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A strong Schottky lemma on n generators for CAT(0) spaces

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Abstract. We give a criterion for a set of n hyperbolic isometries of a CAT(0) metric space X to generate a free group on n generators. This extends a result by Alperin, Farb, and Noskov who proved this for 2 generators under the additional assumption that X is complete and has no fake zero angles. Moreover, when X is locally compact, the group we obtain is also discrete. We then apply these results to Euclidean buildings.

1. Introduction

We generalize the main theorem of [1] as follows.

Theorem A. *Let X be a* CAT(0) *metric space. Let $g_1, \dots, g_n$ be hyperbolic isometries of X with axes $A_1, \dots, A_n$, where $n \geq 2$. Suppose that, for each distinct pair of distinct axes A_i, A_j, either*

(I) *$S_{ij} = A_i \cap A_j$ is a bounded segment, and the two angles $\theta_{ij}^-, \theta_{ij}^+$ between A_i and A_j measured from the two endpoints of S_{ij} are both equal to π (as in the left-hand diagram of Figure 1); or*

(II) *A_i and A_j are disjoint, and there is a geodesic B_{ij} between A_i and A_j such that all four angles between B_{ij} and A_i, A_j are equal to π (as in the right-hand diagram of Figure 1).*

Additionally, suppose that, for each $1 \leq i \leq n$, there is an open segment $D_i \subseteq A_i$ of length equal to the translation length of g_i such that

$$\bigcup_{j \neq i} p_i(A_j) \subseteq D_i,$$

where $p_i : X \to A_i$ is the geodesic projection map. Then the subgroup of $\operatorname{Isom}(X)$ *generated by $g_1, \dots, g_n$ is free of rank n, and when X is locally compact, it is also discrete.*

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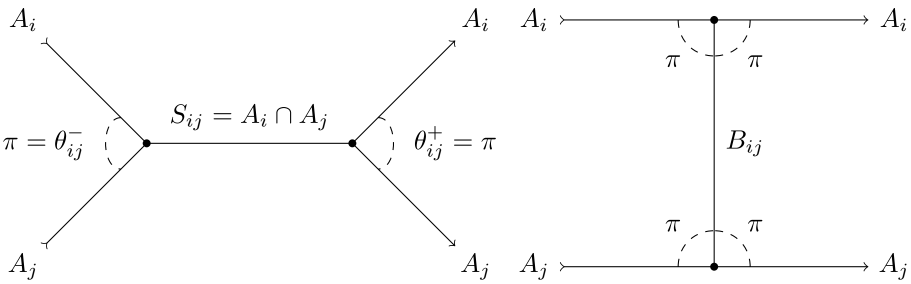

Figure 1. Cases (I) and (II) of Theorem A.

Remark 1.1. By the angle between two geodesic paths, we mean the upper (or Alexandrov) angle, as defined in [3, Chap. I.1, Def. 1.12].

Remark 1.2. We only ever consider a topology on $\mathrm{Isom}(X)$ when X is locally compact. The topology we use is the compact-open topology, which is equivalent to the topology of pointwise convergence in this setting, and this gives $\mathrm{Isom}(X)$ the structure of a topological group which acts continuously on X; see [2, Chap. X, Sec. 2.4, Thm. 1 and Sec. 3.4, Cor. 1].

Theorem A generalizes the theorem stated in [1] as we allow for an arbitrary finite number of generators and we no longer require that X is complete and has no fake zero angles. Moreover, we also prove discreteness when X is locally compact, and this generalizes a result by Lubotzky for isometries of trees [9, Prop. 1.6].

Remark 1.3. There are isometries of locally compact CAT(0) metric spaces which generate groups which are free but not discrete.

For instance, the matrices $A = \left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right]$ generate a free group of rank two (the *Sanov subgroup* [13]). However, viewing them as matrices over the p-adic numbers $\mathbb{Q}_p$, both A and B are infinite-order elliptic isometries of the Bruhat–Tits tree T_p corresponding to $\mathbb{Q}_p$. Hence the corresponding subgroup of $\mathrm{PSL}_2(\mathbb{Q}_p) \leq \mathrm{Isom}(T_p)$ is free of rank two but not discrete.

The main theorem of [1] follows directly from Theorem A when $n = 2$: in case (I), the projection condition implies that S_{ij} has length strictly less than the translation length of both g_1 and g_2, and in case (II), the projection condition always holds since the projection onto each axis is the unique corresponding endpoint of the geodesic B_{12}.

Remark 1.4. Karlsson remarked without proof (see the end of [6, Sec. 6]) that "the condition of no-fake angles in [1] can be removed and the translation lengths do not necessarily have to be strictly greater than the length of S". This is part of what we do here, and our proof is similar to the one in [1].

Without the requirement for completeness, Theorem A may be applied to CAT(0) spaces which are not necessarily complete, such as certain non-discrete Euclidean buildings; see [11] for some background material.

By [7, Prop. 4.6.1 and Cor. 4.6.2], isometries of Euclidean buildings map apartments to apartments, and if the building at infinity is thick, they also map Weyl chambers to Weyl chambers [11, Prop. 2.25 and Prop. 2.27]. As in [1], we call an isometry f *generic* if none of its parallel axes is contained in any wall of any apartment of X. An isometry f is generic if and only if it has a unique invariant apartment $\mathcal{A}_f$ (see [11, Prop. 2.26]). A generic isometry f determines, for any fixed choice of basepoint $x \in \mathcal{A}_f$, a pair of chambers in $\mathrm{link}(x)$. We say that generic isometries f and g are *opposite* if $\mathcal{A}_f \cap \mathcal{A}_g = \{x\}$ and each of the chambers determined by f is opposite in $\mathrm{link}(x)$ to each of the chambers determined by g.

Corollary B. *Let X be a Euclidean building (where X^∞ is thick), and let $f_1, \dots, f_n$ be hyperbolic isometries of X. If $f_1, \dots, f_n$ are pairwise opposite and the pairwise intersection points of their axes are contained in an open ball of radius at most half the minimum of the translation lengths of $f_1, \dots, f_n$, then $f_1, \dots, f_n$ generate a subgroup of* $\mathrm{Isom}(X)$ *which is free of rank n. If X is locally compact, then this subgroup is also discrete.*

Proof. By [11, Prop. 1.12], two halfrays of A_i and A_j emanating from x are contained in an apartment. Thus the projection of A_j onto A_i is equal to their intersection point. By our assumption on the intersection points, and the fact that projection does not increase distances [3, Chap. II.2, Prop. 2.4 (4)], the proof is completed using Theorem A. □

Note that the geometric realization of a simplicial complex (in particular, of a simplicial building) is locally compact if and only if it is locally finite. When G is a linear semisimple algebraic group defined over a nonarchimedean field k, then the Bruhat–Tits building associated to G (see [14]) is locally compact if and only if k is a local field [12, p. 464].

Remark 1.5. Although all simplicial buildings have a metrically complete CAT(0) Davis realization [5, Thm. 11.1], a Euclidean building is not necessarily metrically complete, even if it is a Bruhat–Tits building [10]. Moreover, the Cauchy completion of a Euclidean building is not necessarily a Euclidean building [8, Ex. 6.9]. One can instead use the theory of ultralimits to embed a Euclidean building into a metrically complete Euclidean building [7]. However, to prove Corollary B, we did not need this.

2. Proof of Theorem A

We will use the following statement of the Ping Pong Lemma. This generalizes the version in [4, Lem. 3.3] to an arbitrary finite number of elements. For the discreteness part, we also remove the condition that the topological group G is metrizable.

Lemma 2.1 (The Ping Pong Lemma)**.** *Let G be a group acting on a set X, and let $g_1, \dots, g_n \in G \setminus \{e\}$. Suppose that $X_1^+, X_1^-, \dots, X_n^+, X_n^-$ are nonempty,* *pairwise disjoint subsets of X, which do not cover X and for all $1 \leq i \leq n$ satisfy*

$$g_i(X \setminus X_i^-) \subseteq X_i^+ \quad \textit{and} \quad g_i^{-1}(X \setminus X_i^+) \subseteq X_i^-.$$

Then the subgroup $H = \langle g_1, \ldots, g_n \rangle \leq G$ is free of rank n. In the case that X is a topological space and G is a topological group which acts continuously on X, if each of the subsets $X_1^+, X_1^-, \ldots, X_n^+, X_n^-$ is closed in X, then H is also discrete.

Proof. Set $Y = X_1^+ \cup X_1^- \cup \cdots \cup X_n^+ \cup X_n^-$, and choose $x \in X \setminus Y$. If w is a nontrivial word in $g_1, \ldots, g_n$, then $w(x) \in Y$; therefore $w \neq e$ in G. Hence H is free of rank n.

For the second part, note that H acts continuously on X, that is, the map $H \times X \to X$ is continuous with respect to the product topology. It follows that the inverse image of the open set $X \setminus Y$ is open in $H \times X$. But the intersection of this inverse image with the open set $H \times X \setminus Y$ is $\{e\} \times X \setminus Y$; thus $\{e\}$ is open in H, and hence H is discrete. □

Lemma 2.2. *Let $[x, y]$, $[y, z]$ be geodesics in a* CAT(0) *space. If $\angle_y(x, z) = \pi$, then the concatenation $[x, z] = [x, y] \cup [y, z]$ is a geodesic.*

Proof. By [3, Chap. II.1, Prop. 1.7 (4)] the corresponding angle in the relevant comparison triangle is also π, and thus $d(x, z) = d(x, y) + d(y, z)$. □

Proof of Theorem A. Since geodesics are complete convex subsets in CAT(0) spaces, the projection maps p_i we use are well-defined [3, Chap. II.2, Prop. 2.4].

Note that, for each $1 \leq i \leq n$, the open segment D_i is a fundamental domain for the action of g_i on A_i. Let A_i^+ denote the union of all translates of $\overline{D_i}$ under positive powers of g_i. Similarly, let A_i^- denote the union of all translates of $\overline{D_i}$ under negative powers of g_i. Then A_i^+ and A_i^- are disjoint geodesic rays with $A_i \setminus D_i = A_i^+ \sqcup A_i^-$. Set $X_i^+ = p_i^{-1}(A_i^+)$ and $X_i^- = p_i^{-1}(A_i^-)$ for each i. We will show that these subsets satisfy the hypotheses of the first part of Lemma 2.1.

It is straight-forward to check that the subsets $X_1^\pm, \ldots, X_n^\pm$ are nonempty, closed, and that they do not cover X. Each X_i^+ is also disjoint from X_i^-, so to apply Lemma 2.1, we must show that the sets $X_i^\pm$ are disjoint from $X_j^\pm$ for $i \neq j$. Since we can replace g_i and g_j by their inverses, if necessary, it suffices to show that X_i^+ and X_j^+ are disjoint.

To this end, suppose that $x \in X_i^+ \cap X_j^+$ for some $i \neq j$. Then $p_i(x) \in A_i^+$ and $p_j(x) \in A_j^+$. Note that $p_i(x) \neq p_j(x)$, as otherwise $p_i(x) \in D_i \subseteq A_i \setminus A_i^+$. A similar argument shows that $x \notin A_i \cup A_j$.

In case (I), let y_i and y_j be the (not necessarily distinct) endpoints of S_{ij} which are closest to $p_i(x)$ and $p_j(x)$ respectively. In case (II), let $A_i \cap B_{ij} = \{y_i\}$ and $A_j \cap B_{ij} = \{y_j\}$. By Lemma 2.2, the geodesic $[p_i(x), p_j(x)]$ is the concatenation of geodesics $[p_i(x), y_i] \cup [y_i, y_j] \cup [y_j, p_j(x)]$. In particular,

$$\angle_{p_i(x)}(x, p_j(x)) = \angle_{p_i(x)}(x, y_i) \geq \frac{\pi}{2} \quad \text{and} \quad \angle_{p_j(x)}(x, p_i(x)) = \angle_{p_j(x)}(x, y_j) \geq \frac{\pi}{2}$$

by [3, Chap. II.2, Prop. 2.4 (3)]. But the triangle with distinct vertices x, $p_i(x)$, $p_j(x)$ has a Euclidean comparison triangle with corresponding angles which are also at least $\frac{\pi}{2}$ by [3, Chap. II.1, Prop. 1.7 (4)], and this is a contradiction.

It remains to prove that $g_i(X \setminus X_i^-) \subseteq X_i^+$ and $g_i^{-1}(X \setminus X_i^+) \subseteq X_i^-$ for each $1 \leq i \leq n$. As in [1], we first note that p_i commutes with g_i. Indeed, for $x \in X$, $p_i(g_i(x))$ is the unique point on A_i which realizes the distance $d(g_i(x), A_i)$. It follows that $p_i(g_i(x)) = g_i(p_i(x))$ since

$$d(g_i(x), A_i) = d(g_i(x), g_i(A_i)) = d(x, A_i) = d(x, p_i(x)) = d(g_i(x), g_i(p_i(x))).$$

Hence if $x \in X \setminus X_i^-$, then $p_i(g_i(x)) = g_i(p_i(x)) \in A_i^+$, *i.e.* $g_i(x) \in X_i^+$. Similarly, p_i commutes with g_i^{-1}, and if $x \in X \setminus X_i^+$, then it follows that $p_i(g_i^{-1}(x)) = g_i^{-1}(p_i(x)) \in A_i^-$, *i.e.* $g_i^{-1}(x) \in X_i^-$. Thus $g_1, \ldots, g_n$ generate a free group of rank n by the first part of Lemma 2.1.

Finally, we prove discreteness when X is locally compact. The action of $\mathrm{Isom}(X)$ on X is continuous by Remark 1.2, and each of the subsets $X_i^\pm$ is closed in X by [3, Chap. II.2, Prop. 2.4 (4)]. Hence the second part of Lemma 2.1 completes the proof of Theorem A. □

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