcal{Q}_i \to 0$$

for $i = 0, \ldots, d-2$. Since $H^i(B, \mathcal{O}_{X\times\mathbb{D}^d}) = 0$ for $i = 1, \ldots, d-2$ by Corollary 6.3, we have an isomorphism $H^1(B, \mathcal{G}) = H^{d-2}(B, \mathcal{Q}_{d-3})$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & H^{d-2}(B, \mathcal{Q}_{d-3}) & \xrightarrow{\delta} & H^{d-1}(B, \mathcal{Q}_{d-2}) & \longrightarrow & H^{d-1}(B, \mathcal{O}^{t_{d-3}}_{X\times\mathbb{D}^d}) \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & H^{d-2}(B_U, \mathcal{Q}_{d-3}) & \xrightarrow{\delta_U} & H^{d-1}(B_U, \mathcal{Q}_{d-2}) & \longrightarrow & H^{d-1}(B_U, \mathcal{O}^{t_{d-3}}_{X\times\mathbb{D}^d})
\end{array}$$

Since δ and δ_U are injective, it suffices to show the assertion for $H^{d-1}(B, \mathcal{Q}_{d-2})$. Now the module $\mathcal{B} := \mathcal{Q}_{d-2}$ is reflexive because $\mathcal{B} = \mathcal{B}^{[n+d-2]}$ and $\mathcal{B}|_B$ is locally free. Thus we have exact sequences

$$0 \to \mathcal{K} \to \mathcal{O}^t_{X\times\mathbb{D}^d} \to \mathcal{B}^* \to 0 \quad \text{and} \quad 0 \to \mathcal{B} \to \mathcal{O}^t_{X\times\mathbb{D}^d} \to \mathcal{Q} \to 0,$$

where $\mathcal{Q} \subset \mathcal{K}^*$ is a subsheaf of the dual, the sheaf $\mathcal{Q}|_B$ is locally free and $\mathrm{S}_{n+d-1}(\mathcal{Q}) \subset \mathrm{S}_{n+1}(\mathcal{G})$. By Lemma 6.14, there exists an integer $k \in \mathbb{N}$ with

$$\mathfrak{a}^k \cdot H^{d-2}(B, \mathcal{Q}) = 0.$$

Furthermore, we have the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & H^{d-2}(B, \mathcal{Q}) & \xrightarrow{\delta} & H^{d-1}(B, \mathcal{B}) & \longrightarrow & H^{d-1}(B, \mathcal{O}^{t}_{X\times\mathbb{D}^d}) \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & H^{d-2}(B_U, \mathcal{Q}) & \xrightarrow{\delta_U} & H^{d-1}(B_U, \mathcal{B}) & \longrightarrow & H^{d-1}(B_U, \mathcal{O}^{t}_{X\times\mathbb{D}^d})
\end{array}$$

Then the assertion follows from Lemma 6.16. □

7. Locally free sheaves

In Section 5, we have seen that any line bundle on a rectilinear Hartogs figure $H := (U \times \mathbb{D}^2) \cup X \times \mathbb{D}^2$ extends to a line bundle on $X \times \mathbb{D}^2$. In the case of vector bundles, such an assertion is false; one can only expect that vector bundles on H extend to coherent sheaves on $X \times \mathbb{D}^2$. For example, let F be the second syzygy module of a free resolution of $T_3/(\eta_1, \eta_2, \eta_3)$. Then the coherent sheaf associated to $\mathcal{F} := \tilde{F}$ on $\mathbb{D}^3$ has the properties that $\mathcal{F}$ is locally free on $\mathbb{D}^3 - \{0\}$ and that the homological codimension $\mathrm{cdh}(\mathcal{F}_0) = 2$ at the origin. Thus we have $\mathcal{F} = \mathcal{F}^{[1]}$ by Proposition 6.7, and hence $\mathcal{F}|_{\mathbb{D}^1 \times \partial\mathbb{D}^2}$ is not extendable as a vector bundle on $\mathbb{D}^3$.

Coherence theorem. Whereas the extension of line bundles from the Hartogs figure H to $X \times \mathbb{D}^2$ could be proved by a simple formula, the proof in the case of vector bundles $\mathcal{F}$ of rank $\operatorname{rank}\mathcal{F} \geq 2$ requires more involved techniques. The substantial part of the proof will be the following result.

Theorem 7.1. *Let $(D, \mathcal{O}_D)$ be an affinoid space of pure dimension n, and assume that all local rings $\mathcal{O}_{D,x}$ are Cohen–Macaulay. Set*

$$Y := \{(z_1, z_2) \in \mathbb{P}^1 \times \mathbb{P}^1;\ |z_1| \geq 1 \text{ or } |z_2| \geq 1\}.$$

Denote by $p\colon X := D \times Y \to D$ the canonical projection. Then, for any locally free sheaf $\mathcal{F}$ on X, the direct image $p_*\mathcal{F}$ is a coherent $\mathcal{O}_D$-module.

The proof of this theorem will fill the whole section. For the following, we fix the notion for the whole section right now. We define

T affinoid algebra, Cohen–Macaulay of dimension n,

$D = \mathrm{Sp}(T)$ associated affinoid space assumed to be connected,

$S := T_n \hookrightarrow T$ is a Noether normalization,

$Y_i := \{(z_1, z_2) \in \mathbb{P}^1 \times \mathbb{P}^1;\ |z_i| \geq 1\}$ for $i = 1, 2$,

$X_i := D \times Y_i$ for $i = 1, 2$,

η_i the coordinate function on the i-th factor $\mathbb{P}^1$ of $\mathbb{P}^1 \times \mathbb{P}^1$,

$V := X_1 \cap X_2 \xrightarrow{\sim} D \times \mathbb{D}^2$ with the coordinate functions η_1^{-1}, η_2^{-1} on $\mathbb{D}^2$,

$X_i := D \times Y_i \xrightarrow{\sim} D \times \mathbb{D}^1 \times \mathbb{P}^1$ with the coordinate function η_i^{-1} on $\mathbb{D}^1$.

For the definition of Cohen–Macaulay, see [31, p. IV-18]. Over a local noetherian ring A, an A-module E is Cohen–Macaulay if and only if its completion $\hat{E}$ is Cohen–Macaulay as $\hat{A}$-module; cp. [31, Cor. 2, p. IV-36]. So if $\mathrm{Sp}(T') \subset \mathrm{Sp}(T)$ is an affinoid subdomain and if T is Cohen–Macaulay, then T' is Cohen–Macaulay, too. Namely, the completion of T' at a maximal ideal $\mathfrak{m}'$ coincides with the completion of T at $\mathfrak{m} := \mathfrak{m}' \cap T$.

Furthermore, we remark that T is Cohen–Macaulay if and only if T is Cohen–Macaulay as a finitely generated S-module by [31, Prop. 22, p. IV-18]. Since $S = T_n$ is a regular ring, the finitely generated S-module T is locally free if T is a Cohen–Macaulay ring. This follows from [31, Prop. 10, p. IV-18]. Denote by $\pi\colon D = \mathrm{Sp}(T) \to \mathbb{D}^n := \mathrm{Sp}(S)$ the associated morphism of the affinoid spaces. Then $(\pi \times \mathrm{id})_*\mathcal{F}$ is locally free on $\mathbb{D}^n \times Y$.

Now we start the proof of Theorem 7.1. Due to the finiteness theorem of Kiehl [21], we have the following claim.

Claim 1. *For every $q \in \mathbb{N}$, the $T\langle\eta_i^{-1}\rangle$-module $H^q(X_i, \mathcal{F})$ is finitely generated.*

Due to [27, Satz 1], we may assume that $\mathcal{F}|_{D\times V} = \mathcal{O}_X^r|_{D\times V}$ is free of rank r. We fix further notations

$$G := \Gamma(D \times V, \mathcal{F}) = \bigoplus_{j=1}^{r} T\langle\eta_1^{-1}, \eta_2^{-1}\rangle \cdot e_j,$$

$$F_i := \mathrm{Im}(\rho_i\colon \Gamma(X_i, \mathcal{F}) \to G) = \sum_{\sigma=1}^{s} T\langle\eta_i^{-1}\rangle \cdot g_\sigma^i \subset G,$$

$$G(m_1, m_2) := G/(\eta_1^{-m_1}G + \eta_2^{-m_2}G) \quad \text{for } m_1, m_2 \in \mathbb{N},$$

$$G(m, \infty) := G/(\eta_1^{-m}G) \quad \text{and} \quad G(\infty, m) := G/(\eta_2^{-m}G) \quad \text{for } m \in \mathbb{N},$$

$$p_{m_1,m_2}\colon G \to G(m_1, m_2) \quad \text{residue map for } m_1, m_2 \in \mathbb{N} \cup \{\infty\}.$$

Since $\mathcal{F}$ has no torsion, the restriction maps ρ_1 and ρ_2 are injective.

Claim 2. *In the situation defined above, the following holds.*
(i) *There exists an integer $m_0 \in \mathbb{N}$ such that, for all $m \geq m_0$, the residue maps $p_{\infty,m}|_{F_1}$ and $p_{m,\infty}|_{F_2}$ are injective.*
(ii) *The T-module $F = F_1 \cap F_2 = \Gamma(X, \mathcal{F})$ is finitely generated.*

Proof. (i) We show the assertion for F_1. We set $K_m := F_1 \cap \eta_2^{-m}G \subset F_1$. By Claim 1, we know that $F_1 \supset K_m \supset K_{m+1}$ are noetherian $T\langle\eta_1^{-1}\rangle$-modules. Let $S \to T$ be a noetherian normalization. Note that S is a domain. Then there exists an integer $m_0 \in \mathbb{N}$ such that the $S\langle\eta_1^{-1}\rangle$-ranks $\mathrm{rank}\, K_m = \mathrm{rank}\, K_{m_0}$ for all $m \geq m_0$ become stationary. Then, for any $t \in K_{m_0}$, there exists an element $a = a(m) \in S\langle\eta_1^{-1}\rangle$ with $a \neq 0$ such that $a \cdot t \in K_m$. Now $G/\eta_2^{-m}G$ has no $S\langle\eta_1^{-1}\rangle$-torsion. So we have $t \in K_{m_0}$. Since $\bigcap_{m\leq m_0} K_m = \{0\}$, we see that $p_{\infty,m_0}|_{F_1}$ is injective.

(ii) It suffices to show that $p_{m,\infty}(F)$ is a finitely generated T-module for $m \geq m_0$. By Claim 1, there is a generating system $g_1^1, \ldots, g_s^1$ of F_1 as $T\langle\eta_1^{-1}\rangle$-module. Then the image $p_{m,\infty}(F_1)$ is generated by $p_{m,\infty}(g_1^1), \ldots, p_{m,\infty}(g_s^1)$ as $T\langle\eta_1^{-1}\rangle/(\eta_1^{-m})$-module and hence finitely generated as T-module. Thus we see that $p_{m,\infty}(F) \subset p_{m,\infty}(F_1)$ is finitely generated as T-module. □

Claim 3. *Let $m := m_0$ be as in Claim 2. Then there exists an integer $k \in \mathbb{N}$ such that*

$$F_1 \cap p_{\infty,m}^{-1}(\eta_1^{-\ell-k}G(\infty, m)) \subset \eta_1^{-\ell}F_1 \quad \text{for all } \ell \in \mathbb{N},$$

$$F_2 \cap p_{m,\infty}^{-1}(\eta_2^{-\ell-k}G(m, \infty)) \subset \eta_2^{-\ell}F_2 \quad \text{for all } \ell \in \mathbb{N}.$$

Proof. We show the assertion for F_1. Since $G(\infty, m)$ is a finitely generated $T\langle\eta_1^{-1}\rangle$-module, due to the lemma of Artin–Rees [2, Prop. 10.9], there exists an integer $k \in \mathbb{N}$ such that

$$p_{\infty,m}(F_1) \cap \eta_1^{-\ell-k} G(\infty, m) \subset \eta_1^{-\ell} p_{\infty,m}(F_1) \quad \text{for all } \ell \in \mathbb{N}.$$

Since $p_{\infty,m}|_{F_1}$ is injective, the assertion is clear. □

Claim 4. *There exists an integer $m_1 \in \mathbb{N}$ with $m_1 \geq m_0$ such that, for any nonzero divisor $a_i \in T\langle \eta_i^{-1} \rangle$ with $a_i \neq 0$ for $i = 1, 2$, the residue maps*

$$\psi_1 \colon F_1/a_1F_1 \to G(\infty, m)/a_1G(\infty, m),$$
$$\psi_2 \colon F_2/a_2F_2 \to G(m, \infty)/a_2G(m, \infty)$$

are injective for all $m \geq m_1$.

Proof. We put $M_1(m) := G(\infty, m)/p_{\infty,m}(F_1)$ for $m \geq m_0$ with m_0 as in Claim 2 and let $\rho\colon M_1(m) \to M_1(m_0)$ be the residue map. There exists an $a \in S\langle \eta_1^{-1} \rangle$ with $a \neq 0$ such that a annihilates all the torsion of $M_1(m_0)$. As in Claim 1, there exists an integer $m_1 \in \mathbb{N}$ with $m_1 \geq m_0$ such that

$$\psi\colon F_1/aF_1 \subset \Gamma(X_1, \mathcal{F}/a\mathcal{F}) \to G(\infty, m)/aG(\infty, m)$$

is injective for all $m \geq m_1$. Now consider $x \in F_1$ with $\psi_1(x) = a_1 g$ for some $g \in G(\infty, m)$. Then $a_1 \overline{g} = 0$ in $M_1(m)$, and hence $a_1 \rho(\overline{g}) = 0$ in $M_1(m_0)$. Due to the choice of a, we have $a\rho(\overline{g}) = 0$; *i.e.*, $ag = p_{\infty,m_0}(h)$ for some $h \in F_1$. Since ψ is injective, we can write $h = ah'$ for an $h' \in F_1$. Thus we have that $ag = ap_{\infty,m_0}(h')$ in $G(\infty, m_0)$, and hence $g = p_{\infty,m_0}(h')$ as $G(\infty, m_0)$ has no $S\langle \eta_1^{-1} \rangle$-torsion. So we obtain $p_{\infty,m_0}(x) = p_{\infty,m_0}(a_1 h')$. Since the morphism $p_{\infty,m_0} \colon F_1 \to G(\infty, m_0)$ is injective, we see $x = a_1 h'$. This shows that ψ_1 is injective for $m \geq m_1$. Analogously for ψ_2. □

Claim 5. *There exists an integer $q \in \mathbb{N}$ with the following property: if $h_1 \in F_1$ and $h_2 \in \mathcal{F}_2$ satisfy $h_1 - h_2 \in \eta_1^{-q}\eta_2^{-q} G$, then $h_1 = h_2 \in F$ is a global section of $\mathcal{F}$ over X.*

Proof. Let $m_0 \in \mathbb{N}$ be as in Claim 2. For $m \in \mathbb{N}$ with $m \geq m_0$, we define the T-modules

$$K_m := \{h_1 \in F_1;\, h_1 = h_2 + \eta_1^{-m}\eta_2^{-m} g \text{ with } h_2 \in F_2,\, g \in G\}.$$

Then we claim that K_m is a finitely generated T-module.

Since $p_{\infty,m}|_{F_1}$ is injective, it suffices to show that $p_{\infty,m}(K_m)$ is finitely generated. Because of $\eta_2^{-m} p_{\infty,m}(F_2) = 0$, the T-module $p_{\infty,m}(F_2)$ is a finitely generated T-module. So $p_{\infty,m}(K_m)$ is a finitely generated T-module as submodule of $p_{\infty,m}(F_2)$. Let $S \hookrightarrow T$ be a Noether normalization. Then $p_{\infty,m}(F_2)$ is a finitely generated S-module as well. Thus we see that there exists an integer $q \geq m_0$ such that $\operatorname{rank} K_m = \operatorname{rank} K_q$ as S-module for all $m \geq q$. If $h_1 \in K_q$, then for any $m \geq q$, there exists an $a = a(m) \in S$ with $a \neq 0$ such that $ah_1 \in K_m$. Thus we have

$$h_1 = h_2 + \eta_1^{-q}\eta_2^{-q} g' \qquad \text{with } h_2 \in F_2,\, g' \in G,$$
$$ah_1 = h_2' + \eta_1^{-m}\eta_2^{-m} g'' \quad \text{with } h_2' \in F_2,\, g'' \in G.$$

Since $p_{q,\infty}|_{F_2}$ is injective, we obtain $h'_2 = ah_2$. Then g'' is divisible by a; *i.e.*, $g'' = ag$. Therefore, $h_1 = h_2 + \eta_1^{-m}\eta_2^{-m}g$. Since $m \geq q$ was arbitrary, we have $h_1 = h_2$. □

Claim 6. *Let $k \in \mathbb{N}$ be the integer of Claim 3. There exists an integer $s \in \mathbb{N}$ with the following property: if $h_1 \in F_1$ and $h_2 \in F_2$ satisfy the following conditions:*

$$h_1 - h_2 = \eta_1^{-s}g_1 + \eta_2^{-s}g_2 \in \eta_1^{-s}G + \eta_2^{-s}G,$$

then there exists a global section $h \in F = F_1 \cap F_2$ such that

$$h_1 - h = \eta_1^{-s+k}h'_1 \in \eta_1^{-s+k}F_1 \quad \textit{for some } h'_1 \in F^1,$$
$$h_2 - h = \eta_2^{-s+k}h'_2 \in \eta_2^{-s+k}F_2 \quad \textit{for some } h'_2 \in F^2.$$

Proof. Let $r \geq \max\{m_0 + k, m_1\}$, where m_i is from Claim 1 resp. Claim 4. We define

$$M_m^1 := p_{\infty,r}(F_1) + \eta_1^{-m}G(\infty, r), \quad K_m^1 := M_m^1 \cap p_{\infty,r}(F_2),$$
$$M_m^2 := p_{r,\infty}(F_2) + \eta_2^{-m}G(r, \infty), \quad K_m^2 := M_m^2 \cap p_{r,\infty}(F_1),$$

and then we set

$$M^1 := \coprod_{m=0}^{\infty} M_m^1/M_{m+1}^1 \quad \text{and} \quad M^2 := \coprod_{m=0}^{\infty} M_m^2/M_{m+1}^2.$$

Now M^1 is the graded $S[\eta_1^{-1}]$-module with respect to the ideal η_1^{-1} of the $S\langle\eta_1^{-1}\rangle$-module $G(\infty, r)/p_{\infty,r}(F_1)$. So M^1 is a noetherian $S[\eta_1^{-1}]$-module; cp. [2, Prop. 10.22]. The analog assertion is true for M^2. By Claim 1, it follows that the S-modules $p_{r,\infty}(F_1)$ and $p_{\infty,r}(F_2)$ are finitely generated and hence noetherian. Thus K_m^1 and K_m^2 are finitely generated S-modules, and their ranks are decreasing. So they become stationary. Thus there exists an integer $s \in \mathbb{N}$ such that, for $m \geq s$, the submodules $K_m^1/K_{m+1}^1 \subset M^1$ and $K_m^2/K_{m+1}^2 \subset M^2$ are S-torsion modules. Therefore,

$$K^i := \coprod_{i=s}^{\infty} K_m^i/K_{m+1}^i \subset M^i \quad \text{for } i = 1, 2$$

are S-torsion submodules. Since M^i are noetherian $S[\eta_i^{-1}]$-modules and S is a domain, there exists an $a \in S$ with $a \neq 0$ such that $a \cdot K_m^i \subset K_{m+1}^i$ for $m \geq s$ and $i = 1, 2$.

If $h_1 \in F_1$ and $h_2 \in F_2$ satisfy the identity

$$h_1 - h_2 = \eta_1^{-s}g_1 + \eta_2^{-s}g_2 \quad \text{with } g_1, g_2 \in G, \tag{3}$$

then we have $p_{r,\infty}(h_1) \in K_s^2$ and $p_{\infty,r}(h_2) \in K_s^1$. Thus it follows

$$p_{r,\infty}(h_1) \in \bigcap_{m\in\mathbb{N}} \big(\eta_2^{-m}G(r, \infty) + p_{r,\infty}(F_2)\big)_a,$$
$$p_{\infty,r}(h_2) \in \bigcap_{m\in\mathbb{N}} \big(\eta_1^{-m}G(\infty, r) + p_{\infty,r}(F_2)\big)_a$$

for the localizations with respect to the element a. Due to Claim 4, the $S\langle\eta_2^{-1}\rangle$-module $G(r,\infty)/p_{r,\infty}(F_2)$ is free of torsion, and the $S\langle\eta_1^{-1}\rangle$-module $G(\infty,r)/p_{\infty,r}(F_1)$ is free of torsion, too. Thus, due to Krull's intersection theorem [2, Thm. 10.17], we obtain

$$\begin{aligned} h_1 &= h_2' + \eta_1^{-r} g_1' \quad \text{with } h_2' \in F_2,\ g_1' \in G, \\ h_2 &= h_1' + \eta_2^{-r} g_2' \quad \text{with } h_1' \in F_1,\ g_2' \in G. \end{aligned} \tag{4}$$

By equation (3), this yields

$$\begin{aligned} h_1 - h_1' &= (h_1 - h_2) + (h_2 - h_1') = \eta_1^{-s} g_1 + (\eta_2^{-s} g_2 + \eta_2^{-r} g_2'), \\ h_2 - h_2' &= (h_2 - h_1) + (h_1 - h_2') = \eta_2^{-s} g_2 + (\eta_1^{-s} g_1 + \eta_1^{-r} g_1'). \end{aligned}$$

Since $h_i, h_i' \in F_i$, it follows, by Claim 3,

$$\begin{aligned} h_1 - h_1' &= \eta_1^{-s+k} h_1'' \quad \text{with } h_1'' \in F_1, \\ h_2 - h_2' &= \eta_2^{-s+k} h_2'' \quad \text{with } h_2'' \in F_2. \end{aligned} \tag{5}$$

Then, by combining (4) and (5), we arrive at the equations

$$\begin{aligned} h_1' - h_2' &= (h_1' - h_1) + (h_1 - h_2') = \eta_1^{-r} g_1' - \eta_1^{-s+k} h_1'', \\ h_1' - h_2' &= (h_1' - h_2) + (h_2 - h_2') = \eta_2^{-r} g_2' - \eta_2^{-s+k} h_2''. \end{aligned}$$

Finally, we have that $h_1' - h_2' \in \eta_1^{-t}\eta_2^{-t} G$ for $t := \min\{r, s-k\}$. By Claim 5, it follows

$$h := h_1' = h_2' \in \Gamma(X, \mathcal{F}). \tag{6}$$

Equations (5) and (6) yield the assertion. □

Claim 7. *For any nonzero divisor $a \in T$, there exists an integer $b \in \mathbb{N}$ with the following property: if $h_1 \in F_1$ and $h_2 \in F_2$ satisfy $h_1 - h_2 = a^\ell g \in a^\ell \cdot \Gamma(X_1 \cap X_2, \mathcal{F})$ for some $\ell \geq b$, then there exist $h_1' \in F_1$ and $h_2' \in F_2$ with $h_1 - h_2 = a^{\ell-b}(h_1' - h_2')$.*

Proof. Let $q \in \mathbb{N}$ be the integer of Claim 5 and $s \in \mathbb{N}$ the integer of Claim 6. Due to the lemma of Artin–Rees [2, Prop. 10.9], there exists an integer $j \in \mathbb{N}$ such that, for all $\ell \geq j$,

$$\begin{aligned} (p_{s,s}(F_1) + p_{s,s}(F_2)) \cap a^\ell G(s,s) &\subset a^{\ell-j}(p_{s,s}(F_1) + p_{s,s}(F_2)), \\ (p_{m,\infty}(F_2)) \cap a^\ell G(m,\infty) &\subset a^{\ell-j}(p_{m,\infty}(F_2)), \\ (p_{\infty,m}(F_1)) \cap a^\ell G(\infty,m) &\subset a^{\ell-j}(p_{\infty,m}(F_1)), \end{aligned} \tag{7}$$

where $m = m_0$ is the integer of Claim 2. Thus we can write, for suitable $h_i' \in F_i$ and $g_i \in G$,

$$(h_1 - h_2) = a^{\ell-j}(h_1' - h_2') + \eta_1^{-s} g_1 + \eta_2^{-s} g_2.$$

Due to Claim 6, there exists a global section $h \in \Gamma(x, \mathcal{F})$ such that

$$\begin{aligned} h_1 - a^{\ell-j} \cdot h_1' &= h + \eta_1^{(-s+b)} h_1'' \quad \text{for some } h_1'' \in F_1, \\ h_2 - a^{\ell-j} \cdot h_2' &= h + \eta_2^{(-s+b)} h_2'' \quad \text{for some } h_2'' \in F_2. \end{aligned}$$

Combining with the assumption, we obtain

$$h_2 - h = (h_2 - h_1) + (h_1 - h) = -a^\ell g + a^{\ell-j} h_1' + \eta_1^{-s+k} h_1''.$$

With the similar relation for $h_1 - h$, this yields

$$p_{\infty,m}(h_1 - h) \in a^{\ell-j} G(\infty, m) \quad \text{and} \quad p_{(m,\infty)}(h_2 - h) \in a^{\ell-j} G(m, \infty).$$

Since $p_{\infty,m}|_{F_1}$ and $p_{m,\infty}|_{F_2}$ are injective, due to (7), there exist sections $h_i' \in F_i$ for $i = 1, 2$ such that $h_1 - h = a^{\ell-2j} h_1'$ and $h_2 - h = a^{\ell-2j} h_2'$. With $b := 2j$, we obtain the assertion

$$h_1 - h_2 = (h_1 - h) - (h_2 - h) = a^{\ell-b}(h_1' - h_2). \qquad \square$$

Claim 8. *For any nonzero divisor $a \in T$, there exists an integer $k := k(a) \in \mathbb{N}$ with the following property: if $\xi \in H^1(X, \mathcal{F})$ fulfills $a^\ell \cdot \xi = 0$, then it already holds $a^k \cdot \xi = 0$.*

Proof. Consider the Mayer–Vietoris sequence associated to $X = X_1 \cup X_2$,

$$\begin{aligned} 0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X_1, \mathcal{F}) \oplus \Gamma(X_2, \mathcal{F}) \xrightarrow{\psi} \Gamma(X_1 \cap X_2, \mathcal{F}) \\ \xrightarrow{\delta} H^1(X, \mathcal{F}) \xrightarrow{\phi} H^1(X_1, \mathcal{F}) \oplus H^1(X_2, \mathcal{F}) \longrightarrow 0. \end{aligned}$$

Since $X_1 \cap X_2$ is affinoid, the group $H^1(X_1 \cap X_2, \mathcal{F}) = 0$ vanishes. The $T\langle \eta_i^{-1} \rangle$-modules $H^1(X_1, \mathcal{F})$ are noetherian, so the a-torsion vanishes at finite index; *i.e.*, $a^k \cdot \phi(\xi) = 0$. Then there is a $g \in \Gamma(X_1 \cap X_2, \mathcal{F})$ such that $\delta(g) = a^k \xi$. Since $a^{\ell-k} \delta(g) = 0$, there exist elements $f_i \in \Gamma(X_i, \mathcal{F})$ for $i = 1, 2$ with $a^{\ell-k} g = f_1 - f_2$. Therefore, by Claim 6, we can write $a^{\ell-k} g$ in the form $a^{\ell-k} g = a^{\ell-k-b}(h_1 - h_2)$ for some $h_i \in \Gamma(X_i, \mathcal{F})$ for $i = 1, 2$. Since a is a nonzero divisor, we finally have $a^b g = h_1 - h_2$. Finally, we obtain

$$0 = \delta \circ \psi(h_1, h_2) = \delta(h_1 - h_2) = a^b \delta(g) = a^b a^k \xi.$$

Thus we see that $k(a) := k + b$ satisfies the assertion. $\square$

Finally, we put together all our claims and start the proof of Theorem 7.1.

Proof of Theorem 7.1. After the reduction explained at the beginning, we may assume that $D = \mathrm{Sp}(T)$ is an affinoid space, where T is Cohen–Macaulay of pure dimension n and $\mathcal{F}|_{D \times V}$ is free. Then we have to show the following.

(a) $\Gamma(D \times Y, \mathcal{F})$ is a finitely generated T-module.

(b) For any affinoid subdomain $D' = \mathrm{Sp}(T')$ of D, the canonical morphism

$$\Gamma(D \times Y, \mathcal{F}) \otimes_T T' \to \Gamma(D' \times Y, \mathcal{F})$$

is bijective.

Assertion (a) follows from Claim 1.

(b) We proceed by induction on $n = \dim T$. In the case $n = 0$, there is nothing to show.

Now assume $n \geq 1$. Put $M := \Gamma(D \times Y, \mathcal{F})$ and $M' := \Gamma(D' \times Y, \mathcal{F})$. Since $M \otimes_T T'$ and M' are noetherian T'-modules, it suffices to show that, for any maximal ideal $\mathfrak{n}$ of T', the $\mathfrak{n}$-adic completion $\widehat{M'}$ and $\widehat{M \otimes_T T'}$ are canonically isomorphic. The ideal $\mathfrak{n}$ is of the form $\mathfrak{m}T'$ for a maximal ideal $\mathfrak{m}$ of T. Let

$T_n \hookrightarrow T$ be a Noether normalization. Since T is of pure dimension n, any $a \in T_n$ with $a \neq 0$ gives rise to nonzero divisor of T. Let now $a \in \mathfrak{m} \cap T_d$. Since T/a^iT is Cohen–Macaulay as well and $\dim T/a^iT = n-1$, due to the induction hypothesis, the canonical morphism

$$\Gamma(D \times Y, \mathcal{F}/a^i\mathcal{F}) \otimes_T T' \xrightarrow{\sim} \Gamma(D' \times Y, \mathcal{F}/a^i\mathcal{F})$$

is bijective for all $i \geq 1$. For any $i \in \mathbb{N}$, we have the exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{a^i} \mathcal{F} \longrightarrow \mathcal{F}/a^i\mathcal{F} \longrightarrow 0.$$

For $i \in \mathbb{N}$, we define the modules

$$G_i := \Gamma(D \times Y, \mathcal{F})/a^i\Gamma(D \times Y, \mathcal{F}) \subset F_i := \Gamma(D \times Y, \mathcal{F}/a^i\mathcal{F}),$$
$$Q_i := \operatorname{coker}(G_i \to F_i) = P_i := \{\xi \in H^1(D \times Y, \mathcal{F});\ a^i\xi = 0\}.$$

Denote by G_i', F_i', Q_i', P_i' the corresponding modules over $D' \times Y$. Since $T \to T'$ is flat, we obtain the exact sequences

$$0 \to G_i \to F_i \to Q_i \to 0,$$
$$0 \to G_i' \to F_i' \to Q_i' \to 0,$$
$$0 \to G_i \otimes_T T' \to F_i \otimes_T T' \to Q_i \otimes_T T' \to 0.$$

We have canonical residue maps $G_i \to G_{i-1}$ and $F_i \to F_{i-1}$, resp. $G_i' \to G_{i-1}'$ and $F_i' \to F_{i-1}'$. For the third term $Q_i \to Q_{i-1}$ resp. $Q_i' \to Q_{i-1}'$, the induced map is the multiplication by a. Due to Claim 8, the systems $(Q_i \to Q_{i-1})$ and $(Q_i' \to Q_{i-1}')$ are zero-systems, and hence $(Q_i \otimes_T T' \to Q_{i-1} \otimes_T T')$ as well. By applying the projective limit, we receive the a-adic completions $\widehat{M'}$ of M' resp. $\widehat{M \otimes_T T'}$ of $M \otimes_T T'$,

$$\widehat{M \otimes_T T'} = \varprojlim(G_i \otimes_T T') = \varprojlim(F_i \otimes_T T'),$$
$$\widehat{M'} = \varprojlim(G_i') = \varprojlim(F_i').$$

Due to the induction hypothesis, $F_i \otimes_T T' \xrightarrow{\sim} F_i'$ is bijective, so $\widehat{M \otimes_T T'} \xrightarrow{\sim} \widehat{M'}$ is bijective. This implies by faithfully flat descent the assertion; cp. the proof in [21]. □

Remark 7.2. We would like to remark that Claim 8 is obviously true if the sheaf $\mathcal{F}$ extends to a coherent sheaf $\underline{\mathcal{F}} = \underline{\mathcal{F}}^{[n]}$ on $D \times \mathbb{D}^2$. Indeed, $\underline{\mathcal{F}}$ is reflexive. So there are exact sequences

(8) $\qquad 0 \to \mathcal{K} \to \mathcal{L} \to \mathcal{F}^* \to 0 \quad \text{and} \quad 0 \to \mathcal{F} = \mathcal{F}^{**} \to \mathcal{L}^* \to \mathcal{Q} \to 0$

with a free module $\mathcal{L}$ and a submodule $\mathcal{Q} \subset \mathcal{K}^*$. Moreover, we have $\mathcal{K}^* = (\mathcal{K}^*)^{[n]}$ since $\mathcal{K}^*$ can be represented as the kernel of a morphism between two finitely generated free modules. Hence we have $H^0(D \times \mathbb{D}^2, \mathcal{K}^*) = H^0(B, \mathcal{K}^*)$ for $B = D \times \partial\mathbb{D}^2$ by Proposition 6.8. So $H^0(B, \mathcal{K}^*)$ is a noetherian $T\langle \eta_1, \eta_2 \rangle$-module, and hence the $T\langle \eta_1, \eta_2 \rangle$-submodule $H^0(B, \mathcal{Q})$ of $H^0(B, \mathcal{K}^*)$ is noetherian as well. Now look at the long exact cohomology sequence

$$H^0(B, \mathcal{L}^*) \to H^0(B, \mathcal{Q}) \to H^1(B, \mathcal{F}) \to H^1(B, \mathcal{L}^*).$$

We know that $H^1(B,\mathcal{L}^*)$ has no T-torsion since $\mathcal{L}^*$ is a finitely generated free $T\langle\eta_1,\eta_2\rangle$-module. Namely, for free $T\langle\eta_1,\eta_2\rangle$-modules, we know from Lemma 6.15 that $H^1(B,\mathcal{O}_{D\times\mathbb{D}^2})$ has no T-torsion. Thus the T-torsion of $H^1(B,\mathcal{F})$ is the image of a $T\langle\eta_1,\eta_2\rangle$-submodule of $H^0(B,\mathcal{Q})$ and hence finitely generated. Thus the T-torsion of $H^1(B,\mathcal{F})$ vanishes at finite index.

In the case $D\times Y$, one proceeds similarly. At first, one obtains the analog of the sequences (8) over $D\times\mathbb{P}^1\times\mathbb{P}^1$ after twisting $\mathcal{F}$ by a suitable very ample invertible sheaf. One computes as in Lemma 6.15 that $H^1(D\times Y,\mathcal{O})$ has no T-torsion. As above, one obtains the result for the twisted sheaf $\mathcal{F}(n)$ and hence for $\mathcal{F}$.

Extension of locally free sheaves. Theorem 7.1 provides the essential tool to show the extension for locally free sheaves.

Proposition 7.3. *Keep the situation of Theorem 7.1. Let $U\subset D$ be an open subdomain of D which meets each irreducible component of D. Set $\mathbb{P}:=\mathbb{P}^1\times\mathbb{P}^1$. Set $H:=(D\times Y)\cup(U\times\mathbb{P})$. If $\mathcal{F}=\mathcal{F}^{[n]}$ for $n=\dim D$ is a coherent sheaf on H which is locally free on $D\times Y$, then $\mathcal{F}$ is the restriction of a coherent sheaf $\underline{\mathcal{F}}$ on $D\times\mathbb{P}$ with $\underline{\mathcal{F}}=\underline{\mathcal{F}}^{[n]}$.*

Proof. The Segre embedding $\mathbb{P}:=\mathbb{P}^1\times\mathbb{P}^1\hookrightarrow\mathbb{P}^3$ shows that $\mathbb{P}$ is projective. Let $\mathcal{L}$ denote a relatively ample sheaf on $D\times\mathbb{P}\to D$. Due to the GAGA Theorem, there exists an integer $m\in\mathbb{N}$ such that $\mathcal{F}\otimes\mathcal{L}^{\otimes m}$ is generated by global sections over $U\times\mathbb{P}$. Since $p_*(\mathcal{F}\otimes\mathcal{L}^{\otimes m})$ is coherent by Theorem 7.1, there exist finitely many sections $h_1,\dots,h_r\in\Gamma(D\times Y,\mathcal{F}\otimes\mathcal{L}^{\otimes m})$ generating $\Gamma(U\times Y,\mathcal{F}\otimes\mathcal{L}^{\otimes m})$. The restriction map $\Gamma(U\times\mathbb{P},\mathcal{F}\otimes\mathcal{L}^{\otimes m})\to\Gamma(U\times Y,\mathcal{F}\otimes\mathcal{L}^{\otimes m})$ is bijective due to Proposition 6.8. So the extensions of $h_1,\dots,h_r$ generate $\mathcal{F}\otimes\mathcal{L}^{\otimes m}|_{U\times\mathbb{P}}$. So we obtain a morphism

$$\phi\colon \mathcal{O}^r_{D\times\mathbb{P}}|_H\to\mathcal{F}\otimes\mathcal{L}^{\otimes m}\quad\text{with }\operatorname{coker}(\phi|_{U\times\mathbb{P}})=\varnothing.$$

The kernel $\mathcal{K}:=\ker(\phi)\subset\mathcal{O}^r_{D\times\mathbb{P}}$ is a coherent sheaf on H, and it obviously satisfies $\mathcal{K}=\mathcal{K}^{[n]}$.

Due to Theorem 4.6, the subsheaf $\mathcal{K}$ extends to a subsheaf $\underline{\mathcal{K}}$ on $D\times\mathbb{P}$ satisfying $\underline{\mathcal{K}}=\underline{\mathcal{K}}^{[n]}$. Then put $\underline{\mathcal{Q}}:=(\mathcal{O}^r_{D\times\mathbb{P}}/\underline{\mathcal{K}})\otimes\mathcal{L}^{\otimes -m}$. Now consider the induced morphism $\overline{\phi}\colon\mathcal{Q}\to\mathcal{F}$. Due to Proposition 6.7, the sheaf $\underline{\mathcal{Q}}^{[n]}$ is coherent. Since $\mathcal{F}=\mathcal{F}^{[n]}$, we also have that $\mathcal{Q}:=\underline{\mathcal{Q}}|_H$ is a coherent subsheaf of $\mathcal{F}$. So we may assume $\underline{\mathcal{Q}}=\underline{\mathcal{Q}}^{[n]}$. Now the quotient $\mathcal{F}/\mathcal{Q}$ satisfies $\operatorname{Supp}(\mathcal{F}/\mathcal{Q}|_{U\times\mathbb{P}}))=\varnothing$ due to our construction. Then the support $S:=\operatorname{Supp}(\mathcal{F}/\mathcal{Q}|_{U\times\mathbb{P}}))$ is either empty or has pure dimension $n+1$. Due to Theorem 3.6, the closed analytic subset S extends to a hypersurface $\underline{S}\subset(D\times\mathbb{P})$. Since $\underline{S}\cap(U\times\mathbb{P})=\varnothing$, its projection is a closed analytic subset $p(S)\subset D$ unequal to D. Thus there exists a nonzero divisor $a\in\mathcal{O}_D(D)$ such that $a\cdot\mathcal{F}/\mathcal{Q}=0$. So we can regard $\mathcal{F}\cong a\cdot\mathcal{F}\subset\mathcal{Q}$ as a subsheaf of $\mathcal{Q}=\underline{\mathcal{Q}}|_H$. Then the assertion follows by the subsheaf extension; cp. Theorem 4.6. ☐

By the following lemma, we will reduce the extension property for rectilinear Hartogs figure to the special case just treated.

Lemma 7.4. *Let A be an artinian ring with maximal ideal $\mathfrak{m}$, and let $\mathfrak{n} \subset A$ be an $\mathfrak{m}$-primary ideal. If a locally free sheaf on* $\operatorname{Spec}(A/\mathfrak{m}) \times \mathbb{P}^1$ *is trivial, then $\mathcal{F}$ is already trivial.*

Proof. Since $\mathcal{F}/\mathfrak{m}\mathcal{F}$ is trivial, the restriction of the sheaves $\mathcal{F}/\mathfrak{n}\mathcal{F}$ onto affine parts of $\operatorname{Spec}(A) \times \mathbb{P}^1$ is trivial. Thus $\mathcal{F}$ is given by an invertible matrix $I + M$, where M is a matrix with entries in $\mathfrak{m}A[\eta^{-1}, \eta]$. Since there is an integer ℓ with $\mathfrak{m}^\ell \subset \mathfrak{n}$, we can split the matrix in the form $I + M = (I + M^+) \cdot (I + M^-)$, where M^+ has entries in $\mathfrak{m}A[\eta]$ and M^- has entries in $\mathfrak{m}A[\eta^{-1}]$. □

Lemma 7.5. *Let $D := \operatorname{Sp}(A)$ be an affinoid space, and let $\mathcal{F}$ be a locally free sheaf on $D \times \mathbb{P}^1$. Let $x \in D$ be a point. Assume that $\mathcal{O}_{D,x}$ is Cohen–Macaulay and that $\mathcal{F}(x) := \mathcal{F}|_{x \times \mathbb{P}^1}$ is trivial on the fiber $\{x\} \times \mathbb{P}^1$. Then there exists a function $h \in A$ with $h(x) \neq 0$ such that $\mathcal{F}|_{(D - V(h)) \times \mathbb{P}^1}$ is trivial over $D - V(h)$.*

Proof. Let $\mathfrak{m} \subset A$ be the maximal ideal associated to x. For any $\mathfrak{m}$-primary ideal $\mathfrak{n} \subset A$, the sheaf $\mathcal{F} \otimes_A (A/\mathfrak{n})$ is trivial as well by Lemma 7.4. Since $A_\mathfrak{m}$ is Cohen–Macaulay, after a localization of A by an element $a \in A$ with $a(x) \neq 0$, there exists a sequence $t_1, \dots, t_d \in A$ of successive nonzero divisors which generates an $\mathfrak{m}$-primary ideal $\mathfrak{n} \subset A$. After localization by a, we may assume $a = 1$. Denote by $p \colon D \times \mathbb{P}^1 \to D$ the canonical projection. Now we want to show that $\Gamma(D \times \mathbb{P}^1, \mathcal{F}/\mathfrak{n}\mathcal{F})$ is generated by global sections of $\Gamma(D \times \mathbb{P}^1, \mathcal{F})$. We show this by induction on the length d of the sequence $(t_1, \dots, t_d)$. For $d = 1$, consider the exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\cdot t_1} \mathcal{F} \longrightarrow \mathcal{F}/t_1\mathcal{F} \longrightarrow 0.$$

Since $\mathcal{F}/t_1\mathcal{F} = \mathcal{F}/\mathfrak{n}\mathcal{F}$ is trivial, we have that $H^1(D \times \mathbb{P}^1, \mathcal{F}/t_1\mathcal{F}) = 0$ vanishes and $\Gamma(D \times \mathbb{P}^1, \mathcal{F}/t_1\mathcal{F}))$ generates each stalk in the fiber $p^{-1}(x)$. Then consider the associated long exact cohomology sequence. By the lemma of Nakayama, we have $(R^1p_*\mathcal{F})_x = 0$, and hence the sections of $p_*(\mathcal{F}/t_1\mathcal{F})$ are induced by sections $f_1, \dots, f_r$ of $\Gamma(D \times \mathbb{P}^1, \mathcal{F})$, where r is the rank of $\mathcal{F}$. Now the support of $\mathcal{F}/(f_1, \dots, f_r)$ is a closed analytic subset of $D \times \mathbb{P}^1$ and does not meet $\{x\} \times \mathbb{P}^1$. Due to the proper mapping theorem [21], the image of this support is a closed analytic subset which does not contain x. Then there is an element $h \in A$ with $h(x) \neq 0$ such that the support is contained in the locus $V(h)$ of h. For the induction step, one argues in a similar way. Indeed, due to the induction hypothesis, we have a basis of $\mathcal{F}/t_1\mathcal{F}$. This can be lifted due to the same argument as above. □

Lemma 7.6. *Let $D := \operatorname{Sp}(A)$ be an affinoid space, and let $\mathcal{F}$ be a locally free sheaf on $D \times \partial\mathbb{D}^2$. Then, after a finite field extension, there exists a finite rational covering $\{V_1, \dots, V_s\}$ such that each $\mathcal{F}|_{V_\sigma \times \partial\mathbb{D}^2}$ is the restriction of locally free sheaf $\mathcal{F}_\sigma$ on $V_\sigma \times Y$, where Y is as in Theorem 7.1.*

Proof. Due to [27, Satz 2], we may assume that $\mathcal{F}|_{D \times \mathbb{D}^1 \times \partial\mathbb{D}^1}$ is free. Then, by trivial extension, we can enlarge the domain of definition of $\mathcal{F}$ to the domain

$D \times \mathbb{D}^1 \times A(1,c)$, where $A(1,c)$ is the annulus with radii 1 and $c \in |K^\times|$ with $1 < c$. Again by [27, Satz 2], we may assume that $\mathcal{F}|_{D\times\partial\mathbb{D}^1\times\mathbb{D}^1(c)}$ is trivial as well.

Let $\underline{f} := (f_1, \dots, f_r)$ be a basis of $\mathcal{F}|_{D\times\partial\mathbb{D}^1\times\mathbb{D}^1(c)}$, and let $\underline{e} := (e_1, \dots, e_r)$ be a basis of $\mathcal{F}|_{D\times\mathbb{D}^1\times A(1,c)}$. Now let $w := (x, z_2) \in D \times A(1,c)$. Then there is an invertible matrix M in $\mathrm{GL}(r, k(w)\langle \eta_1, \eta_1^{-1}\rangle)$ such that $M \cdot \underline{f} = \underline{e}$ for the restriction on $\{w\} \times \partial\mathbb{D}^1$. We may assume that w is a rational point; otherwise, we perform a base field extension. Then we can regard M as an invertible matrix on $D \times \partial\mathbb{D}^1 \times \mathbb{D}^1(c)$, and hence the basis $\underline{f}$ can be transformed by M. If $M = I$ is the unit matrix, then we extend $\mathcal{F}$ by the free sheaf with the basis $\underline{f}$ on $D \times \mathbb{D}^1_- \times \mathbb{D}^1(c)$, where $\mathbb{D}^1_- := \{z_1 \in \mathbb{P}^1,\ |z_1| \geq 1\}$. Due to Lemma 7.5, for this new locally free sheaf, there exists a function

$$h = \sum_{\nu\in\mathbb{Z}} a_\nu \eta_2^\nu \in \Gamma(D \times A(1,c), \mathcal{O}) \quad \text{with } h(w) \neq 0$$

such that $\mathcal{F} = \mathcal{O}^r|_{(D\times\mathbb{P}^1\times A(1,c))-V(h)}$ is free. Furthermore, we may assume that the coefficients $(a_\nu;\ \nu \in \mathbb{Z})$ have no common zeros. Otherwise, we pick a new point in the zero set of the coefficients and start the same process; by reasons of dimension, we will find a finite Zariski-open covering $\mathfrak{U} := \{U_1, \dots, U_n\}$ of D and for each U_i a function h_i as above such that its coefficients have no common zeros on U_i. Due to [23, Folgerung 1.3], we can shrink the Zariski-open covering $\mathfrak{U}$ by an affinoid covering. Thus we see that the asserted reduction is justified. Now, due to Lemma 3.19, there exists a finite rational covering $\{V_1, \dots, V_s\}$ and numbers ε_σ with $1 < \varepsilon_\sigma < c$ such that $h|_{V_\sigma\times\partial\mathbb{D}^1(\varepsilon_\sigma)}$ is invertible since $\mathcal{F}|_{V_\sigma\times\mathbb{P}^1\times A(\varepsilon_\sigma,\varepsilon_\sigma)}$ is free. Then, by trivial extension, we enlarge the domain of definition of $\mathcal{F}$ to $V_\sigma \times Y$. □

Proposition 7.7. *Let D be a connected affinoid space of dimension n, and let $U \subset D$ be a nonempty open subdomain. Assume that D is smooth.*

Consider the Hartogs figure $H := (U \times \mathbb{D}^2) \cup (D \times \partial\mathbb{D}^2)$, and let $\mathcal{F}$ be a coherent sheaf on H. If $\mathcal{F}$ satisfies that $\mathcal{F} = \mathcal{F}^{[n]}$ and $\mathrm{cdh}(\mathcal{F}|_{D\times\partial\mathbb{D}^2}) = n+2$, then $\mathcal{F}$ is the restriction of a coherent sheaf $\underline{\mathcal{F}}$ on $D \times \mathbb{D}^2$ with $\underline{\mathcal{F}} = \underline{\mathcal{F}}^{[n]}$.

Proof. Since D is smooth, its local rings $\mathcal{O}_{D,x}$ are regular. Then $\mathcal{F}|_{D\times\partial\mathbb{D}^2}$ is a locally free sheaf; cp. [31, Cor. 2, p. IV-36]. Let $K' \supset K$ be the finite field extension of Lemma 7.6. Then $D' := D \otimes_K K'$ is Cohen–Macaulay, and there exists a finite affinoid covering $\{V_1, \dots, V_s\}$ of D' such that $\mathcal{F}' := \mathcal{F}|_{D'\times\partial\mathbb{D}^2}$ is the restriction of a coherent sheaf $\mathcal{F}'_\sigma$ on $V_\sigma \times Y$. Then it follows by Proposition 7.3 that $\mathcal{F}'_\sigma$ extends to a coherent sheaf $\underline{\mathcal{F}}'_\sigma$ on $V_\sigma \times \mathbb{P}$ such that $\underline{\mathcal{F}}'_\sigma = \underline{\mathcal{F}}'^{[n]}_\sigma$. Due to Proposition 6.8, these sheaves fit together to build a coherent sheaf $\mathcal{F}'$ on $D' \times \mathbb{D}^2$.

Since $\Gamma(D \times \partial\mathbb{D}^2, \mathcal{F}) \otimes_K K' = \Gamma(D' \times \partial\mathbb{D}^2, \mathcal{F}')$ and $\Gamma(D' \times \partial\mathbb{D}^2, \mathcal{F}')$ generates every stalk of $\underline{\mathcal{F}}'|_{D'\times\mathbb{D}^2}$, the same is true for $\mathcal{F}$. Thus we have a surjective morphism $\phi\colon \mathcal{O}^t_X|_H \to \mathcal{F}$. The kernel $\mathcal{K} := \ker(\phi)$ satisfies $\mathcal{K} = \mathcal{K}_{[n]\mathcal{O}^t_X}$, and hence it extends to a coherent subsheaf $\underline{\mathcal{K}}$ of $\mathcal{O}^t_X$ due to Theorem 4.6. Thus the quotient $\underline{\mathcal{F}} := \mathcal{O}_X^t/\underline{\mathcal{K}}$ is a coherent sheaf on $D \times \mathbb{D}^2$ with $\underline{\mathcal{F}}|_H = \mathcal{F}$. Due to Proposition 6.7, the sheaf $\underline{\mathcal{F}}^{[n]}$ is coherent and extends $\mathcal{F}$. □

Remark 7.8. In the case of a ball figure, *e.g.* $D \times \partial\mathbb{D}^3$, the analog assertion of Proposition 7.7 is much easier to prove. Indeed, at first, one shows that the locally free sheaf $\mathcal{F}$ can be extended to a locally free sheaf $\mathcal{F}'$ on the subset $\mathbb{D} \times (\mathbb{D}^3(c) - \mathbb{D}^3_+)$, where $c > 1$ and $\mathbb{D}^3_+$ is the open polydisc. Then the restriction map $H^1(D \times (\mathbb{D}^3(c) - \mathbb{D}^3_+), \mathcal{F}') \to H^1(D \times (\mathbb{D}^3(1) - \mathbb{D}^3_+), \mathcal{F}')$ is bijective by Proposition 6.8 and properly continuous as an $\mathcal{O}_{D\times(\mathbb{D}^3(c))}(D \times \mathbb{D}^3(c))$-module. Thus $H^1(D \times (\mathbb{D}^3(c) - \mathbb{D}^3_+), \mathcal{F}')$ is a noetherian $\mathcal{O}_{D\times(\mathbb{D}^3(c)}(D \times \mathbb{D}^3(c))$-module; cp. [21, Kor. 2.5]. Then it is clear that the a-torsion for every nonzero divisor is of finite index. This replaces the hard part, Claim 8, in the proof of Theorem 7.1. Then one can continue as above.

8. Coherent sheaves

We have all the tools at our disposal. So it remains to put things together.

Extension properties of ball figures. Let us start with the ball figures.

Lemma 8.1. *Let X be an affinoid space of pure dimension $n + d$ with $d \geq 3$. Let $B \subset X$ be a ball figure of dimension n; then the couple (B, X) has property $(\mathrm{G}(n + 1))$.*

Proof. Let $\mathcal{G} = \mathcal{G}^{[n+1]}$ be a coherent sheaf on B with $0_{[m+1]\mathcal{G}} = 0$, $0_{[m+2]\mathcal{G}} = \mathcal{G}$ for some $m \geq n + 1$. Then $\mathcal{F} := \mathcal{G}^{[m]}$ is coherent by Proposition 6.7 (a) and (d), and the subsheaf $\mathcal{G} \subset \mathcal{F}$ fulfills the condition $\mathcal{G}_{[n+1]\mathcal{F}} = \mathcal{G}$. Because (B, X) has property (U_{n+1}) by Theorem 4.6, it suffices to show the assertion for $\mathcal{F}$. In particular, we have $\dim \mathrm{S}_{m+1}(\mathcal{F}) \leq m - 1$ by Proposition 6.7 (c).

The support $S := \mathrm{Supp}(\mathcal{F})$ is of pure dimension $m + 2$. By Proposition 3.3, the closed analytic set S extends to a pure dimensional closed analytic subset $\underline{S}$ of X. Due to Lemma 1.5, there exists a morphism $\psi\colon X \to \mathbb{D}^{m+2}$ such that $\psi|_{\underline{S}}\colon \underline{S} \to \mathbb{D}^{m+2}$ is finite and such that it holds $B' := \psi^{-1}(\mathbb{D}^{m-1} \times \partial\mathbb{D}^3) \subset B$ and $B' \cap \mathrm{S}_{m+1}(\mathcal{F}) = \varnothing$. Then $\psi_*(\mathcal{F}|_{B'})$ is coherent on $\mathbb{D}^{m-1} \times \partial\mathbb{D}^3$ and locally free because of $B' \cap \mathrm{S}_{m+1}(\mathcal{F}) = \varnothing$. Then we know by Proposition 7.7 that $\psi_*(\mathcal{F}|_{B'})$ extends to a coherent sheaf on $\mathbb{D}^{m-1} \times \mathbb{D}^3$. This implies that $\Gamma(B', \mathcal{F})$ generates each fiber $\mathcal{F}_x$ for $x \in B'$. Since $(\underline{S} \cap B', \underline{S})$ is a ball figure of dimension $m - 1 \geq n$, the extension of $\mathcal{F}|_{B'}$ follows by Theorem 4.6. The final step follows from Lemma 8.2 below. □

Lemma 8.2. *Let $B \subset X$ be a ball figure as in Lemma 8.1, and let $\mathcal{F} = \mathcal{F}^{[n]}$ be a coherent sheaf on B. If there exists a ball figure $B' \subset B$ of dimension n in X such that $\Gamma(B', \mathcal{F})$ generates any stalk $\mathcal{F}_x$ for $x \in B'$, then there exists a coherent sheaf $\underline{\mathcal{F}} = \underline{\mathcal{F}}^{[n]}$ on X and an isomorphism $\underline{\mathcal{F}}|_B \xrightarrow{\sim} \mathcal{F}$.*

Proof. By the assumptions, there is a surjective morphism $\phi\colon \mathcal{O}_X^t|_{B'} \to \mathcal{F}|_{B'}$. Because of $0_{[n+1]\mathcal{F}} = 0$, the kernel $\ker(\phi)$ satisfies the property $\mathcal{K} = \mathcal{K}_{[n+1]\mathcal{O}_X^t}$. By Theorem 4.6, the subsheaf $\mathcal{K}$ extends to a coherent subsheaf $\underline{\mathcal{K}}$ with the property $\underline{\mathcal{K}} = \underline{\mathcal{K}}_{[n+1]\mathcal{O}_X^t}$. Then $\underline{\mathcal{G}} := \mathcal{O}_X^t/\underline{\mathcal{K}}$ is a coherent sheaf which coincides with $\mathcal{F}|_{B'}$ on B'. Then $\underline{\mathcal{G}}|_B$ and $\mathcal{F}$ coincide on B as well; cp. Corollary 6.13. $\square$

Lemma 8.3. *Let D be a regular connected affinoid space of dimension n, and let $U \subset D$ be a nonempty open subset. Set $X := D \times \mathbb{D}^d$ with $d \geq 3$. Consider the rectilinear Hartogs figure*

$$H := T \cup B \quad \text{with } T := (U \times \mathbb{D}^d) \text{ and } B := D \times \partial\mathbb{D}^d.$$

Let $\mathcal{F}$ be a coherent sheaf on H satisfying $\mathcal{F} = \mathcal{F}^{[n]}$. Set $\mathcal{G} = 0_{[m+2]\mathcal{F}}$ for some $m \geq n$ and $\mathcal{R} := \mathcal{F}/\mathcal{G}$. Assume that $\mathrm{cdh}(\mathcal{G}|_B) \geq n+2$ and $\mathcal{R}|_B = \mathcal{R}^{[n]}|_B$.

If the coherent sheaves $\mathcal{G} = \mathcal{G}^{[n]}$ resp. $\mathcal{H} := \mathcal{R}^{[n]}$ extend to coherent sheaves $\underline{\mathcal{G}} = \underline{\mathcal{G}}^{[n]}$ resp. $\underline{\mathcal{H}} = \underline{\mathcal{H}}^{[n]}$ on $D \times \mathbb{D}^d$, then there exists a coherent sheaf $\underline{\mathcal{F}} = \underline{\mathcal{F}}^{[n]}$ on $D \times \mathbb{D}^d$ such that $\underline{\mathcal{F}}|_H = \mathcal{F}$ over H.

Proof. Because of $0_{[m+2]\mathcal{R}} = 0$, the sheaf $\mathcal{R}^{[n]}$ is coherent due to Proposition 6.7 (d). Now we set $B_T := B \cap T$. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\Gamma(B,\underline{\mathcal{G}}) & \longrightarrow & \Gamma(B,\mathcal{F}) & \longrightarrow & \Gamma(B,\mathcal{R}) & = & \Gamma(B,\underline{\mathcal{H}}) & \xrightarrow{\delta} & H^1(B,\underline{\mathcal{G}}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Gamma(B_T,\underline{\mathcal{G}}) & \longrightarrow & \Gamma(B_T,\mathcal{F}) & \longrightarrow & \Gamma(B_T,\mathcal{R}) & = & \Gamma(B_T,\underline{\mathcal{H}}) & \xrightarrow{\delta'} & H^1(B_T,\underline{\mathcal{G}})
\end{array}$$

The identifications are due to the assumption $\mathcal{R}|_B = \mathcal{H}|_B$. Since T is affinoid and hence the Γ-functor is exact, the exact sequence $\mathcal{F} \to \mathcal{H} \to \mathcal{H}/\mathcal{R} \to 0$ gives rise to the exact sequence

$$\begin{array}{ccccccc}
\Gamma(T,\mathcal{F}) & \longrightarrow & \Gamma(T,\mathcal{H}) & \longrightarrow & \Gamma(T,\mathcal{H}/\mathcal{R}) & \longrightarrow & 0 \\
\downarrow\cong & & \downarrow\cong & & \hookdownarrow & & \\
\Gamma(B_T,\mathcal{F}) & \longrightarrow & \Gamma(B_T,\mathcal{R}) & \xrightarrow{\delta'} & H^1(B_T,\underline{\mathcal{G}}) & &
\end{array}$$

The first two vertical down-arrows are isomorphism due to Proposition 6.8 because of $\underline{\mathcal{H}} = \underline{\mathcal{H}}^{[n]}$ and $\underline{\mathcal{H}}|_B = \mathcal{R}|_B$. Put $A := \mathrm{Supp}(\mathcal{H}/\mathcal{R}) \cap T$. Then we have $A \cap B_T = \varnothing$ because of $\mathcal{R}|_B = \mathcal{H}|_B$. Thus there even exist nonzero elements $f_j \in \mathcal{O}_D(U)\langle\eta_j\rangle$ with $f_j \cdot \mathrm{Im}(\delta') = 0$ for $j = 1, \ldots, d$. Due to Proposition 6.17, there exist nonzero polynomials $p_j \in \mathcal{O}_D(D)[\eta_j]$ for $j = 1, \ldots, d$ such that $\mathfrak{a}^k p_j \,\mathrm{Im}(\delta) = 0$ and $V(p_j) \cap (D \times \partial\mathbb{D}^1) = \varnothing$, where $\mathfrak{a} := \mathrm{Id}(\mathrm{S}_{n+1}(\underline{\mathcal{G}}))$ is the vanishing ideal of $\mathrm{S}_{n+1}(\mathcal{G})$. Note $\mathrm{S}_{n+1}(\underline{\mathcal{G}}) \cap B = \varnothing$. So, for $x \in B$, the following sequences are exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(B,\mathcal{G}) \otimes \mathcal{O}_x & \longrightarrow & \Gamma(B,\mathcal{F}) \otimes \mathcal{O}_x & \longrightarrow & \Gamma(B,\mathcal{R}) \otimes \mathcal{O}_x & \longrightarrow & 0 \\
 & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \\
0 & \longrightarrow & \mathcal{G}_x & \longrightarrow & \mathcal{F}_x & \longrightarrow & \mathcal{R}_x & \longrightarrow & 0
\end{array}$$

Since $\mathcal{G}$ extends to $D \times \mathbb{D}^d$, the canonical morphism α is bijective. Since $\mathcal{R}|_B = \underline{\mathcal{H}}|_B$, the morphism γ is bijective. Then β has to be bijective. Now the assertion follows from Lemma 8.2. $\square$

Theorem 8.4. *Every ball figure $B \subset X$ of dimension n has property* (G_{n+1}).

Proof. The assertion on the uniqueness follows by Corollary 6.13.

Now consider a coherent sheaf $\mathcal{F} = \mathcal{F}^{[n+1]}$. Let $m < \dim X$ be the greatest integer with $0_{[m+1]\mathcal{F}} = 0$. Obviously, we have $n+1 \leq m$ since, for $n+1 \geq m$, we have $\mathcal{F} = 0$. Otherwise, set $\mathcal{G} := 0_{[m+2]\mathcal{F}}$ and $\mathcal{R} := \mathcal{F}/\mathcal{G}$. By Lemma 8.1, we know that $\mathcal{G} = \mathcal{G}^{[n+1]}$ extends to a coherent sheaf on X. Since $0_{[m+2]\mathcal{R}} = 0$, the sheaf $\mathcal{R}^{[n+1]}$ is coherent by Proposition 6.7 (d) and satisfies the induction hypothesis. So $\mathcal{R}^{[n+1]}$ extends to a coherent sheaf on X. Put $S := \mathrm{S}_{n+2}(\mathcal{R})$ and $T := \mathrm{S}_{n+2}(\mathcal{G})$. Then we have $\dim S \leq n+1$ and $\dim T \leq n$; cp. Proposition 6.7. Due to Lemma 1.5, there exists a finite morphism $\psi \colon X \to \mathbb{D}^{n+d}$ with

$$\psi^{-1}(\mathbb{D}^n \times \partial\mathbb{D}^d) =: B' \subset B \quad \text{and} \quad T \cap B' = \varnothing,$$
$$\psi^{-1}(\mathbb{D}^{n+1} \times \partial\mathbb{D}^{d-1}) \cap S = \varnothing.$$

By Lemma 8.3, we obtain that $\psi_*(\mathcal{F}|_{B'})$ extends to a coherent sheaf on $\mathbb{D}^{n+d}$. Then $\Gamma(B', \mathcal{F})$ generates each stalk $\mathcal{F}_x$ for $x \in B'$. The assertion follows by Lemma 8.2. □

Theorem 8.5. *Let X be a rigid space, and let $S \subset X$ be a closed analytic subset of X of dimension n. Then the couple $(X - S, X)$ has all the properties* (E_{n+1}), (M_{n+1}), (A_{n+1}), (U_{n+1}), (G_{n+1}).

Proof. We may assume that X is affinoid. Then $B := X_{\tilde{S}} \subset X$ is a ball figure of dimension n. Due to property (E_{n+1}), established in Theorem 4.6, and due to Lemma 8.2, it suffices to know that (B, X) has all the asserted properties. So the assertion follows from Theorem 4.6 and Theorem 8.4. □

Extension properties of Hartogs figures. The first step towards the extension property $(\mathrm{G}(n))$ for the n-dimensional Hartogs figure is the following lemma which settles the case of the special Hartogs figure discussed in Proposition 1.14. In contrast to Proposition 1.14, for technical reasons, we will denote the coordinate functions of $\mathbb{D}^{n+d}$ by $\zeta_1, \ldots, \zeta_n, \eta_1, \ldots, \eta_d$; *i.e.*, we switch the factors of $\mathbb{D}^{n+d} = \mathbb{D}^n \times \mathbb{D}^d$. The tube of the Hartogs figure is always given by the first coordinate functions $\zeta_1, \ldots, \zeta_n$.

Lemma 8.6. *Denote by $(\zeta_1, \ldots, \zeta_n, \eta_1, \eta_2)$ the coordinate functions on $\mathbb{D}^{n+2}$. Consider the following figure of Hartogs type of dimension n in $\mathbb{D}^{n+2}$:*

$$H = T \cup B \quad \text{with } T := (\mathbb{D}^n_+ \times \mathbb{D}^2)_{\tilde{h}} \text{ and } B := \mathbb{D}^{n+2}_{\tilde{N}},$$

where $\mathbb{D}^n_+ := \{x \in \mathbb{D}^n;\ |\zeta_i(x)| < 1 \text{ for } i = 1, \ldots, n\}$. Assume that the following is satisfied.

(o) *$\tilde{N} \subset \mathbb{A}^{n+2}$ is of dimension n.*
(i) *$\tilde{N} \cap V(\tilde{\zeta}_1, \ldots, \tilde{\zeta}_n)$ consists of finitely many points $\{0, \tilde{y}_1, \ldots, \tilde{y}_r\}$.*
(ii) *Every irreducible component of $\tilde{N}$ contains the origin $\{0\}$.*
(iii) *$\tilde{h} \in k[\eta_1]$ is a polynomial with $\tilde{h}(0) \neq 0$ and $\tilde{h}(\tilde{y}_i) = 0$ for $i = 1, \ldots, r$.*

Let $\mathcal{F}$ be a coherent sheaf on H with $\mathcal{F} = \mathcal{F}^{[n]}$, and assume that $\mathcal{F}$ is locally free on $\mathbb{D}^{n+2}_{\tilde{N}}$. Then $\mathcal{F}$ extends to coherent sheaf $\underline{\mathcal{F}}$ on $\mathbb{D}^{n+2}$ satisfying $\underline{\mathcal{F}} = \underline{\mathcal{F}}^{[n]}$.

Proof. Since $\dim \tilde{N} = n < n+1$, there exists a polynomial $f \in K[\zeta, \eta_2]$ with $|f| = 1$ and $\tilde{N} \subset V(\tilde{f})$. At first, we perform a transformation of the coordinates $\zeta_i \mapsto \zeta_i - \eta_2^{t_i}$ and $\eta_2 \mapsto \eta_2$ such that $\tilde{f}$ is monic (up to a unit) in η_2. Moreover, we can arrange the transformation in such a way that $V(\tilde{\zeta}_1 - \tilde{\eta}_2^{t_1}, \dots, \tilde{\zeta}_n - \tilde{\eta}_2^{t_n})$ intersects $\tilde{N}$ in finitely many points. Note that the origin always belongs to this intersection. Then f gives rise to a finite map $\phi\colon \mathbb{D}^{n+2} \to \mathbb{D}^{n+2}$ such that $\tilde{\phi}(\tilde{N}) \subset \mathbb{A}^{n+1} \times (\mathbb{A}^1 - \{0\})$. Then we have $\tilde{\phi}(\tilde{N}) = V(\tilde{g}, \tilde{\eta}_2)$, where $\tilde{g} \in k[\zeta, \eta_1]$.

After a linear transformation $\eta_2 \mapsto \eta_2$ and $\eta_1 \mapsto \eta_2 + \eta_1^{s_1}$, we may assume that conditions (o) to (iii) are fulfilled for the target situation. So we may assume that $\tilde{N} \subset V(\tilde{\eta}_2)$ since it suffices to show the extension of $\phi_*\mathcal{F}$. Namely, then $\mathcal{F}$ is generated by global sections, and the extension follows from Theorem 4.6. Moreover, we may have that $\tilde{N} \subset V(\tilde{g}, \tilde{\eta}_2)$, where $\tilde{g} \in k[\zeta, \eta_1]$ and all irreducible factors of $\tilde{g}$ have a zero at the origin of $\mathbb{A}_k^{n+1}$. Furthermore, we remind that $\phi_*\mathcal{F}$ is locally free on $\mathbb{D}^{n+1} \times \partial\mathbb{D}^1$ due to [31, Cor. 2, p. IV-36].

Then we can extend $\mathcal{F}$ to a coherent sheaf $\overline{\mathcal{F}}$ on $\mathbb{D}_{\tilde{g}}^{n+1} \times \mathbb{P}^1$ by trivial extension since $\mathcal{F}$ is associated to a free module over $\mathbb{D}^{n+1} \times \partial\mathbb{D}^1$ due to [27, Satz 2]. Due to GAGA, we know that $\overline{\mathcal{F}}|_{\mathbb{D}_{\tilde{g}}^{n+1} \times \mathbb{A}_K^1}$ is algebraic over $T_n\langle \eta_1, 1/g \rangle[\eta_2]$. Due to the famous theorem of Quillen [28], we have that $\overline{\mathcal{F}}|_{\mathbb{D}_{\tilde{g}}^{n+1} \times \mathbb{A}_K^1}$ is extended from a projective $T_n\langle \eta_1, 1/g \rangle$-module P of finite type. Now we apply Lemma 8.7 below. So there exists a function $a \in T_n$ with $|a| = 1$ such that P is free over $\{x \in \mathbb{D}^n;\ |a(x)| \geq c\}$ for a suitable $c \in |K^\times|$ with $c < 1$.

As in the proof of Lemma 2.10, there is an étale extension $(\tilde{D}, \tilde{x}) \to (\mathbb{A}_k^n, \tilde{0})$ such that $\tilde{g}$ decomposes into factors $\tilde{g} = \tilde{g}_1 \cdot \tilde{g}_2$ over $\tilde{D} \times \mathbb{A}_k^1$ such that $V(\tilde{g}_1) \to \tilde{D}$ is finite, $V(\tilde{g}_1)$ contains $(\tilde{x}, \tilde{0})$ and $V(\tilde{g}_2)$ does not contain $(\tilde{x}, \tilde{0})$; cp. [12, Prop. 2.3/8]. After shrinking $\tilde{D}$, we may assume that $V(\tilde{g}_1)$ and $V(\tilde{g}_2)$ are disjoint. Let $D \to \mathbb{D}^n$ be a lifting of $\tilde{D} \to \mathbb{A}_k^n$.

Let $\mathcal{P}$ be the sheaf associated to P. Since $\mathcal{P}$ is free over the subdomain $D_{a,c} := \{x \in D;\ |a(x)| \geq c\}$, we can extend $\mathcal{P}$ and hence $\mathcal{F}$ across $V(\tilde{g}_2)$ by the free sheaf to a locally free sheaf $\mathcal{F}_1$ on

$$H_1 := ((D_{a,c} \cap D_+^n) \times \mathbb{D}^2) \cup ((D_{a,c} \times \mathbb{D}^2)_{\tilde{g}_1} \cup (D_{a,c} \times \mathbb{D}^1 \times \partial\mathbb{D}^1)).$$

Note that $(D_{a,c} \cap D_+) \neq \varnothing$, where D_+ is the formal fiber of D at $\tilde{x}$. Now we obtain the extension of $\mathcal{F}_1$ on $D \times \mathbb{D}^2$ with $\mathcal{F}_1 = \mathcal{F}_1^{[n]}$ by Proposition 7.7. Indeed, we perform the finite projection $\psi\colon H_1 \to H_2$ by g_1 onto the Hartogs figure H_2 as in Proposition 7.7 satisfying $\psi^{-1}(H_2) \subset H_1$. This shows that $\mathcal{F}_2 := \psi_*\mathcal{F}_1$ is generated by global sections. So the extension of $\mathcal{F}_1$ to $D_{a,c} \times \mathbb{D}^2$ follows by the extension property for coherent subsheaves, Theorem 4.6. By the extension property for ball figures, Theorem 8.4, we see that $\mathcal{F}_1$ extends to a coherent sheaf $\underline{\mathcal{F}}_1$ on $D \times \mathbb{D}^2$ satisfying the property $\underline{\mathcal{F}}_1 = \underline{\mathcal{F}}_1^{[n]}$.

It remains to descend the coherent sheaf $\underline{\mathcal{F}}_1$ to $\mathbb{D}^{n+2}$. Let $X \subset \mathbb{D}^{n+1}$ be the image of $X' := (D \times \mathbb{D}^1)_{\tilde{g}_2}$. This is a formal open subset of $\mathbb{D}^{n+1}$. Due to the conditions on the irreducible components of $\tilde{g}$, we have that X contains a dense formally open part of $V(\tilde{g})$. Set $Y := X \times \mathbb{D}^1$ and $Y' := X' \times \mathbb{D}^1$. Then we will consider a descent datum with respect to $p\colon Y' \to Y$ on the coherent sheaf $\mathcal{G}' := \underline{\mathcal{F}}_1|_{Y'}$. Denote by $p_i\colon Y'' := Y' \times_Y Y' \to Y'$ the i-th projection for $i = 1, 2$. We have a canonical descent datum

$$\varphi\colon p_1^*\mathcal{G}'|_{Y'_{\tilde{g}}} \xrightarrow{\sim} p_2^*\mathcal{G}'|_{Y'_{\tilde{g}}}$$

which is also defined on $D_+ \times \mathbb{D}^2$. Then, due to Proposition 6.10, the map φ extends to a descent datum

$$\underline{\varphi}\colon p_1^*\mathcal{G}'|_{Y'} \xrightarrow{\sim} p_2^*\mathcal{G}'|_{Y'}.$$

By [11, Thm. 3.1], this descent is effective. So there exists a coherent sheaf $\underline{\mathcal{F}}$ on Y which extends $\mathcal{F}|_Y$. Since $\tilde{Y}$ contains $V(\tilde{g}_1)$ except for a lower-dimensional closed subset, $\tilde{Y}$ contains $\tilde{N}$ except for a lower-dimensional closed subset. Finally, the assertion follows from Theorem 8.4. □

Lemma 8.7. *Let $A := K\langle\zeta_1, \dots, \zeta_n\rangle$ and $B := A\langle\eta\rangle\langle 1/g\rangle$, where $g \in A\langle\eta\rangle$ with $|g| = 1$. Let P be a finitely generated projective B-module of rank r. Then there exist a free submodule $F \subset P$ of rank r and an element $a \in A$ with $|a| = 1$ such that the support of the quotient P/F is contained in $\{x \in \mathrm{Sp}(A);\ |a(x)| \le c\}$ for some $c \in |K^\times|$ with $c < 1$.*

Proof. We remark that B is factorial in any dimension of A as it follows from [24, Lem. 6.2.3].

In the case of $\dim A = 1$, we can prove more; namely, the assertion is true for $c = 0$. So let us explain this case first. We successively choose elements $t_1, \dots, t_r \in P$ such that $F := Bt_1 + \dots + Bt_r$ satisfies our assertion. We start with a nonzero element $s_1 \in P$. Since B is factorial, we can write $s_1 = b_1 t_1$ such that the vanishing locus $V(t_1)$ is of codimension 2. Since B is 2-dimensional, $V(t_1)$ consists of finitely many points. Thus there exists a nonzero $a_1 \in A$ such that $V(t_1) \subset V(a_1)$. Now we look at $P_1 := P/Bt_1$ which is projective after localizing by a_1. Since B_{a_1} is factorial, by the same argument as above, there exists $t_2 \in P$ such that the locus of t_2 in P_1 is of codimension 2. Then we choose a nonzero element $a_2 \in A$ such that $P_2 := P/(Bt_1 + Bt_2)$ localized by $a_1 a_2$ is projective. We continue in this way, and hence we arrive at sequence $t_1, \dots, t_r \in P$ and $a_1 \cdot \ldots \cdot a_r \in A$ such that $F := Bt_1 + \dots + Bt_r$ satisfies our assertion with $a := a_1, \dots, a_r$.

Now we consider the general case. At first, we proceed in the same way as above, but instead of the affine localization, we use the affinoid localization. Since the reduction of the projection of $V(t_1)$ is contained in an algebraic subset of codimension 1, there is an $a_1 \in A$ with $|a_1| = 1$ such that $V(t_1) \subset \{x \in X;\ |a_1(x)| < 1\}$. Then we look at the subset $\{y \in \mathrm{Sp}(B);\ |a_1(y)| = 1\}$ instead of the affine localization B_{a_1}. So, by a procedure similar to above, we end up with elements $t'_1, \dots, t'_r \in \Gamma(\mathrm{Sp}(B)_{\tilde{a}}, \tilde{P})$, where $\tilde{P}$ is the coherent sheaf associated to P on $\mathrm{Sp}(B)$ such that $F' := B_{\tilde{a}} t'_1 + \dots + B_{\tilde{a}} t'_1 = \Gamma(\mathrm{Sp}(B)_{\tilde{a}}, \tilde{P})$ is free. Now we can approximate $t'_1, \dots, t'_r$ by elements $t_1, \dots, t_r$ in P_a. After multiplying these sections by a certain power of a, we may assume that $t_1, \dots, t_r \in P$ belong to P and generate P over $\mathrm{Sp}(B_{\tilde{a}})$. The function a takes its maximum on the support of P/F with $F := Bt_1 + \cdots + Bt_r$ which is a certain number $c \in |K^\times|$ with $c < 1$. Thus the assertion is clear. □

Lemma 8.8. *Let $X = \mathrm{Sp}(A)$ be an affinoid space of pure dimension $(n+d)$ with $d \geq 2$, and let $H = T \cup B \subset X$ be a Hartogs figure of dimension n as in Definition 1.10. Then (H, X) has property $(\mathrm{G}(n))$.*

Proof. Let $\mathcal{G} = \mathcal{G}^{[n]}$ be a coherent sheaf on H with $0_{[m+1]\mathcal{G}} = 0$, $0_{[m+2]\mathcal{G}} = \mathcal{G}$ for some $m \geq n$. Then $\mathcal{F} := \mathcal{G}^{[m]}$ is coherent by Proposition 6.7 (d), and the subsheaf $\mathcal{G} \subset \mathcal{F}$ fulfills the condition $\mathcal{G}_{[n]\mathcal{F}} = \mathcal{G}$. If $m \geq n+1$, then the assertion for $\mathcal{F}$ is covered by Lemma 8.1 since the Hartogs figure contains a ball figure of dimension n. Because (H, X) has property (U_n) by Theorem 4.6, the assertion for $\mathcal{G}$ is clear. The extension of an isomorphism $\varphi\colon \mathcal{G}|_B \to \underline{\mathcal{G}}|_B$ to an isomorphism over the tube T follows from Proposition 6.8 since $B \cap T$ is a ball figure of dimension n in T by Lemma 1.11.

So we assume that $m = n$ and $0_{[n+1]\mathcal{F}} = 0$ and $0_{[n+2]\mathcal{F}} = \mathcal{F}$. In particular, we have $\dim \mathrm{S}_{n+1}(\mathcal{F}) \leq n-1$. The support $S := \mathrm{Supp}(\mathcal{F})$ is of pure dimension $n+2$. By Theorem 3.6, the closed analytic set S extends to a pure dimensional closed analytic subset $\underline{S}$ of X.

We want to project the subset $\underline{S}$ to a Hartogs figure in $\mathbb{D}^{n+2}$ such that we can apply our result Lemma 8.6. For doing so, we proceed as in the proof of Theorem 3.6. At first, assume that we have to consider only one tube. Then we may assume that this tube is maximal. Namely, by using Lemma 1.11, we can directly reduce the assertion to the special Hartogs figure $(\mathbb{D}^n(\varepsilon) \times \mathbb{D}^2) \cup (\mathbb{D}^n \times \partial\mathbb{D}^2)$ in $\mathbb{D}^{n+2}$, and we are done by Lemma 8.6 and Theorem 4.6 as usual. After that, due to Proposition 1.14, we have a finite morphism $\phi\colon X \to \mathbb{D}^{n+d}$ and a polynomial $\tilde{h} \in k[\eta_1]$ with $\tilde{h}(0) \neq 0$ satisfying the following properties.

(o) Set $H' := T \cup B \subset \mathbb{D}^{n+d}$ with $T := (\mathbb{D}^n(\underline{\varepsilon}) \times \mathbb{D}^d)_{\tilde{h}}$ and $B := \mathbb{D}^{n+d}_{\tilde{N}}$.

(i) $H' \subset \mathbb{D}^{n+d}_{\tilde{h}}$ is a Hartogs figure of dimension n in $\mathbb{D}^{n+d}_{\tilde{h}}$ with $\phi^{-1}(H') \subset H$.

(ii) The induced map $\phi\colon \phi^{-1}(H') \to H'$ is finite.

At this point, it is eventually necessary to extend the base field. Since we only need to know that $\mathcal{F}$ is generated by global sections, the extension of the base field poses no problem. Furthermore, it suffices to show that $\phi_*\mathcal{F}$ is generated by global sections. Namely, then the assertion follows from the extension property for subsheaves, Theorem 4.6. Now we proceed as in the proof of Lemma 3.5 where we enlarge the radii $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$ step by step to $\varepsilon_\nu = 1$ for $1 \leq \nu \leq n$. In each step, we obtain the extension of $\phi_*\mathcal{F}$ by applying Lemma 8.6. Namely, as exercised in Lemma 3.5, at each step, there is a finite covering map of S to a Hartogs figure of dimension n in a space of dimension $n+2$.

For the general case, we have to reduce the number of tubes. For that, one can choose a function $a \in A$ with $|a| = 1$ such that $\tilde{a}$ vanishes on all the irreducible components which do not meet the given complete intersection defining one fixed tube. Then we look at $X_{\tilde{a}}$. So we arrive at a situation where we can apply the procedure of above. Doing so for all tubes, we obtain the extension on a ball figure of dimension $n-1$. Finally, we succeed by Theorem 8.5. □

The proof of the extension property (G_n) for Hartogs figures follows in a similar way as in the case of ball figures, Theorem 8.4. As preparation, we need an analog of Lemma 8.3 adjusted to the special Hartogs figure we mentioned in Lemma 8.6.

Lemma 8.9. *As in Lemma 8.6, consider the following figure of Hartogs type of dimension n in $\mathbb{D}^{n+d}$:*

$$H = T \cup B \quad \text{with } T := (\mathbb{D}^n_+ \times \mathbb{D}^d)_{\tilde{h}} \text{ and } B := \mathbb{D}^{n+d}_{\tilde{N}},$$

where $\tilde{h} \in k[\eta_1]$ is a monic polynomial. Let $h \in K[\eta_1]$ be a monic lifting of $\tilde{h}$.

Let $\mathcal{F}$ be a coherent sheaf on H satisfying $\mathcal{F} = \mathcal{F}^{[n]}$. Set $\mathcal{G} = 0_{[m+2]\mathcal{F}}$ for some $m \geq n$ and $\mathcal{R} := \mathcal{F}/\mathcal{G}$. Assume that $\operatorname{cdh}(\mathcal{G}|_B) \geq n+2$ and $\mathcal{R}|_B = \mathcal{R}^{[n]}|_B$. If the coherent sheaves $\mathcal{G} = \mathcal{G}^{[n]}$ resp. $\mathcal{H} := \mathcal{R}^{[n]}$ extend as coherent sheaves $\underline{\mathcal{G}} = \underline{\mathcal{G}}^{[n]}$ resp. $\underline{\mathcal{H}} = \underline{\mathcal{H}}^{[n]}$ on $\mathbb{D}^{n+d}$, then there exists a coherent sheaf $\underline{\mathcal{F}} = \underline{\mathcal{F}}^{[n]}$ on $\mathbb{D}^{n+d}$ such that $\underline{\mathcal{F}}|_H = \mathcal{F}$ over H.

Proof. Due to Theorem 8.4, we may assume $m = n$. Set $B_T := B \cap T$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
\Gamma(B,\underline{\mathcal{G}}) & \longrightarrow & \Gamma(B,\mathcal{F}) & \longrightarrow & \Gamma(B,\mathcal{R}) & = & \Gamma(B,\underline{\mathcal{H}}) & \xrightarrow{\delta} & H^1(B,\underline{\mathcal{G}}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Gamma(B_T,\underline{\mathcal{G}}) & \longrightarrow & \Gamma(B_T,\mathcal{F}) & \longrightarrow & \Gamma(B_T,\mathcal{R}) & = & \Gamma(B_T,\underline{\mathcal{H}}) & \xrightarrow{\delta'} & H^1(B_T,\underline{\mathcal{G}})
\end{array}$$

The identifications are due to the assumption $\mathcal{R}|_B = \mathcal{H}|_B$. Since T is affinoid, the exact sequence $\mathcal{F} \to \mathcal{H} \to \mathcal{H}/\mathcal{R} \to 0$ gives rise to the exact sequence

$$\begin{array}{ccccccc}
\Gamma(T,\mathcal{F}) & \longrightarrow & \Gamma(T,\mathcal{H}) & \longrightarrow & \Gamma(T,\mathcal{H}/\mathcal{R}) & \longrightarrow & 0 \\
\downarrow\cong & & \downarrow\cong & & \downarrow & & \\
\Gamma(B_T,\mathcal{F}) & \longrightarrow & \Gamma(B_T,\mathcal{R}) & \xrightarrow{\delta'} & H^1(B_T,\underline{\mathcal{G}}) & &
\end{array}$$

The vertical down-arrows are isomorphisms due to Proposition 6.10 because of $\underline{\mathcal{H}} = \underline{\mathcal{H}}^{[n]}$ and $\underline{\mathcal{H}}|_B = \mathcal{R}|_B$. Set $S := \operatorname{Supp}(\mathcal{H}/\mathcal{R}) \cap T$. Then we have $S \cap B_T = \varnothing$ because of $\mathcal{R}|_B = \mathcal{H}|_B$. So there exist nonzero elements $f'_j \in \mathcal{O}_Y(U)\langle\eta_j\rangle$ with $f'_j \cdot \operatorname{Im}(\delta') = 0$ and $V(f_j) \cap (U \times \partial\mathbb{D}^1) = \varnothing$ for $j = 1, \ldots, d$.

Now we apply Lemma 8.10 from below. So there is a formal étale neighborhood $(Y, \tilde{y}) \to (\mathbb{D}^n, \tilde{0})$ satisfying all the properties mentioned there. Due to Proposition 6.17, there exist nonzero polynomials $p_j \in \mathcal{O}_Y(Y)[\eta_j]$ for $j = 1, \ldots, d$ such that $\mathfrak{a}^k p_j \operatorname{Im}(\delta) = 0$ and $V(p_j) \cap (Y \times \partial\mathbb{D}^1) = \varnothing$, where $\mathfrak{a} := \operatorname{Id}(\mathrm{S}_{n+1}(\underline{\mathcal{G}}))$ is the vanishing ideal of $\mathrm{S}_{n+1}(\underline{\mathcal{G}})$. Note $\mathrm{S}_{n+1}(\underline{\mathcal{G}}) \cap B = \varnothing$. So, for

$x \in B$, the following sequences are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(B,\mathcal{G})\otimes\mathcal{O}_x & \longrightarrow & \Gamma(B,\mathcal{F})\otimes\mathcal{O}_x & \longrightarrow & \Gamma(B,\mathcal{R})\otimes\mathcal{O}_x & \longrightarrow & 0 \\ & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \\ 0 & \longrightarrow & \mathcal{G}_x & \longrightarrow & \mathcal{F}_x & \longrightarrow & \mathcal{R}_x & \longrightarrow & 0 \end{array}$$

Since $\mathcal{G}$ extends to $Y_K \times \mathbb{D}^d$, the canonical morphism α is bijective. Since $\mathcal{R}|_B = \underline{\mathcal{H}}|_B$, the morphism γ is bijective. Then β has to be bijective. Thus $\mathcal{F}$ is generated by global sections over $(Y_K \times \mathbb{D}^d)$, and hence $\mathcal{F}$ extends to a coherent sheaf $\mathcal{F}_Y$ on $(Y_K \times \mathbb{D}^d)$ due to the extension property for coherent subsheaves, Theorem 4.6.

Since $Y_K \to \mathbb{D}^n$ is quasi-compact and faithfully flat onto a formally dense open subset $\mathbb{D}^n_{\tilde{a}}$ with $\tilde{a}(0) \neq 0$ of $\mathbb{D}^n$, by a descent argument, it follows that $\mathcal{F}$ extends to $(\mathbb{D}^n_{\tilde{a}} \times \mathbb{D}^n)$. Indeed, the canonical descent datum on $\mathcal{F}$ extends to a descent datum on $\mathcal{F}_Y$ as seen by an argument similar to the one applied in the proof of Lemma 8.6. This descent is effective due to [11, Thm. 3.1]. So we obtain an extension of $\mathcal{F}$ to the $(n-1)$-dimensional ball figure $\mathbb{D}^{n+d}_{\tilde{N}\cap V(\tilde{a})}$ in $\mathbb{D}^{n+d}$. Finally, the assertion follows from Theorem 8.4. □

Lemma 8.10. *Keep the situation of Lemma 8.9. Let* $\underline{\mathcal{G}} = \mathcal{G}^{[n]}$ *be a coherent sheaf on* $\mathbb{D}^{n+d}$ *with* $\mathrm{cdh}(\underline{\mathcal{G}}|_B) \geq n+2$. *Let* $\mathfrak{a} := \mathrm{Id}(\mathrm{S}_{n+1}(\underline{\mathcal{G}}))$ *be the vanishing ideal of* $\mathrm{S}_{n+1}(\underline{\mathcal{G}})$. *Then there exist a formal étale neighborhood* $(Y,\tilde{y}) \to (\mathbb{D}^n, \tilde{0})$ *and a constant* $k \in \mathbb{N}$ *with the following properties.*

If the restriction of a cohomology class $\xi \in H^1(B,\underline{\mathcal{G}})$ *onto* B_T *is annihilated by monic polynomials* $f'_j \in \mathcal{O}(\mathbb{D}^n_+)\langle\eta_j\rangle$ *for* $j = 1,\dots,d$, *then there exist functions* $f_j \in \mathcal{O}(Y_K)\langle\eta_j\rangle$ *for* $j = 1,\dots,d$ *satisfying*

(i) $V(f_j) \cap (Y_K \times \partial\mathbb{D}^1) = \varnothing$ *for* $j = 1,\dots,d$,

(ii) $f_j \cdot \mathfrak{a}^k \cdot \xi = 0$ *in* $H^1(B_Y, \underline{\mathcal{G}})$, *where* $B_Y \subset B \times_{\mathbb{D}^n} Y_K$ *is a ball figure of dimension* n *in* $Y_K \times_{\mathbb{D}^n} \mathbb{D}^{n+d}$.

Proof. Since $\tilde{N} \to \mathbb{A}^n_k$ is quasi-finite at the origin $\{0\}$, there exists an étale neighborhood $\varphi\colon \tilde{Y} \to \mathbb{A}^n_k$ of 0 such that the pullback $\varphi^*\tilde{N} \to \tilde{Y}$ is finite due to [12, Prop. 2.3/8]. Then the coordinate functions $\tilde{\eta}_1,\dots,\tilde{\eta}_d$ satisfy monic polynomial minimal equations

$$\tilde{P}_j(\tilde{\eta}_j) := \tilde{\eta}_j^{t_j} + \tilde{a}_{j,t_j-1}\tilde{\eta}_j^{t_j-1} + \cdots + \tilde{a}_{j,0} = 0$$

with coefficients $\tilde{a}_{j,i} \in \mathcal{O}_{\tilde{Y}}(\tilde{Y})$. Then we obtain a finite morphism

$$\tilde{\Phi} := (\tilde{P}_1(\tilde{\eta}_1),\dots,\tilde{P}_1(\tilde{\eta}_1))\colon \tilde{Y} \times_{\mathbb{A}^n_k} \mathbb{A}^{n+d}_k \to \tilde{Y} \times \mathbb{A}^d_k$$

satisfying $\tilde{\Phi}^{-1}(\{\tilde{y}_0\} \times (\mathbb{A}^d_k - \{0\})) \supset \tilde{N}$. Then we lift all the data to the formal level, and hence we obtain a finite map $\Phi\colon Y \times_{\mathbb{D}^n} \mathbb{D}^{n+d} \to Y \times \mathbb{D}^d$ on the affinoid site.

We have to compute the cohomology group $H^{d-1}((Y \times \mathbb{D}^d)_{\tilde{N}}, \mathcal{O}_{Y\times\mathbb{D}^d})$. Set $X := Y \times \mathbb{D}^d$ and set $C := \mathcal{O}_Y(Y)$. Then we have $A := \mathcal{O}_X(X) = C\langle\eta_1,\dots,\eta_d\rangle$. Let $h_j \in C[\eta_j]$ be a monic polynomial of degree t_j for $j = 1,\dots,d$ such that

$\tilde{h}_j \in \tilde{C}[\eta]$ is the minimal polynomial of $\tilde{\eta}_j|_{\tilde{N}}$. Now consider the ball figure $B_Y := X_{V(\tilde{h}_1,\ldots,\tilde{h}_d)}$ of dimension n in X. We want to compute $H^{d-1}(B_Y, \mathcal{O}_X)$.

At first, we remark the following. If $h \in C[\eta]$ is a monic Weierstraß polynomial of degree $t \geq 1$, then any element $f \in \mathcal{O}_X(X_{\tilde{h}})$ has a unique representation $f = \sum_{\nu \leq 0} r_\nu \cdot h^\nu$, where $r_\nu \in C[\eta]$ is a polynomial of $\deg r_\nu \leq t-1$ for all $\nu < 0$ and $r_0 \in A$. Then, as in Lemma 6.15, we have the following description of $H^{d-1}(B, \mathcal{O}_X)$:

$$H^{d-1}(B_Y, \mathcal{O}_X) = A\langle h_1^{-1}, \ldots, h_d^{-1}\rangle \Big/ \bigoplus_{j=1}^{d} A\langle h_1^{-1}, \ldots, \widehat{h_j^{-1}}, \ldots, h_d^{-1}\rangle.$$

So any $\xi \in H^{d-1}(B_Y, \mathcal{O}_X)$ has unique representation

$$\xi = \sum_{\nu_1 < 0, \ldots, \nu_d < 0} c_{\nu_1,\ldots,\nu_d} \cdot h_1^{\nu_1} \ldots h_d^{\nu_d},$$

where $c_{\nu_1,\ldots,\nu_d} \in C[\eta_1, \ldots, \eta_d]$ are polynomials with $\deg_{\eta_j} c_{\nu_1,\ldots,\nu_d} < t_j$ for $j = 1, \ldots, d$.

Let $Y' \subset Y$ be a nonempty open affinoid subdomain which reduces to $\tilde{y}_0 \in \tilde{Y}$, which lies over the origin in $\mathbb{A}_k^n$. Set $B'_Y := (Y' \times \mathbb{D}^d) \cap B_Y$ and $T' := (Y' \times \mathbb{D}^d) \cap X_{\tilde{h}}$; regard $\tilde{h}$ as a factor of $\tilde{h}_1$. Set $C' := \mathcal{O}_Y(Y')$. Consider a cohomology class $\xi \in H^{d-1}(B, \mathcal{O}_X)$ such that the restriction $\xi|_{C'}$ is annihilated by some nonzero $f'_j \in C'\langle \eta_j \rangle$ for some $j \in \{1, \ldots, d\}$ such that $V(f'_j) \cap (Y \times \mathbb{D}^1)_{\tilde{h}_j} = \varnothing$ and $f'_j \cdot \xi = 0$. Then there exists an $f_j \in C[\eta_j]$ such that $V(f_j) \cap (Y \times \mathbb{D}^1)_{\tilde{h}_j} = \varnothing$ and $f_j \cdot \mathfrak{a}^k \cdot \xi = 0$. Indeed, as in Lemma 6.17, this follows by the extension property of meromorphic functions on Hartogs figures, Theorem 2.11. □

Theorem 8.11. *Let X be an affinoid space of pure dimension $n + d$, and let $H \subset X$ be a Hartogs figure of dimension n in X in the sense of Definition 1.10. Then (H, X) has property (G_n).*

Proof. The assertion on the uniqueness follows by Corollary 6.12 and Theorem 4.6. Indeed, let $\mathcal{F}$ be a coherent sheaf with $\mathcal{F} = \mathcal{F}^{[n]}$ on X, and let $\mathcal{H} = \mathcal{H}^{[n-1]}$ be another coherent sheaf such that there is an isomorphism $\phi\colon \mathcal{H}|_H \to \mathcal{F}|_H$ over an Hartogs figure H of dimension n. Then we consider $\mathrm{S}_{n+1}(\mathcal{H})$. Since $\dim \mathrm{S}_{n+1}(\mathcal{F}) \leq n-1$ by Proposition 6.7 (c), we have $\dim \mathrm{S}_{n+1}(\mathcal{H}|_H) \leq n-1$ due to the isomorphism. By property (E_n) for Hartogs figures, we obtain $\dim \mathrm{S}_{n+1}(\mathcal{H}|_H) \leq n-1$. Thus $\mathcal{H}$ satisfies $\mathcal{H} = \mathcal{H}^{[n]}$ due to Proposition 6.7 (c) as $\mathcal{H} = \mathcal{H}^{[n-1]}$. Then the uniqueness follows from Corollary 6.12.

Then the proof of the extension follows similarly to the proof of Theorem 8.4. Consider a coherent sheaf $\mathcal{F} = \mathcal{F}^{[n]}$. Let $m < \dim X$ be the greatest integer with $0_{[m+1]\mathcal{F}} = 0$. Obviously, we have $n \leq m$ since, for $n \geq m$, we have $\mathcal{F} = 0$. Otherwise, set $\mathcal{G} := 0_{[m+1]\mathcal{F}}$ and $\mathcal{R} := \mathcal{F}/\mathcal{G}$. Due to Lemma 8.8, we know that $\mathcal{G} = \mathcal{G}^{[n]}$ extends to a coherent sheaf on X. Since $0_{[m+1]\mathcal{F}} = 0$, the sheaf $\mathcal{R}^{[n]}$

is coherent by Proposition 6.7 (d) and satisfies the induction hypothesis. So $\mathcal{R}^{[n]}$ extends to a coherent sheaf on X.

Put $A := \mathrm{S}_{n+1}(\mathcal{R})$ and $B := \mathrm{S}_{n+1}(\mathcal{G})$. Then we have $\dim A \leq n$ and $\dim B \leq n-1$; cp. Proposition 6.7. Therefore, due to Proposition 1.14, there exists a finite morphism $\psi\colon X \to \mathbb{D}^{n+d}$ with $\psi^{-1}(H') \subset H$, where

$$H' := T' \cup B' \quad \text{with } T' := (\mathbb{D}^n_+ \times \mathbb{D}^d)_{\tilde{h}} \text{ and } B' := \mathbb{D}^{n+d}_{\tilde{N}}$$

is a figure of Hartogs type as in Lemma 8.6. By Lemma 8.9, we obtain that $\psi_*(\mathcal{F}|_{B'})$ extends to a coherent sheaf on $\mathbb{D}^{n+d}$. Then $\Gamma(B', \mathcal{F})$ generates each stalk $\mathcal{F}_x$ for $x \in B'$, where $B' := \mathbb{D}^{n+d}_{\tilde{N}}$. The assertion now follows by Lemma 8.2. □

Extension properties of rectilinear Hartogs figures. In the following, we consider a rectilinear Hartogs figure

$$H := (V \times \mathbb{D}^d) \cup (Y \times \partial\mathbb{D}^d) \subset X := (Y \times \mathbb{D}^d),$$

where Y is an irreducible affinoid space and $V \subset Y$ is a nonempty open subset. We will only sketch the procedure of the proof and leave details to the reader.

Theorem 8.12. *The rectilinear Hartogs figure of dimension n satisfies all the extension properties* (E_n), (M_n), (A_n), (U_n), (G_n).

Proof. We already know that the rectilinear Hartogs figure satisfies properties (E_n), (M_n), (A_n) resp. (U_n); cp. Proposition 1.12, Proposition 2.6, Proposition 3.14, resp. Theorem 4.6.

For the proof of (G_n), we proceed as in the proof of Proposition 3.14 by reducing the assertion to the case where Y is a curve, which follows from the case of Hartogs figures, Theorem 8.11, in the sense of Definition 1.10 by using the stable reduction theorem of curves. Indeed, instead of restricting to a closed analytic subset of Y of dimension 1, one can use tubular neighborhoods of such curves in the sense of Proposition 1.15. By this method, one obtains nonempty open subsets $V_i \subset Y_i'$, where the reduction of Y_i' is a formal open part of an irreducible component of $\tilde{Y}$ such that $Y_1' \cup \cdots \cup Y_r'$ is a formal dense open part of $\tilde{Y}$. Moreover, as in the proof of Proposition 3.14, one can assume that there exist finite projections $S \cap (Y_i' \times \mathbb{D}^d) \to Y_i' \times \mathbb{D}^t$ for $t := \dim S - \dim Y$ whenever a closed analytic subvariety $S \subset Y \times \mathbb{D}^d$ shows up in the several reduction steps like in Lemma 8.8 and Lemma 8.9. So we obtain the extension of coherent sheaves $\mathcal{F} = \mathcal{F}^{[n]}$ to a ball figure of dimension $n-1$. The latter case is handled by Theorem 8.5. □

We add a historical note. The crucial point for proving (G_n) is the coherence result, Theorem 7.1, which was published in [27]. In an unpublished paper of the author, this result was used to show property (G_n) for the rectilinear Hartogs figure $(\mathbb{D}^n(\varepsilon) \times \mathbb{D}^d) \cup (\mathbb{D}^n \times \partial\mathbb{D}^d)$. Moreover, the technicalities of § 4 and § 6 were provided in that notes. Later, Bartenwerfer used these latter results in [8] to generalize the results for the full assertions of this section.

Unfortunately, he avoided the use of Theorem 7.1, and so it appeared a fatal gap in his paper just at the beginning.

APPENDIX A

In this appendix, we provide Theorem A.17 which allows us to give a smooth proof of Theorem 3.15 (b). For this purpose, we reprove some results of Bartenwerfer which are difficult to access; cp. [6, 7]. There is also a contribution by van der Put [35]. They follow different methods, but they both make essential use of enlarged coverings; see below. So they get the vanishing result, Theorem A.7, only for $\mathcal{O}_X^\vee$ but not for $\mathcal{O}_X^\circ$ as we do. Our method is completely different from theirs. In the following, we assume that K is *algebraically closed.* We will use the following notations: let $c \in \sqrt{|K^\times|}$ and $c < 1$.

Let $\mathcal{O}_X^\circ$ be the subsheaf of $\mathcal{O}_X$ consisting of the functions f with spectral norm $|f| \leq 1$. Let $\mathcal{O}_X^\vee$ be the subsheaf of $\mathcal{O}_X$ consisting of the functions f with spectral norm $|f| < 1$. Let $\mathcal{O}_X(c)$ be the subsheaf of $\mathcal{O}_X$ consisting of the functions f with spectral norm $|f| < c$.

Metric cohomology of the polydisc. Let us start by recalling some standard definitions. A *rational covering* $\mathfrak{U} = \{U_0, \ldots, U_n\}$ of X is given by function $f_0, \ldots, f_n \in A$ without a common zero such that

$$U_i = \{x \in X;\ |f_j(x)| \leq |f_i(x)| \text{ for } j = 0, \ldots, n\}.$$

Such subsets are called *rational domains.*

If $\mathfrak{U} = \{U_0, \ldots, U_n\}$ is a rational covering as defined above, an *enlargement* of $\mathfrak{U}$ is a covering $\mathfrak{U}^\varrho := \{U_0^\varrho, \ldots, U_n^\varrho\}$, where $\varrho \in \sqrt{|K^\times|}$ and $\varrho > 1$ and

$$U_i^\varrho = \{x \in X;\ |f_j(x)| \leq \varrho \cdot |f_i(x)| \text{ for } j = 0, \ldots, n\}.$$

We cite the result of Gerritzen and Grauert [16], which is often used.

Lemma A.1. *Let $\mathfrak{X}$ be a finite covering of $X = \mathrm{Sp}(A)$ by open affinoid subdomains. Then there exists a rational covering $\mathfrak{U}$ of X which is a refinement of the covering $\mathfrak{X}$.*

Proof. See [9, Thm. 4.2/10]. □

Proposition A.2. *Let X be an affinoid space, and let $\mathfrak{U}$ be a finite covering by affinoid subdomains of X. Then there exists an element $t \in K^\times$ such that $t \cdot H^q(\mathfrak{U}, \mathcal{O}_X^\circ) = 0$ for all $q \in \mathbb{N}$ with $q \geq 1$.*

Proof. The spaces $C^q(\mathfrak{U}, \mathcal{O}_X^\circ)$ and $Z^q(\mathfrak{U}, \mathcal{O}_X^\circ)$ are Banach spaces. Since it holds that $H^q(\mathfrak{U}, \mathcal{O}_X) = 0$ for $q \geq 1$, the coboundary map

$$\partial^{q-1} \colon C^{q-1}(\mathfrak{U}, \mathcal{O}_X) \to Z^q(\mathfrak{U}, \mathcal{O}_X)$$

is surjective for $q \geq 1$. So the map is open due to Banach's theorem. Thus there exists an element $t \in K^\times$ such that $t \cdot Z^q(\mathfrak{U}, \mathcal{O}_X^\circ)$ is contained in the image of $\partial^{q-1}|C^{q-1}(\mathfrak{U}, \mathcal{O}_X^\circ)$. Thus we see that $t \cdot H^q(\mathfrak{U}, \mathcal{O}_X^\circ) = 0$ for all $q \in \mathbb{N}$ with $q \geq 1$. Note that $H^q(\mathfrak{U}, \mathcal{O}_X^\circ) = 0$ if q is larger than the number of members of the covering $\mathfrak{U}$; cp. Remark A.3. □

Remark A.3. In the situation of Proposition A.2, we have an exact sequence

$$\begin{aligned} H^q(\mathfrak{U},\mathcal{O}_X^\circ) &\xrightarrow{\cdot t} H^q(\mathfrak{U},\mathcal{O}_X^\circ) \xrightarrow{\varrho} H^q(\mathfrak{U},\mathcal{O}_X^\circ/t\cdot\mathcal{O}_X^\circ) \\ &\xrightarrow{\delta} H^{q+1}(\mathfrak{U},\mathcal{O}_X^\circ) \xrightarrow{\cdot t} H^{q+1}(\mathfrak{U},\mathcal{O}_X^\circ), \end{aligned}$$

where the maps $\cdot t$ are zero for $q \geq 1$ due to Proposition A.2 and hence ϱ is injective and δ is surjective for $q \geq 1$. Thus the vanishing of $H^q(\mathfrak{U},\mathcal{O}_X^\circ)$ for all $q \geq 1$ is equivalent to the vanishing of $H^q(\mathfrak{U},\mathcal{O}_X^\circ/t\cdot\mathcal{O}_X^\circ)$ for all $q \geq 1$. So, if we later work with formal models, the vanishing of $H^q(\mathfrak{U},\mathcal{O}_X^\circ)$ becomes a question in algebraic geometry of finite presentation over the ring R/Rt.

Next we concentrate on the vanishing of $H^1(\mathbb{D}^d,\mathcal{O}_{\mathbb{D}^d}^\circ)$. In contrast to Proposition A.2, we also have to deal with the limit of all rational coverings of X.

Proposition A.4. *Let $X = \mathbb{D}^d = \mathrm{Sp}(T_d)$ be the unit polydisc. For any rational covering $\mathfrak{V}$ of X, we have the short exact sequence*

$$0 \longrightarrow T_d \underset{\pi}{\overset{\iota}{\rightleftarrows}} C^0(\mathfrak{V},\mathcal{O}_X) \xrightarrow{\partial^0} Z^1(\mathfrak{V},\mathcal{O}_X).$$

The map ι has a left inverse π on the submodule $\ker(\partial^0) \subset C^0(\mathfrak{V},\mathcal{O}_X)$, and $\partial^0|_{\ker\pi}$ is an isometry with respect to the spectral norm.

Proof. Assume that $\mathfrak{V}$ is a rational covering given by $g_1,\dots,g_n \in T_d$. After a suitable transformation of the variables, we may assume that each $g_i \in T_d$ is a Weierstraß divisor. So we can write $g_i = u_i \cdot \omega_i$ for $i = 1,\dots,n$, where $\omega_i \in T_{d-1}[\eta]$ is a Weierstraß polynomial and $u_i \in T_d^\times$ is a unit. The coordinate functions of $\mathbb{D}^d$ are named by $\zeta_1,\dots,\zeta_{d-1},\eta$. The unit u_i can be written in the form $u_i = c_i \cdot e_i$, where e_i is a unit with absolute value $|e_i| = 1$ and $c_i \in K^\times$ is a constant. Note that the units $e_1,\dots,e_n$ have constant absolute value functions 1. Furthermore, we may assume that $|c_1| = 1 \geq \max\{|c_1|,\dots,|c_n|\}$.

A typical member V_i of $\mathfrak{V}$ has the form

$$V_i := \{x \in X;\, |c_1\omega_1(x)| \leq |c_i\omega_i(x)|,\dots,|c_n\omega_n(x)| \leq |c_i\omega_i(x)|\}.$$

We put $\omega := \omega_1 \cdot \ldots \cdot \omega_n$. So we have

$$V_0 := \{x \in X;\, |\omega(x)| = 1\} = \{x \in X;\, |\omega_1(x)| = 1,\dots,|\omega_n(x)| = 1\}.$$

Note that $V_0 \subset V_1$ is a subset of V_1 and it is connected and not empty since the polynomials are monic with absolute value $|\omega_i| = 1$ for all $i = 1,\dots,n$. Since we are free to refine the covering $\mathfrak{V}$, we add V_0 to our covering $\mathfrak{V}$. By abuse of notation, we denote the new covering by $\mathfrak{V}$, too. Any $f_0 \in \mathcal{O}_X(V_0)$ has a unique representation

$$f_0 = h + \sum_{\nu=1}^{\infty} a_\nu/\omega^\nu, \quad \text{where } h \in T_d,\ a_\nu \in T_{d-1}[\eta] \text{ with } \deg a_\nu < \deg\omega,$$

$$|f_0|_{V_0} = \max\{|h|,|\ell_0|\} \quad \text{with Laurent tail } \ell_0 := \sum_{\nu=1}^{\infty} a_\nu/\omega^\nu,$$

$$|\ell_0| = \max\{|a_\nu|;\, \nu \in \mathbb{N}\}.$$

Then we define the section $\pi\colon C^0(\mathfrak{V},\mathcal{O}_X)\to T_d$ for $f=(f_i)$ via

$$\pi((f_i))=\pi(f_0)=h\in T_d.$$

We will show that $|f|=|\partial^0(f)|$ for $f\in\ker(\pi)$. It suffices to work on the fibers $\{y\}\times\mathbb{D}^1$ for $y\in\mathbb{D}^{d-1}$. Thus, from now on, we may assume that $d=1$ and that the base field K is algebraically closed. In this case, we have a precise description of the affinoid subdomains; cp. [24, Prop. 2.4.8]. Moreover, we know its formal models and its reductions. The reduction is a tree-like configuration of smooth projective lines which meet transversally, and it has a refinement such that the components meet in ordinary double points. Then the assertions follow from the lemma below. □

Lemma A.5. *Let k be algebraically closed. Let P be a reduced connected algebraic curve whose irreducible components are smooth projective lines which meet transversally in ordinary double points and constitute a tree-like configuration. Let $x_0\in P$ be a closed point which is smooth. Set $D:=P-\{x_0\}$.*

(a) *For any open affine covering $\mathfrak{V}=\{V_1,\dots,V_n\}$ of P, the sequence*

$$0\longrightarrow k\underset{\pi}{\overset{\iota}{\rightleftarrows}} C^0(\mathfrak{V},\mathcal{O}_P)\xrightarrow{\partial^0} Z^1(\mathfrak{V},\mathcal{O}_P)\longrightarrow 0$$

is exact. The map ι maps $c\in k$ to the constant function $(c|_{V_i})\in C^0(\mathfrak{V},\mathcal{O}_P)$.

There is a left inverse π on $\ker(\partial^0)$ of the map ι by sending (f_i) to $f_1(x_1)$, where $x_1\in V_1$ is a closed smooth point of P.

(b) *Denote by η the coordinate function on the projective line L_0 which contains x_0. Assume that η has a pole at x_0. Then, for any open affine covering $\mathfrak{V}=\{V_1,\dots,V_n\}$ of D, the sequence*

$$0\longrightarrow k[\eta]\underset{\pi}{\overset{\iota}{\rightleftarrows}} C^0(\mathfrak{V},\mathcal{O}_D)\xrightarrow{\partial^0} Z^1(\mathfrak{V},\mathcal{O}_D)\longrightarrow 0$$

is exact. Any polynomial $f\in k[\eta]$ gives rise to a regular function $f|_{L_0}$ on L_0 which determines values at the intersection points of L_0 with the remaining components. By each of these values, one extends $f|_{L_0}$ onto the other components on the subsequent subtree following the given intersection point by constant functions, and hence one gets a global function h on D. The map ι maps a polynomial $f\in k[\eta]$ to the cocycle $(h|_{V_i})\in C^0(\mathfrak{V},\mathcal{O}_D)$.

There is a left inverse π on $\ker(\partial^0)$ of the map ι by sending an element $f:=(f_i)\in\ker(\partial^0)$ to $\pi(f|_{L_0})$, where $\pi(f|_{L_0})\in k[\eta]$ is the polynomial defined by the partial fraction decomposition of the rational function $f|_{L_0}$.

Proof. We know that ι is an isomorphism to $\ker(\partial^0)$ in both cases. Moreover, we have that $H^1(\mathfrak{V},\mathcal{O}_P)=0$ and $H^1(\mathfrak{V},\mathcal{O}_D)=0$ since it holds for the projective line resp. the affine line and hence for a tree-like configuration where all the components are projective lines except for the initial component L_0 which is an affine line, respectively. □

Remark A.6. Actually, one can generalize this method. It also works for a relative annulus $X:=Y\times A(r,1)$ if there is a covering $\mathfrak{X}$ of $X:=Y\times A(r,1)$ such that $(Y\times\partial\mathbb{D}^1(r))_{\omega_1}$ resp. $(Y\times\partial\mathbb{D}^1(1))_{\omega_0}$ is contained in a member V_1

resp. V_0 of our covering, where ω_1 resp. ω_0 are Weierstraß polynomials in r/η resp. $\eta/1$. In that case, one defines the section π via

$$\pi\colon C^0(\mathfrak{X}, \mathcal{O}_X^\circ) \to \mathcal{O}_X^\circ(X); \quad \pi((f)) = h_0 + \ell_r,$$

where $f_r \in \mathcal{O}_X(V_1)$ is written as $f_1 = h_1 + \ell_r$, where ℓ_r is the Laurent tail given by the hole $|\eta| < r$, and h_0 is given by the part of f_0 belonging to $\mathcal{O}_X(Y \times \mathbb{D}^1)$.

Proposition A.4 yields the first step of an induction process for proving the following vanishing theorem.

Theorem A.7. *Let $X = \mathbb{D}^d$ be the d-dimensional unit disc, and let $\mathcal{O}_X^\circ$ be the sheaf of holomorphic functions with $|f| \leq 1$. Then we have $H^q(X, \mathcal{O}_X^\circ) = 0$ for all $q \geq 1$.*

Proof. The vanishing of $H^1(X, \mathcal{O}_X^\circ) = 0$ follows directly from Proposition A.4.

For $q \geq 2$, we proceed by induction on d to show the assertion. In the 1-dimensional case, consider a rational covering $\mathfrak{V}$ of X. Then there exists a flat formal R-model $\mathcal{X}$ of X such that $\mathfrak{V}$ is induced by an open covering of that model; cp. [24, Thm. 3.3.4]. Then look at the sheaf of power bounded functions $\mathcal{O}_{\mathcal{X}}^\circ$. Due to Remark A.6, it suffices to show $H^q(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^\circ/t \cdot \mathcal{O}_{\mathcal{X}}^\circ) = 0$ for all $q \geq 2$ and a suitable $t \in K^\times$. Since $\mathcal{X}$ is 1-dimensional, the assertion follows by a theorem of Grothendieck; cp. [17] or [19, Thm. III, 2.7]. So we are done in the case $d = 1$.

Now consider the case $d \geq 2$, and assume that the assertion is true in the lower-dimensional case. As in the proof of Proposition A.4, we may assume that we have to consider a rational covering given by functions $g_0, \ldots, g_n \in T_d$. We may assume that $|g_i| \leq |g_0| = 1$ for $i = 1, \ldots, n$ and that g_0 is a Weierstraß polynomial of positive degree. Note that $\{V_0, \ldots, V_n, V_\infty\}$ with

$$\begin{aligned} V_\infty &:= \{x \in \mathbb{D}^{d-1} \times \mathbb{P}^1;\ |g_0(x)| \geq 1\}, \\ V_i &:= \{x \in \mathbb{D}^{d-1} \times \mathbb{D}^1;\ |g_0(x)| \leq |g_i(x)|, \ldots, |g_n(x)| \leq |g_i(x)|\} \end{aligned}$$

gives rise to an admissible covering of $P := \mathbb{D}^{d-1} \times \mathbb{P}^1$ by affinoid domains. Then any cocycle $(f) \in C^q(\mathfrak{V}, \mathcal{O}_X^\circ)$ of degree $q \geq 1$ can be regarded as a cocycle on P. Let $p\colon P \to Y := \mathbb{D}^{d-1}$ and $p'\colon X := \mathbb{D}^{d-1} \times \mathbb{D}^1 \to Y := \mathbb{D}^{d-1}$ be the projections. We have

$$p_*\mathcal{O}_X^\circ = \mathcal{O}_Y^\circ \quad \text{and} \quad p'_*\mathcal{O}_X^\circ = \widehat{\bigoplus_{\nu \in \mathbb{N}}} \mathcal{O}_Y^\circ \cdot \eta^\nu,$$

where η is the coordinate function on $\mathbb{P}^1$. Since H^q commutes with the formation of $\widehat{\bigoplus}$, we obtain $H^q(Y, p_*\mathcal{O}_P^\circ) = 0$ and $H^q(Y, p'_*\mathcal{O}_X^\circ) = 0$ for all $q \geq 1$ by the induction hypothesis.

Next we want to show $R^q p_*\mathcal{O}_P^\circ = 0$ for all $q \geq 1$. It suffices to show that, for any open affinoid subdomain $U \subset Y$ and any finite open affinoid covering $\mathfrak{V} = \{V_1, \ldots, V_m\}$ of $p^{-1}(U)$, there exists a finite open affinoid covering $\mathfrak{U} = \{U_1, \ldots, U_n\}$ such that $H^q(\mathfrak{V}_{U_i}, \mathcal{O}_P^\circ) = 0$, where $\mathfrak{V}_{U_i}$ denotes the covering $\mathfrak{V}$ restricted to $U_i \times \mathbb{P}^1$. For doing so, we choose an R-model $\pi\colon \mathcal{P} \to \mathcal{Y}$ of

$p\colon P \to Y = \mathbb{D}^{d-1}$ such that $\mathfrak{V}$ is induced by formal open subsets of $\mathcal{P}$ and such that π is flat; cp. [13, Cor. 5.10]. Since π is flat, the fibers of π are of dimension 1. Let $\mathcal{O}^\circ_{\mathcal{P}}$ be the normalization of $\mathcal{O}_{\mathcal{P}}$. Then $\mathcal{O}^\circ_{\mathcal{P}}$ is a coherent $\mathcal{O}_{\mathcal{P}}$-module. Namely, for any open affine subset $\mathcal{V}$ of $\mathcal{P}$, the normalization $\mathcal{O}^\circ_{\mathcal{P}}(\mathcal{V})$ is a finite $\mathcal{O}_{\mathcal{P}}(\mathcal{V})$-module due to the theorem of Grauert and Remmert; cp. [10, Cor. 6.4.1/5]. Moreover, the formation of normalization is compatible with formal localizations. So, due to the theorem of Grothendieck *loc. cit.*, the cohomology groups $H^q(\pi^{-1}(y), \mathcal{O}^\circ_{\mathcal{P}}|_{\pi^{-1}(y)}) = 0$ vanish for all $q \geq 2$ and for all closed points of $\mathcal{Y}$ since the fibers are of dimension 1. Thus we have $R^q p_* \mathcal{O}^\circ_P = 0$ for all $q \geq 2$ due to the theorem on cohomology and base change.

Now we consider the case $q = 1$. We want to show that $R^1 p_* \mathcal{O}^\circ_P = 0$. Keep the notation of above. Since we have that $H^0(P, \mathcal{O}_P) = \mathcal{O}_Y(U)$ and $H^q(P, \mathcal{O}_P) = 0$ for all $q \geq 1$, the sequence

$$0 \longrightarrow \mathcal{O}_Y(U) \xrightarrow{\iota} C^0(\mathfrak{V}, \mathcal{O}_P) \xrightarrow{\partial^0} Z^1(\mathfrak{V}, \mathcal{O}_P) \longrightarrow 0$$

is exact. The map ι maps an $h \in \mathcal{O}_Y(U)$ to the 0-chain $(h|_{V_j}) \in C^0(\mathfrak{V}, \mathcal{O}_P)$ which is constant on the fibers. The map ι admits local sections π_i with respect to the finite affinoid covering $\mathfrak{U} := \{\sigma^*(V_1), \dots, \sigma^*(V_n)\}$, where $\sigma\colon U \to \mathbb{P}^1$ is the point at infinity. The section π_i pulls back a 0-chain f by evaluating at infinity. The kernel $\ker(\pi_i)$ is mapped to $Z^1(\mathfrak{V}_i, \mathcal{O}_P)$ by ∂^0 in an isometric way, as follows from Lemma A.5. Thus we see $H^1(\mathfrak{V}_{U_i}, \mathcal{O}^\circ_P) = 0$. This in turn implies $R^1 p_* \mathcal{O}^\circ_P = 0$.

Concluding, we obtained $R^q p_* \mathcal{O}^\circ_P = 0$ for all $q \geq 1$. By the Leray spectral sequence, we finally obtain $H^q(P, \mathcal{O}^\circ_P) = 0$ for all $q \geq 1$. As we said at the beginning of the proof, $H^q(P, \mathcal{O}^\circ_P) = H^q(X, \mathcal{O}^\circ_X)$ for all $q \geq 1$. Finally, we obtain $H^q(X, \mathcal{O}^\circ_X) = 0$ for all $q \geq 1$. □

More generally, we have the following result.

Theorem A.8. *Let X be an affinoid space. The following holds for all $q \geq 1$.*
(a) *If X is smooth, there exists a constant $c \in K^\times$ with $c \cdot H^q(X, \mathcal{O}^\circ_X) = 0$.*
(b) *If X admits a smooth formal model, then $H^q(X, \mathcal{O}^\circ_X) = 0$.*

For the proof of Theorem A.8, we need preliminary results. Assertion (a) will follow from Proposition A.11 (iii). Assertion (b) will be shown at the end of this subsection.

Lemma A.9. *Let $\phi\colon X_K := \mathrm{Sp}(A) \to Y_K = \mathrm{Sp}(B)$ be a finite flat morphism. Then, for every finite affinoid covering $\mathfrak{U}$ of X, there exists a finite affinoid covering $\mathfrak{V}$ of Y_K such that $\phi^*\mathfrak{V}$ is finer than $\mathfrak{U}$. Here $\phi^*\mathfrak{V}$ denote the covering induced by the connected components of all $\phi^{-1}(V)$ for $V \in \mathfrak{V}$.*

Proof. Let $\varphi\colon X := \mathrm{Spf}(A^\circ) \to Y = \mathrm{Spf}(B^\circ)$ be the morphism of the associated affine formal models. There exists a formal blowing up $X'' \to X$ such that the covering $\mathfrak{U}$ is induced by a formal covering of X''; cp. [24, Thm. 3.3.4]. Since φ is flat, there exists an admissible formal blowing up $Y' \to Y$ such that the induced morphism $\varphi'\colon X' \to Y'$ is flat and finite [13, Cor. 5.10]. As a flat morphism φ' is open, for every point $x \in X'$, there is an open neighborhood V of $\varphi'(x)$ in Y such that each connected component of $\varphi^{-1}(V)$ is contained in some U which belongs to the covering $\mathfrak{U}$. □

Lemma A.10. *Let $X = \mathrm{Sp}(A)$ be a smooth irreducible affinoid space of dimension d. Consider a finite morphism $\varphi\colon X = \mathrm{Sp}(A) \to \mathbb{D}^d = \mathrm{Sp}(T)$ of rank n.*

(i) *Then there exists a T-basis $e_1, \dots, e_n$ of A with $e_1, \dots, e_n \in A^\circ$. Set*

$$F := \varphi^* T^\circ \cdot e_1 \oplus \cdots \oplus \varphi^* T^\circ \cdot e_n.$$

(ii) *For any small perturbation ψ of φ, see Proposition 1.16, we also have*

$$F = \psi^* T^\circ \cdot e_1 \oplus \cdots \oplus \psi^* T^\circ \cdot e_n.$$

(iii) *If the perturbation ψ is étale over $\mathbb{D}^d - V(h)$ for some nonzero function h on $\mathbb{D}^d$, then there exist a $c \in K^\times$ and an exponent $\alpha \in \mathbb{N}$ such that*

$$c \cdot h^\alpha \cdot \psi_* \mathcal{O}_X^\circ \subset \mathcal{O}_{\mathbb{D}^d}^\circ \cdot e_1 \oplus \cdots \oplus \mathcal{O}_{\mathbb{D}^d}^\circ \cdot e_n.$$

(iv) *With the notation of* (iii)*, we have $c \cdot \psi^* h^\alpha \cdot H^q(X, \mathcal{O}_X^\circ) = 0$ for all $q \geq 1$.*

(v) *One can choose $c \in K^\times$ such that $c \cdot A^\circ \subset F$.*

Proof. (i) The map φ is flat since the local rings of the source and the target are regular, and hence the homological dimension is 0 due to [31, Prop. 21, p. IV-35]. So $\varphi_* \mathcal{O}_X$ is locally free of rank n. Since locally free T-modules are free due to [27, Satz 1], there exists such a basis.

(ii) Since pullbacks of the coordinate functions $\zeta_1, \dots, \zeta_d$ by φ and ψ satisfy the inequality $|\varphi^* \zeta_i - \psi^* \zeta_i| < 1$ for $i = 1, \dots, d$, we obtain the assertion by iteration.

(iii) Let U be an open subdomain of $\mathbb{D}^d$. At first, assume that $\psi^{-1}(U)$ is isomorphic to $V(\omega) \subset U \times \mathbb{D}^1$, where $\omega \in T[\eta]$ is a Weierstraß polynomial. Any $f \in \psi_* \mathcal{O}_X(U)$ can be represented in the form

$$f = a_1 \eta^0 + \cdots + a_n \eta^{n-1} \quad \text{modulo } \omega.$$

Thus the values of f on the fiber of $\psi^{-1}(y)$ are given by

$$f = (f(\beta_1), \dots, f(\beta_n))^t = M \cdot (a_1(y), \dots, a_n(y))^t,$$

where M is the van de Monde matrix. Thus we see that the coefficients $a_1, \dots, a_n$ can be bounded by spectral norm of f via the adjoint matrix M^* of M and the determinant of M^{-1},

$$(a_1(y), \dots, a_n(y))^t = (\det M)^{-1} \cdot M^* \cdot (f(\beta_1), \dots, f(\beta_n))^t.$$

The basis $\eta^0, \dots, \eta^{n-1}$ mod ω is related to our basis $e_1, \dots, e_n$ by a matrix N,

$$(\eta^0, \dots, \eta^{n-1})^t = N \cdot (e_1, \dots, e_n)^t.$$

Starting with a representation $f = b_1 \cdot e_1 + \cdots + b_n \cdot e_n$, we have

$$(b_1, \dots, b_n)^t = (\det(NM))^{-1} \cdot N^* \cdot M^* \cdot (f(\beta_1), \dots, f(\beta_n))^t.$$

Note that N has bounded entries since $e_1, \dots, e_n$ is basis of $\phi_* \mathcal{O}_X$. The determinant $\det(NM)$ is invertible on $\mathbb{D}^d - V(h)$. Moreover, there exists a power h^α such that $(\det(NM))^{-1} \cdot h^\alpha$ is bounded by 1. Due to Proposition 1.16 (b), over $\mathbb{D}^d_{h,\varepsilon}$, there are finitely many representation of X as $V(\omega_i) \subset U_i \times \mathbb{D}^1$, where $\{U_1, \ldots, U_n\}$ is an open affinoid covering of $\mathbb{D}^d_{h,\varepsilon}$. Thus we can bound the factors $\det(NM), M^*, N^*$. This shows that there exist an element $c \in K^\times$ and an exponent $\alpha \in \mathbb{N}$ as claimed.

(iv) We have the exact sequence

$$0 \to (\mathcal{O}^\circ_{\mathbb{D}^d})^n \to \psi_*\mathcal{O}^\circ_X \to \psi_*\mathcal{O}^\circ_X/(\mathcal{O}^\circ_{\mathbb{D}^d})^n \to 0.$$

This sequence induces the long exact sequences, for $q \geq 1$,

$$\begin{aligned} &\to H^q(\mathbb{D}^d, (\mathcal{O}^\circ_{\mathbb{D}^d})^n) \to H^q(\mathbb{D}^d, \psi_*\mathcal{O}^\circ_X) \to H^q(\mathbb{D}^d, \psi_*\mathcal{O}^\circ_X/(\mathcal{O}^\circ_{\mathbb{D}^d})^n) \\ &\to H^{q+1}(\mathbb{D}^d, (\mathcal{O}^\circ_{\mathbb{D}^d})^n) \to . \end{aligned}$$

Due to Theorem A.7, we have $H^q(\mathbb{D}^d, \mathcal{O}^\circ_{\mathbb{D}^d}) = 0$ for $q \geq 1$, and hence both exterior terms vanish. The third term is annihilated by $c \cdot \psi^* h^\alpha$, and hence the second term is also. Due to Lemma A.9, we have $H^q(\mathbb{D}^d, \psi_*\mathcal{O}^\circ_X) = H^q(X, \mathcal{O}^\circ_X)$, and hence $H^q(X, \mathcal{O}^\circ_X)$ is annihilated by $c \cdot \psi^* h^\alpha$.

(v) If $c \cdot A^\circ \subset F$, the entries of N have spectral norm less than or equal to 1. The entries of M also have spectral norm less than or equal to 1. □

Proposition A.11. *Let X be a smooth irreducible affinoid space of dimension d. Fix a finite morphism $\varphi\colon X = \mathrm{Sp}(A) \to \mathbb{D}^d = \mathrm{Sp}(T)$ and a T-basis $(e_1, \ldots, e_n)$ of A. Then there exist finitely many small perturbations $\psi_i\colon X \to \mathbb{D}^d$ of φ and functions h_i on $\mathbb{D}^d$ for $i = 1, \ldots, n$ such that*
(i) *$\psi_1^* h_1, \ldots, \psi_n^* h_n$ have no common zeros,*
(ii) *(ψ, h_i) satisfy the assertion of Lemma A.10 with c_i, α_i,*
(iii) *there exists an element $c \in K^\times$ such that $c \cdot H^q(X, \mathcal{O}^\circ_X) = 0$ for $q \geq 1$.*

Proof. Since X is smooth, for any point $x \in X$, there exists a perturbation $\psi\colon X \to \mathbb{D}^d$ of φ such that ψ is étale over $\psi(x)$; cp. Proposition 1.16 (a). Since the étale locus of a morphism is open, there exists a function h on $\mathbb{D}^d$ such that ψ is étale over $\mathbb{D}^d - V(h)$. In particular, we have $\psi^* h(x) \neq 0$. Then, by a noetherian argument, there are finitely many perturbations satisfying (i) and (ii).

(iii) Due to (i), there exist functions $g_1, \ldots, g_n \in A^\circ$ such that

$$b = g_1 \psi^* h_1^{\alpha_1} + \cdots + g_n \psi^* h_n^{\alpha_n}$$

for some $b \in K^\times$. Then, by Lemma A.10 (iv), we can conclude

$$c \cdot b \cdot H^q(X, \mathcal{O}^\circ_X) = 0 \quad \text{for all } q \geq 1.$$ □

Now we turn to the proof of Theorem A.8 (b).

Proof of Theorem A.8 (b). We proceed as in the proof of (a) by using formal coverings. Due to Proposition 1.17 (a), there exist morphisms $\phi_i\colon X \to \mathbb{D}^d$ for $i = 1, \ldots, n$ which are finite and formally étale over $\mathbb{D}^d_{\tilde{g}_i}$ with $g_i \in T_d$ and $|g_i| = 1$ such that the subdomains $X_i = X_{\phi_i^* \tilde{g}_i}$ for $i = 1, \ldots, n$ cover X. In particular, $\tilde{\phi}_i\colon \tilde{X} \to \mathbb{A}^d_k$ is finite and flat. Indeed, the local rings of the source and the target are regular, and hence the homological dimension is 0 due to

[31, Prop. 21, p. IV-35]. So the direct image $\tilde{\phi}_{i,*}\mathcal{O}_{\tilde{X}}$ is locally free and hence free due to [28]. Since we can lift the basis of $\tilde{\phi}_{i,*}\mathcal{O}_{\tilde{X}}$ to a basis of $\mathcal{O}_X^\circ(X)$, we have

$$A^\circ = T^\circ \cdot e_1 \oplus \ldots \oplus T^\circ \cdot e_n.$$

As in the proof of Lemma A.10, we obtain

$$g_i^{\alpha_i} \cdot \phi_* \mathcal{O}_X^\circ(X) \subset \mathcal{O}_{\mathbb{D}^d}^\circ \cdot e_1 \oplus \cdots \oplus \mathcal{O}_{\mathbb{D}^d}^\circ \cdot e_n. \tag{9}$$

The constant c in Lemma A.10 is equal to 1 in this case due to Proposition A.11 (v). Actually, one has to replace g_i by the determinant h_i of the van de Monde matrix associated to ϕ_i. Since ϕ_i is formally étale over $\mathbb{D}^d_{\tilde{g}_i}$, we have that $|h_i| = 1$ with $V(\tilde{h}_i) \subset V(\tilde{g}_i)$. So we can directly assume that $h_i = g_i$ without loss of generality. As in Lemma A.10, relation (9) implies $\phi^* g_i^{\alpha_i} \cdot H^q(X, \mathcal{O}_X^\circ) = 0$ for all $q \geq 1$. Since $g_1, \ldots, g_n$ is a formal covering, there exist functions $f_1, \ldots, f_n \in O_{\mathbb{D}^d}^\circ(\mathbb{D}^d)$ with $1 = f_1 \cdot g_1^{\alpha_1} + \cdots + f_n \cdot g_n^{\alpha_n}$. So we finally get

$$H^q(X, \mathcal{O}_X^\circ) = f_1 \cdot g_1^{\alpha_1} \cdot H^q(X, \mathcal{O}_X^\circ) + \cdots + f_n \cdot g_n^{\alpha_n} \cdot H^q(X, \mathcal{O}_X^\circ) = 0$$

for all $q \geq 1$. This completes the proof of Theorem A.8. □

Theorem A.12. *Let $Y := \mathrm{Sp}(B)$ be an affinoid space which admits a smooth formal model. Let $X = Y \times \prod_{i=1}^n A(r_{1,i}, r_{2,i})$ be a product of Y and n annuli $A(r_{1,i}, r_{2,i})$, where $r_{1,i} \leq r_{2,i}$ for $i = 1, \ldots, n$. Then we have the vanishing $H^q(X, \mathcal{O}_X^\circ) = 0$ for all $q \geq 1$.*

Proof. We proceed by induction on the number n of the involved annuli. The case $n = 0$ is settled by Theorem A.8 (b). Now assume $n \geq 1$ and that the assertion is true for all numbers less than n. Denote $\overline{A} = A(r_{1,1}, r_{1,1})$. Then the affinoid space $Y \times \overline{A}$ has smooth reduction. Now look at a cocycle $(f_{i,j}) \in Z^1(\mathfrak{U}, \mathcal{O}_X^\circ)$. At first, we look at the restriction of this cocycle onto

$$\overline{X} := Y \times A(r_{1,1}, r_{1,1}) \times \prod_{i=2}^n A(r_{1,i}, r_{2,i}) = (Y \times \overline{A}) \times \prod_{i=2}^n A(r_{1,i}, r_{2,i}).$$

In order to keep notation simple, set $P := \prod_{i=2}^n A(r_{1,i}, r_{2,i})$. Due to the induction hypothesis, we have $H^q(\overline{X}, \mathcal{O}_X^\circ) = 0$. Thus there exists a 0-chain $h \in C^0(\mathfrak{U}|_{\overline{X}}, \mathcal{O}_X^\circ)$ with $\partial^0(h) = (f_{i,j})|_{\overline{X}}$. Now we define a new cocycle on $Y \times \mathbb{D}^1(r_{2,1}) \times P$ with respect to the covering $\mathfrak{W} := \mathfrak{U} \cup \{U_0\}$ with $U_0 := Y \times \mathbb{D}^1(r_{1,1}) \times P$ by setting $g_{i,0} = h_i$ on $U_i \cap U_0$ for $i \neq 0$ and $g_{i,j} := f_{i,j}$ if $i \neq 0$ and $j \neq 0$. Then $g := (g_{i,j}) \in Z^1(\mathfrak{W}, \mathcal{O}_X^\circ)$. Thus, by the induction hypothesis, we can solve $(g_{i,j})$ by a 0-chain $\ell \in C^0(\mathfrak{W}, \mathcal{O}_X^\circ)$. This settles the assertion for $H^1(X, \mathcal{O}_X^\circ)$. The assertion for $H^q(X, \mathcal{O}_X^\circ)$ with $q \geq 2$ follows similarly. □

Analytic Picard group of $X \times \mathbb{A}_K^1$. The aim of this subsection is the rigid analytic version of the following well-known result in commutative algebra; cp. [14, Chap. VII, § 1, Prop. 18].

Proposition A.13. *Let A be a normal noetherian ring, and let $A[\xi]$ be the polynomial ring over A in one variable ξ. Then the canonical morphism $\operatorname{Pic}(A) \xrightarrow{\sim} \operatorname{Pic}(A[\xi])$ from the Picard group of A to the one of the polynomial ring over A is bijective. The canonical morphism $\operatorname{Pic}(A) \xrightarrow{\sim} \operatorname{Pic}(A[\xi,\xi^{-1}])$ is also bijective.*

The prime ideals $\mathfrak{p}$ of $A[\xi]$ resp. of $A[\xi,\xi^{-1}]$ of height 1 are either induced by prime ideals of height 1 from A or $\mathfrak{p}\cap A=(0)$ and $\mathfrak{p}$ is generated by a primitive polynomial. In particular, any divisor ideal $\mathfrak{a}$ of $A[\xi]$ which contains a monic polynomial is principal.

Proof. The assertion follows by the Lemma of Gauß for normal rings. □

Let us first concentrate on the statement for $\operatorname{Pic}(X\times\mathbb{D}^1)$ resp. the one for $\operatorname{Pic}(X\times\partial\mathbb{D}^1)$; cp. [15] for similar results.

Theorem A.14. *Let $X=\operatorname{Sp}(A)$ be an affinoid space which admits a smooth formal model. Then the following holds.*
(a) *The canonical morphism $\operatorname{Pic}(X)\to\operatorname{Pic}(X\times\mathbb{D}^1)$ is bijective.*
(b) *The canonical morphism $\operatorname{Pic}(X)\to\operatorname{Pic}(X\times\partial\mathbb{D}^1)$ is bijective.*
(c) *$H^1(X,\mathbb{Z})=0$ and $H^1(X,K^\times)=0$.*

Proof. Due to [24, Lem. 6.2.4], any line bundle $\mathcal{L}_K$ on $X\times\mathbb{D}^1$ resp. on $X\times\partial\mathbb{D}^1$ extends to a formal line bundle $\mathcal{L}$ on the smooth model of $X\times\mathbb{D}^1$ resp. $X\times\partial\mathbb{D}^1$. So the reduction $\tilde{\mathcal{L}}$ gives rise to a line bundle on $\tilde{X}\times\mathbb{A}^1_k$ resp. on $\tilde{X}\times\mathbb{G}_{m,k}$. The canonical maps $\operatorname{Pic}(\tilde{X})\xrightarrow{\sim}\operatorname{Pic}(\tilde{X}\times\mathbb{A}^1_k)$ and $\operatorname{Pic}(\tilde{X})\xrightarrow{\sim}\operatorname{Pic}(\tilde{X}\times\mathbb{G}_{m,k})$ are bijective by Proposition A.13 since $\tilde{X}$ is smooth. So the reduction $\tilde{\mathcal{L}}$ is locally trivial over $\tilde{X}$. Since one can lift a generator of $\tilde{\mathcal{L}}|_{\tilde{U}}$ to a true generator of $\mathcal{L}|_U$ for any formal affine open subset U of X, we see that $\mathcal{L}_K$ trivializes over an open formal covering of X. Thus any line bundle $\mathcal{L}$ on $X\times\mathbb{D}^1$ resp. on $X\times\partial\mathbb{D}^1$ is equivalent to a cocycle given by transition functions $\lambda_{i,j}\in\mathcal{O}_X(U_i\cap U_j)\langle\eta\rangle^\times$ resp. by $\lambda_{i,j}\in\mathcal{O}_X(U_i\cap U_j)\langle\eta,1/\eta\rangle^\times$, where $\mathfrak{U}=\{U_1,\dots,U_n\}$ is a formal open covering of X.

Due to the unique decomposition of units, we can write

$$\lambda_{i,j}=c_{i,j}\cdot(1+h_{i,j}),$$

where $(c_{i,j})\in Z^1(\mathfrak{U},\mathcal{O}_X^\times)$ and $(1+h_{i,j})\in Z^1(\mathfrak{U},\mathcal{O}^\times_{X\times\mathbb{D}^1})$, where $|h_{i,j}|<1$ with $h_{i,j}(0)=0$. Moreover, any line bundle $\mathcal{L}$ on $X\times\partial\mathbb{D}^1$ is equivalent to a cocycle

$$\lambda_{i,j}=c_{i,j}\cdot\eta^{n_{i,j}}\cdot(1+h_{i,j}),$$

where $(c_{i,j})\in Z^1(\mathfrak{U},\mathcal{O}_X^\times)$ and $(n_{i,j})\in Z^1(\mathfrak{U},\mathbb{Z}^\times)$ and $(1+h_{i,j})\in Z^1(\mathfrak{U},\mathcal{O}^\times_{X\times\partial\mathbb{D}^1})$, where $|h_{i,j}|<1$ with $h_{i,j}(1)=0$. Since $\tilde{X}$ has irreducible connectedness components, we have $H^1(\mathfrak{U},\mathbb{Z}^\times)=0$. The units of the form $1+h$ with $h\in\mathcal{O}_{X\times\partial\mathbb{D}^1}(U\times\partial\mathbb{D}^1)$ with $|h|<1$ can be decomposed into two factors

$$1+h=(1+h^+)\cdot(1+h^-),$$

where $h^*\in\mathcal{O}_X(U)\langle\eta^{*1}\rangle)$ with $|h^*|<1$. The vanishing of $H^1(\mathfrak{U},1+\mathcal{O}^\vee_{X\times\mathbb{D}^1})$ will be shown in the following lemma by using Theorem A.8. □

Now you will apply our result Theorem A.8 to show a similar vanishing statement for certain cocycles of invertible functions. In the following, we denote by $\mathcal{O}_X^\times(r)$ the subsheaf of $\mathcal{O}_X^\times$ consisting of all the invertible functions f which can be written as $f = 1 + h$, where $|h| < r$. Furthermore, we denote by $\mathcal{O}_X(r_1, r_2)$ the quotient $\mathcal{O}_X(r_1)/\mathcal{O}_X(r_2)$ for $0 < r_2 < r_1 < 1$.

Lemma A.15. *Let X be a smooth affinoid space. Assume that there exists a $c \in K^\times$ with $s := |c| \leq 1$ such that $c \cdot H^q(X, \mathcal{O}_X(1)) = 0$ for $q = 1, 2$. Then the canonical map $H^1(\iota)\colon H^1(X, \mathcal{O}_X^\times(s^2 r)) \to H^1(X, \mathcal{O}_X^\times(r))$ induced by the inclusion $\iota\colon \mathcal{O}_X^\times(s^2 r) \hookrightarrow \mathcal{O}_X^\times(r)$ vanishes for all $r \in (0, s^2)$.*

In particular, for $s = 1$, we have $H^1(X, \mathcal{O}_X^\times(1)) = 0$. Due to Theorem A.8, the latter is fulfilled if X has a smooth formal model.

Proof. Consider $c \in K^\times$ with $s := |c|$. For $0 < r^2 \leq r' < r$, we have the following isomorphism:

$$\mathcal{O}_X^\times(r)/\mathcal{O}_X^\times(r') = \mathcal{O}_X(r)/\mathcal{O}_X(r') =: \mathcal{O}_X(r, r').$$

Put $r_0 := r < s^2$ and $\varepsilon := r/s^2$. Then put $r_n = \varepsilon^n r$. We have the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
H^1(X, \mathcal{O}_X(r_n)) & \longrightarrow & H^1(X, \mathcal{O}_X(r_n, r_{n+1})) & \xrightarrow{\partial} & H^2(X, \mathcal{O}_X(r_{n+1})) \\
\downarrow{\scriptstyle \cdot c} & & \downarrow{\scriptstyle \cdot c} & & \downarrow{\scriptstyle \cdot c} \\
H^1(X, \mathcal{O}_X(r_n)) & \xrightarrow{\rho} & H^1(X, \mathcal{O}_X(r_n, r_{n+1})) & \xrightarrow{\partial} & H^2(X, \mathcal{O}_X(r_{n+1}))
\end{array}$$

Consider a cocycle $\xi = (1 + c^2 \cdot h_{i,j}) \in Z^1(\mathfrak{U}, \mathcal{O}^\times(s^2 r_0))$, and set $\overline{h} := (\overline{h}_{i,j}) \in Z^1(\mathfrak{U}, \mathcal{O}_X(r_0, s r_1))$. Thus we obtain $\partial(c \cdot \overline{h}) = 0$ in $H^2(\mathfrak{U}, \mathcal{O}_X(r_1))$ since the last vertical map is 0 due to our assumption. So there exists an f in $H^1(\mathfrak{U}, \mathcal{O}_X(r_0))$ such that $c \cdot \overline{h} = \rho(f)$. Since $c \cdot f$ vanishes in $H^1(\mathfrak{U}, \mathcal{O}_X(r_0))$ due to our assumption, there exist functions $g_i \in \mathcal{O}_X(U_i)(r_0)$ such that

$$c \cdot f_{i,j} = g_i - g_j.$$

Then we obtain

$$c^2 \cdot \overline{h}_{i,j} = \rho(c \cdot f_{i,j}) = \rho(g_i - g_j) \quad \text{in } \mathcal{O}_X(U_i \cap U_j)(r_0, r_1).$$

Now we have

$$\begin{aligned}
\xi \cdot (1 - g_i) \cdot (1 + g_j) &= (1 + c^2 h_{i,j}) \cdot (1 - g_i) \cdot (1 + g_j) \in Z^1(\mathfrak{U}, \mathcal{O}^\times(r_0)) \\
&= 1 + (c^2 h - (g_i - g_j)) \mod \mathcal{O}_X(r_0^2) \\
&= 1 + h' \in Z^1(\mathfrak{U}, \mathcal{O}^\times(r_0^2))
\end{aligned}$$

which is homologous to $H^1(\iota)(\xi) \in H^1(\mathfrak{U}, \mathcal{O}^\times(r_1))$. Moreover, we have

$$1 + h' = 1 + c^2 h^{(1)} \quad \text{with } h^{(1)} \in C^0(\mathfrak{U}, \mathcal{O}_X(\varepsilon \cdot r)$$

since $r_0^2/c^2 = (r/c^2) \cdot r = \varepsilon \cdot r$.

Now we repeat this procedure with r_n instead of r_0 and r_{n+1} instead of r_1 and a cycle $\xi^{(0)} := \xi = (1 + c^2 h) \in Z^1(\mathfrak{U}, \mathcal{O}^\times(s^2 r_0))$. By induction, we obtain that the image of $H^1(\iota)(\xi)$ is homologous to a cocycle $\xi_n := (1 + c^2 h_{i,j}^{(n)})$ in $H^1(X, \mathcal{O}^\times(, r_n))$, where

$$\xi_{n+1} = (1 + c^2 h_{i,j}^{(n+1)}) = (1 + c^2 h_{i,j}^{(n)}) \cdot (1 + g_i^{(n)}) \cdot (1 - g_j^{(n)})$$

with $g_i^{(n)} \in \mathcal{O}_X(r_n)(U_i)$. The sequences $g_i^{(n)} \in \mathcal{O}_X(r^n)(U_i)$ converge for $n \to \infty$, and in the limit, we obtain that

$$H^1(\iota)(\xi) = \prod_{n=1}^{\infty} (1 + g_i^{(n)}) \cdot \prod_{n=1}^{\infty} (1 - g_j^{(n)}).$$

The infinite products converge in $Z^1(\mathfrak{U}, \mathcal{O}^\times(r))$ to a coboundary. For the additional statement, it obviously suffices to treat the case $r < 1$ due to the very definition of $\mathcal{O}_X^\times(1)$. □

With this result, we are more or less done by the following observation.

Lemma A.16. *Let Y be a smooth affinoid space. Then there exists an $r \in |K^\times|$ with $r \geq 1$ such that the following holds. Let $\mathcal{L}$ be any holomorphic invertible sheaf on $Y \times \mathbb{D}^1(r)$. If $\mathcal{L}|_{Y \times 0}$ is trivial, then the restriction $\mathcal{L}|_{Y \times \mathbb{D}^1(1)}$ to the subdomain $Y \times \mathbb{D}^1(1)$ is trivial.*

Proof. Any line bundle on $Y \times \mathbb{D}^1$ trivializes locally over Y; cp. the proof of Corollary 3.18. So $\mathcal{L}$ can be given by a cocycle

$$(1 + h_{i,j}) \in H^1(X, 1 + \mathcal{O}_{X \times \mathbb{D}(\rho)}(r)) \quad \text{with } h_{i,j}(0) = 0$$

since $\mathcal{L}|_{X \times 0} = \mathcal{O}_X$. The functions h_{ij}, which are defined on $(U_i \cap U_j) \times \mathbb{D}^1(r)$, satisfy $|h_{i,j}|_{(U_i \cap U_j) \times \mathbb{D}(1)} < 1/r$ when restricted to the subset $(U_i \cap U_j) \times \mathbb{D}(1)$. So the $h_{i,j}$ have small absolute value. Now the assertion follows from Lemma A.15 with $r := 1/s^2$, where $s := |c|$ is provided by Theorem A.8 (a). □

Now our main result about the analytic Picard group follows immediately.

Theorem A.17. *Let Y be a smooth affinoid space. Then the canonical map of the analytic Picard groups $\operatorname{Pic}(Y) \to \operatorname{Pic}(Y \times \mathbb{A}_K^1)$ is bijective, which pulls back line bundles via the projection $Y \times \mathbb{A}_K^1 \to Y$.*

Proof. We can assume that the invertible sheaf $\mathcal{L}$ on $X \times \mathbb{A}_K^1$ satisfies $\mathcal{L}|_{X \times 0} = \mathcal{O}_X$. By Lemma A.16, we see that $\mathcal{L}|_{X \times \mathbb{D}_K^1(r)}$ is trivial for any $r \in |K^\times|$. Now choose an increasing sequence of radii $(r_i;\, i \in \mathbb{N})$ tending to ∞. Due to the above result, we find generators ℓ_i of $\mathcal{L}|_{X_K \times \mathbb{D}_K^1(r_i)}$. We normalize them at the zero section by $\ell_0|_{X_K \times \{0\}} = \ell_i|_{X_K \times \{0\}}$. Then $\ell := \lim_{i \to \infty} \ell_i \in \mathcal{L}(X_K \times \mathbb{A}_K^1)$ exists since $\ell_j = e_{i,j} \ell_i$ for $j \geq i$ for invertible functions $e_{i,j}$ of type $1 + f_{i,j}$ with $f_{i,j}(0) = 0$ and $|f_{i,j}|_{X_K \times \mathbb{D}_K^1(r)} < r/r_i$ for all $r \leq r_i$. So ℓ yields a global generator of $\mathcal{L}$. □

There is another interesting application of Lemma A.15.

Proposition A.18. *Let X be an affinoid space which admits a smooth formal model. Then there is a canonical isomorphism $\operatorname{Pic} X \xrightarrow{\sim} \operatorname{Pic} \tilde{X}$ from the Picard group of X to the one of its reduction $\tilde{X}$.*

Proof. We may assume that X is connected. Due to [24, Lem. 6.2.3], any line bundle $\mathcal{L}_K$ on X has a formal model $\mathcal{L}$ defined over the smooth model of X. It follows from [24, Prop. 3.4.1] that the extension $\mathcal{L}$ is unique up to a principal divisor (t), where $t \in K^\times$ is a nonzero constant, since X has a smooth formal model and X is connected. This gives rise to a well-defined map $\operatorname{Pic} X \to \operatorname{Pic} \tilde{X}$. This map is injective since a generator of the reduction $\tilde{\mathcal{L}}$ can be lifted to a true generator of $\mathcal{L}$. For showing the surjectivity, one has to show that $H^2(\mathfrak{U}, \mathcal{O}_X^\times(1)) = 0$ for any formal covering $\mathfrak{U}$ of X. This follows in a way similar to the one exercised in the proof of Lemma A.15 due to the vanishing of $H^q(\mathfrak{U}, \mathcal{O}_X^\circ)$ for all $q \geq 1$. Of course, the latter is used only for formal coverings, and then it is much easier to establish. □

Note that the assertion does not hold if X is not affinoid. For example, even for a nontrivial abelian variety with good reduction, the assertion of Proposition A.18 fails to be true.

Finally, we want to mention that the last result is due to Kerz, Saito and Tamme; cp. [20]. In that paper, the authors give a different proof of Theorem A.8 (a) following ideas of van der Put [35]. But their methods are much more complicated and do not suffice to prove Theorem A.8 (b). Theorem A.8 (b) was mentioned by Bartenwerfer in [7], but it is difficult to follow his proof; see also the English translation of [6] by Shizhang Li, especially his 30 footnotes and appendix.

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