Combinatorial Complexity in Henselian Valued Fields

Pushing Anscombe-Jahnke up the Ladder

Blaise Boissonneau

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Combinatorial Complexity in Henselian Valued Fields

Pushing Anscombe-Jahnke up the Ladder

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Dekanin/Dekan:	Prof. Dr. Xiaoyi Jiang
Erste Gutachterin/Erster Gutachter:	Prof. Dr. Franziska Jahnke
Zweite Gutachterin/Zweiter Gutachter:	Prof. Dr. Artem Chernikov
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Abstract

In this thesis, we study the links between combinatorial complexity of first-order theories of fields and algebraic properties of fields, and most notably, the role henselianity plays.

To start, we study valuations, and we expose several classical methods to define them with a first-order formula in specific cases. If a valuation happens to be definable in a pure field, then the first-order theories of the valued field and the pure field have the same complexity; this is notably the case in algebraic extensions of $\mathbb{Q}_p$.

We continue by studying links between Artin-Schreier extensions of fields in positive characteristic and combinatorial complexity. The study of these links started with a result by Thomas Scanlon on stable fields, and was later on studied in other complexity classes in several papers by Thomas Scanlon himself, as well as by Artem Chernikov, Nadja Hempel, Itay Kaplan, Pierre Simon and Frank Wagner. We reformulate these results by exhibiting explicit formulas witnessing combinatorial patterns; this gives us new ways to witness complexity in henselian valued fields of mixed characteristic, and to conclude that fields which lie in some classes have strong algebraic properties. Most notably, we extend results of Franz-Viktor Kuhlmann and Anna Rzepka to prove that NTP2 henselian valued fields are tame, semitame, or finitely ramified by parts, and that NIP_n henselian valued fields are separably algebraically maximal Kaplansky or finitely ramified by parts, exactly as NIP henselian valued fields.

Conversely, we also study so-called transfer theorems, which state that henselian valued fields with certain algebraic properties are not more complex than their residue fields. The method we use to obtain transfer was developed by Artem Chernikov and Martin Hils in the NTP2 context and extended to the NIP context by Franziska Jahnke and Pierre Simon. We, in turn, extend it to the NIP_n context, and deduce a complete classification of NIP_n henselian valued fields down to their residue fields, generalizing a result of Sylvy Anscombe and Franziska Jahnke for NIP henselian valued fields. We also state a transfer theorem for NTP2 henselian valued fields as a first step towards a complete classification.

All these results are illustrated in algebraic extensions of $\mathbb{Q}_p$, where we provide examples and classifications of all usual complexity classes.

Notations

- We write "$\subset$" for subsets, proper or not. We write "$\subsetneq$" for proper subsets and "$\not\subset$" for non-subsets.
- Given two sets A and B, we let $A + B = \{a + b \mid a \in A \wedge b \in B\}$, assuming we work in a structure where "$+$" makes sense, and similarly for $A - B$.
- We write $\mathrm{ch}(K)$ for the characteristic of the field K.
- We do not use different fonts to distinguish a structure and its base set, in fact, we do not distinguish them.
- When we do not specify, "definable" means "definable with parameters".
- Lowercase letters $x, y, z, \ldots$ can represent single variables or tuples of variables, and $a, b, c, \ldots$ can represent fixed single elements or tuples of elements. We almost never write $\overline{x}$ to represent a tuple, as we prefer to use the overline to denote the residue of an element.
- We write "we" for "long rainy nights in Münster, long uncomfy nights in commute, long though short-seaming nights in Paris, many other nights and I".

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During these years, I had the chance to interact with many researchers across the world, including Artem Chernikov, thanks to whom I developed a better understanding of the behavior of NTP2 theories, Pierre Simon, without whom I couldn't have developed the methods used to obtain NIP_n transfer, Nadja Hempel, without whom I wouldn't have started working on NIP_n theories whatsoever, and Sylvy Anscombe, with whom I discussed to great length definability of valuations, the theory of $\mathbb{F}_p((\mathbb{Q}))$, Artin-Schreier lifting, and valued fields in general. It is an understatement to say that most of the results in this thesis would not exist without her.

I want to emphasize as well the role my teachers played, and notably, Hedi Chakroun, who showed me the way towards mathematical research, and Adrien Deloro and Tamara Servi, who taught me model theory.

Finally, I would like to thank Hanane for their love and support.

Der Schoß ist fruchtbar noch, aus dem das kroch.

La jeunesse emmerde le front national.

Contents

Introduction

The broad subject of this dissertation is the study of model theory of valued fields. More precisely, we study the links between model-theoretic complexity of first-order theories of valued fields and the algebraic properties of these valued fields.

0.1 Combinatorial complexity

Dating back to the 70s and the work of Saharon Shelah in [She78], model theorists have found that more often than not, meaningful dividing lines between somewhat easy-to-study theories and more complex ones can be expressed in terms of combinatorial configurations that may or may not be encoded in these theories. The prototypical example of this phenomenon is stability: at first studied in terms of the number of different types a theory can have, an equivalent definition is to say that stable theories can not encode an infinite linear order.

This global-local duality between the behavior of the whole theory and the combinatorial properties of individual formulas gives rise to different approaches to study these notions of complexity. One of these approaches is to study the links with algebraic structures. This goes both ways: given an algebraic structure, we want to know how complex it is, a contrario, if we know that some structure has a certain complexity, we want to describe it algebraically.

We like to think about all these notions as a ladder that we try to climb in order to understand theories which are more and more complex. A nice example of this ladder-climbing is the study of Artin-Schreier extensions, we will say more later.

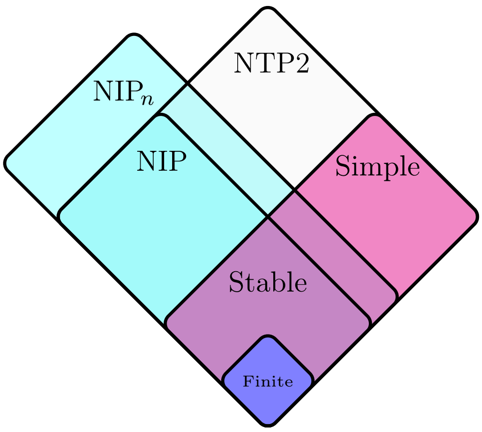


Figure 1: The ladder.

In order to gain some understanding of the model theory of fields, we quote some classical results:

- infinite ω-stable fields and superstable fields are algebraically closed [Mac71, CS80];
- separably closed fields are stable [Woo79];
- pseudo-algebraically closed field are simple iff they are bounded [Hru02, Cha99];
- non separably closed PAC fields have IP_n for all n [Dur80, Hem14];
- $\mathbb{Q}_p$ is NIP.

0.2 Complexity of fields and Artin-Schreier extensions

This story starts in 1999, with the following remarkable result:

Fact 1 ([Sca00]). *Infinite stable fields of characteristic $p > 0$ have no Artin-Schreier extensions.*

It is in fact conjectured that infinite stable fields have no separable extensions whatsoever; this result tells us that, in characteristic p, they at least have no separable extension of degree p.

In 2011, this result was pushed up the ladder:

Fact 2 ([KSW11]). *Infinite NIP fields of characteristic $p > 0$ have no Artin-Schreier extensions; simple fields of characteristic $p > 0$ have finitely many distinct Artin-Schreier extensions.*

We see here a good example of ladder-climbing; starting with a result in the stable context, it can be extended, sometimes exactly as it is, sometime to a slightly weaker result.

But the ladder continues:

Fact 3 ([CKS12]). *NTP2 fields of characteristic $p > 0$ have finitely many distinct Artin-Schreier extensions.*

Fact 4 ([Hem14]). *Infinite NIP_n fields of characteristic $p > 0$ have no Artin-Schreier extensions.*

In Chapter 2, we study in detail those results, explaining the proof strategy. We then "localize" them, in the following sense: given a field, if it has an Artin-Schreier extension, then it has IP_n, and if it has infinitely many, then it has TP2; by definition, this then means that there is a formula witnessing IP_n or TP2 in this field. This fact is not explicit in the original papers. We exhibit these formulas, and prove in fact the following:

Theorem. *Let K be an infinite field of characteristic $p > 0$. Then*

$$\varphi(x, y_1, \cdots, y_n) : \exists t\; x = y_1 \cdots y_n(t^p - t)$$

has IP_n iff K has an Artin-Schreier extension, and

$$\psi(x, yz) : \exists t\; x + z = y(t^p - t)$$

has TP2 iff it has infinitely many distinct Artin-Schreier extensions.

This formula is positive existential, therefore, in a henselian valued field, we know that it witnesses a combinatorial pattern in the field iff it does so in the residue field.

0.3 Valued fields

It is sometimes useful to study augmented structures in order to then deduce results for restricted structures. In our case, the study of valued fields, notably henselian, has proven useful to the study of pure fields, notably towards the NIP side of the ladder.

We will open this dissertation with a collection of definitions and classical facts about valuation theory, with a model-theoretic point of view. We notably study definability of valuations: when one adds a valuation to a field, it naturally comes with a definable topology, a residue field, and other new definable sets. However, if the valuation ring is already definable in the pure field, then adding valuation does not yield new definable sets. This is famously the case in $\mathbb{Q}_p$.

To this regard, henselian valuations are of much more interest. Henselianity allows to lift simple roots of polynomials from the residue field to the original field. This fact gives us a lot of control over the definable sets of the valued field. Since many pure fields define a henselian valuation – it is in fact conjectured that most unstable NIP fields do so –, the studies of henselian valued fields and of pure fields are intertwined.

The most famous example is the Ax-Kochen/Ershov (AKE) transfer principle. Discovered independently at the same time, it states that henselian valued fields of residue characteristic 0 are elementarily equivalent iff their residue fields and value groups are elementarily equivalent. In particular, non principal ultraproducts of $\mathbb{Q}_p$ and of $\mathbb{F}_p((t))$ are elementary equivalent, which means that for all prime p but finitely many, a given first order formula holds for $\mathbb{Q}_p$ iff it holds for $\mathbb{F}_p((t))$.

0.4 Complexity of henselian valued fields

In the spirit of the aforementioned AKE transfer principle, more transfer theorems have been established in different settings. They are of the form "if we know enough about the residue field and the value group, then we also know a lot about the valued field".

NIP transfer theorems have been established as early as 1980, and little by little in more and more cases. They culminated in 2019, with Anscombe-Jahnke's classification of NIP henselian valued fields, that we repeat here and will study precisely in Chapter 3:

Theorem (Anscombe-Jahnke, [AJ19a]). *Let (K, v) be a henselian valued field. Then (K, v) is NIP iff the following holds:*

1. *k is NIP, and*
2. *either*
   (a) *(K, v) is of equicharacteristic and is either trivial or SAMK, or*
   (b) *(K, v) has mixed characteristic $(0, p)$, (K, v_p) is finitely ramified, and $(k_p, \overline{v})$ checks 2a, or*
   (c) *(K, v) has mixed characteristic $(0, p)$ and $(k_0, \overline{v})$ is AMK.*

This is as good as it can get; since it is an equivalence, establishing NIP transfer theorems in cases outside of this list is not needed.

Now that we know what the optimal NIP transfer theorem is, we aim to push it up the ladder. Some key ingredients of the proof have already been pushed up: the Artin-Schreier closure of NIP fields, which we already mentioned, and the method used for transfer; in fact, this is rather a case of trickling down the ladder, since this method was first established for NTP2 transfer, and only afterwards adapted for NIP transfer.

One other key ingredient is Shelah's expansion theorem, which does not hold outside of NIP theories, as we will recall in appendix section B. It is used in mixed characteristic together with the standard decomposition: a mixed characteristic valuation can be decomposed as an equicharacteristic 0 part, a small – rank 1 – jump from characteristic 0 to characteristic p, and an equicharacteristic p part. This decomposition is externally definable, thus, adding it to the structure preserves NIP by Shelah's expansion theorem. We can then argue part by part to obtain the result.

It is however possible to bypass this argument: instead of trying to prove that each part is NIP, we can use the explicit formula witnessing IP in fields with Artin-Schreier extensions, and lift complexity to the field. This way, there's no need to add intermediate valuations to the language, at least to prove that relevant part are p-closed or p-divisible.

This strategy can then be adapted to NIP$_n$ and to NTP2 henselian valued fields. Together with a NIP$_n$ transfer argument, we generalize Anscombe-Jahnke to NIP$_n$ fields in Chapter 3:

Theorem. *Let (K, v) be a henselian valued field. Then (K, v) is NIP$_n$ iff the following holds:*

1. *k is NIP$_n$, and*
2. *either*
   (a) *(K, v) is of equicharacteristic and is either trivial or SAMK, or*
   (b) *(K, v) has mixed characteristic $(0, p)$, (K, v_p) is finitely ramified, and $(k_p, \overline{v})$ checks 2a, or*
   (c) *(K, v) has mixed characteristic $(0, p)$ and $(k_0, \overline{v})$ is AMK.*

This gives, among others, the following corollary:

Corollary. *Let (K, v) be a NIP$_n$ henselian valued field. If k is NIP$_m$ for some $m \leqslant n$, then (K, v) is NIP$_m$. In particular, if k is NIP, (K, v) is NIP.*

As for NTP2 henselian valued fields, we prove in Section 2.5 on the one hand that NTP2 transfer holds in the same algebraic configuration as in Anscombe-Jahnke's theorem:

Proposition. *Let (K, v) be henselian. Suppose k is NTP2. If either*

1. *(K, v) is of equicharacteristic and SAMK or trivial, or*
2. *(K, v) is of mixed characteristic with v_p finitely ramified and $(k_p, \overline{v})$ SAMK or trivial, or*
3. *(K, v) is of mixed characteristic with $(k_0, \overline{v})$ AMK;*

then (K, v) is NTP2.

On the other hand, using explicit formulas, we prove that NTP2 henselian valued fields obey strong tameness conditions:

Proposition. *Let K be NTP2 and v be henselian. Then (K, v) is either*

1. *of equicharacteristic 0, hence tame, or*
2. *of equicharacteristic p and semitame, or*
3. *of mixed characteristic with $(k_0, \overline{v})$ semitame, or*
4. *of mixed characteristic with v_p finitely ramified and $(k_p, \overline{v})$ semitame.*

In particular, (K, v) is gdr.

This is as far as we could push Anscombe-Jahnke for NTP2 henselian valued fields. Though this allows us to give new examples of TP2 or NTP2 algebraic extensions of $\mathbb{Q}_p$, open questions remain, most notably, whether $\mathbb{F}_p((\mathbb{Q}))$ is NTP2.

0.5 Algebraic extensions of $\mathbb{Q}_p$

In Chapter 4, we study algebraic extensions of $\mathbb{Q}_p$, and by applying the previously obtained results, we classify them in terms of their complexity. Of course, the classifications we discussed are classification of henselian valued fields, and not of pure fields. Fortunately, the p-adic valuation is definable in any non-algebraically closed algebraic extension of $\mathbb{Q}_p$, a fact that is well-known but for which we provide a self-standing argument. On the NIP$_n$ side of the spectrum, we obtain the following:

Theorem. *An algebraic extension of $\mathbb{Q}_p$ is NIP$_n$ iff it is NIP, and iff it is in one of the following mutually exclusive cases:*

1. *it is a finite extension of $\mathbb{Q}_p$,*
2. *it is a finite extension of $\mathbb{Q}_p^t$,*
3. *it is an algebraic extension of $(\mathbb{Q}_p^t)^k$, where $(\mathbb{Q}_p^t)^k$ is any $\mathbb{Q}_p^t$-complement of $\mathbb{Q}_p^v$.*

We also study in detail other classes of complexity and present in Section 4.3 a state of art of complexity of algebraic extensions of $\mathbb{Q}_p$.

Chapter 1

Algebra and Model Theory of Valued Fields

Many textbooks on valued fields exist, whether it be of algebraic, number-theoretic or model-theoretic flavor. We write this chapter in order to lay the ground for what comes next, assembling definitions and theorems from, among other sources, [EP10, Hil18, Jah18].

1.1 Valuation theory

This section is mainly a rewriting of [EP10, sec. 4] and [Jah18]. We will often omit proofs, especially when the argument follows directly from definitions.

1.1.1 Dictionary of valuations

In order to properly define valuations, we should first define ordered abelian groups and how to extend them by a point at infinity; we let that aside for now and will come back to it in Section 1.3.1, when we study their model theory in details.

Definition 1.1.1. Let K be a field and $(\Gamma, +, <)$ an ordered abelian group. A valuation on K is a group epimorphism $v\colon (K^\times, \times) \twoheadrightarrow (\Gamma, +)$ such that $v(x+y) \geqslant \min(v(x), v(y))$ for all $x, y \in K^\times$ with $x + y \neq 0$.

We extend v to K by setting $v(0) = \infty$ and we sweep details under the carpet.

Example 1.1.2.

- On any field K, we can define the trivial valuation by setting $v(x) = 0$ for any $x \in K^\times$.
- The p-adic valuation $v_p\colon \mathbb{Q} \to \mathbb{Z}$ is defined by $v_p(p) = 1$ and $v_p(q) = 0$ for $q \neq p$ prime; this uniquely determines it by multiplicity – in particular, $v_p(1) = 0$ and $v_p(0) = \infty$.
- Given a field K and an irreducible polynomial $P \in K[t]$, we define the P-adic valuation $v_P\colon K(t) \to \mathbb{Z}$ by letting $v_P(P) = 1$ and $v_P(Q) = 0$ for $Q \in K[t]$ coprime with P. When $P = t$, we have the special case of the t-adic valuation.
- Finally, given a field K and an ordered abelian group Γ, we consider the Hahn field $K((\Gamma))$ of formal series $\sum_{\gamma\in\Gamma} a_\gamma t^\gamma$ such that $\{\gamma \in \Gamma \mid a_\gamma \neq 0\}$ is a well-ordered set. The t-adic valuation $v_t\colon K((\Gamma)) \to \Gamma$ is defined by letting $v(\sum_{\gamma\in\Gamma} a_\gamma t^\gamma) = \min\{\gamma \mid a_\gamma \neq 0\}$.

Definition 1.1.3. A valuation ring is a subring of a field such that for any non-zero x in the field, either x or x^{-1} is in the valuation ring.

Proposition 1.1.4.

- *Given a valuation v, we define the set $\mathcal{O}_v = \{x \in K \mid v(x) \geqslant 0\}$; it is a valuation ring.*
- *Given a valuation ring $\mathcal{O}$, we define a relation on $K^\times$ by writing $x \sim y$ iff $x^{-1}y \in \mathcal{O}^\times$; it is an equivalence relation, and the projection $v\colon K^\times \to (K^\times/{\sim})$ is a valuation – with value group $(\Gamma, +) = (K^\times/{\sim}, \times)$, ordered by writing $x/{\sim} \leqslant y/{\sim}$ iff $x^{-1}y \in \mathcal{O}$.*
- *Let v and w be two valuations on the same field. We have $\mathcal{O}_v = \mathcal{O}_w$ iff v and w are equivalent, that is, $v(x) > v(y)$ iff $w(x) > w(y)$ for all $x, y \in K$.*

The proof is omitted; it follows quite directly from definitions.

Definition 1.1.5. Given a valuation ring $\mathcal{O}$ and the corresponding valuation v, we define $\mathcal{M} = \mathcal{O} \setminus \mathcal{O}^\times = \{x \in K \mid v(x) > 0\}$. It is the unique maximal ideal of $\mathcal{O}$, thus $k = \mathcal{O}/\mathcal{M}$ is a field; we call it the residue field of the valuation. We call the projection from $\mathcal{O}$ to k the residue map of the valuation. We can extend it to a map from K to $k \cup \{\infty\}$ by sending any element $x \notin \mathcal{O}$ to ∞, we call this extended residue map a place.

Thus, any valuation comes with a valuation ring $\mathcal{O}$, a value group Γ and a residue field k – we will denote them be these letters usually, sometimes with v as a subscript to prevent ambiguity.

Example 1.1.6. We encourage readers not familiar with valuations to determine the value groups, the valuation rings and the residue fields of valuations defined in Example 1.1.2.

Lemma 1.1.7. *Let (K, v) be a valued field with residue field k. If $\mathrm{ch}(K) = p$, then $\mathrm{ch}(k) = p$. If $\mathrm{ch}(K) = 0$, then for any prime p, $\mathrm{ch}(k) = p$ iff $v(p) > 0$; in particular $\mathrm{ch}(k) = 0$ iff $v(p) = 0$ for all primes p.*

When $\mathrm{ch}(K) = \mathrm{ch}(k)$, we say (K, v) is of equicharacteristic, and specify 0 or p. If $\mathrm{ch}(K) \neq \mathrm{ch}(k)$, we know that $\mathrm{ch}(K) = 0$ and $\mathrm{ch}(k) = p > 0$, and we say (K, v) is of mixed characteristic.

1.1.2 Extensions of valuations

Theorem 1.1.8 (Chevalley's extension theorem)**.** *Let K be a field and $R \subset K$ a subring with a prime ideal $I \subset R$. Then there is a valuation ring $\mathcal{O} \subset K$ such that $R \subset \mathcal{O}$ and $\mathcal{M} \cap R = I$.*

Corollary 1.1.9. *Let $K \subset L$ be fields and v a valuation on K, then there is a valuation w on L extending v – in the sense that $w|_K$ is equivalent to v.*

Furthermore, the value group and residue field of (K, v) can be naturally embedded in those of (L, w).

Proposition 1.1.10. *Let L/K be algebraic and w be a valuation on L. Let $v = w|_K$. Then we have $\Gamma_v \subset \Gamma_w \subset \mathrm{Div}(\Gamma_v)$ and $k_v \subset k_w \subset k_v^{\mathrm{alg}}$.*

$\mathrm{Div}(\Gamma)$ is the divisible hull of Γ, it is constructed by adding to Γ, if necessary, γ/n for every $\gamma \in \Gamma$ and every $n \in \mathbb{N} \setminus \{0\}$. See section 1.3.1 for more details.

Theorem 1.1.11 (Fundamental equality)**.** *Let (K, v) be a valued field and L/K be a finite extension. Up to equivalence, there are only finitely many different extensions of v to L. Let $(w_i)_{i<n}$ be all the extensions of v to L. For each i let $e_i = (\Gamma_{w_i} : \Gamma_v)$ and $f_i = [k_{w_i} : k_v]$. Then there is $(d_i)_{i<n} \in \omega^n$ such that:*

$$[L\colon K] = \sum_{i<n} d_i e_i f_i.$$

Furthermore:

- *In residue characteristic 0,* $d_i = 1$*;*
- *In residue characteristic* p*, there is* $(m_i) \in \omega^n$ *such that* $d_i = p^{m_i}$*;*
- *If* L/K *is a Galois extension, the quantities* d_i*,* e_i *and* f_i *do not depend on* i*, and* $[L\colon K] = ndef$*.*

The quantities d_i, e_i and f_i are called respectively the defect, the ramification index and the inertia degree of the extension $(L, w_i)/(K, v)$. For a proof of the fundamental equality in the form stated here, we refer to [Kuh11, Lem. 11.2].

Definition 1.1.12. An extension $(L, w)/(K, v)$ is called immediate if $\Gamma_w = \Gamma_v$ and $k_w = k_v$. If a valued field has no proper immediate extension, it is called maximal. Similarly, if a valued field has no algebraic or separable algebraic immediate extension, we call it algebraically or separably algebraically maximal.

Definition 1.1.13. A valued field of residue characteristic $p > 0$ is Kaplansky if its value group is p-divisible and its residue field is p-closed and perfect, that is, it admits no extension (separable or not) of p-power degree.

Combining these two notions give (separably) algebraically maximal Kaplansky fields, abridged as (S)AMK, which will play an important role in Chapter 3.

1.1.3 Decomposition of valuations

Definition 1.1.14. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two valuation rings on the same field K. We call them comparable if one is included in the other and incomparable otherwise. If $\mathcal{O}_1 \subset \mathcal{O}_2$, we call $\mathcal{O}_1$ a refinement of $\mathcal{O}_2$ and $\mathcal{O}_2$ a coarsening of $\mathcal{O}_1$.

Note that two valuation rings always have a smallest common coarsening, namely, their product.

Definition 1.1.15. Let (K, v) be a valued field and w be a valuation on k_v. The lift of $\mathcal{O}_w$ to K, that is, $\{x \in K \mid \overline{x} \in \mathcal{O}_w \subset k_v\}$ is a valuation ring; we call the corresponding valuation the composition of w and v and we denote it $w \circ v$.

Since each valuation comes with a place, we usually draw the following diagram, where we denote by the same letter the place and the valuation:

$$K \xrightarrow{v} k_v \xrightarrow{w} k_w$$

The residue field of $(K, w \circ v)$ is k_w. The valuation ring $\mathcal{O}_{w\circ v}$ is a refinement of $\mathcal{O}_v$, and we have $\Gamma_{w\circ v} \equiv \Gamma_v \oplus \Gamma_w$, where $\oplus$ is the lexicographic sum:

Definition 1.1.16. Let Γ and Δ be ordered abelian groups. $\Gamma \oplus \Delta$ is the group $(\Gamma \times \Delta, +)$ ordered lexicographically: $(\gamma, \delta) > (\gamma', \delta')$ if and only if either $\gamma > \gamma'$ and δ, δ' are arbitrarily ordered, or $\gamma = \gamma'$ and $\delta > \delta'$.

Proposition 1.1.17. *Refinements of valuation rings correspond exactly to valuations on the residue field.*

Coarsenings of valuation rings correspond exactly to convex subgroups of the value group.

Proof. We've seen that a valuation on a residue field gives a refinement by composition. Now, given a refinement $\mathcal{O} \subset \mathcal{O}_v$, take the residue $\overline{\mathcal{O}} \subset k_v$, it is a valuation ring. This correspondence gives a bijection.

Now for coarsenings. Given a convex subgroup Δ of Γ_v, the valuation w sending x to $v(x)/\Delta$ is a coarsening of v. Finally, given a coarsening w of v, the set $v(\mathcal{O}_w^\times) = \{\gamma \in \Gamma_v \mid \exists x\ w(x) = 0 \wedge v(x) = \gamma\}$ is a convex subgroup of Γ_v. This correspondence is again bijective. □

With this, we can now decompose valuations; given a convex subgroup Δ of Γ_v, we can see a valuation v as a composition of two valuations:

$$K \xrightarrow{\Gamma_v/\Delta} k_\Delta \xrightarrow{\Delta} k_v$$

Definition 1.1.18. Let (K, v) be a valued field and fix $t \in \mathcal{O}_v$. The standard decomposition around t is defined by fixing two convex subgroups:

$$\Delta_0 = \bigcap_{\substack{v(t)\in\Delta \\ \Delta\subset\Gamma \text{ convex}}} \Delta \qquad \& \qquad \Delta_t = \bigcup_{\substack{v(t)\notin\Delta \\ \Delta\subset\Gamma \text{ convex}}} \Delta$$

And performing the following decomposition:

$$K \xrightarrow{\Gamma_v/\Delta_0} k_0 \xrightarrow{\Delta_0/\Delta_t} k_t \xrightarrow{\Delta_t} k_v$$

We immediately remark that Δ_0/Δ_t is of rank 1 – meaning that it has no non-trivial proper convex subgroup – and that $\bar{t} \neq 0$ in k_0 and $\bar{t} = 0$ in k_t. Most notably, when (K, v) is of mixed characteristic, we will perform this decomposition with $t = p$, decomposing a valuation into two equicharacteristic parts and a mixed characteristic rank 1 part in the middle.

Because the jump from characteristic 0 to characteristic p happens exactly at the value of p, the properties of the interval $[0, v(p)]$ play an important role.

Definition 1.1.19. Let (K, v) be a mixed characteristic valued field. We call it unramified if $v(p)$ is the minimal positive element of Γ, finitely ramified if the interval $[0, v(p)] \subset \Gamma$ is finite, and inifinitely ramified otherwise.

If the interval $[0, v(p)]$ is n-divisible, we call Γ roughly n-divisible.

1.2 Henselianity

1.2.1 Hensel's lemma

Similarly to an absolute value, a valuation induces a topology on a field, called a V-topology. In the same way as $\mathbb{R}$ is the completion of $(\mathbb{Q}, <)$, one can for each prime p define $\mathbb{Q}_p$ as the completion of $(\mathbb{Q}, v_p)$; that is, any sequence which is Cauchy according to the topology induced by v_p converges in $\mathbb{Q}_p$.

However, completeness is not a first-order property. A first-order counterpart of completion is henselianity:

Definition 1.2.1. A valuation v on a field K is called *henselian* if it extends uniquely – up to equivalence – to the algebraic closure K^{alg}. We also say that a field is henselian if it admits a non-trivial henselian valuation.

Henselianity can be expressed via several equivalent properties:

Proposition 1.2.2 (Hensel's lemma). *For a valued field (K, v), the following are equivalent:*

1. *v is henselian;*

2. *For all $P \in \mathcal{O}_v[X]$, if $\overline{P}$ has a simple zero $\alpha \in k$, then P has a zero $a \in \mathcal{O}_v$ with $\overline{a} = \alpha$;*

3. *For all $P \in \mathcal{O}_v[X]$, if there is $a \in \mathcal{O}_v$ such that $v(P(a)) > 2v(P'(a))$, then there is a unique $b \in \mathcal{O}_v$ with $P(b) = 0$ and $v(b - a) > v(P'(a))$;*

4. *For any $P \in \mathcal{M}_v[X]$ of degree d, $X^{d+2} + X^{d+1} + P$ has a root in K.*

The proof is once more omitted, see for example [EP10, Thm. 4.1.3].

Example 1.2.3.

- The trivial valuation is always henselian. Since $\mathbb{F}_p$ can only be trivially valued, algebraic extensions of $\mathbb{F}_p$ themselves can only be trivially valued.

- As stated, the p-adic valuation on $\mathbb{Q}_p$ is henselian, but not on $\mathbb{Q}$; for example, $X^2 + 1$ admits a simple root in $\mathbb{F}_5$ but not in $\mathbb{Q}$.

- Somewhat similarly, the t-adic valuation is henselian on $K((t))$ – i.e. $K((\mathbb{Z}))$ – but not on $K(t)$.

Corollary 1.2.4 (Henselian fundamental equality). *Let (K, v) be a henselian valued field, let L/K be finite and w be the unique extension of v to L; then:*

$$[L\colon K] = d(\Gamma_w\colon \Gamma_v)[k_w\colon k_v]$$

where $d = 1$ if $\mathrm{ch}(k) = 0$ and $d = p^m$ for some $m \in \omega$ when $\mathrm{ch}(k) = p$.

1.2.2 The canonical henselian valuation

We highlight the following facts; they are not hard to prove and will be useful, we refer to [EP10, sec. 4.2].

Proposition 1.2.5. *Let (K, v) be a valued field.*

1. *If $K = K^{\mathrm{sep}}$ (and v is non-trivial), then $k_v = k_v^{\mathrm{alg}}$;*

2. *If v is the composition of two valuations v_1 and v_2, then v is henselian iff v_1 and v_2 are henselian.*

Since any two valuation rings are included in a bigger valuation ring, namely their product, and coarsenings of a valuation are linearly ordered, valuations are always arranged in a tree. Henselian valuation rings are well behaved regarding this tree structure, forming two meaningful components. There, in the middle, lies one ring; one ring to compare them all and in the darkness define them.

All of the results of this subsection are based on [EP10, sec. 4.4].

Proposition 1.2.6. *We call two valuation rings of K independent if they are incomparable and if their only common coarsening is K itself.*

If a field K admits two independent henselian valuation rings, then K is separably closed.

Note that by Proposition 1.2.5 any coarsening of a henselian valuation is still henselian. Now split the set H of all henselian valuation rings of K in two:

$$H_1 = \{\mathcal{O}_v \mid v \text{ is henselian and } k_v \neq k_v^{\text{sep}}\}$$
$$H_2 = \{\mathcal{O}_v \mid v \text{ is henselian and } k_v = k_v^{\text{sep}}\}$$

Since K itself is a valuation ring H is never empty.

Proposition 1.2.7. *Let K be a field, then H_1 is linearly ordered by inclusion; furthermore, for any $\mathcal{O}_1 \in H_1$ and $\mathcal{O}_2 \in H_2$, we have $\mathcal{O}_2 \subset \mathcal{O}_1$.*

It is now clear what we ought to define as the canonical henselian valuation: the finest valuation which does not branch, see Figure 1.1.

Definition 1.2.8. The canonical henselian valuation of a field K, denoted by v_K, is the coarsest valuation of H_2 if H_2 is non-empty, and the finest valuation of H_1 if H_2 is empty.

Note that this valuation exists by Zorn's lemma in the first case and is just the intersection of all valuations of H_1 in the second case.

Proposition 1.2.9. *It follows from the definition:*

1. *Every henselian valuation is comparable with v_K and with every coarsening of it;*
2. *v_K is non-trivial iff $K \neq K^{\text{sep}}$ and K is henselian;*
3. *No proper coarsening of v_K has separably closed residue field;*
4. *All proper refinements of v_K have separably closed residue field.*

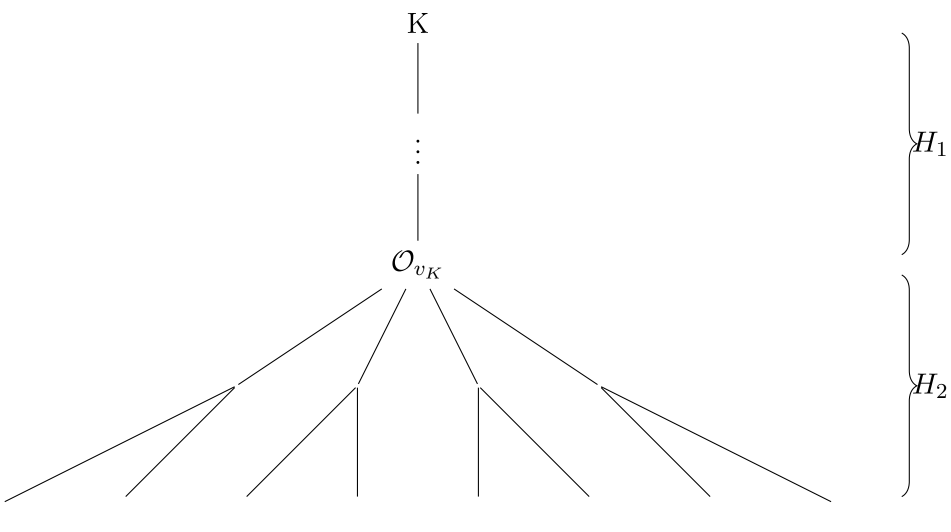


Figure 1.1: The tree structure of H.

1.3 Model theory of valued fields

Most of the results of this section can be found in [Hil18].

1.3.1 Ordered abelian groups

We call OAG the first-order theory of ordered abelian groups in the language $\mathcal{L}_{\mathrm{OAG}} = \{0, +, <\}$. It is axiomatized by:

- Abelian group axioms,
- Linear order axioms,
- $\forall x \forall y \forall z (x < y \to x + z < y + z)$.

When talking about valued fields, we need to augment the structure of the value group by a point at infinity; the theory OAG$^+$ in the language $\mathcal{L}_{\mathrm{OAG}^+} = \mathcal{L}_{\mathrm{OAG}} \cup \{\infty\}$ is OAG (on all $x \neq \infty$) plus $\forall x (x + \infty = \infty)$ and $\forall x (x \neq \infty \to x < \infty)$.

Because these two theories are bi-interpretable, we jump from one to the other without too much care; we will work with OAG when talking about pure ordered abelian groups and OAG$^+$ when talking about the structure of a value group of a valued field.

Finally, we consider the theory DOAG – or DOAG$^+$ – of divisible ordered abelian groups, consisting of OAG and $\forall x \exists y (x = \underbrace{y + \cdots + y}_{n})$ for each $n > 1$.

We usually allow the trivial group $\{0\}$ to be a model of both OAG and DOAG, especially since we need to be able to talk about trivial valuations; however, this trivial structure is sometimes an annoying counterexample, so we exclude it when needed by adding $\exists x (x \neq 0)$ or $\exists x (x \neq 0 \wedge x \neq \infty)$ to the theories.

As $a < \frac{a+b}{2} < b$ for any $a < b$, we have – except for the trivial group – the following:

Lemma 1.3.1. DOAG $\vdash$ DLO

Lemma 1.3.2. *Any ordered abelian group has a divisible hull unique up to isomorphism.*

By the divisible hull of $\Gamma \vDash$ OAG, we mean $\mathrm{F} \vDash$ DOAG such that F/Γ is torsion. The existence and uniqueness of such a divisible hull is an easy exercise (mimic the construction of $\mathbb{Q}$ from $\mathbb{Z}$).

We write $\mathrm{Div}(\Gamma)$ for the divisible hull.

Proposition 1.3.3. DOAG *has QE and is complete.*

Completeness follows from QE, and QE can be seen easily by extending partial embeddings.

Corollary 1.3.4. DOAG *is o-minimal.*

Finally, let us mention the following, which will be usefull later:

Theorem 1.3.5 (Gurevich-Schmitt, [GS84])**.** OAG *is NIP.*

1.3.2 Languages of valued fields

There are many different languages for valued fields. The four we present here do not contain more than what is needed, and expressing the theory of valued fields in those four languages result in four bi-interpretable theories.

- $\mathcal{L}_{\mathcal{O}}$ is $\mathcal{L}_{ring}$ plus a unary predicate $\mathcal{O}$ for a valuation ring,
- $\mathcal{L}_{div}$ is $\mathcal{L}_{ring}$ plus a binary relation $|$ which is to be interpreted as $x|y$ iff $v(x) \leqslant v(y)$,

- $\mathcal{L}_\Gamma$ is a two-sorted language with a sort K equipped with $\mathcal{L}_{ring}$ and a sort Γ equipped with $\mathcal{L}_{\text{OAG}}$, plus a function symbol $v\colon K \to \Gamma$,
- $\mathcal{L}_{\Gamma k}$ is a three-sorted language containing the previous one as well as a copy of $\mathcal{L}_{ring}$ for the third sort k and a function $R\colon K^2 \to k$ which is to be interpreted as the function sending (a, b) to the residue of $b^{-1}a$ if it is in $\mathcal{O}$, and 0 otherwise.

The reason why we specifically consider these languages is a QE result (see Theorem 1.3.9).

Most of the time, a valued field has more structure than the same field without valuation – in the sense that more sets are definable. However, it might happen that the valuation is definable, in the sense that $\mathcal{O}$ is a ring-definable set; this is true for example for v_p in $\mathbb{Q}_p$ or v_t in $K((t))$.

Definition 1.3.6. We define open balls of center $a \in K$ and of radius $\gamma \in \Gamma_v$ as $B_{>\gamma}(a) = \{x \in K \mid v(x-a) > \gamma\}$ and closed balls as $B_{\geqslant\gamma}(a) = \{x \in K \mid v(x-a) \geqslant \gamma\}$.

We can consider singletons as closed balls of infinite radius and K as an open ball of negative infinite radius.

Open balls are a basis for a field topology – that is, a topology making field operations continuous.

Note that $\mathcal{O} = B_{\geqslant 0}(0)$ and $\mathcal{M} = B_{>0}(0)$.

Since balls form a definable family, this topology is said to be definable in a valued field. This topology is discrete iff the valuation is trivial. Topologies induced by valuations or archimedean absolute values are called V-topologies. They can be characterized in first-order:

Lemma 1.3.7. *Let τ a collection of open neighborhoods of 0 forming a basis for a topology. Then τ is a V-topology iff:*

- $\bigcap_{U\in\tau} U = \{0\}$ *but* $\{0\} \notin \tau$*;*
- *For any pair $U, V \in \tau$, there is $W \in \tau$ such that $W \subset U \cap V$;*
- *For every $U \in \tau$ there is $W \in \tau$ for which $W - W \subset U$;*
- *For all $U \in \tau$ and every pair $x, y \in K$ there is $W \in \tau$ satisfying $(x+W)(y+W) \subset xy+U$;*
- *For $U \in \tau$ and $x \in K^\times$, there exists $W \in \tau$ such that $(x+W)^{-1} \subset x^{-1}+U$;*
- *For all $W \in \tau$ there exists $U \in \tau$ such that $x, y \in K$ and $xy \in U$ implies $x \in W$ or $y \in W$.*

If τ is a definable family, then these conditions are first-order.

See [EP10, Annex B] for more details on V-topologies, including a proof of the result above.

1.3.3 Algebraically closed valued fields

Definition 1.3.8. We call ACVF the valued-field-theory of algebraically closed valued fields. We write $\text{ACVF}_{(\text{ch}(K),\text{ch}(k))}$ for the theory ACVF with a fixed characteristic for the field and the residue field.

Theorem 1.3.9 (Robinson)**.** ACVF *has QE in any of the four languages. Its completions are* ACF_0 *and* ACF_p *with trivial valuations,* $\text{ACntVF}_{(0,0)}$, $\text{ACVF}_{(0,p)}$ *and* $\text{ACntVF}_{(p,p)}$ *when "ntVF" stands for non-trivially valued fields.*

No need to specify "nt" for mixed characteristic since trivial valuations are always equicharacteristic. Again, we refer to [Hil18] for details and proof of the result above and all the corollaries below.

Corollary 1.3.10. ACVF *is C-minimal: every definable subset of* $K \models$ ACVF *is a finite disjoint union of Swiss Cheeses, that is, sets of the form* $B \setminus \cup_{i<n} B_i$ *for balls* B *and* $(B_i)_{i<n}$.

Corollary 1.3.11. *(Any completion of)* ACVF *is NIP.*

Note that (any completion of) ACF is stable, even strongly minimal; adding a valuation gives a definable order and thus instability, but nonetheless remains NIP. See Chapter 2 for definitions of these notions, precisely Definitions 2.1.1 and 2.2.1.

Corollary 1.3.12. *In* $K \models$ ACVF*, the value group and the residue fields are stably embedded and orthogonal (that is, definable subsets of* $\Gamma^n \times k^m$ *are of the form* $X \times Y$ *for* $X \subset \Gamma^n$ *definable and* $Y \subset k^m$ *definable).*

1.3.4 The AKE principle

A cornerstone in the model theory of valued fields is the following result, obtained independently by Yuri Ershov and by James Ax and Simon Kochen:

Theorem 1.3.13 (Ax-Kochen [AK65], Ershov [Ers65])**.** *Let* (K, v) *and* (L, w) *be henselian valued fields of equicharacteristic 0, then* $(K, v) \equiv (L, w)$ *iff* $\Gamma_v \equiv \Gamma_w$ *and* $k_v \equiv k_w$.

In particular, this implies the following:

Corollary 1.3.14. *Let* φ *be a first-order sentence of valued fields, then for almost all* p, $\mathbb{Q}_p \models \varphi$ *iff* $\mathbb{F}_p((t)) \models \varphi$.

Indeed, an ultraproduct of $\mathbb{Q}_p$ and of $\mathbb{F}_p((t))$ – regarding the same non-principal ultrafilter on the set of primes numbers – have the same residue field and same value group, and are of equicharacteristic 0, and we conclude by Łoś' theorem.

At the time of publication of this result, in 1965, it was considered as a mere lemma, used to obtain the main theorem, namely, an approximate answer to Artin's conjecture. Nowadays, AKE-like results are an important part of the model-theoretic study of valued fields. The AKE-philosophy can be summarized by: if I know many properties of the residue field and the value group, what can I say about the valued field itself?

Modern proofs of the AKE principle can be found in [Sim15, app. A.2] and [Hil18, sec. 6], we give a summary of these arguments here.

Definition 1.3.15. Let (K, v) be a valued field. An angular component is a map $\mathrm{ac}\colon K \to k$ such that $\mathrm{ac}(xy) = \mathrm{ac}(x)\,\mathrm{ac}(y)$ and extending the residue map, in the sense that if $v(x) = 0$, $\mathrm{ac}(x) = \mathrm{res}(x)$.

Angular components can be obtained by cross sections of the valuation: given a group homomorphism $s\colon \Gamma \to K^\times$ such that $v \circ s = \mathrm{id}$, the map $\mathrm{ac}_s(x) = \mathrm{res}(\frac{x}{s(v(x))})$ is an angular component.

Not all valued fields have angular components, however, any $\aleph_1$-saturated valued field does so – because its valuation then has a cross section.

The three-sorted language of valued fields with an angular component instead of the usual residue map is called the Denef-Pas language, and was proven to be an interesting language for the model theory of valued fields by Johan Pas:

Theorem 1.3.16 (Pas, [Pas89])**.** *Let (K, v, ac) be a henselian valued field of equicharacteristic 0 equipped with an angular component. Then (K, v, ac) eliminates K-quantifiers in the Denef-Pas language.*

The AKE principle then follows. Note that this modern proof does not say anything about other characteristics, whereas, as noted by Françoise Delon in [Del80], Ershov's proof actually does apply to any algebraically maximal Kaplansky henselian valued field; in particular, it applies to equicharacteristic 0 fields, but not only.

1.4 Definability of valuations

In general, adding a valuation to a pure field can significantly change its model-theoretic properties, by adding new definable sets; as we've seen, a general algebraically closed valued field is NIP and unstable if the valuation is non-trivial, whereas an algebraically closed field is strongly minimal.

However, in some cases, adding the valuation does not change anything, because it was already definable: we call a valuation definable when the set $\mathcal{O}_v \subset K$ is a definable set in the pure field structure $(K, 0, 1, +, -, \times)$. Whether it is definable with or without parameters is mostly not relevant for us. The best example, also significant historically, is the p-adic field, on which the p-adic valuation is definable by a formula of Julia Robinson.

In this section, we compile definability results which will be useful for the rest of our work; none of them are new.

1.4.1 Robinson's formula

A very beautiful formula dating back to Julia Robinson can define v_p in $\mathbb{Q}_p$:

$$\varphi(x) \colon \exists y \; 1 + px^q = y^q$$

Here q is a prime number different from p. Indeed, if $v_p(x) < 0$ then $v_p(1 + px^q) = v_p(px^q) = qv_p(x) + 1$; hence $v(1 + px^q) \neq v(y^q)$ for all y and $\mathbb{Q}_p \not\models \varphi(x)$. On the other hand, if $v_p(x) \geqslant 0$ then $v_p(px^2) > 0$ and $X^2 - (1 + px^2)$ has a root by Hensel's lemma, so $\mathbb{Q}_p \models \varphi(x)$.

This formula works mainly because there is an element of minimum positive valuation, namely p, and because there is a polynomial to which we can easily apply Hensel's lemma.

Proposition 1.4.1. *Let (K, v) be a henselian valued field. Suppose there is $t \in \mathcal{O}_v$ such that $v(t)$ is not p-divisible for some $p \neq \mathrm{ch}(k)$. Then the formula*

$$\varphi_p(x, t) \colon \exists y \; 1 + tx^p = y^p$$

defines the set $I = B_{>-v(t)/p}(0)$, and the formula

$$\psi_p(x, t) \colon \forall y (\varphi_p(y, t) \to \varphi_p(xy, t))$$

defines a non-trivial coarsening of $\mathcal{O}_v$.

This is a quite well-known generalization of Robinson's formula which can be attributed to Thomas Scanlon, see for example [Jah19, Prop. 3.6].

Proof. Let $x \in K$ be such that $v(t) + pv(x) > 0$. Then $v(1 + tx^p) = 0$, and the polynomial $Y^p - (1 + tx^p)$ has a root by Hensel's lemma. Note: this is why we need $p \neq \mathrm{ch}(k)$. Now let $x \in K$ such that $v(t) + pv(x) < 0$. Then $v(1 + tx^p) = v(t) + pv(x)$ is not p-divisible, hence x cannot realize φ_p. The case $v(t) + pv(x) = 0$ cannot happen since $v(t)$ is not p-divisible.

Therefore we have that $\varphi_p(x,t)$ holds iff $pv(x) > -v(t)$. In particular, it holds for all elements of $\mathcal{O}_v$.

The formula ψ_p defines the multiplicative stabilizer of I. This gives us a ring, containing $\mathcal{O}_v$, but not equal to the whole field since $\psi_p(t^{-1},t)$ doesn't hold. $\square$

We will be more precise as to exactly which coarsening this formula defines.

Lemma 1.4.2. *Let (K,v) and t be as above, and consider a convex subgroup Δ of Γ. Let $R = \psi_p(x,t)$. Then $\Delta \subset vR$ iff $\overline{v(t)} \in \Gamma/\Delta$ is not p-divisible, that is, there is no $\varepsilon \in \Delta$ such that $v(t)+\varepsilon$ is p-divisible (in Γ).*

Proof. Let $\varepsilon \in \Delta$ be such that $v(t)+\varepsilon$ is p-divisible. If ε is negative, then $v(t)+\varepsilon-p\varepsilon$ is also p-divisible and thus, possibly replacing ε with $(1-p)\varepsilon$, we might assume $\varepsilon > 0$. Now, $v(t)-(p-1)\varepsilon$ is also p-divisible, and is smaller than $v(t)$. Let $\delta = -\frac{v(t)-(p-1)\varepsilon}{p}$. We have $p\delta > -v(t)$ so $\delta \in vI$. But $p(\delta-\varepsilon) = -v(t)-\varepsilon < -v(t)$, so $\delta-\varepsilon \notin vI$; hence $-\varepsilon \notin vR$.

Now assume that there is $\varepsilon \in \Delta_{>0}$ such that $-\varepsilon \notin vR$. This means that there is $\delta \in \Gamma$ such that $p\delta > -v(t)$ but $p(\delta-\varepsilon) < -v(t)$. Hence $0 < p\delta + v(t) < p\varepsilon$: $v(t)+p\delta \in \Delta$ and $v(t)-(v(t)+p\delta)$ is p-divisible. $\square$

Here's a tentative explanation of the phenomenon: replace your parameter t by at in ψ_p for some a with $v(a) = \varepsilon \in \Delta$; if it remains non-$p$-divisible, well then this shifted parameter should define the same set, so there is no way to tell ε and 0 apart in the value group.

Corollary 1.4.3. *Let (K,v) be henselian. If Γ has a rank 1 convex subgroup which is not p-divisible for some $p \neq \mathrm{ch}(k)$, then v is definable.*

In particular, if Γ has a minimal positive element, then v is definable.

1.4.2 Fehm's method

Robinson's formula gives, in particular, an existential definition of the t-adic valuation in $K((t))$ using one parameter, namely, t. In 2013, Sylvy Anscombe and Jochen Koenigsmann gave an existential parameter-free formula defining the t-adic valuation in $\mathbb{F}_q((t))$, using the fact that the residue field, being finite, is qf-$\emptyset$-definable. Arno Fehm then generalized their result to other cases. Their motivation was to ensure existential $\emptyset$-definability, however, here we will allow parameters, making the argument more direct but weaker. All of the results below come from Fehm's paper [Feh15].

Lemma 1.4.4. *Let (K,v) be henselian, and suppose the residue field k is not separably closed. Let $f \in \mathcal{O}[X]$ be a monic polynomial such that $\overline{f} \in k[X]$ has no root and $\overline{f}'$ is not zero. Then the set $U = \frac{1}{f(K)} - \frac{1}{f(K)}$ is definable and we have $\mathcal{M} \subset U \subset \mathcal{O}$.*

Proof. U is definable via the formula $\varphi(x,a)\colon \exists y \exists z f(y) - f(z) = xf(y)f(z)$, where a are the coefficients of f. To prove the inclusions, note that:

- If $v(x) \geqslant 0$, then $\overline{f}(\overline{x}) \neq 0$ by the choice of f; this means $f(x) \in \mathcal{O}^\times$ and thus $\frac{1}{f(x)} \in \mathcal{O}^\times$.
- If $v(x) < 0$, then $v(f(x)) < 0$, so $\frac{1}{f(x)} \in \mathcal{M}$.

Hence, if we take $u \in U$, u is a sum of two elements of $\mathcal{O}$ and thus is in $\mathcal{O}$.

Now, by our choice of f there is $a \in \mathcal{O}$ such that $\overline{f}'(\overline{a}) \neq 0$. Now fix $m \in \mathcal{M}$ and consider the polynomial $P(X) = mf(a)f(X) + f(X) - f(a)$. Its residue is $\overline{P}(X) = \overline{f}(X) - \overline{f(a)}$. $\overline{a}$ is a simple root of $\overline{P}$; by henselianity P has a root $b \in \mathcal{O}$, and thus $m = \frac{f(a)-f(b)}{f(a)f(b)} \in U$. $\square$

It remains to add to this set U a lift of the residue field, that is, a set $T \subset \mathcal{O}$ such that $\overline{T} = k$. Indeed, given such a T, fix any $x \in \mathcal{O}$; there is $t \in T$ such that $\bar{t} = \overline{x}$. Now $(x - t) \in \mathcal{M} \subset U$, so $x = (x - t) + t \in U + T$, and we have $U + T = \mathcal{O}$.

Lemma 1.4.5. *Let (K, v) be henselian with residue field k not separably closed. If k is finite or PAC*, then there is a definable $T \subset \mathcal{O}$ such that $\overline{T} = k$.*

Proof. If k is finite, then we can find a finite T, which is then definable – with parameters.

If k is PAC, then we fix a monic polynomial $f \in \mathcal{O}[X]$ such that $\overline{f}$ is square-free, has no root, and $\overline{f}' \neq 0$. Then the set $T = \frac{1}{f(K)} \cdot \frac{1}{f(K)}$ works: indeed, as above, $T \subset \mathcal{O}$. Now fix $c \in k^{\times}$; the polynomial $\overline{f}(X)\overline{f}(Y) - \frac{1}{c}$ is absolutely irreducible in $k[X, Y]$, therefore it has a rational point, that is to say, $\frac{1}{c} \in \overline{f}(k) \cdot \overline{f}(k)$, or equivalently, $c \in \overline{\frac{1}{f(K)} \cdot \frac{1}{f(K)}}$, which means $k = \overline{T}$. □

Corollary 1.4.6. *Let (K, v) be henselian with residue field k. If k is finite or PAC and not separably closed, then v is definable.*

1.4.3 Non-explicit definability

In 2014, Franziska Jahnke and Koenigsmann proved that canonical p-adic valuations are definable in a wide class of fields, using a method that gives no explicit formula, or rather, that cannot give one. We do not define here v_K^p, the canonical p-henselian valuation, or its spin-off v_K^{2*}, as we will do so later in section A, where all notations are explained.

Theorem 1.4.7 (Jahnke-Koenigsmann, [JK14])**.** *For any prime number $p \neq 2$, there is a $\emptyset$-ring-formula φ_p such that if either* $\mathrm{ch}(K) = p$ *or K contains a p^{th}-root of unity, then $\varphi_p(K) = \mathcal{O}_{v_K^p}$; and for $p = 2$ there is a $\emptyset$-ring-formula φ_2 such that if $K \vDash T_2$, then $\varphi_2(K) = \mathcal{O}_{v_K^2}$ when $k_{v_K^2}$ is not euclidean and $\varphi_2(K) = \mathcal{O}_{v_K^{2*}}$ when $k_{v_K^2}$ is euclidean.*

This result was first announced in a 1995 paper by Koenigsmann alone, however, the proof was incomplete. Jahnke and Koenigsmann, not yet aware of the incompleteness, used the result to define henselian valuations in fields with non-universal Galois groups. Afterwards, they published a paper fixing the proof, and ended up publishing those two papers in reverse order, making sure everything is well-founded. Nevertheless, a slight gap remained in a jump from equicharacteristic p to mixed characteristic; this was spotted and fixed by Zoé Chatzidakis and Milan Perera.

Thus, the complete and (hopefully) correct proof of the definability of canonical p-henselian valuations can be found in [Koe95, JK14, JK15, CP17]; one should read them in antichronological order to know which result from the previous papers are being fixed.

Because of this sinuous literature, we assembled a complete proof, as well as definitions and discussion around p-henselianity, which we will provide in appendix A.

For our purpose, what matters truly is the definability of henselian valuations, rather than p-henselian valuations. We will use the following theorem by Jahnke and Koenigsmann, of which we only give a proof heuristic here.

Corollary 1.4.8 ([JK15, Thm. 3.15])**.** *If the Galois group of a field K is non-universal and if k_{v_K} is neither separably closed nor real closed, then v_K is $\emptyset$-definable in K.*

*A field is called pseudo-algebraically closed, or PAC, if it is existentially closed in each of its regular extensions. This is equivalent to saying that every irreducible variety has a rational point.

Proof heuristic. The assumptions are here to make sure that v_K is the finest henselian valuation on K. Now, if the residue field has only Galois extensions of p-power degree, then we will have $v_K^p = v_K$, and we can then check that K satisfies the needed assumptions and ensure that v_K^p is $\emptyset$-definable.

If the residue field has Galois extension of degree divisible by 2 different primes $p < q$, then – potentially going there and back again as in section 4.1.3 – v_K^p and v_K^q are definable. Because the residue is neither p nor q closed, both v_K^p and v_K^q are refinements of v_K. It turns out that v_K^p and v_K^q are comparable, and that the coarsest one is henselian; thus, because v_K is the finest henselian valuation, it must be equal to one of v_K^p and v_K^q and is thus definable.

It might seem that the non-universality of the Galois group is not used in the proof. But saying "it turns out that v_K^p and v_K^q are comparable" is not a proof; and the reason why it turns out like that involves non-universality. We refer to the original paper for more details, most notably [JK15, Prop. 3.13]. □

Chapter 2

Artin-Schreier Extensions & Combinatorial Complexity

The study of combinatorial complexity and its link with algebraic properties of fields can be traced back to the 70's, when Angus Macintyre showed that infinite ω-stable fields are separably closed [Mac71]. This result has since been extended to superstable fields [CS80] and recently to large stable fields [JTWY21].

The study of NIP fields has gained more interest in the recent years. To the extent of current knowledge, unstable NIP fields seem to be o-minimal or henselian. This is known for dp-finite fields by the work of Will Johnson [Joh20], furthermore, NIP henselian fields are classified by a result of Anscombe and Jahnke [AJ19a], as we will see in Theorem 3.1.1.

On the other hand, it is believed that fields are NIP_n exactly when they are NIP. Many properties of NIP fields can be generalized to NIP_n fields, for example by work of Nadja Hempel and Artem Chernikov [Hem14, CH21].

Finally, NTP2 fields have seen many recent developments, including transfer from residue field in equicharacteristic 0 by Chernikov [Che14], extensive study of valued difference NTP2 fields by Chernikov and Martin Hils [CH12], and a proof of NTP2 for bounded PRC and PpC fields by Samaria Montenegro [Mon17].

This chapter introduces relevant complexity classes of first-order structure and presents the most important results and the main conjectures. It focuses in detail on the relationship between Artin-Schreier extensions and these complexity classes. The starting point is a well-known result by Itay Kaplan, Scanlon and Frank Wagner, stating that infinite NIP fields of characteristic $p > 0$ have no Artin-Schreier extension [KSW11]. This has been shown to also hold for NIP_n (n-dependent) fields by Hempel [Hem14], and Chernikov, Kaplan and Pierre Simon extended this result to the NTP2 setting, proving that an NTP2 field of characteristic $p > 0$ has finitely many Artin-Schreier extensions [CKS12].

These conditions can be used to check whether a given field fails to be $\mathrm{NIP}_{(n)}$ or NTP2: $\mathbb{F}_p((\mathbb{Z}))$ has TP2, $\mathbb{F}_p((\mathbb{Q}))$ has $\mathrm{IP}_{(n)}$. But this is rather unsatisfying: being $\mathrm{NIP}_{(n)}$ or NTP2 is a global property, whereas proving that some theory has $\mathrm{IP}_{(n)}$ or TP2 should be done by exhibiting a specific formula witnessing it. Such explicit formulas can be found by reversing the original arguments, they are exposed in Corollary 2.2.14, Corollary 2.3.12 and Corollary 2.5.9.

2.1 Stable fields

In this section we present stable theories as a first example of a combinatorial complexity class of first-order structure. Historically, stable theories have been extensively studied before introducing most other complexity classes, and mathematically, most of these classes are generalizations of stable theories. Thus, even though as we will quickly note, valued fields are always unstable, and therefore nothing in this section is used elsewhere in the text; it makes sense for us to at least define them and develop some understanding of their behavior.

Definition 2.1.1. Fix a complete theory T and a monster model $\mathcal{M} \vDash T$.

A formula $\varphi(x, y)$ is said to have the order property (OP) in T if there are $(a_i)_{i<\omega}$, $(b_i)_{i<\omega}$ in $\mathcal{M}$ such that

$$\mathcal{M} \vDash \varphi(a_i, b_j) \Leftrightarrow i < j.$$

A formula which doesn't have the order property is called NOP, and a theory is called NOP if all formulas are.

Example 2.1.2. Any infinite ordered theory, such as $(\mathbb{N}, <)$ or DLO, and any theory with a definable order such as (the theory of the pure fields) $\mathbb{Q}$ or $\mathbb{R}$ have OP. In particular, (non-trivial) OAG and (non-trivially) valued fields have OP.

On the other hand, ACF is a NOP theory – it is even strongly minimal.

Many other classes of complexity can be defined by studying whether or not a given theory expresses some pattern, here an infinite linear order, but in other cases a random graph, or a infinite tree. In order to understand why this is a relevant notion, we will present another equivalent definition of stability and a forking calculus result.

Definition 2.1.3. A complete theory is called κ-stable if over any set of parameters $A \subset \mathcal{M}$ of size κ and any $n < \omega$, $|S_n(A)| = \kappa$.

A theory is called stable if it is κ-stable for some κ and superstable if it κ-stable for all κ large enough.

Proposition 2.1.4 ([Che15, Prop. 2.10]). *A theory is NOP iff it is stable.*

Usually, "stable" is a more standard term than "NOP", despite its lack of consistency with the rest of the important classes, and its vagueness – other mathematical objects can be called stable, whereas as far as our knowledge go, NOP has only one meaning. Do not be afraid, we follow the standard notation here; à charge de revanche.

Theorem 2.1.5. *T is stable iff there is a ternary relation $\mathop{\smile\hspace{-0.9em}|\hspace{0.5em}}$ (defined on subsets of a sufficiently saturated model $\mathcal{M}$) such that:*

- *$\mathop{\smile\hspace{-0.9em}|\hspace{0.5em}}$ is invariant under automorphism;*
- *$A \mathop{\smile\hspace{-0.9em}|\hspace{0.5em}}_C B$ iff $a \mathop{\smile\hspace{-0.9em}|\hspace{0.5em}}_C b$ for any finite tuples $a \in A$ and $b \in B$;*
- *For $A \mathop{\smile\hspace{-0.9em}|\hspace{0.5em}}_C B$ and any D, there is $\sigma \in \mathrm{Aut}(\mathcal{M})$ such that $\sigma(A) \mathop{\smile\hspace{-0.9em}|\hspace{0.5em}}_C BD$;*
- *For any tuple a and set B, there is $C \subset B$ with $|C| \leqslant |T|$ such that $a \mathop{\smile\hspace{-0.9em}|\hspace{0.5em}}_C B$;*
- *$A \mathop{\smile\hspace{-0.9em}|\hspace{0.5em}}_C B$ and $A \mathop{\smile\hspace{-0.9em}|\hspace{0.5em}}_{CB} D$ iff $A \mathop{\smile\hspace{-0.9em}|\hspace{0.5em}}_C BD$;*
- *$A \mathop{\smile\hspace{-0.9em}|\hspace{0.5em}}_C B$ iff $B \mathop{\smile\hspace{-0.9em}|\hspace{0.5em}}_C A$;*
- *If $A \mathop{\smile\hspace{-0.9em}|\hspace{0.5em}}_C A$ then $A \subset \mathrm{acl}^{eq}(C)$;*
- *If $\mathrm{stp}(a/A) = \mathrm{stp}(b/A)$ and $a \mathop{\smile\hspace{-0.9em}|\hspace{0.5em}}_A B$ and $b \mathop{\smile\hspace{-0.9em}|\hspace{0.5em}}_A B$, then $\mathrm{stp}(a/AB) = \mathrm{stp}(b/AB)$.*

Furthermore, $\mathop{\smile\hspace{-0.9em}|\hspace{0.5em}}$ then corresponds to the usual non-forking independence relation.

We refer to [Pal18, Thm. 4.14] for a proof and a definition of the strong type "stp".

Conjecture 2.1.6 (Stable fields conjecture). *A pure field is stable iff it is finite or separably closed.*

Theorem 2.1.7.

- *Separably closed fields are stable (Wood, [Woo79]).*
- *ω-stable fields are finite or algebraically closed (Macintyre, [Mac71]).*
- *Superstable fields are finite or algebraically closed (Cherlin-Shelah, [CS80]).*
- *Large* stable fields are finite or separably closed (Johnson-Tran-Walsberg-Ye, [JTWY21])*

Finally, let us mention one important result:

Theorem 2.1.8. *Infinite stable fields of characteristic p are Artin-Schreier closed.*

This was shown by Scanlon in a short note on his personal website, see [Sca00]. This sparked interest in the research on links between Artin-Schreier extensions and combinatorial complexity.

However, looking at stability in a combinatorial manner yields the following line of reasoning: take an infinite field of characteristic p with Artin-Schreier extensions, we thus know that it is unstable, hence there exists a formula with the order property. However, this formula is not explicited in Scanlon's work. Later, we will settle this in the study of NIP fields, by giving a formula which has IP – and thus also the order property – in fields with Artin-Schreier extensions.

2.2 NIP fields

Consider DLO. It is clearly unstable, since it has the order property – one can also see that $|S_1(A)| = \text{ded}(|A|) > |A|$. But it is a very well-behaved theory. In order to study theories "like DLO", another property which they do not witness has been introduced, namely the independence property. Definitions go back to Shelah, but we present them in a modern form, mostly based on Simon's book [Sim15].

2.2.1 The independence property

Definition 2.2.1. Let T be a complete theory and $\mathcal{M} \vDash T$ a monster. A formula $\varphi(x, y)$ is said to have the independence property (IP) if there are $(a_i)_{i<\omega}, (b_J)_{J\subset\omega}$ such that $\mathcal{M} \vDash \varphi(b_J, a_i)$ iff $i \in J$.

A formula is said to be NIP if it doesn't have IP, and a theory is called NIP if all formulas are NIP.

It is easy to see that NOP formulas are NIP.

As for stability, this definition can seem arbitrary. Therefore, we give some general results to try and build intuition, and to see how they can, or can't, be extended later to NIP_n theories.

Definition 2.2.2. The Vapnik-Chervonenkis dimension ($\dim_{\text{VC}}$) of a formula $\varphi(x, y)$ (in a given complete theory T) is $\geqslant N$ if we can find $(a_i)_{i<N}$ and $(b_J)_{J\subset N}$ in a model $\mathcal{M} \vDash T$ such that $\mathcal{M} \vDash \varphi(b_J, a_i)$ iff $i \in J$.

If $\dim_{\text{VC}}(\varphi) \geqslant N$ for all N, we write $\dim_{\text{VC}}(\varphi) = \infty$. It is then clear that $\dim_{\text{VC}}(\varphi) = \infty$ iff φ has IP.

*A field is called large if any curve having at least one smooth rational point has infinitely many rational points. Henselian fields and PAC fields are examples of large fields, see [Pop14].

Lemma 2.2.3 ([Sim15, Lem. 2.5 & 2.9]). *NIPity is preserved under boolean combinations and under swapping x and y.*

In particular, this means one could define IP with a_i on the first tuple of variables, and b_J on the second one; this is what is usually done. The reason why we set it up in reverse order is to prepare the terrain for a generalization of IP, called IP_n, that we will define in the next section.

Lemma 2.2.4 ([Sim15, Lem. 2.7]). *A formula $\varphi(x, y)$ has IP iff there exists $(a_i)_{i<\omega}$ indiscernible (over $\emptyset$) and b such that $\varphi(b, a_i)$ holds iff i is even.*

Lemma 2.2.5 ([Sim15, Lem. 2.11]). *A theory is NIP iff all formulas $\varphi(x, y)$ with $|x| = 1$ are NIP.*

Lemma 2.2.6 ([Sim15, Prop. 2.43]). *A theory T is called* dependent *if and only if for all $\mathcal{M} \models T$ and for all $p \in S_n(M)$, p has at most $2^{|M|+|T|}$ global coheirs.*

T is dependent iff it is NIP.

Thus, once again, two names cohabit for these theories. "NIP" is the most common name in today's literature, and the one we use. We think of this notion and all other in terms of truth patterns that can or can't be witnessed, and it thus makes more sense to us. Maybe the best practice would be to use "NIP", "NOP", "NTP" etc. when working with those truth patterns, and "dependent", "stable", "simple" etc. when working globally, doing for example type-counting or forking-calculus.

2.2.2 NIP fields

Conjecture 2.2.7 (Shelah's conjecture). *NIP fields are either finite, separably closed, real closed, or henselian.*

We will discuss some aspects of this conjecture later in Section 2.2.6. For now, we state some results which have been obtained by Johnson and which make this conjecture somewhat believable.

Definition 2.2.8. A complete theory is said to have dp-rank $\geqslant n$ if there are formulas $\varphi_1(x, y) \cdots \varphi_n(x, y)$ and mutually indiscernible sequences $(a_i^1)_{i<\omega} \cdots (a_i^n)_{i<\omega}$ such that for any $i_1 \cdots i_n$, the type $\left\{\varphi_k(x, a_{i_k}^k) \mid k \leqslant n\right\} \cup \left\{\neg\varphi_k(x, a_i^k) \mid k \leqslant n, i \neq i_k\right\}$ is consistent.

The theory is called dp-minimal if it has dp-rank 1 and dp-finite if it has finite dp-rank.

Proposition 2.2.9. *dp-finite theories are NIP.*

Proof. Let $\varphi(x, y)$ have IP. Then by Lemma 2.2.4 there is $(a_i)_{i<\omega}$ indiscernible and b such that $\varphi(b, a_i)$ holds iff i is even. By Ramsey and compactness, we can find another sequence with the same property but of length $n\omega$, thus thinking of it as n mutually indiscernible sequences. Fix N, fix $i_0, \cdots, i_{n-1}$, and consider the finite type $\{\varphi(x, a_{k\omega+2i_k}) \mid k = 0, \cdots, n\} \cup \{\neg\varphi(x, a_{k\omega+2i+1}) \mid k = 0, \cdots, n, i \neq i_k, i < N\}$. It is satisfied by b. Thus, moving the indices to i_k and i respectively, by indiscernability the finite type $\{\varphi(x, a_{k\omega+i_k}) \mid k = 0, \cdots, n\} \cup \{\neg\varphi(x, a_{k\omega+i}) \mid k = 0, \cdots, n, i \neq i_k, i < N\}$ is satisfiable, and the theory has dp-rank $\geqslant n$ for any n. □

Theorem 2.2.10 (Johnson in [Joh20]). *dp-finite fields are either finite, algebraically closed, real closed, or henselian.*

Remember that separably closed fields are NIP (even stable); but dp-infinite by this theorem, providing a counterexample to the converse of Proposition 2.2.9.

Theorem 2.2.11 (Johnson again but this time in [Joh21])**.** *Let (K, v) be NIP and* $\mathrm{ch}(K) = p$*. Then v is henselian.*

The main course of this chapter concerns Artin-Schreier extensions. Building from Scanlon's work, Kaplan and Wagner studied Artin-Schreier extensions of NIP and simple fields.

Theorem 2.2.12 (Kaplan-Scalon-Wagner in [KSW11])**.** *Infinite NIP fields of characteristic p are Artin-Schreier closed.*

Once again, no formula having IP is explicited in their paper. In order to find one, let's summarize the argument: In a NIP theory, definable families of subgroups check a certain chain condition, namely, Baldwin-Saxl's. In an infinite field of characteristic $p > 0$, the family $\{a\wp(K) \mid a \in K\}$, where $\wp(X)$ is the Artin-Schreier polynomial $X^p - X$, is a definable family of additive subgroups; thus it checks Baldwin-Saxl, and this is only possible if $\wp(K) = K$. The complexity of this argument is mainly hidden in the very last affirmation; it needs a whole paper to prove it, namely [KSW11]. One can also look at [CH21, Appendix] for a more explicit proof.

2.2.3 Baldwin-Saxl condition for NIP formulas

Let T be an $\mathcal{L}$-theory, we work in a monster $\mathcal{M} \vDash T$. Let G be a type-definable set, and $\cdot$ be a definable group law on G. Example: in a field K, we can take $G = K$ and $\cdot = +$.

Let $\varphi(x, y)$ be an $\mathcal{L}$-formula, and let $A \subset M$ be a set of parameters such that $H_a = \varphi(M, a) \cap G$ is a subgroup of G for any $a \in A$.

Proposition 2.2.13 (Baldwin-Saxl)**.** *The VC-dim of φ is finite*[†] *iff the family $(H_a)_{a\in A}$ checks the BS-condition: there is N (depending only on φ) such that for any finite $B \subset A$, there is a $B_0 \subset B$ of size $\leqslant N$ such that:*

$$\bigcap_{a\in B} H_a = \bigcap_{a\in B_0} H_a$$

That is, the intersection of finitely many H's is the intersection of at most N of them.

This is a classical result first studied in [BS76]. Modern versions can be found in many model theory textbooks, for example [Sim15]; however, it is usually not stated as an equivalence, since "in a NIP theory, all definable families of groups check a specific chain condition" is much more useful than "if a specific family checks this hard-to-check chain condition, a specific formula is NIP, but some others might have IP". We give a proof here for convenience.

Proof.

⇒: Suppose that the family $(H_a)_{a\in A}$ fails to check the BS-condition for a certain N, that is, we can find $a_0, \cdots, a_N \in A$ such that:

$$\bigcap_{0\leqslant i\leqslant N} H_i \subsetneq \bigcap_{0\leqslant i\leqslant N \,\&\, i\neq j} H_i$$

for all $j \leqslant N$, and where we write H_i for H_{a_i}. We take $b_j \notin H_j$ but in every other H_i and we define $b_I = \prod_{j\in I} b_j$, where the product denote the group law of G – the order of operations doesn't matter. We have $\mathcal{M} \vDash \varphi(b_I, a_i)$ iff $i \notin I$, so the VC-dim of $\neg\varphi^{\mathrm{opp}}$ is $> N$.

Thus, if the VC-dim of φ is finite, the VC-dim of $\neg\varphi^{\mathrm{opp}}$ is also finite, and there is a maximal such N.

[†]Precisely, the VC-dim of $\varphi|_{y\in A}$, which is φ with the range of y restricted to A (which need not be a definable set). We can either do this by adding a predicate for A, adding a sort for A, or even by restricting to the case where A is the whole model, which is our case in the rest of the section. If we do not restrict, left-to-right still holds, but right-to-left might fail, which is also a reason why it's usually not stated.

$\Leftarrow$: Suppose that $(H_a)_{a\in A}$ checks the BS-condition for a given N, and suppose that we can find $a_0,\cdots,a_N \in A$ and $(b_I)_{I\subset\{0,\cdot\cdot,N\}} \in G$ such that $\mathcal{M} \vDash \varphi(b_I, a_i)$ iff $i \in I$. Now by BS, $\bigcap_{0\leqslant i\leqslant N} H_i = \bigcap_{0\leqslant i<N} H_i$ (maybe reindexing it). But now, let $b = b_{\{0,\cdot\cdot,N-1\}}$; we know that $\mathcal{M} \vDash \varphi(b, a_i)$ for $i < N$, which means that $b \in \bigcap_{0\leqslant i<N} H_i$, thus $b \in H_N$, and thus $\mathcal{M} \vDash \varphi(b, a_N)$, which contradicts the choice of a and b. □

2.2.4 Artin-Schreier closure and local NIPity

We can now state the original result by Kaplan-Scanlon-Wagner as an equivalence:

Corollary 2.2.14 (Local KSW). *In an infinite field K of characteristic $p > 0$, the formula $\varphi(x, y)\colon \exists t\ x = y(t^p - t)$ is NIP iff K has no AS-extension.*

Proof. Apply previous result with $(G,\cdot) = (K,+)$ and $A = K$: φ is NIP iff φ has finite VC-dim iff the family $H_a = a\wp(K)$ checks the BS-condition. This then implies that K is AS-closed as discussed in the paragraph following Theorem 2.2.12. The opposite direction is quite trivial: if K is AS-closed, then $\wp(K) = K$, so the BS-condition is obviously checked. □

2.2.5 Lifting

The formula we obtained is existential, so if it witnesses IP in the residue field of a henselian valued field, we can lift this pattern to the field itself.

Lemma 2.2.15. *Let (K, v) be henselian and suppose k_v is infinite, of characteristic p, and not AS-closed; then K has IP as a pure field witnessed by $\varphi(x, y)\colon \exists t\, x = (t^p - t)y$.*

Proof. By assumption and by Corollary 2.2.14, there are $(a_i)_{i<\omega}$ and $(b_J)_{J\subset\omega}$ such that $k_v \vDash \varphi(b_J, a_i)$ iff $i \in J$, that is, $P_{i,J}(T) = a_i(T^p - T) - b_J$ has a root in k_v iff $i \in J$. But by henselianity, taking any lift α_i, β_J of a_i and b_J, $P_{i,J}(T) = \alpha_i(T^p - T) - \beta_J$ has a root in K iff $i \in J$, thus $K \vDash \varphi(\beta_J, \alpha_i)$ iff $i \in J$. □

This gives explicit IP formulas in some fields, for example, in complements of $\mathbb{Q}_p^v$ over $\mathbb{Q}_p$, see Proposition 4.2.6 and Theorem 4.2.7: they have residue $\mathbb{F}_p$, value group $\mathbb{Z}[\frac{1}{p^\infty}]$, and are defectless; going to a sufficiently saturated extension, we can find a non-trivial proper coarsening w of v with residue characteristic p, thus $(k_w, \overline{v})$ is a non-trivial valued field of equicharacteristic p with residue $\mathbb{F}_p$, thus it is not AS-closed, and we apply the previous Lemma to (K, w): K has IP as a pure field.

Apart from an explicit formula, this is not new, and can be argued in a slightly more general way, to the cost of explicitness:

Lemma 2.2.16 (Jahnke). *Let K be NIP and v be henselian, then (K, v) is NIP.*

Corollary 2.2.17. *Let (K, v) be henselian, if (K, v) has IP, then K has IP as a pure field. In particular, if k has IP, K has IP.*

At heart of Jahnke's result is Shelah's expansion theorem, since her strategy was to prove that, in most cases, v is externally definable. We refer to [Jah19] for details.

So, in fact, the main interest of explicit Artin-Schreier lifting is that it skips Shelah's expansion theorem, which only works for NIP theories. We will study Shelah's expansion in more details in appendix B.

We end this section by discussing Shelah's conjecture, Conjecture 2.2.7, with a specific goal, namely, to explicitly define a henselian valuation on NIP fields under Shelah's conjecture; to do so, we use Anscombe-Jahnke's classification of NIP henselian valued fields.

2.2.6 Explicit Shelah's conjecture

Shelah's conjecture state that if a pure field is NIP, then it is finite, SCF, RCF, or henselian. As a first remark, note that this is not an equivalence, as henselian valued fields can have IP. It can however be turned into an equivalence by using the following theorem:

Theorem 2.2.18 (Anscombe-Jahnke). *Let (K, v) be a henselian valued field. Then (K, v) is NIP iff the following holds:*

1. *k is NIP, and*
2. *either*
   - *(a) (K, v) is of equicharacteristic and is either trivial or SAMK, or*
   - *(b) (K, v) has mixed characteristic $(0, p)$, (K, v_p) is finitely ramified, and $(k_p, \overline{v})$ checks 2a, or*
   - *(c) (K, v) has mixed characteristic $(0, p)$ and $(k_0, \overline{v})$ is AMK.*

It is the main theorem of [AJ19a]. We will discuss it in more details in Chapter 3, for now, we only apply it.

A second remark is that in Shelah's conjecture, we assume that a pure field is NIP, and we end up – when it's not finite, SCF or RCF – with a henselian valuation; morally, if it always exists, it should be definable. This holds; it is a consequence of Corollary 1.4.8. Here we will prove that we can explicitly define it, by using Robinson's generalized formula:

Conjecture 2.2.19 (Explicit Shelah's conjecture). *Let K be a NIP field, then K is either finite, RCF, SCF, or there is a prime q and a parameter $t \in K$ such that $\psi_q(K, t)$, as defined in Proposition 1.4.1, is a non-trivial henselian valuation ring.*

Proposition 2.2.20. *Conjecture 2.2.7 $\Leftrightarrow$ Conjecture 2.2.19.*

Proof. Let's take K NIP, not SCF nor RCF nor finite. We want to use Robinson's generalized formula to define a non-trivial henselian valuation on K – assuming Shelah's conjecture.

By the definition of the canonical henselian valuation, see Definition 1.2.8, K admits a non-trivial valuation iff v_K is non-trivial. Let k and Γ be the residue field and value group of v_K.

By Lemma 2.2.16, (K, v_K) is NIP.[‡] Let k be the residue of (K, v_K). We know that k is NIP, so we can apply Conjecture 2.2.7 to it: k is SCF, RCF, or finite. It can also be henselian; but by definition of v_K this can only happen in the case $k \models$ SCF, so it's redundant. We now study the cases in Theorem 3.1.1; our goal is to prove that there is an element $t \in K$ which has value not divisible (by some prime $q \neq \mathrm{ch}(k)$). We can then use this element as the parameter for Robinson's generalized formula.

- If (K, v_K) is of equicharacteristic $(0, 0)$, then k is SCF or RCF. Assume Γ is divisible and take L/K algebraic. Then $[L\colon K] = [k_L\colon k] = 1$ or 2, and K itself is SCF or RCF, which can't be by assumption. Thus, Γ is not divisible.

- If (K, v_K) is of equicharacteristic (p, p), then k is SCF or finite. But by Theorem 3.1.1, (K, v_K) is SAMK; in particular k is perfect and not finite, so ACF, and Γ is p-divisible. If Γ is divisible, then K^{sep}/K is immediate, but since K is separably algebraically maximal this yields $K = K^{\mathrm{sep}}$, which again can't happen by assumption; hence Γ is not divisible by some $q \neq p$.

[‡]Let us note that apart from this fact, our argument about definability could be generalized to NIP_n fields; as of now, we could only obtain a NIP_n version of Lemma 2.2.16 when the residue characteristic is p, see Corollary 3.4.10.

- If (K, v_K) is of mixed characteristic $(0, p)$ and (K, v_p) is finitely ramified, then Γ/Δ_p admits a smaller positive element $\overline{\gamma}$; any lift γ of which checks that $\gamma + n\delta \notin \Delta_p$ for all $\delta \in \Gamma$ and all $n > 1$. In particular γ is not n-divisible for all n.

- If (K, v_K) is of mixed characteristic $(0, p)$ and $(k_0, \overline{v_K})$ is AMK, then again k is ACF and Δ_0 is p-divisible. If Δ_0 is not divisible, we are done. Otherwise, $(k_0)^{\text{alg}}/k_0$ is immediate, hence trivial by algebraic maximality; so k_0 is ACF. Now $(K, v_0) \cong (k_0((\Gamma/\Delta_0)), v_{\Gamma/\Delta_0})$, and we are back to the equicharacteristic $(0, 0)$ case.

In summary, under our assumptions:

- If (K, v_K) is of equicharacteristic $(0, 0)$, then Γ is not divisible.

- If (K, v_K) is of equicharacteristic (p, p), then Γ is p-divisible but not divisible.

- If (K, v_K) is of mixed characteristic $(0, p)$, then either Γ is not q-divisible for all q, or Δ_0 is p-divisible but not divisible, or Γ/Δ_0 is not divisible.

In all cases, we can find a parameter t and a prime q such that $\psi_q(K, t)$ is a non-trivial coarsening of v_K. □

2.2.7 Consequences on other conjectures

There are many classical conjectures around NIP fields, most of them can be seen as intermediate steps towards Shelah's conjecture. We quote some of them here and see how they depend on each other.

Conjecture 2.2.21 (Henselianity conjecture (HC))**.** *Let (K, v) be NIP. Then v is henselian.*

Conjecture 2.2.22 (Weak henselianity conjecture (WHC))**.** *Let K be NIP, and v be definable. Then v is henselian.*

Conjecture 2.2.23 (Definability conjecture (DC))**.** *Let K be NIP. Then K is SCF, RCF, finite, or admits a non-trivial definable valuation.*

Since we proved that SC (Conjecture 2.2.7) is equivalent to an explicitly definable version of it, it is clear that SC implies DC. Thus, we formulate here an explicit version of DC:

Conjecture 2.2.24 (Explicit definability conjecture (EDC))**.** *Let K be NIP. Then K is SCF, RCF, finite, or there is a prime q and a parameter $t \in K$ such that $\psi_q(K, t)$ is a non-trivial valuation ring.*

The logical relations between those conjectures are shown in Figure 2.1.

Of all these relations, the non-trivial ones are "SC $\Leftrightarrow$ ESC" and "ESC $\Rightarrow$ HC". We already proved the first one. The second one was obtained in [HHJ20], we summarize the proof here:

Proof. Assume SC holds, thus also ESC. Let (K, v) be NIP. If K is SCF, RCF or finite, then v is henselian – for RCF, see [HHJ20, Prop. 5.7]. Now assume K is not one of these. Let $v^\dagger$ be the intersection of all henselian valuation rings which are externally definable in (K, v). We use the following result:

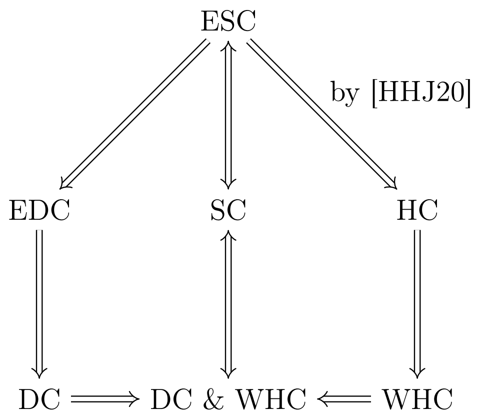


Figure 2.1: Relations between conjectures on NIP fields.

Fact [HHJ20, Cor. 5.4]. If K is a field, v_1 is a henselian valuation, v_2 is a valuation and v_1 and v_2 are incomparable; then (K, v_1, v_2) has IP.

(K, v) is NIP and $v^\dagger$ is henselian.

If $v^\dagger$ is externally definable, then $(K, v, v^\dagger)$ is NIP by Shelah's expansion; but then v and $v^\dagger$ are comparable by the fact above.

Now assume that $v^\dagger$ is not externally definable and that v and $v^\dagger$ are incomparable. Consider their common coarsening: it is a coarsening of v, thus externally definable in (K, v), and henselian. But since $v^\dagger$ is a strict refinement, there must be an externally definable henselian valuation between them; this valuation is incomparable with v, which contradicts the fact above. Thus, v and $v^\dagger$ must be comparable.

If v is a coarsening of $v^\dagger$ then it is henselian. If it is a refinement, then $\overline{v}$ is a valuation on $k_{v^\dagger}$. But $k_{v^\dagger}$ is NIP:

If $k_{v^\dagger}$ is not RCF, SCF, nor finite, then it admits a non-trivial definable valuation w by Robinson's generalized formula – remember that we assumed Shelah's conjecture. Now if $\overline{v}$ is a coarsening of w, $\overline{v}$ is henselian and thus v as well. If $\overline{v}$ is a refinement, then v is a refinement of the lift of w to K and thus w is externally definable in (K, v), despite being finer than $v^\dagger$ – that is not possible.

Lastly, if $k_{v^\dagger}$ is RCF, SCF or finite, then $\overline{v}$ is henselian on $k_{v^\dagger}$, and v is henselian. □

Around EDC We care about EDC because the class of fields which are neither finite, RCF, SCF and such that no $\varphi_q(x, t)$ defines a valuation is elementary; but adding "henselian" to it makes it non-elementary – and even more when not giving a formula to define a valuation.

So, a counterexample of this conjecture is much better defined than any other conjecture, and proving that it can't have IP seem to be a much more achievable task – even though it remains to be done.

2.2.8 dp-finite fields

As mentioned before, it is known, by the work of Johnson, that Shelah's conjecture holds for dp-finite fields; that is, a dp-finite field is SCF, RCF, finite, or admits a non-trivial henselian valuation.

The proof above can be easily adapted to see that dp-finite fields have explicitly definable henselian valuations. Right now, the proof reads:

- Let K be NIP but not SCF, RCF or finite; then (assuming SC) it is henselian.

- By Lemma 2.2.16, (K, v_K) is NIP, hence the residue k is NIP.
- k itself is thus either SCF, RCF or finite, and with the help of Theorem 2.2.18, we obtain that in all these cases a valuation is definable.

If we naively try to adapt the proof for dp-finite, the only roadblock is that from knowing that K is dp-finite, we only get that (K, v_K) is NIP, and thus k is NIP. However, in order to say that k is SCF, RCF or finite, we need to know that it is dp-finite. So, the only thing we need is the following lemma:

Lemma 2.2.25. *If K is a dp-finite (pure) field and v is henselian, then the residue field k is dp-finite.*

Proof. We adapt the original argument of Jahnke, itself building from an argument of Scanlon, to prove that adding any henselian valuation to any field preserves dp-finiteness in some cases, and then to crush the other cases.

First, note that the Shelah expansion of a dp-finite theory remains dp-finite – in fact, it keeps exactly the same dp-rank. This is a direct corollary of quantifier elimination in the Shelah expansion of a NIP theory, see [Sim15, Prop. 3.23].

Let K be any field and v any henselian valuation on K.

Case 1: k is not SCF or RCF k thus has a Galois extension of degree $d > 1$. Choose p prime dividing d. We go there and back again, as in section 4.1.3: we find a finite extension of the residue field that is not p-closed. Let (L, w) be a finite extension of (K, v) having this non p-closed residue. L itself is non p-closed. v_L^p is thus definable on L, and because its residue is non p-closed, we have $v_L^p \leqslant w$. The restriction of v_L^p to K is also definable, and thus there is a definable refinement of v. v is then externally definable in K.

Therefore if K is dp-finite, then (K, v) also.

Case 2: k RCF v_K^{2*} is thus definable, and its residue is euclidean. If we can prove that the induced valuation $\overline{v}$ on this residue is externally definable, then we will have proven that v is externally definable in K. Thus we may restrict to the case where K is euclidean. Now the unique order on K is defined by saying that the positive elements are exactly the squares. Any valuation ring must contain the convex hull of $\mathbb{Z}$, which is itself an externally definable valuation ring; hence v is externally definable in (K, w), and if K is dp-finite, (K, v) also.

Case 3: k SCF We use Lemma 3.2.2: if v has a proper coarsening of residue characteristic p, then k is perfect. Thus, if v is non-trivial and $\mathrm{ch}(K) = p$, k is perfect, thus ACF and dp-minimal. If v of mixed characteristic, then either it is finitely ramified, in which case it is definable in K by Corollary 1.4.3 and thus (K, v) is dp-finite, or it is infinitely ramified. Going to a sufficiently saturated elementary extension $(K^*, v^*) \succcurlyeq (K, v)$, we see that v^* has a non-trivial coarsening of residue characteristic p, and thus k^* is perfect; because this is first-order, k itself is perfect, hence ACF and dp-minimal. Note that here, we only obtain that k is dp-minimal, not (K, v). □

Now it is just a matter of re-reading the proof above but replacing NIP by dp-finite, and we obtain explicit definability of valuations in dp-finite fields.

2.3 NIP$_n$ fields

2.3.1 The n-independence property

NIP$_n$ theories are the most natural generalization of NIP. They were first defined and studied for $n = 2$ by Shelah in [She05]. Their behavior is erratic, sometimes very similar to NIP theories, sometimes wildly different.

Definition 2.3.1. Let T be a complete theory and $\mathcal{M} \vDash T$ a monster model. A formula $\varphi(x; y_1, \ldots, y_n)$ is said to have the independence property of order n (IP$_n$) if there are $(a_i^k)_{i<\omega}^{1\leqslant k\leqslant n}$ and $(b_J)_{J\subset\omega^n}$ such that $\mathcal{M} \vDash \varphi(b_J, a_{i_1}^1, \ldots, a_{i_n}^n)$ iff $(i_1, \ldots, i_n) \in J$. A formula is said to be NIP$_n$ if it doesn't have IP$_n$, and a theory is called NIP$_n$ if all formulas are NIP$_n$. We also write "strictly NIP$_n$" for "NIP$_n$ and IP$_{n-1}$".

Note that having IP$_{n+1}$ implies having IP$_n$ and that IP$_1$ corresponds to the usual IP.

Example 2.3.2. The random graph is strictly NIP$_2$. The random n-hypergraph, which is the Fraïssé limit of the class of all finite n-hypergraphs – which are sets of vertices equipped with a symmetrical irreflexive n-ary relation –, is strictly NIP$_n$.

As for NIP, the study of NIP$_n$ formulas can be reduced significantely by considering only atomic formulas with one singleton variable, and can also be reformulated in terms of indiscernibles – though we only quote that result in Proposition 3.3.3.

Proposition 2.3.3 ([CPT19, Prop. 6.5])**.** *Being NIP$_n$ is preserved under boolean combinations: if $\varphi(x; y_1, \ldots, y_n)$ and $\psi(x; y_1, \ldots, y_n)$ are NIP$_n$, so are $\varphi \wedge \psi$ and $\neg\varphi$.*

Being NIP$_n$ is preserved under permutation of the variables, as long as we keep the same partitioning – recall that x and each y_i can be tuples.

Finally, a theory is NIP$_n$ iff all formulas $\varphi(x, y_1, \cdots, y_n)$ with x a singleton are NIP$_n$.[§]

Random n-hypergraphs have quantifier elimination and no function symbol, thus only the hyperedge relation needs to be studied. Because it is n-ary, it can't have IP$_n$.

On the other hand, fix N and consider the finite hypergraph composed of $(n-1)N + 2^{N^{n-1}}$ vertices, named $(a_i^k)_{i<N}^{1\leqslant k<n}$ and $(b_J)_{J\subset N^{n-1}}$, with a hyperedge relation that holds for n points iff one of them is b_J, the others are $a_{i_1}^1, \ldots, a_{i_{n-1}}^{n-1}$, and $(i_1, \ldots, i_{n-1}) \in J$. We can embed this graph into the random hypergraph, hence, it has IP$_{n-1}$.

However, these random hypergraphs are simple – another complexity notion that we will study in Section 2.4 –, thus, contrary to the NIP case, when $n \geqslant 2$ a theory can be NIP$_n$, simple, and unstable. The NIP$_n$ hierarchy overlaps with other complexity classes in all possible ways:

Example 2.3.4.

- For $n \geqslant 2$, the random n-hypergraph R_n is simple, unstable, and strictly NIP$_n$.
- Let $\mathcal{M}$ be a NIP unstable $\mathcal{L}$-structure, such as DLO. Then the disjoint pair $(\mathcal{M}, R_n)$ is strictly NIP$_n$, non-simple, and NTP2 – see Section 2.5.
- The triangle-free random graph is strictly NIP$_2$ and has TP2. Adjoining R_n to it as above, we get a TP2 and strictly NIP$_n$ structure.

These structures might seem artificial; they are specifically constructed to fit in all this cases. However, with the Mekler construction, we can interpret all these structures in a pure group:

[§]In fact, one can reduce further, and only consider the formulas with all but one variable being singletons, see [CH21, Thm. 2.12]; however, we only use this weaker version in this dissertation.

Theorem 2.3.5 (Mekler, Baudisch-Pentzel, Chernikov-Hempel; see [CH17]). *Let $\mathcal{M}$ be a structure in a finite relational language. There is a pure group $G(\mathcal{M})$ which is 2-nilpotent, of exponent p, and which interprets $\mathcal{M}$. Furthermore, G is κ-stable (resp. NIP, NIP_n, simple, NTP2) iff $\mathcal{M}$ is κ-stable (resp. NIP, NIP_n, simple, NTP2).*

Thus, by doing this construction on previously exhibited structures, we know that strictly NIP_n pure groups exist; and they can be NIP_n, unstable and simple, or NIP_n, non-simple and NTP2, or NIP_n and TP2 – for $n \geqslant 2$.

Let us also note an important example: consider the 2-sorted structure with one sort for the group $(\mathbb{F}_p^{<\omega}, +)$, one sort for the field $\mathbb{F}_p$, together with a function $\cdot\colon \mathbb{F}_p^{<\omega} \to \mathbb{F}_p$ which is to be interpreted as to $a \cdot b = \sum a_i b_i$. This structure appears in Wagner's book on Simple Theories, [Wag00], where he proves that it has IP. Hempel later proved that it was NIP_2, giving the first example of an explicit algebraic structure strictly which is NIP_2, and demonstrating that understanding NIP_n theories can help study these structures.

2.3.2 NIP_n fields

Hopefully by now we managed to convince you that NIP_n is a proper generalization of NIP, definitely not reducible to the NIP case... right?

Conjecture 2.3.6. *For $n \geqslant 2$, strictly NIP_n pure fields do not exist; that is, a pure field is NIP_n iff it is NIP.*

This is for pure fields. Augmenting fields with structure – for example by adding a relation for a random hypergraph – will of course break this conjecture, however, natural extensions of field structure such as valuation or distinguished automorphism are believed to preserve it. Let us state this conjecture:

Conjecture 2.3.7. *Strictly NIP_n henselian valued fields do not exist.*

It is clear that Conjecture 2.3.7 implies Conjecture 2.3.6 since the trivial valuation is henselian.

We quote some results which make this conjecture somewhat believable:

Proposition 2.3.8 (Duret [Dur80], Hempel [Hem16]). *Let K be PAC and not separably closed. Then, K has IP_n for all n.*

Proposition 2.3.9 (Chernikov-Hempel, [CH21]). *Let (K, v) be NIP_n and $\mathrm{ch}(K) = p$, then v is henselian.*

Theorem 2.3.10 (Hempel, [Hem14]). *Infinite NIP_n fields of characteristic p are Artin-Schreier closed.*

Overall, as soon as interesting results are obtained about or in the context of NIP fields, some people (mostly Hempel and Chernikov as you can see) work hard to sneakily add $_n$ after NIP in these results. They succeed most of the time, though not always taking a straightforward route. Conjecture 2.3.6 arose naturally from their work and can be attributed to Hempel, in duo with Chernikov.

Going back to Theorem 2.3.10, as for NIP fields, we want to know the formula witnessing IP_n in infinite fields with Artin-Schreier extensions; and, that is a promise, this time there will be a nice application; namely, Theorem 3.1.1.

The proof of Theorem 2.3.10 is similar to Kaplan-Scanlon-Wagner's argument, as one expects: in a NIP_n theory, definable families of subgroups check a certain analog of Baldwin-Saxl's condition. In characteristic p, $\{a_1 \cdots a_n \wp(K) \mid \overline{a} \in K^n\}$ is a definable family of additive subgroups. In order for it to check the aforementioned chain condition, we must have $\wp(K) = K$.

2.3.3 Baldwin-Saxl-Hempel's condition for NIP$_n$ formulas

Let T be an $\mathcal{L}$-theory, $M \vDash T$ a monster. Let G be a type-definable set, and $\cdot$ be a definable group law on G. Example: if K is a field, $G = K$ and $\cdot = +$.

Let $\varphi(x, y_1, \cdots, y_n)$ be an $\mathcal{L}$-formula, and let $A \subset M$ be a set of parameters such that $H_{a_1, \cdots, a_n} = \varphi(M, a_1, \cdots, a_n) \cap G$ is a subgroup of G for any $(a_1, \cdots, a_n) \in A$.

Proposition 2.3.11 (Hempel). *A formula φ is said to check the BSH$_n$-condition if there is N (depending only on φ) such that for any d greater or equal to N and any array of parameters $(a^i_j)^{1 \leqslant i \leqslant n}_{j \leqslant d}$, there is $\overline{k} = (k_1, \cdots, k_n) \in \{0, \cdots, N\}^n$ such that:*

$$\bigcap_{\overline{j}} H_{\overline{j}} = \bigcap_{\overline{j} \neq \overline{k}} H_{\overline{j}}$$

with $H_{\overline{j}} = H_{a^1_{j_1}, \cdots, a^n_{j_n}}$.

The formula φ checks the BSH$_n$ condition iff φ is NIP$_n$.¶

Proof. This is a very natural NIP$_n$ version of Baldwin-Saxl, first stated by Hempel in [Hem14]. However, as for Baldwin-Saxl, it is usually not stated as an equivalence.

$\Leftarrow$: Suppose that the BSH$_n$ condition is not checked for N, so one can find $(a^i_j)^{1 \leqslant i \leqslant n}_{j \leqslant N} \in A$ such that

$$\bigcap_{\overline{j}} H_{\overline{j}} \subsetneq \bigcap_{\overline{j} \neq \overline{k}} H_{\overline{j}}$$

for any $\overline{k} \in \{0, \cdots, N\}^n$.

We take $b_{\overline{j}} \notin H_{\overline{j}}$ but in every other $H_{\overline{k}}$. Then for any $J \subset \{0, \cdots, N\}^n$, we define $b_J = \prod_{\overline{j} \in J} b_{\overline{j}}$, where the product denotes the group law of G – the order of operation doesn't matter. We have $M \vDash \varphi(b_J, a^1_{j_1}, \cdots, a^n_{j_n})$ iff $b_J \in H_{\overline{j}}$ (by definition of H), and it is the case iff $\overline{j} \notin J$. If this were to hold for arbitrarily large N, we would have IP$_n$ for φ. Thus, if φ is NIP$_n$, there is a maximal such N.

$\Rightarrow$: Suppose that φ checks the BSH$_n$ condition for N, and suppose we can find $(a^i_j)^{1 \leqslant i \leqslant n}_{j \leqslant N} \in A$ and $(b_J)_{I \subset \{0, \cdots, N\}^n} \in G$ such that $M \vDash \varphi(b_J, a^1_{j_1}, \cdots, a^n_{j_n})$ iff $\overline{j} \in J$. Now by assumption, there is $\overline{k}$ such that $\bigcap_{\overline{j}} H_{\overline{j}} = \bigcap_{\overline{j} \neq \overline{k}} H_{\overline{j}}$. But now, let $b = b_{J \setminus \{\overline{k}\}}$; we know that $M \vDash \varphi(b, a^1_{j_1}, \cdots, a^n_{j_n})$ iff $\overline{j} \neq \overline{k}$, which means that $b \in \bigcap_{\overline{j} \neq \overline{k}} H_{\overline{j}}$. But this means $b \in H_{\overline{k}}$, which yields $M \vDash \varphi(b, a^1_{k_1}, \cdots, a^n_{k_n})$ and contradicts the choice of b. □

2.3.4 Artin-Schreier closure of NIP$_n$ fields

Corollary 2.3.12 (Local KSWH). *In an infinite field K of characteristic $p > 0$, the formula $\varphi(x; y_1, \cdots, y_n)\colon \exists t\, x = y_1 y_2 \cdots y_n(t^p - t)$ is NIP$_n$ iff K has no AS-extension.*

Proof. Apply the previous result with $(G, \cdot) = (K, +)$ and $A = K$: φ is NIP$_n$ iff the family $H_{a_1, \cdots, a_n} = a_1 a_2 \cdots a_n \wp(K)$ checks the BSH$_n$ condition. This then implies that K is AS-closed, see [Hem14] – again, this is the hard part of the proof. The opposite direction is quite trivial: if K is AS-closed, then $\wp(K) = K$, so the BSH$_n$ condition is obviously checked. □

¶As before, we technically need to restrict the domain of φ to A. If we do not restrict, right to left still holds.

2.3.5 Lifting

Ideally, we would like a NIP$_n$ version of Corollary 2.2.17. But this relies on Lemma 2.2.16, the proof of which needs Shelah's expansion theorem, which fails in general for NIP$_n$ structures; notably, it fails for the random graph. We will study the Shelah expansion of the random graph in appendix B and provide details about this fact.

However, thanks to the explicit formula obtained before and with the help of henselianity, we can lift IP$_n$ in the case where it is witnessed by Artin-Schreier extensions:

Lemma 2.3.13. *Suppose (K, v) is henselian and has a residue field k infinite, of characteristic p, and not AS-closed; then K has IP$_n$ witnessed by $\varphi(x; y_1, \cdots, y_n)$: $\exists t\, x = y_1 \cdots y_n(t^p - t)$.*

Proof. By assumption, there are $(a^i_j)_{1\leqslant i\leqslant n, j<\omega}$ and $(b_J)_{J\subset\omega^n}$ such that $k \vDash \varphi(b_J, a^1_{j_1}, \cdots, a^n_{j_n})$ iff $\bar{j} \in J$, that is, $P_{\bar{j},J}(T) = a^1_{j_1} \cdots a^n_{j_n}(T^p - T) - b_J$ has a root in k iff $\bar{j} \in J$. But by henselianity, since roots of this polynomial are all simple, taking any lift α^i_j, β_J of a^i_j and b_J, $P_{\bar{j},J}(T) = \alpha^1_{j_1} \cdots \alpha^n_{j_n}(T^p - T) - \beta_J$ has a root in K iff $\bar{j} \in J$, thus $K \vDash \varphi(\beta_J, \alpha^1_{j_1}, \cdots, \alpha^n_{j_n})$ iff $\bar{j} \in J$. □

So, in this specific case, we don't need the valuation to witness IP$_n$. This fact will have fruitful applications in Section 3.2.

2.4 Simple fields

It would be possible for us to ignore simple theories completely. Indeed, as we will note later, they are completely orthogonal to NIP theories, and are irrelevant to the study of valued fields; however, both historically and mathematically, they play an important role. Thus, we quickly introduce them and the main conjecture on simple fields.

Definition 2.4.1. A formula $\varphi(x, y)$ has the tree property (TP) if there are $(a_s)_{s\in\omega^{<\omega}}$ and some k such that for each $\sigma \in \omega^\omega$, $\{\varphi(x, a_{\sigma|_0}), \varphi(x, a_{\sigma|_1}), \varphi(x, a_{\sigma|_2}), \dots\}$ is consistent, but for any $s \in \omega^{<\omega}$, $\{\varphi(x, a_{s0}), \varphi(x, a_{s1}), \varphi(x, a_{s2}) \dots\}$ is k-inconsistent.

A formula without the tree property is said to be NTP and a complete theory is NTP if all formulas are.

Note that NTP is not preserved under boolean combinations.

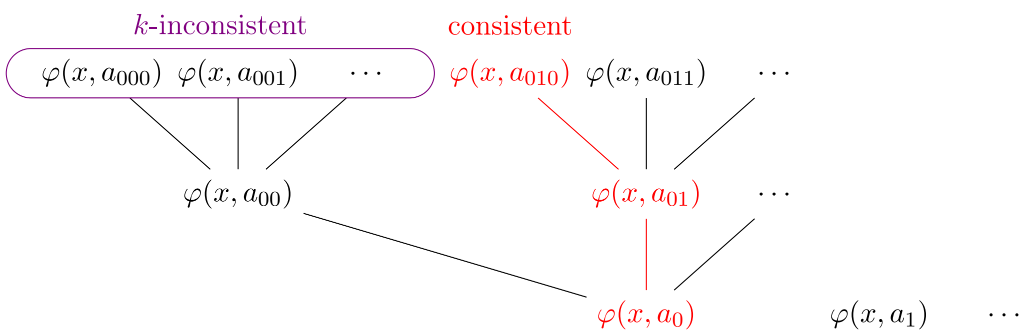


Figure 2.2: A TP pattern.

Lemma 2.4.2 ([TZ12, Prop. 7.2.5]). *A theory T is called* simple *if for all sets of parameters B and for all types $p \in S_n(B)$ there is $A \subset B$ with $|A| \leqslant |T|$ such that p does not divide over A.*

A theory is simple iff it is NTP.

Again, these two names are competing, but "simple" dominates. We argue, as before, that good usage should be to prefer one or the other depending on whether one thinks locally or globally. We are however too weak to object against the entire literature, and settle on "simple" – for now.

Proposition 2.4.3 (Shelah). *If a theory is both simple and NIP, then it is stable.*

Shelah proved it with a larger class than simple, namely, NSOP. We refer to [Che15, Thm. 2.4].

Note that this doesn't work locally: a formula can be NIP, NTP, and have OP, for example, $x < y$ in DLO. However, $y_1 < x < y_2$ has TP in DLO; which also gives an example of a boolean combination of NTP formulas which has TP.

Note also that some theories can be unstable, simple and NIP_n for some $n \geqslant 2$, as discussed before.

Proposition 2.4.4 ([KP97, Thm. 4.2]). *Let T be a complete theory. If there exists a ternary relation $\mathop{\smile\!\!\!\!\!\!|}$ between sets with the following properties:*

- *$\mathop{\smile\!\!\!\!\!\!|}$ is invariant under automorphisms.*
- *$A \mathop{\smile\!\!\!\!\!\!|}_B CD$ iff $A \mathop{\smile\!\!\!\!\!\!|}_B C$ and $A \mathop{\smile\!\!\!\!\!\!|}_{BC} D$.*
- *$A \mathop{\smile\!\!\!\!\!\!|}_B C$ iff $C \mathop{\smile\!\!\!\!\!\!|}_B A$.*
- *$A \mathop{\smile\!\!\!\!\!\!|}_B C$ iff $a \mathop{\smile\!\!\!\!\!\!|}_B C$ for all finite $a \subset A$.*
- *There is a cardinal κ such that for any finite A and any B, there is $B_0 \subset B$ of size $\leqslant \kappa$ and such that $A \mathop{\smile\!\!\!\!\!\!|}_{B_0} B$.*
- *For all tuple a and sets B and C there is a' such that $\mathrm{tp}(a'/B) = \mathrm{tp}(a/B)$ and $a' \mathop{\smile\!\!\!\!\!\!|}_B C$.*
- *Let $\mathcal{M}$ be a model. If $a \mathop{\smile\!\!\!\!\!\!|}_{\mathcal{M}} b$, $a \mathop{\smile\!\!\!\!\!\!|}_{\mathcal{M}} a'$ and $b \mathop{\smile\!\!\!\!\!\!|}_{\mathcal{M}} b'$, and if $\mathrm{tp}(a'/\mathcal{M}) = \mathrm{tp}(b'/\mathcal{M})$; then there is some c such that $\mathrm{tp}(c/\mathcal{M}a) = \mathrm{tp}(a'/\mathcal{M}a)$, $tp(c/\mathcal{M}b) = \mathrm{tp}(b'/\mathcal{M}b)$ and $c \mathop{\smile\!\!\!\!\!\!|}_{\mathcal{M}} ab$.*

Then T is simple and $\mathop{\smile\!\!\!\!\!\!|}$ is the usual forking independence.

We now give a (very brief) overview of the study of simple fields, starting with the main conjecture:

Conjecture 2.4.5. *A pure field K is simple iff it is bounded and PAC (or finite).*

As far as we know, this conjecture can be traced back to Anand Pillay – although it is usually stated only for supersimple fields.

Note that Jean-Louis Duret's result, Proposition 2.3.8, implies that PAC fields which are not SCF are unstable, therefore this conjecture implies the stable fields conjecture.

Proposition 2.4.6 ([Hru02, Cha99]). *PAC fields are simple iff they are bounded.*

The other direction of the conjecture is still open, though some intermediate results are known, most notably:

Proposition 2.4.7 (Kaplan-Scanlon-Wagner, [KSW11])**.** *Simple fields of characteristic p have finitely many Artin-Schreier extensions.*

This was obtained in the same paper as for NIP fields, still generalizing Scanlon's note. Indeed, no formula having TP was explicited. We will expose one when studying NTP2 fields, for which the same result holds.

Simple pure fields are very interesting, but simple valued fields are boring:

Lemma 2.4.8. *Let (K, v) be simple, then v is trivial.*

Indeed, since any OAG is a NIP theory, if (K, v) is simple then its value group is both simple and NIP, thus stable; but this means it is trivial.

2.5 NTP2 fields

We now turn towards the very last relevant notion of combinatorial complexity. Many more exist which we do not study here.

2.5.1 The tree property of order 2

Definition 2.5.1. A formula $\varphi(x, y)$ is said to have the tree property of order 2 (TP2) if there is $(a_{ij})_{(i,j)\in\omega^2}$ and k such that for any $i < \omega$, $\{\varphi(x, a_{ij}) \mid j < \omega\}$ is k-inconsistent, but for any $f : \omega \to \omega$, $\{\varphi(x, a_{if(i)}) \mid i < \omega\}$ is consistent.

A formula is NTP2 if it doesn't have TP2, and a theory is NTP2 if all its formulas are NTP2.

Note that each of stable, NIP, and simple implies NTP2. Also, as for NTP, NTP2 is not preserved under boolean combinations.

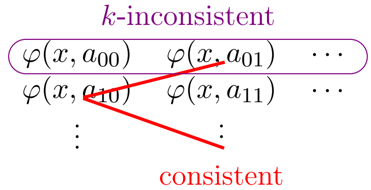


Figure 2.3: A TP2 pattern

Example 2.5.2. Bounded PAC, PRC and PpC fields are NTP2, see [Mon17].

As pure rings, $\mathbb{Z}$ and thus also $\mathbb{Q}$ have TP2: in $\mathbb{Z}$, the formula "x divides y and $x \neq 1$" has TP2. However its negation does not, since rows can't be k-inconsistent.

2.5.2 NTP2 fields

No standard conjecture for NTP2 fields is known, but based on known examples, we formulate the following:

Conjecture 2.5.3 (Pseudoconjecture)**.** *NTP2 fields are either pseudofinite, pseudo algebraically closed, pseudo real closed, or "pseudohenselian".*

Note that "pseudostuff" usually means "existentially closed in extensions compatible with stuff"; here, "K is pseudohenselian" means "K admits a valuation v such that (K, v) is existentially closed in (K^h, v^h)". However, "pseudofinite" means "a model of the theory of all finite fields". For the purpose of stating this conjecture concisely, finite fields are considered pseudofinite – and thus not completely redundant with PAC fields.

If you accept Shelah's conjecture, the pseudoconjecture then reads "an NTP2 field is pseudo-NIP".

Theorem 2.5.4 (Chernikov-Kaplan-Simon in [CKS12])**.** *NTP2 fields of characteristic p are AS-finite, also called p-bounded – they have only finitely many distinct Artin-Schreier extensions.*

The Chernikov-Kaplan-Simon argument is very similar to Kaplan-Scanlon-Wagner. First, one needs to find a suitable chain condition for definable families of subgroups in NTP2 theories, and then apply it to the Artin-Schreier additive subgroup. Namely, instead of saying that the intersection of $N+1$ subgroups is the same as just N of them, this condition is saying that the intersection of all but one of them is not quite the whole intersection, but is of finite index in it. Then, one shows that in a field K with infinitely many Artin-Schreier extensions, the family $a\wp(K)$ fails this condition.

2.5.3 Chernikov-Kaplan-Simon condition for NTP2 formulas

Theorem 2.5.5 ([CKS12, Lem. 2.1])**.** *Let T be NTP2, $M \vDash T$ a monster and suppose that $(G, \cdot)$ is a definable group. Let $\varphi(x, y)$ be a formula, for $i \in \omega$ let $a_i \in M$ be such that $H_i = \varphi(M, a_i)$ is a normal subgroup of G. Let $H = \bigcap_{i\in\omega} H_i$ and $H_{\neq j} = \bigcap_{i\neq j} H_i$. Then there is an i such that $[H_{\neq i} \colon H]$ is finite.*

It turns out that, once again, we do not need T to be completely NTP2: the proof goes by contradiction and shows that if this finite index condition is not respected, the formula $\psi(x; y, z) \colon \exists w\, (\varphi(w, y) \wedge x = w \cdot z)$ has TP2. Thus we need only to assume NTP2 for this ψ. As in the NIP case for Baldwin-Saxl, we establish an equivalence between one specific formula being NTP2 and this condition.

Remark 2.5.6. This condition says that in a given family of subgroups, one of them has finitely many distinct cosets witnessed by elements which lie in the intersection of every other subgroup. By compactness, we can cap this finite number, and consider only finite families: there is k and N, depending only on φ, such that given k many subgroups defined by φ, one of them has no more than N cosets witnessed by elements in the intersection of the $k-1$ other subgroups.

Porism 2.5.7 (CKS-condition for fomulas)**.** *Let T be an $\mathcal{L}$-theory, $M \vDash T$ a monster and $(G, \cdot)$ a definable group. Let $\varphi(x, y)$ be a formula such that for any $a \in M$, $H_a = \varphi(M, a)$ is a normal subgroup of G. Let $\psi(x; y, z)$ be the formula $\exists w\, (\varphi(w, y) \wedge x = w \cdot z)$. We will suppose for more convenience that $\cdot$, or rather, the formula defining $\{x, y, z \mid x \cdot y = z\}$ contains, or at least implies, $x, y, z \in G$; thus ψ doesn;t hold if $z \notin G$. Then $\psi(x; yz)$ is NTP2 iff the CKS-condition holds: for any $(a_i)_{i\in\omega}$, there is i such that $[H_{\neq i} \colon H]$ is finite, where $H = \bigcap_{i\in\omega} H_i$ and $H_{\neq j} = \bigcap_{i\neq j} H_i$.*

Note that since $^{-1}$ is definable, $\psi(x; y, z)$ is equivalent to $\varphi(x \cdot z^{-1}, y)$.

Proof. The formula $\psi(x; yz)$ holds iff $x \in H_y \cdot z$. Also, we use H_i to denote H_{a_i} and later H_i^j to denote $H_{a_{ij}}$ because it is much more convenient.

We work in 4 steps, but truly, only the 4th step is an actual proof, and it is technically self-sufficient. The raison d'être of step 1 to 3 is to – hopefully – make the proof strategy clearer.

Step 1: true equivalence, from CKS. In their paper, Chernikov, Kaplan and Simon prove that given some $(a_i)_{i\in\omega}$, if the family H_i does not check the CKS-condition, then ψ has TP2. They do this by explicitly witnessing TP2 by $c_{ij} = (a_i, b_{ij})$, with a for y and b for z, and with $b_{ij} \in H_{\neq i}$. Reversing their argument, we prove the following equivalence:

ψ has TP2 witnessed by some $c_{ij} = (a_i, b_{ij})$ with $b_{ij} \in H_{\neq i}$ iff the family H_i does not check the CKS-condition.

Right-to-left is exactly given by the original paper. Now let a_i and b_{ij} be as wanted. $\psi(x; c_{ij})$ says that $x \in H_i \cdot b_{ij}$. So the TP2-pattern is as follows:

$$\begin{array}{ccccc} H_0b_{00} & H_0b_{01} & H_0b_{02} & H_0b_{03} & \cdots \\ H_1b_{10} & H_1b_{11} & H_1b_{12} & H_1b_{13} & \cdots \\ H_2b_{20} & H_2b_{21} & H_2b_{22} & H_2b_{23} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

For a given i, k-inconsistency of the rows says that a given coset of H_i might only appear $k-1$ times. So there are infinitely many cosets of H_i, witnessed by elements $b_{ij} \in H_{\neq i}$. This means that $H \cdot b_{ij} = H \cdot b_{ij'}$ iff $H_i \cdot b_{ij} = H_i \cdot b_{ij'}$. But that gives infinitely many cosets of H in $H_{\neq i}$, for any i, proving that CKS-condition is not checked.

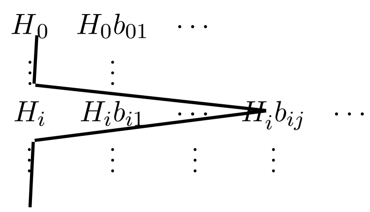
Note that we did not use at any time consistency of the vertical paths. We can use it to loosen our assumption. Let's keep in mind that our final goal is to prove this equivalence with a depending on i and j (right now it depends only on i) and with b_{ij} not necessarily lying in $H_{\neq i}$.

Step 2: going outside $H_{\neq i}$. We now want to prove:

ψ has TP2 witnessed by some $c_{ij} = (a_i, b_{ij})$ with iff the family H_i does not check the CKS-condition.

We already know right-to-left. Let $c_{ij} = (a_i, b_{ij})$ witness TP2 for ψ. Consistency of the vertical paths implies that there is $\lambda \in \bigcap_{i\in\omega} H_i \cdot b_{i0}$. Now write $b'_{ij} = b_{ij} \cdot \lambda^{-1}$. Replacing b by b' won't alter TP2, but will ensure that $H_i b_{i0} = H_i$. So we might as well take $b'_{i,0}$ to be the neutral element of G.

Fix i, j. Consider the vertical path $f = \delta_{ij} \colon \omega \to \omega$ such that $\delta_{ij}(i) = j$ and $\delta_{ij}(i') = 0$ for $i' \neq i$. Consistency yields: $H_i \cdot b'_{ij} \cap \bigcap_{i'\neq i} H_{i'} = H_i \cdot b'_{ij} \cap H_{\neq i} \neq \emptyset$. Thus we can witness this coset of H_i by an element $b''_{ij} \in H_{\neq i}$. Thus $c''_{ij} = (a_i, b''_{ij})$ still witnesses TP2.



Thus, we reduced to the case in step 1, and we can drop the assumption on b. We still have to drop the assumption on a. We used k-inconsistency of rows in step 1, we used consistency of (some) vertical paths in step 2, we didn't yet use normality.

Step 3: arbitrary a, 2-inconsistency. An example of such a TP2 pattern in $\mathbb{Z}$:

$$\begin{array}{ccccc} 2\mathbb{Z} & 4\mathbb{Z}+1 & 8\mathbb{Z}+3 & 16\mathbb{Z}+7 & \cdots \\ 3\mathbb{Z} & 9\mathbb{Z}+1 & 27\mathbb{Z}+4 & 81\mathbb{Z}+13 & \cdots \\ 5\mathbb{Z} & 25\mathbb{Z}+1 & 125\mathbb{Z}+6 & 625\mathbb{Z}+31 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

Note that none of these subgroups have infinitely many cosets, let alone in the intersection of the others! But, for any N, some of them will have more cosets than N.

We aim to prove the following, of which once again we know right to left:

There is some $c_{ij} = (a_{ij}, b_{ij})$ forming a TP2 pattern for ψ with 2-inconsistency of the rows iff the family H_i does not check the CKS-condition.

Let H_i^j be the subgroup $\varphi(M, a_{ij})$. Suppose ψ has TP2, witnessed by $c_{ij} = (a_{ij}, b_{ij})$. As noted before, by compactness we do not need to find an infinite family such that every subgroup has infinitely many cosets in the intersection of the rest, but merely for each finite m and N, a family of m sugroups such that each of them has at least N cosets in the intersection of the rest.

First, we apply the reduction as before: by consistency of vertical paths, we may take b_{i0} to be the neutral element for each i. Then, looking at the path $f = \delta_{ij}$, we may assume $b_{ij} \in H^0_{\neq i}$.

Claim. Let $N \in \omega$. For each i, there is j such that $(b_{ij'})_{j'<\omega}$ witnesses at least N cosets of H_i^j: $\#\left\{H_i^j b_{ij'} \;\middle|\; j' \in \omega\right\} \geqslant N$.

Before proving this claim, let's see why it is enough for our purpose: let $N \in \omega$. For a fixed i, we find j_i such that $H_i^{j_i}$ has $\geqslant N$ cosets witnessed by some b_{ij}. Now by vertical consistency, considering the path δ_{ij_i}, we find an element $\lambda \in H^0_{\neq i} \cap H_i^{j_i} b_{ij_i}$. Compose everything by λ^{-1}, re-index the sequence by switching c_{i0} and c_{ij_i}; this makes it so we can assume that H_i^0 has $\geqslant N$ many cosets in $H^0_{\neq i}$. When we compose by λ, nothing changes: b and b' generate the same coset of H iff $b'b^{-1} \in H$ iff $(b'\lambda)(b\lambda)^{-1} \in H$. So we do this row by row, and we might assume that for any i, H_i^0 has $\geqslant N$ many cosets witnessed by elements from $H^0_{\neq i}$. This implies that some family will fail the CKS condition by compactness.

Now to prove the claim, fix i and N. If there is j such that H_i^j has infinitely many cosets, witnessed in the row i, then we're done. Otherwise, for each j, all H_i^j have finitely many cosets. We will reduce the problem in the following way:

H_i^0 has finitely many cosets in an infinite row, so by pigeonhole, one of them appears infinitely many times. Ignore all the rest, rename them; we may thus assume that $H_i^0 b_{ij} = H_i^0 b_{i1}$ for any $j \geqslant 1$. We can do the same thing with any j, ensuring that $H_i^j b_{ik} = H_i^j b_{i,j+1}$ for any $k > j \in \omega$. Note that we only assume that cosets of a given H_i^j witnessed by b appearing after j are identical, not before, since we already modified things before. In short, we have $b_{ij} b_{ik}^{-1} \in H_i^{j-1}$ for any i, j, and $k > j$.

Up to this point, we didn't use 2-inconsistency, so everything will still hold for the k-inconsistent case.

Because of 2-inconsistency, cosets of H_i^j appearing before j cannot be the same: let $j_1 < j_2 < j_3$. By our reduction, we have $b_{ij_3} b_{ij_2}^{-1} \in H_i^{j_1}$. Suppose furthermore that $b_{ij_2} b_{ij_1}^{-1} \in H_i^{j_3}$, so 2 cosets of $H_i^{j_3}$ appearing before j_3 are the same. Now $b_{ij_3} b_{ij_2}^{-1} b_{ij_1} = (b_{ij_3} b_{ij_2}^{-1}) b_{ij_1} \in H_i^{j_1} b_{ij_1}$ on one hand, and $b_{ij_3} b_{ij_2}^{-1} b_{ij_1} = b_{ij_3}(b_{ij_2}^{-1} b_{ij_1}) \in b_{ij_3} H_i^{j_3} = H_i^{j_3} b_{ij_3}$ by normality on the other hand, contradicting 2-inconsistency.

Thus, if we take $j \geqslant N$, we are sure that H_i^j has $\geqslant N$ many cosets witnessed in the row i, proving the claim.

Step 4: k-inconsistency. We now are ready to prove Porism 2.5.7. We already know one direction, so we now prove that if ψ has TP2 witnessed by some $c_{ij} = (a_{ij}, b_{ij})$, then the family H_i does not check the CKS condition.

We follow the argument of step 3 until the point where 2-inconsistency enters the party. We aim to prove the claim. First, we fix i; since the argument now does not depend on i, we stop writing the subscripts i; readers attached to formal correctness are invited to take a pen and scribble them back in place.

Let $j_1 < j_2 < \cdots < j_{2k-1} \in \omega$. Suppose that b_{j_1} and b_{j_2} spawn the same coset of $H^{j_3}, H^{j_5}, \cdots, H^{j_{2k-1}}$, so $b_{j_1} b_{j_2}^{-1} \in H^{j_3} \cap H^{j_5} \cap \cdots \cap H^{j_{2k-1}}$. Similarly, suppose b_{j_3} and b_{j_4} spawn the same coset of all the odd indexed groups above them, and again for all the rest. Let $b = b_{j_1} b_{j_2}^{-1} b_{j_3} b_{j_4}^{-1} \cdots b_{j_{2k-3}} b_{j_{2k-2}}^{-1} b_{j_{2k-1}}$. We claim that $b \in H^{j_1} b_{j_1} \cap H^{j_3} b_{j_3} \cap \cdots \cap H^{j_{2k-1}} b_{j_{2k-1}}$, contradicting k-inconsistency: Fix $n \in \{1, 3, \cdots, 2k-1\}$. By the reduction, all the products $b_j b_{j'}^{-1}$ on the right of b_{j_n} are in H^{j_n}, and by assumption, all the products on the left also. Thus $b = h b_{j_n} h'$, where $h, h' \in H^{j_n}$. So $b \in H^{j_n} b_{j_n} H^{j_n}$, and by normality we conclude.

Therefore, we know that as soon as $j_1 < j_2 < \cdots < j_{2k-1}$, there is a pair b_{j_n}, $b_{j_{n+1}}$, with odd n, that do not spawn the same coset of some $H^{j'_n}$, $j_{n'} > j_{n+1}$. We want to show that some H^{j_n} must have at least N many different cosets, for arbitrary $N \in \omega$.

Fix N. Let $j_{2k-1} > C$, where C is a big enough constant we will explicit later. We construct a graph with N vertices, which are the j such that $j_{2k-1} - (N+1) < j < j_{2k-1}$, and j, j' are connected iff b_j and $b_{j'}$ generate *different* cosets of $H^{j_{2k-1}}$. This forces $C \geqslant N$. If it is a complete graph, then $H^{j_{2k-1}}$ has at least N many pairwise disjoint cosets, so we are done. Otherwise, there are $j_{2k-1} - (N+1) < j_{2k-3} < j_{2k-2} < j_{2k-1}$ such that $b_{j_{2k-3}}$ and $b_{j_{2k-2}}$ generate the same coset of $H^{j_{2k-1}}$.

We now look back $R_2(N)$ points before j_{2k-3}. Here we call $R_r(s)$ the smallest number $V \in \mathbb{N}$ such that if a complete colored graph with r many colors has at least V many vertices, there's a monochromatic s-clique. $R_r(s)$ is guaranteed to exist for any $r, s \in \mathbb{N}$ by Ramsey's theorem, see [Ram30].

Since $j_{2k-3} > C - N$, we take $C \geqslant N + R_2(N)$. We construct a bi-colored graph with $R_2(N)$ vertices, which are the j such that $j_{2k-3} - (R_2(N)+1) < j < j_{2k-3}$. j, j' are connected by a blue edge iff b_j and $b_{j'}$ generate 2 different cosets of $H^{j_{2k-3}}$, and they are connected by a red edge iff they generate different cosets of $H^{j_{2k-1}}$. They might be connected by both a red and blue edge at the same time, this does not break the argument. If you don't like when edges coincide, choose one color arbitrarily. As before, if this graph is complete, then by Ramsey's theorem, there must be a monochromatic N-clique, ensuring that one of $H^{j_{2k-1}}$ or $H^{j_{2k-3}}$ have at least N many different cosets. Otherwise, we find a pair $j_{2k-5} < j_{2k-4}$ generating the same coset of both $H^{j_{2k-1}}$ and $H^{j_{2k-3}}$, we fix them, and continue.

We now construct a tri-colored graph with $R_3(N)$ vertices, corresponding to the $R_3(N)$ indices preceding j_{2k-5}, blue edge between vertices if they generate different cosets of $H^{j_{2k-1}}$, red if they generate different cosest of $H^{j_{2k-3}}$, green if they generate different cosets of $H^{j_{2k-5}}$. Again, by Ramsey's theorem, we either can find an N-clique, in which case we stop here, or we can find j_{2k-7} and j_{2k-6} not connected (hence generating the same coset of all of the previously fixed groups). This construction is illustrated in Figure 2.4.

We continue doing this strategy for as long as we can; either we stop when we find a monochromatic N-clique, or we end up with $j_1 < j_2 < \cdots < j_{2k-1}$ such that all consecutive pairs generate the same coset of all subgroups above them; but as seen before, this contradicts k-inconsistency. Therefore, this process must stop before, which means we found a clique at some point, and that guarantees a subgroup with at least N many different cosets.

As for the value of C, the construction requires $C \geqslant N + R_2(N) + R_3(N) + \cdots + R_k(N)$, and any such C works. □

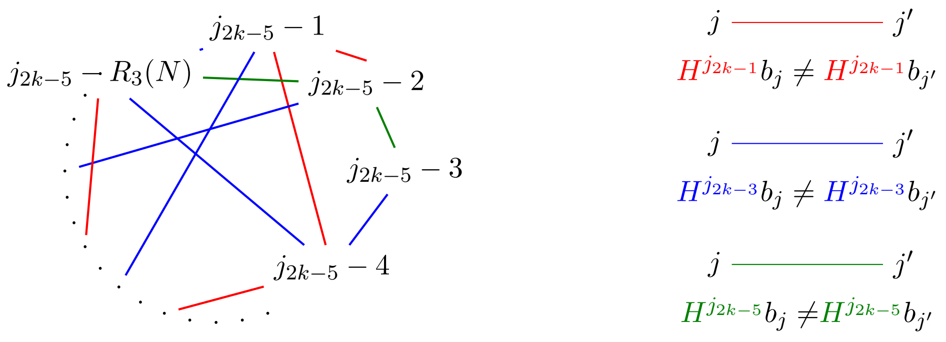


Figure 2.4: After finding $j_{2k-5}, \cdots, j_{2k-1}$, we connect the $R_3(N)$ many points $j_{2k-5} - 1, \cdots, j_{2k-5} - R_3(N)$ with edges colored as indicated; we seek either a monochromatic N-clique or two non-connected points that we then name j_{2k-6} and j_{2k-7}.

Remark 2.5.8. CKS asked whether normality is a necessary assumption. In our proof as well as in theirs, it is useful to assume it, and doesn't seem avoidable. It seems to us that this assumption is necessary, but as of yet, no argument exists to assert or refute this claim.

2.5.4 Artin-Schreier finiteness of NTP2 fields

Corollary 2.5.9 (Local CKS). *In a field K of characteristic $p > 0$, the formula*

$$\psi(x; y, z) \colon \exists t \; x - z = y(t^p - t)$$

is NTP2 iff K has finitely many AS-extensions.

Proof. Apply Porism 2.5.7 with $(G, \cdot) = (K, +)$ and with $\varphi(x, y) \colon \exists t \; x = (t^p - t)y$, which means "$x \in y\wp(K)$". If the formula is NTP2 then it checks CKS and thus K has finitely many AS-extensions, by the original CKS argument – which goes by contraposition, and again, takes a whole paper to be properly done. Now if K has finitely many AS-extensions, then $[K \colon \wp(K)]$, as additive groups, is finite. Thus any additive subgroup of the form $a\wp(K)$ has finitely – and boundedly – many cosets in the whole K, so in particular in any intersection of any family. Thus CKS is checked and ψ is NTP2. □

Remark 2.5.10. As Philip Dittmann pointed out, "finitely many" is an optimal bound, since NTP2 fields with an arbitrarily large number of AS-extensions exist: given a profinite free group with n generators, there exists a PAC field of characteristic p having this group as absolute Galois group. Such a field will have finitely many Galois extension of each degree, that is, it is bounded and hence simple; but if one takes n large enough, it will have an arbitrarily large number of Artin-Schreier extensions.

We now discuss two applications of local CKS: one is, as for NIP_n, lifting complexity, and the other one is only a potential programme to obtain NTP2 of some fields, most notably, $\mathbb{F}_p((\mathbb{Q}))$.

2.5.5 Lifting

Let (K, v) be henselian of residue characteristic $p > 0$. Shelah's expansion doesn't work in general in NTP2 theories, so adding coarsenings to the language might disturb NTP2. Note however that some weaker versions hold, for example [MOS18, Annex A], where one needs to ensure that the value group is NIP and stably embedded before adding coarsenings to the theory. Meanwhile, we can apply the same trick as above to lift complexity and derive some conditions on NTP2 fields.

Lemma 2.5.11. *Let (K, v) be henselian of residue characteristic p and suppose k has infinitely many AS-extensions, then K has TP2 witnessed by $\psi(x; y, z)\colon \exists t\ x - z = y(t^p - t)$.*

Proof. Since k has infinitely many AS-extensions, we know that there are $(a_{ij}, b_{ij})_{i,j<\omega}$ in k witnessing TP2 for ψ. Take any lift α_{ij}, β_{ij} in K, we claim that they witness a TP2 pattern for ψ in K.

Vertical consistency: Let $f\colon \omega \to \omega$ be a vertical path. We know that there is c in k such that $k \vDash \psi(c; a_{if(i)} b_{if(i)})$ for all i.‖ This means $a_{if(i)}(T^p - T) - c - b_{if(i)}$ has a root in f. Take any lift γ of c, then $\alpha_{if(i)}(T^p - T) - \gamma - \beta_{if(i)}$ has a root in K by henselianity, which means $K \vDash \psi(\gamma; \alpha_{if(i)}, \beta_{if(i)})$.

Horizontal m-inconsistency: let's name $P_{ij}(T, x) = a_{ij}(T^p - T) - b_{ij} - x$. Now the residue field $k \vDash \psi(c; a_{ij}, b_{ij})$ iff $P_{ij}(T, c)$ has a root. Fix i and $j_1, \cdots, j_m$. m-inconsistency means that for any choice of $t_1, \cdots, t_m$ and c, one of $P_{ij_l}(t_l, c)$ is not 0. Instead of fixing x and pondering at T, let's fix t_1 to t_m and name $f_l(x) = P_{ij_l}(t_l, x)$. m-inconsistency is equivalent to saying that for any choice of t_l, the family $(f_l)_{1\leqslant l\leqslant m}$ of polynomials can't have a common root.

Since k is not AS-closed, we can find a separable polynomial d with no root in k. Write $d(z) = r_n z^n + \cdots + r_1 z + r_0$, and fix a lift $\delta(z) = \rho_n z^n + \cdots + \rho_1 z + \rho_0$ to K. δ also has no root in K. Let $D(z_1, z_2) = r_n z_1^n + r_{n-1} z_1^{n-1} z_2 + \cdots + r_1 z_1 z_2^{n-1} + r_0 z_2^n$ be the homogenized version of d and similarly $\Delta(z_1, z_2)$ be the homogenized version of δ.

Now $D(z_1, z_2) = 0$ iff $z_1 = 0 = z_2$ by the choice of d, and same goes for Δ. Let f, g be two polynomials. Then f, g have a common root iff $D(f(x), g(x))$ has a root. Thus we have m-inconsistency in k iff the family $(f_l)_{1\leqslant l\leqslant m}$ has no common root in k iff $D(f_1(x), D(f_2(x), \cdots))$ has no root in k iff, by henselianity, $\Delta(f_1(x), \Delta(f_2(x, \cdots))$ has no root in K iff the family $(f_l)_{1\leqslant l\leqslant m}$ has no common root in K, the latter exactly giving m-inconsistency of the pattern in K. □

Thus, given an NTP2 henselian field (K, v), if we take a coarsening of v with residue characteristic p, we know its residue field has finitely many AS-extensions, without having to ponder at external definability or anything.

2.5.6 Tame & semitame

Recently, Franz-Viktor Kuhlmann proved in [Kuh21] that valued fields of characteristic p with finitely many Artin-Schreier extensions are *semitame*, which is a notion he studied in detail in a joint paper with Anna Rzepka. In particular, contrary to the NIP case, where AS-closure implies defectlessness, NTP2 fields could have defect, only, no *dependent* defect:

Definition 2.5.12. Let $(L, w)/(K, v)$ be a purely defect Galois extension of degree p. Let $\sigma \in \mathrm{Gal}(L/K) \setminus \{\mathrm{id}\}$. Consider the set $\Sigma = \left\{ w(\frac{\sigma(x)-x}{x}) \,\middle|\, x \in L^\times \right\}$. If there is a convex

‖This is only true if K is $\aleph_1$-saturated, so let's assume it is.

subgroup $\Delta \subset \Gamma$ such that $\Sigma = \{\gamma \in \Gamma \mid \gamma > \Delta\}$, we call $(L, w)/(K, v)$ an independent defect extension. Otherwise, we call it a dependent defect extension.

Definition 2.5.13. A non-trivially valued field (K, v) of residue characteristic p is called tame if Γ is p-divisible, k is perfect, and (K, v) is defectless. Valued fields of residue characteristic 0 are also called tame. Here we will furthermore let trivially valued fields, of any characteristic, be called tame.

A non-trivially valued field (K, v) of residue characteristic p is called semitame if Γ is p-divisible, k is perfect, and (K, v) has no dependent defect extension. Again, we also call trivially valued fields and equicharacteristic 0 fields semitame.

Note that tame and semitame are first-order properties. Equivalent definitions can be found in [Kuh21], as well as a proof of the following result:

Theorem 2.5.14. *Let (K, v) be a valued field of equicharacteristic p. If K is AS-finite, then (K, v) is semitame.*

We will also need the following lemma:

Lemma 2.5.15 ([KR21, Prop. 1.4])**.** *A composition of two (semi)tame henselian valuation, each of residue characteristic p, is (semi)tame.*

Note that the statement by Kuhlmann and Rzepka that we reference is formulated for "generalized deeply ramified" fields (gdr) without restricting to residue characteristic p, and is then claimed to also hold in the (semi)tame context; as stated, it is slightly wrong, as one needs to avoid some stupid counterexample: if (K, v) is of equicharacteristic 0 with a non-divisible value group, say, $\mathbb{Z}$, and (k_v, w) is mixed-characteristic tame; then $(K, w \circ v)$ is not tame, nor semitame, because its value group is not p-divisible. Thus, Kuhlmann and Rzepka's proof appears to have a hidden assumption, namely, residue characteristic p, that we made explicit here.

This very fact, that compositions of (semi)tame fields are not always (semi)tame, together with the non-(semi)tameness of finitely ramified fields which are noughtwithstanding very well behaved, lead to the definition of gdr fields. We will not define what they are here, instead, we refer to the aforementionned paper [KR21].

We prove a quick but very useful lemma – an NTP2 version of Lemma 3.2.2:

Lemma 2.5.16. *Let K be NTP2, let v be henselian of residue characteristic p, and suppose k_v is imperfect; then v is the coarsest valuation with residue characteristic p. In particular, there is at most one imperfect residue of characteristic p.*

Proof. Suppose w is a non-trivial proper coarsening of v with residue characteristic p. Then $(k_w, \overline{v})$ is a non-trivial equicharacteristic p henselian valued field with imperfect residue. By Theorem 2.5.14, since semitame fields have residue perfect, k_w is not semitame and thus has infinitely many AS-extensions. But, by AS-lifting, that means K has TP2. Thus v can't have any proper coarsening of residue characteristic p. □

We combine all this with the standard decomposition around p, written in terms of places $K \xrightarrow{v_0} k_0 \xrightarrow{\overline{v_p}} k_p \xrightarrow{\overline{v}} k_v$ as in Definition 1.1.18, and obtain:

Proposition 2.5.17. *Let K be NTP2 and v be henselian. Then (K, v) is either*

1. *of equicharacteristic 0, hence tame, or*
2. *of equicharacteristic p and semitame, or*

3. *of mixed characteristic with $(k_0, \overline{v})$ semitame, or*

4. *of mixed characteristic with v_p finitely ramified and $(k_p, \overline{v})$ semitame.*

In particular, (K, v) is gdr.

We will discuss afterwards how close – or rather, how far – this is to an Anscombe-Jahnkish equivalence.

Proof. Most cases follow directly from Theorem 2.5.14 and Artin-Schreier lifting, we only give details for case 3.

Let (K, v) be of mixed characteristic such that v_p is infinitely ramified, that is, Δ_0/Δ_p is dense. This is an elementary statement, that is, going to $(K^*, v^*) \succcurlyeq (K, v)$ sufficiently saturated and doing the standard decomposition in this new structure, Δ_0^*/Δ_p^* remains dense; see [AJ19a, Lem. 2.6]. Furthermore, $(k_0^*, \overline{v_p^*})$ is defectless and has value group $\mathbb{R}$. These facts come directly from saturation, see [AK16].

By Artin-Schreier lifting, k_p is AS-finite, and thus $(k_p, \overline{v})$ is semitame. Finally, an argument similar to the aforementioned proof allows us to obtain perfection of k_p: going to yet another sufficiently saturated elementary extension (L, u) of $(k_0, \overline{v_p})$ – in a language of valued fields –, we know that the value group has a proper convex subgroup below $u(p)$; thus there is a non-trivial coarsening of u with residue characteristic p, and by Lemma 2.5.16 k_u is perfect. This is a first-order statement, so k_p is also perfect.

So, $(k_0, \overline{v_p})$ is defectless, has divisible value group, and perfect residue, that is, it is tame; and $(k_p, \overline{v})$ is semitame. By Lemma 2.5.15, $(k_0, \overline{v})$ is semitame, as wanted. □

Corollary 2.5.18. *Let (K, v) be henselian, of mixed characteristic, and infinitely ramified. If K is NTP2, then (K, v) is roughly p-divisible, of perfect residue, and has no dependent defect extension.*

2.5.7 Towards a classification

As far as we know, Proposition 2.5.17 is as strong as it gets. It is unclear how we could get anything stronger than semitame for equicharacteristic p. The question now is to try and determine if semitame fields have transfer, which is far beyond what can be done now.

We know that transfer happens at least in the same cases as in NIP fields, that is:

Proposition 2.5.19. *Let (K, v) be henselian. Suppose k is NTP2. If either*

1. *(K, v) is of equicharacteristic and SAMK or trivial, or*

2. *(K, v) is of mixed characteristic with v_p finitely ramified and $(k_p, \overline{v})$ SAMK or trivial, or*

3. *(K, v) is of mixed characteristic with $(k_0, \overline{v})$ AMK;*

then (K, v) is NTP2.

Proof. We only sketch the proof, since it is just a reinterpretation of the NIP case done by Anscombe-Jahnke; indeed, if the value group and residue field are stably embedded, and if the type of immediate extensions is determined by NTP2 formulas, then we have NTP2 transfer. This was obtained by Chernikov and Hils in [CH12]. We will say more about this result and how it has been extended to other contexts in section 3.1.1. By the work of Jahnke-Simon and Anscombe-Jahnke, see [AJ19a, Prop. 4.1 & Lem. 4.4], we know SAMK fields and unramified fields satisfy those conditions – in fact, the types of immediate extensions are even implied by NIP formulas. It is now just a matter of composing valuations, which can be done in a slight different way than for NIP_n fields:

Claim. Let (K,v) be a valued field and w a coarsening of v. Assume that k_w is stably embedded (as a pure field) in (K,w). Then $(k_w,\overline{v})$ is stably embedded in (K,w).

This is a direct consequence of [CS15, Lem. 46]. In the NIP$_n$ context, we then proceed to prove that, under stable embeddedness assumptions, a composition of NIP$_n$ valuations is NIP$_n$. We could not achieve such a powerful result in NTP2, however, we can take a shortcut and argue directly by transfer; sadly, this shortcut ends up being more tedious:

- If (K,v) is of mixed characteristic and $(k_0,\overline{v})$ is AMK, then $(k_0,\overline{v})$ has NTP2 transfer, as noted above. Now (K,v_0) is of equicharacteristic 0 and thus trivially SAMK, therefore, it also has transfer; specifically, it has stably embedded value group and (pure) residue field, and the type of immediate extensions is implied by NTP2 formulas. By the claim, $(k_0,\overline{v})$ is still stably embedded, and thus the structure $(K,v_0,(k_0,\overline{v}))$ also has transfer. Thus, if k is NTP2, (K,v) is NTP2.

- If (K,v) is of mixed characteristic, $(k_p,\overline{v})$ is SAMK or trivial and (K,v_p) is finitely ramified, then v_p is definable by Robinson's generalized formula, see Corollary 1.4.3. Moving to $(K^*,v^*)\succcurlyeq(K,v)$ sufficiently saturated, v_p^* is still definable and $(k_p^*,\overline{v^*})$ is still SAMK or trivial. Now $(k_0^*,\overline{v_p^*})$ can be seen as a finite extension of an unramified field (L,w) with same residue k_p^*, see [War93, Thm. 22.7]. Unramified fields have transfer, and as above, it is a resplendent transfer. Thus, if k is NTP2, $(k_p^*,\overline{v^*})$ is NTP2 by SAMK transfer, $(L,w,(k_p^*,\overline{v^*}))$ is NTP2 by unramified transfer, $(k_0^*,\overline{v_p^*},(k_p^*,\overline{v^*}))$ is NTP2 since it is a finite extension of an NTP2 field, and finally $(K,\overline{v_0^*},(k_0^*,\overline{v^*}))$ is NTP2 by equicharacteristic 0 transfer.

□

In short, the current situation is: if a henselian valued field is AS-closed – or rather, if its standard decomposition is AS-closed or finitely ramified by parts –, then it has transfer. In NIP or NIP$_n$ settings, this is enough, as NIP$_n$ fields are known to be AS-closed (or finitely ramified by parts); however, we only know that NTP2 fields are AS-finite (or finitely ramified by parts), and thus tame or semitame (or finitely ramified by parts), and this gives not enough information to ensure transfer under the known theorems.

We ask two questions with no answer yet:

- Can an NTP2 field have defect?
- Do SAMAK henselian valued fields have NTP2 transfer?

Here "SAMAK" means sep. alg. max. *almost* Kaplansky; we still assume that Γ is p-divisible and k is perfect, but we only assume that k is AS-finite, and not necessarily AS-closed.

Regarding defect, we know that semitame fields can not have dependent defect. It is likely that NTP2 fields indeed *can* have defect, but an example remains to be exhibited – and proving transfer theorems for defect fields is yet another task.

2.5.8 What about $\mathbb{F}_p((\mathbb{Q}))$?

If we had NTP2 transfer for SAMAK fields, we would know that $\mathbb{F}_p((\mathbb{Q}))$ is NTP2. This is believable; the only reason why this field has IP is because it has exactly 1 Artin-Schreier extension, other than that, it has divisible value group, finite residue field, is henselian and maximal.

We give a three step programme to try and prove this *without* using transfer. Note that the t-adic valuation is definable in $\mathbb{F}_p((\mathbb{Q}))$ by Corollary 1.4.6.

Step 1: $\mathbb{F}_p((\mathbb{Q}))$ has QE down to predicates for roots of additive polynomials. To be precise, we let S_n be an $n+1$-ary predicate, to be interpreted in $K \equiv \mathbb{F}_p((\mathbb{Q}))$ as $K \vDash S_n(a_0,\cdots,a_{n-1},b) \leftrightarrow \exists x(x^{p^n} + a_{n-1}x^{p^{n-1}} + \cdots + a_0 x = b)$. We would like to prove that $\mathbb{F}_p((\mathbb{Q}))$ has QE in the language of valued fields augmented with these predicates. We give three reasons to believe that this language is exactly the one we need: first of all, the predicates S_n cannot be equivalent to a quantifier free formula in any valued field language, because S_n has IP in $\mathbb{F}_p((\mathbb{Q}))$ – and because quantifier free formulas of valued fields are NIP in any henselian valued field, because ACVF is NIP. For example, $S_1(-1, \frac{x}{y})$ is equivalent to $\exists t\, x = y(t^p - t)$, which has IP by Corollary 2.2.14.

Secondly, we believe the study of all polynomials can be reduced to an argument about additive polynomials. This is mainly because of the following standard result:

Lemma 2.5.20 (Ore's lemma, [Lis21, Lem. 2.1.2])**.** *Let K be a field of characteristic p and fix $P(X) \in K[X]$. Then there is an additive polynomial $Q(X) \in K[X]$ which is divisible by P.*

Then, if we know that P has a root, we know that Q has a root. However, we're more interested by the other direction; we would like to reduce the study of the roots of an arbitrary polynomial P to the study of the roots of this additive polynomial divisible by P. In some sense, this is possible, for example, it was used by Victor Lisinski in [Lis21] to obtain decidability of, among other, $\mathbb{F}_p((\mathbb{Q}))$ – with a constant symbol for t.

The third and last reason is that we believe $\mathbb{F}_p((\mathbb{Q}))$ has QE down to predicates for root of all polynomials: indeed, Ingo Brigandt proved this exact result for tame extensions of $\mathbb{Q}_p$, and their method can be adapted to tame extensions of $\mathbb{F}_p((\mathbb{Z}))$, since it only relies on tameness and on certain properties of residue fields and value groups which are exactly the same in those settings. We refer to [Bri01] for the proof above $\mathbb{Q}_p$.

Step 2: These predicates have TP2 iff the field has infinitely many Artin-Schreier extensions. Indeed, write $H_{a_0,\cdots,a_{n-1}} = \{b \in K \mid K \vDash S_n(a_0,\cdots,a_{n-1},b)\}$. Then H is a definable family of subgroups of K. In the proof of Corollary 2.5.9, apart from additivity, there's nothing special about the Artin-Schreier polynomial, it probably could be replaced by any other additive polynomial; however, the devil is in the details, and a more general – furthermore local – version of the proof needs to be carefully done.

Let us be more precise: keeping notation consistent, the exact formulas which would then be known to be NTP2 would be $S_n(y_0,\cdots,y_{n-1},x-z)$ – with the variable partition $(x;yz)$. Setting z to 0 can only decrease complexity, so this gives NTP2 for S_n with variables as inputs, but it does not say anything if we input arbitrary terms.

Step 3: Term inputs and boolean combination are well-behaved. This is the most ill-defined of the steps. It is of course necessary, since atomic formulas can comprise arbitrary terms, and since boolean combinations do not preserve NTP2; but there's no clear method on how to do it. It is possible that the methods developed during the first two steps can significantly reduce the complexity of this step, but it is also possible that it would have to be tackled frontally, without any sleight of hand.

Chapter 3

NIP_n Henselian Valued Fields

3.1 Overview

3.1.1 Transfer theorems: from world 0-0 to the final boss

Our main result is the following extension Anscombe & Jahnke's characterization of NIP henselian valued field:

Theorem 3.1.1 (The Final Boss of transfer theorems)**.** *Let (K, v) be a henselian valued field. Then (K, v) is NIP_n iff the following holds:*

1. *k is NIP_n, and*
2. *either*
   (a) *(K, v) is of equicharacteristic and is either trivial or SAMK, or*
   (b) *(K, v) has mixed characteristic $(0, p)$, (K, v_p) is finitely ramified, and $(k_p, \overline{v})$ checks 2a, or*
   (c) *(K, v) has mixed characteristic $(0, p)$ and $(k_0, \overline{v})$ is AMK.*

The case $n = 1$ is the main theorem of [AJ19a], which we already stated as Theorem 2.2.18. For definitions of (S)AMK see Definition 1.1.12 and Definition 1.1.13 and for a definition of v_0 and v_p see Definition 1.1.18. We give a short history of transfer theorems.

Transfer theorems are of the form "Assume (K, v) has good properties; if k (and Γ) are *noice*, then (K, v) is *noice*". In our case, *noice* would be NIP or NIP_n, but in general, it could be any model-theoretic property. The goal is to assume as little as possible for (K, v), but often "henselian" is a minimum, as well usually as tameness assumptions in residue characteristic p such as p-divisibility, Kaplansky, defectlessness...

Normally, one needs to assume that both the residue and the value group are *noice*, but bear in mind that all OAG are NIP.

In some sense, one can see AKE as a transfer theorem, transfering elementary equivalence. But if we focus on combinatorial complexity, the first transfer theorem was proven in 1981 by Delon in [Del79]: Assume (K, v) is henselian of equicharacteristic 0, then it has NIP transfer. Because we assume so little, it is already optimal, in some sense.

In 1999 in [Bé99], Luc Bélair obtained transfer in two cases: when (K, v) is henselian, equicharacteristic p, and AMK – so in particular perfect; and when (K, v) is henselian, mixed characteristic, unramified and of residue perfect.

In 2012 in [Che14], Chernikov extended Delon's result to obtain NTP2 transfer in henselian valued field of equicharacteristic 0; this sparked interest towards NTP2 valued fields, and conducted Chernikov and Hils to determine 2 conditions, later named (SE) and (Im), which were shown to be sufficient to have NTP2 transfer in [CH12]. These conditions were then studied in the NIP context by Jahnke and Simon in [JS20], and were used to extend Bélair's results to non-perfect fields, partly in the same article, and also by Anscombe and Jahnke in [AJ19a]. The unramified case was then generalized into a finitely ramified case by Anscombe and Jahnke, using the study of Cohen rings, which they themselves did in [AJ19b], and obtaining the aforementioned theorem for NIP.

We give a summary of the proof before generalizing it to the NIP$_n$ case:

⇒: In equicharacteristic 0, the requirement is empty. In equicharacteristic p, it is a consequence of KSW, see Theorem 2.2.12.

In mixed characteristic, with the help of the standard decomposition – see Definition 1.1.18 –, we reduce to the equicharacteristic case and to a dichotomy on the rank 1 part: either it is discrete or dense. This gives the two cases in mixed characteristic. This doesn't disturb NIP thanks to Shelah's expansion theorem.

⇐: Equicharacteristic 0 has been done by Delon. One now uses Chernikov-Hils conditions to prove transfer in the remaining cases:

- separably algebraically maximal Kaplansky fields check Chernikov-Hils conditions,
- unramified fields check Chernikov-Hils conditions,
- finitely ramified fields can be seen, with some caveats, as finite extension of unramified fields.

To conclude that transfer holds in all cases of the theorem, we need to be able to compose valuations without disturbing NIP, which is possible when the residue field is stably embedded.

Towards arbitrary n: As mentioned before, KSW has been generalized to arbitrary n by Hempel in [Hem14], see Theorem 2.3.10, so "⇒" holds for equicharacteristic p – and for equicharacteristic 0 alike. In mixed characteristic however, we can't use the standard decomposition as such, since the argument uses Shelah's expansion theorem, which fails in general for NIP$_n$ theories, though some weaker version holds, similar to what exists for NTP2, see [MOS18, Annex] and Corollary 3.4.6. But, we saw before than some p-closure results can be obtained via explicit Artin-Schreier lifting, and this will allow us to skip Shelah's expansion. Then we will proceed to prove that Chernikov-Hils conditions, adapted to the NIP$_n$ setting, are sufficient to have NIP$_n$ transfer, and that composing valuations preserves NIP$_n$. This will yield transfer in all cases of the theorem.

3.2 Left-to-right

Let (K, v) be henselian, and suppose it is NIP$_n$ (as a valued field). Since the residue field is interpretable in a NIP$_n$ structure, it is also NIP$_n$. In equicharacteristic 0, there is nothing to prove. We do the equicharacteristic p case in the same way as for NIP fields:

Lemma 3.2.1. *If (K, v) is NIP$_n$, henselian, and of equicharacteristic p, then it is SAMK or trivial.*

This is a NIP$_n$ version of [AJ19a, 3.1].

Proof. If v is trivial, then we're done. Assume not. By Theorem 2.3.10, K is AS-closed; this implies that it has no separable algebraic extension of degree divisible by p (see [KSW11, 4.4]). Then it is clearly separably defectless, it has p-divisible value group, and AS-closed residue. Remains to prove that the residue is perfect. Suppose $\alpha \in k$ has no p^{th}-root in k, and consider $X^p - mX - a$, where $v(m) > 0$ (but non-zero; remember than v is non-trivial) and where a is a lift of α. Then this polynomial has no root, thus K is not AS-closed. □

Now, for the mixed characteristic case, we will follow Anscombe-Jahnke's proof for the most part, except we swap Shelah's expansion for explicit Artin-Schreier lifting.

Lemma 3.2.2. *Let (K, v) be a NIP$_n$ henselian valued field. Then v has at most one coarsening with imperfect residue field. If such a coarsening exists, then it is the coarsest coarsening w of v with residue characteristic p.*

This is a NIP$_n$ version of [AJ19a, 3.4].

Proof. Let w be a proper coarsening of v, name k_w its residue. Suppose k_w is of characteristic p. Then $(k_w, \overline{v})$ is a non-trivial equicharacteristic p henselian valued field. If its residue is imperfect, then k_w is not AS-closed by the proof of Lemma 3.2.1; then K has IP$_n$ as a pure field by explicit Artin-Schreier lifting.

So, if v has a coarsening with imperfect residue field, this coarsening can't in turn have any proper coarsening of residue characteristic p; thus the only coarsening of v that could possibly have imperfect residue is the coarsest coarsening of residue characteristic p (possibly trivial). □

Proposition 3.2.3. *Let (K, v) be a NIP$_n$ henselian valued field of mixed characteristic. Then either 1. (K, v_p) is finitely ramified and $(k_p, \overline{v})$ is SAMK or trivial, or 2. $(k_0, \overline{v})$ is AMK.*

This is a NIP$_n$ version of [AJ19a, 3.1].

Proof. Consider $(k_p, \overline{v})$. If its valuation is non-trivial, k_p must be AS-closed, otherwise K would have IP$_n$ by explicit Artin-Schreier lifting. So, $(k_p, \overline{v})$ is either SAMK or trivial by (the proof of) Lemma 3.2.1.

We now make the following case distinction: if Δ_0/Δ_p is discrete, then (K, v_p) is finitely ramified, and since we already know that $(k_p, \overline{v})$ is SAMK or trivial, case 1 holds. Otherwise, Δ_0/Δ_p is dense. We go to an $\aleph_1$-saturated extension (K^*, v^*) of (K, v), and redo the standard decomposition there. Δ_0^*/Δ_p^* is still dense (see [AJ19a, Lem. 2.6]), and by saturation, it is equal to $\mathbb{R}$; in particular, Δ_0^*/Δ_p^* is p-divisible. Now, as before, if $(k_p^*, \overline{v^*})$ is non-trivial, then it is SAMK. It is clearly non-trivial by saturation, since we assumed (K, v_p) was infinitely ramified. Thus, $(k_0^*, \overline{v^*})$ is Kaplansky. We can state this in first order by saying that k is perfect and AS-closed (the valuation v is in our language for now), and that Γ is roughly p-divisible, i.e. if $\gamma \in [0, v(p)] \subset \Gamma$, then γ is p-divisible.

Remains to prove that $(k_0, \overline{v})$ is algebraically maximal. First, we prove that k_p is perfect. Consider the henselian valued field (K^*, v_p^*) (so this time we have v_p^* in the language, and not v^*) and an $\aleph_1$-saturated extension (K', u') of it. Since (K^*, v_p^*) is infinitely ramified, by saturation u' admits a proper coarsening of residue characteristic p, so by Lemma 3.2.2, its residue field is perfect; going down to (K^*, v_p^*), this means k_p^* is perfect. Since we already know that $(k_p^*, \overline{v^*})$ is separably algebraically maximal, because it is perfect we now know it is algebraically maximal.

Now by saturation $(k_0^*, \overline{v_p^*})$ is maximal; in particular it is defectless, see [AK16]. Now v^* is a composition of defectless valuations, thus it is defectless (see [AJ19a, Lem. 2.8]). By [AJ19a, Lem. 2.4], defectlessness is a first-order property, so (K, v) is also defectless, and thus $(k_0, \overline{v})$ is defectless. Because defectlessness implies algebraic maximality, we conclude. □

3.3 Transfer theorems

As we will discuss later, the first step for transfer is to study usual characterizations of NIP$_n$ in terms of indiscernible sequences and obtain an array extension lemma.

3.3.1 NIP$_n$ & generalized indiscernibles

Definition 3.3.1. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $\mathcal{I}$ be an $\mathcal{L}_0$-structure, where $\mathcal{L}$ and $\mathcal{L}_0$ are possibly different languages. A sequence $(a_i)_{i\in I}$ of tuples of M is said to be $\mathcal{I}$-indiscernible over a set $A \subset \mathcal{M}$ if for any $i_0,\cdots,i_n$ and $j_0,\cdots,j_n$ in $\mathcal{I}$, $\text{qftp}_{\mathcal{L}_0}(i_0,\cdots,i_n) = \text{qftp}_{\mathcal{L}_0}(j_0,\cdots,j_n)$ implies $\text{tp}_{\mathcal{L}}(a_{i_0},\cdots,a_{i_n}/A) = \text{tp}_{\mathcal{L}}(a_{j_0},\cdots,a_{j_n}/A)$.

Remark 3.3.2. We call two tuples of elements of a structure a and b "of the same mould" if they are of the same length n and if for all $i < n$, a_i and b_i are in the same sort. Given a tuple a, we say that a tuple of variable x is "a mould" of a if they are of the same length n and for all $i < n$, x_i is a variable on the sort containing a_i. A contrario, given a tuple of variables x, we say that a tuple of elements a is "a cast" of x if x is a mould of a, and we say similarly that 2 tuples of variables x and y are "identical as moulds".

The reason we care about it is that there's no need for $(a_i)_{i\in I}$ to be a sequence of tuples of the same mould; and for a generalized indiscernible sequence, we do not need to compare the type of a_i and a_j if i and j have different types; so they might be of different lengths and of different sorts.

We denote by G_n an ordered random n-partite n-hypergraph; it is a structure in the language $\{<, P_1,\cdots,P_n, R\}$ and is axiomatized as follows:

- $G_n = P_1 \sqcup \cdots \sqcup P_n$,
- $<$ is a dense linear order without endpoints on each P_i,
- $P_1 < \cdots < P_n$,
- R is an n-ary relation on $P_1 \times\cdots\times P_n$ – the hyperedge relation,
- For any finite disjoint $A_0, A_1 \subset P_1 \times\cdots\times P_{j-1} \times P_{j+1} \times\cdots\times P_n$ and for any $b_0 < b_1 \in P_j$, there is $b \in P_j$ such that $b_0 < b < b_1$ and if you fix $(g_1,\cdots,g_{j-1},g_{j+1},\cdots,g_n) \in A_0$, then $(g_1,\cdots,g_{j-1},b,g_{j+1},\cdots,g_n)$ is an edge; and same goes for A_1 with non-edges.

We name O_n the reduct of G_n to the language $\{<, P_1,\cdots,P_n\}$, that is, we ignore the edges; O_n is thus just $\{1,\cdots,n\} \times \text{DLO}$, lexicographically ordered.

Proposition 3.3.3 ([CPT19, Prop. 5.2], [CH21, Prop. 2.8])**.** *A formula $\varphi(x; y_0,\cdots,y_n)$ has IP$_n$ iff there exists (in a sufficiently saturated model $\mathcal{M}$) a tuple b and a sequence $(a_g)_{g\in G_n}$ which is O_n-indiscernible over $\emptyset$ and G_n-indiscernible over b such that $\varphi(b; y)$ encodes the edges of the graph; that is:*

$$\mathcal{M} \models \varphi(b, a_{g_1},\cdots,a_{g_n}) \textit{ iff } G_n \models R(g_1,\cdots,g_n).$$

Note that considering a sequence indexed by G_n which is O_n-indiscernible is the same as considering n mutually indiscernible sequences indexed by each P_i.

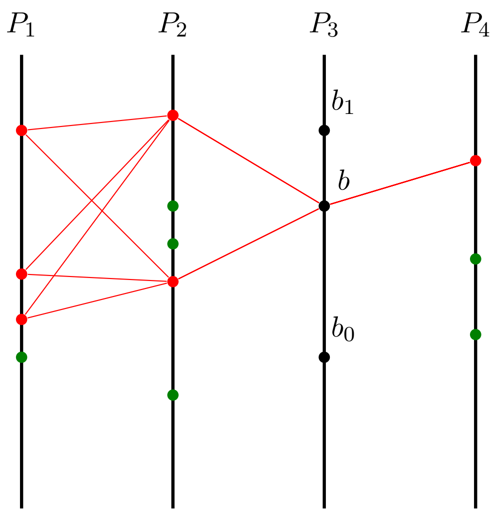


Figure 3.1: An ordered random 4-hypergraph. Each P_i is represented by a vertical line. Sets A_0 and A_1 are represented in red and in green respectively. Such a graph will have many more edges which are not drawn here.

3.3.2 Chernikov-Hils' Im Plus SE conditions

In 2014, Chernikov and Hils defined 2 conditions under which NTP2 transfer happens. We already mentioned these conditions in section 2.5.7. Later, they were studied by Jahnke and Simon in the NIP settings, and proved to be interesting conditions to consider, in general, to obtain transfer theorems. They are the following:

(SE): The residue field and the value group are stably embedded.

(Im): For any small model K and any singleton b (from a monster model) such that $K(b)/K$ is immediate, we have that $\mathrm{tp}(b/K)$ is implied by instances of NTP2 formulas, that is, there is $p \subset \mathrm{tp}(b/K)$ closed under conjunctions and such that:

- any formula $\varphi(x, y) \in p$ – where x is the cast for b and y for (a finite subtuple of) K – is NTP2,
- for any $\psi(x, y) \in \mathcal{L}$, $\psi(b, K)$ holds iff $p \vdash \psi(x, K)$.

We say a valued field has NTP2 CHIPS if it checks Chernikov-Hils' Im Plus SE conditions. These conditions are sufficient to obtain transfer:

Theorem 3.3.4 (NTP2 CHIPS transfer, [CH12, Thm. 4.1])**.** *Let (K, v) have NTP2 CHIPS, then (K, v) is NTP2 iff k is NTP2.*

In 2018, Jahnke and Simon proved a NIP CHIPS transfer, using a modified condition (Im) with NIP formulas instead of NTP2, see [JS20]. Our goal is to prove a NIP$_n$ CHIPS transfer.*

We give a heuristic about why CHIPS is sufficient to obtain transfer: most of the time, combinatorial complexity can be witnessed by indiscernibles, so if a formula φ has TP2, IP or IP$_n$, there's a (potentially generalized) indiscernible sequence $(a_i)_{i \in I}$ and a singleton b such that $\varphi(b, a_i)$ witnesses some pattern. By Ramsey and compactness, we can extend $(a_i)_{i \in I}$

*This joke has been partially funded by the Sylvy Anscombe Foundation on Acronymic Research and Interaction.

until each a_i is a small model K_i. Now, of course, φ is in the type of b over some K_i, say K_0 (otherwise it's always false, and that's not a pattern), but $K_0(b)/K_0$ might not be immediate. Well, whatever; let's append an enumeration of the value groups and residue field of $K_0(b)$ to K_0. We would like to also be able to append to the rest of the sequence K_i so that the now augmented sequence keeps the indiscernability properties it had before; because the value group and residue field are stably embedded, this can be done via an array extension lemma. In the end, we have indeed that $K_0(b)/K_0$ is immediate, so φ is implied by NTP2, NIP or NIP$_n$ formulas, and thus is itself NTP2, NIP or NIP$_n$– which contradicts the choice of φ.

3.3.3 NIP$_n$ CHIPS transfer

In this section, we put in place the argument summarized above in the NIP$_n$ context. As such, we first need to obtain an array extension lemma. We do so in an arbitrary complete theory T with a given monster model $\mathcal{M}$.

Definition 3.3.5. Let D be a $\emptyset$-definable set. We say that D is n-hanced stably embedded if for all formulas $\varphi(x, y_1, \cdots, y_n)$ and for all sequences $(a_i^k)_{i\in I}^{1\leqslant k\leqslant n} \in \mathcal{M}$ such that each a_i^k is a cast of y_k, there is a formula $\psi(x, z_1, \cdots, z_n)$ and a sequence $(b_i^k)_{i\in I}^{1\leqslant k\leqslant n} \in D$ – with each b_i^k a cast of z_k – such that:

$$\varphi(D, a_{i_1}^1, \cdots, a_{i_n}^n) = \psi(D, b_{i_1}^1, \cdots, b_{i_n}^n).$$

The usual definition of stable embeddedness is that any $\mathcal{M}$-definable subset of D is D-definable. A priori, this D-definition depends wildly on the original $\mathcal{M}$-definition, however, with compactness and coding tricks, this can be strengthened to a uniform version. This is discussed in great details in [Tou20, sec. 1].

Our version is semi-uniform – ψ depends on φ and also on the choice of the sequence $(a_i)_{i\in I}$, but does not change when going from a_i to $a_{i'}$ –, and more importantly, it works on n variables at once. It might be that this is equivalent to being stably embedded, assuming D is infinite, via a coding trick and a compactness argument; but it remains to be proved. We note the following:

Lemma 3.3.6. *If every automorphism of D^n lifts to an automorphism of $\mathcal{M}^n$, then D is n-hanced stably embedded.*

Proof. This can be obtained by adapting the proof of [CH99, App. Lem. 1], specifically, the proof of (6) implies (5). Note that if D is not n-hanced stably embedded, then there exists an $\mathcal{M}$-definable family $S_{a_1,\cdots,a_n} = \{b \in D \mid \mathcal{M} \vDash \varphi(b, a_1, \cdots, a_n)\}$ which is not a D-definable family. Following the original proof with this definable family instead of a mere definable set yields the wanted result. □

In order to study n-hanced stable embeddedness in more detail, we ideally would want an n-hanced version of the aforementioned lemma [CH99, App. Lem. 1], this has not been achieved as of yet.

Lemma 3.3.7. *Let $(a_g)_{g\in G_n}$ be O_n-indiscernible over a set A. Suppose D is a $\emptyset$-definable set which is n-hanced stably embedded and fix $d \in D$. If the induced structure on D is NIP$_n$, then no formula with parameters in Ad can encode the hyperedges of $(a_g)_{g\in G_n}$.*

This is a NIP$_n$ version of [JS20, Lem. 2.1].

Proof. Let $\varphi(d; y_1, \cdots, y_n)$ be a formula with non-written parameters in A and encoding the hyperedges of $(a_g)_{g\in G_n}$. By n-hanced stable embeddedness, we can find $\psi(x, z_1, \cdots, z_n)$ and $(b_g)_{g\in G_n} \in D$ such that $\varphi(D; a_{g_1}, \cdots, a_{g_n}) = \psi(D; b_{g_1}, \cdots, b_{g_n})$ for all $\overline{g}$.

Claim. For any $J \subset P_1 \times \cdots \times P_n$, we can find $d_J \in D$ such that $\varphi(d_J; a_{g_1}, \cdots, a_{g_n})$ holds iff $(g_1, \cdots, g_n) \in J$.

Given such d_J, we immediately have that $\psi(d_J; b_{g_1}, \cdots, b_{g_n})$ holds iff $(g_1, \cdots, g_n) \in J$, which yields IP_n on D; thus proving the claim is enough to prove the lemma.

To prove the claim, let f enumerate $P_1 \times \cdots \times P_n$ in such a way that $f(i)$ and $f(i+1)$ always differ in exactly one coordinate – here we take G_n countable. We will prove that one can find a $d_N \in D$ such that $\varphi(d_N; a_{f(i)_1}, \cdots, a_{f(i)_n})$ holds iff $f(i) \in J$ for $i < N$. For $N = 1$, either $f(0)$ is in J or not. We can find $(g_1, \cdots, g_n) \in P_1 \times \cdots \times P_n$ such that $\varphi(d; a_{g_1}, \cdots, a_{g_n})$ holds (or not), so $M \vDash \exists x \in D (\neg)\varphi(d; a_{g_1}, \cdots, a_{g_n})$, and by O_n-indiscernability, $M \vDash \exists x \in D(\neg)\varphi(d; a_{f(0)_1}, \cdots, a_{f(0)_n})$.

Now assume such a d_N exists for some N. We do the case $f(N) \in J$, the other one is similar. We need to find $(g_1, \cdots, g_n)$ in the same place as $f(N)$ regarding $f(i), i < N$ (that is, $g_j < f(i)_j$ iff $f(N)_j < f(i)_j$, etc.), forming a hyperedge, and not colluding with previous choices.

By our choice of f, $f(N-1)_i = f(N)_i$ for all i but 1. Then we take $g_i = f(N-1)_i$, and for the remaining g_j, we use the properties of G_n:

If $f(N)_j$ has never appeared before, we just need to choose a g_j in the correct place such that $(g_1, \cdots, g_n)$ forms a hyperedge. This is possible.

If $f(N)_j$ has appeared before, then fixing $g_j = f(N)_j$ might cause trouble, since $(g_1, \cdots, g_n)$ might not be connected. Instead, we let I be the set of i such that $f(i)_j = f(N)_j$. We let $b_0 = \max\{f(i)_j \mid f(i)_j < f(N)_j\}$ and $b_1 = \min\{f(i)_j \mid f(i)_j > f(N)_j\}$. We let A_0 be the set of $(f(i)_1, \cdots, f(i)_{j-1}, f(i)_{j+1}, \cdots, f(i)_n)$ such that $(f(i)_1, \cdots, f(i)_{j-1}, f(N)_j, f(i)_{j+1}, \cdots, f(i)_n)$ forms a hyperedge, for $i \leqslant N$, and A_1 be the counterpart with non-hyperedges. Then by the properties of G_n, there is b between b_0 and b_1, forming edges with all points of A_0 and no points of A_1; we now let $f'(i) = f(i)$ for $i \notin I$, $f'(i)_k = f(i)_k$ for $k \neq j$, and $f'(i)_j = b$ for $i \in I$. We conclude by indiscernibility as before. □

Lemma 3.3.8 (NIP_n array extension lemma)**.** *Let D be n-hanced stably embedded and let $(a_g)_{g \in G_n}$ be O_n-indiscernible over $\emptyset$ and G_n-indiscernible over some tuple b. Fix an edge $(g_1, \cdots, g_n) \in P_1 \times \cdots \times P_n$. For each g_i let $c_{g_i} \in D$ be a small tuple. Then, we can interpolate the rest of the sequence, that is, we can find $(c_g)_{g \neq g_i}$ and $(a'_g)_{g \in G}$ such that:*

- $a'_{g_i} = a_{g_i}$,
- $\mathrm{tp}((a'_g)_{g \in G_n}/b) = \mathrm{tp}((a_g)_{g \in G_n}/b)$,
- $(a'_g c_g)_{g \in G_n}$ *is O_n-indiscernible over $\emptyset$ and G_n-indiscernible over b.*

This is a NIP_n version of [JS20, Lem. 2.2] and [CH12, Lem. 3.8].

Proof. We do it part by part, mimicking the strategy of the NIP case. We fix an edge $\overline{g} = (g_1, \cdots, g_n) \in G_n$, and we fix i. In the NIP case, we do even and odd separately; here we define the set of "even" indices to be $E_i = \{g \in P_i \mid (g_1, \cdots g_{i-1}, g, g_{i+1}, \cdots, g_n) \text{ is an edge}\}$. Because $(a_g)_{g \in G_n}$ is G_n-indiscernible over b, we can find c_g for each $g \in E_i$ such that $a_g c_g \equiv_{b, a_{g_1}, \cdots, a_{g_{i-1}}, a_{g_{i+1}}, \cdots, a_{g_n}} a_{g_1} c_{g_1}$. Now, by Ramsey, we may assume $(a_g c_g)_{g \in G_n, g \notin (P_i \setminus E_i)}$ is O_n-indiscernible over $\emptyset$ and G_n-indiscernible over b.

Now, because this is true for any sequence with these properties, we move to a new sequence where P_i is now P_i^* and is very long. Any "even" element of P_i^* has already been extended by a c.

For each element of $g \in E_i$ (the original, short version) we chose a representation $\lambda_g \in P_i^*$. We make sure to take them very far apart from each other.

$P_i \setminus E_i$ injects into the set of cuts of E_i. Fix an "odd" index h, and look at the corresponding cut C_h (in P_i^*) of $\{\lambda_g \mid g \in E_i\}$. Now $P_1 \sqcup \cdots \sqcup P_{i-1} \sqcup C_h \sqcup P_{i+1} \sqcup \cdots \sqcup P_n$ is itself a random graph.

Take a formula $\varphi(a_{g_i}, c_{g_i}) \in \operatorname{tp}(a_{g_i} c_{g_i}/b(a_g)_{g \notin P_i})$. By the previous lemma, $\varphi(a_k, c_k)_{k \in C_h}$ can't encode all the random graph; so except for discretely many points, it's either always true or always false.

If we exclude those discretely many points from C_h, after having done that for all formulas, we still have points, because P_i^* is really long. Chose any "even" point in what remains; we will call it λ_h.

Now we take an automorphism σ over $b(a_g)_{g \notin P_i}$ taking each $a_{\lambda_g} c_{\lambda_g}$ to $a_g c_g$. We define $a'_h c_h = \sigma(a_{\lambda_h} c_{\lambda_h})$. Now the sequence with extended points in the ith part and a' for "odd" indices satisfy the theorem. $\square$

We now suppose T is a complete theory of valued fields (possibly with additional structure), and we consider the following properties:

(SE)$_n$: The residue field and the value group are n-hanced stably embedded.

(Im)$_n$: For any models $K_1, \cdots, K_n \vDash T$, writing L for the compositum of all of them, and for any singleton $b \in \mathcal{M}$, if $L(b)/K_i$ is immediate for all i, then we have that $\operatorname{tp}(b/K_1, \cdots, K_n)$ is implied by instances of NIP$_n$ formulas, that is, there is a $p \subset \operatorname{tp}(b/K_1, \cdots, K_n)$ such that:

- any formula $\varphi(x; y_1, \cdots, y_n) \in p$ – where x is the cast for b and y_i for K_i – is NIP$_n$, and
- $\psi(b, K_1, \cdots, K_n)$ holds iff $p \vdash \psi$.

We say that (the complete theory of) a valued field, potentially with augmented structure, has NIP$_n$ CHIPS if it checks these two conditions.

Theorem 3.3.9 (NIP$_n$ CHIPS transfer)**.** *If T is a complete theory of valued fields with NIP$_n$ CHIPS, then T has NIP$_n$ transfer; that is, T is NIP$_n$ iff the theories of the residue field and the value group are NIP$_n$.*

This is a NIP$_n$ version of [JS20, 2.3]. Let us also note that in the case where the structure is augmented, when checking whether a theory has CHIPS – whether it be of NIP, NIP$_n$ or NTP2 flavour –, we need to be careful on exactly what is the structure we consider on the residue field and on the value group; if for example k is NIP$_n$ as a pure field, but we only know that an augmented structure of k is (SE)$_n$, augmented structure for which we don't know NIP$_n$, then this theorem does not guarantee transfer.

Proof. Assume T has IP$_n$. Then we can find a formula $\varphi(x; y_1, \cdots, y_n)$ with x unary, a singleton b and a sequence $(a_g)_{g \in G_n}$ O_n-indiscernible over $\emptyset$ and G_n-indiscernible over b, such that $\varphi(b; a_{g_1}, \cdots, a_{g_n})$ holds iff $G_n \vDash R(g_1, \cdots, g_n)$.

By Ramsey and compactness, we can extend each a_g until it enumerates a small model K_g. We refer to [CPT19], specifically the appendix, for the study of Ramsey properties in NIP$_n$ theories.

We fix an edge $\overline{g} = (g_1, \cdots, g_n) \in P_1 \times \cdots \times P_n$. Let k' and Γ' be the residue and value group of $K_{g_1} \cdots K_{g_n}(b)$, let c_{g_i} and d_{g_i} be enumerations of $k' \setminus k_{g_i}$ and $\Gamma' \setminus \Gamma_{g_i}$. Apply the previous lemma twice to obtain a sequence $(a'_g c_g d_g)_{g \in G_n}$ such that:

- $a'_{g_i} = a_{g_i}$,
- $\mathrm{tp}((a'_g)_{g \in G_n}/b) = \mathrm{tp}((a_g)_{g \in G_n}/b)$,
- $(a'_g c_g d_g)_{g \in G_n}$ is O_n-indiscernible over $\emptyset$ and G_n-indiscernible over b.

We now start over: we extend each $(a'_g c_g d_g)$ to enumerate a model, add the residue and value group of this model plus b, and interpolate. After ω iterations, we have a sequence (N_g) of small models, O_n-indiscernible over $\emptyset$, G_n-indiscernible over b, such that $\mathrm{tp}((N_g)_{g \in G_n}/b)$, restricted to the correct subtuple, equals $\mathrm{tp}((a_g)_{g \in G_n}/b)$, and such that $N_{g_1} \cdots N_{g_n}(b)/N_{g_i}$ is immediate. Now by $(\mathrm{Im})_n$, $\mathrm{tp}(b/N_{g_1}, \cdots, N_{g_n})$ is implied by instances of NIP_n formulas. By G_n-indiscernability, such a formula will also hold for any edge. But by NIP_n-ity, it can't also not hold for all non-edges, in fact it can only not hold for finitely many of them. Hence we must have a non-edge $(g'_1, \cdots, g'_n)$ such that all the NIP_n formulas implying $\mathrm{tp}(b/N_{g_1}, \cdots, N_{g_n})$ hold, and thus $\varphi(b, a_{g'_1}, \cdots, a_{g'_n})$ holds, which contradicts the initial choices of φ, b, and a. □

3.4 Right-to-left

We now use Theorem 3.3.9 to get NIP_n transfer in all cases. We still follow Anscombe-Jahnke's strategy.

Proposition 3.4.1. *SAMK henselian valued fields have (SE)$_n$.*

Proof. By Lemma 3.3.6, it is enough to show that every automorphism of Γ^n lifts to K^n, and similarly for every automorphism of k^n. This follows directly from adapting the proof of Anscombe-Jahnke in the case $n = 1$, see [AJ19b, Thm. 12.6]. □

Proposition 3.4.2. *If (K, v) is SAMK with NIP$_n$ residue, then it is NIP$_n$.*

Proof. For $n = 1$, this was done by Jahnke and Simon in the case of finite degree of imperfection, and Anscombe and Jahnke for the rest; see [JS20, Thm. 3.3] and [AJ19a, Prop. 4.1].

The previous proposition tells us (K, v) has $(\mathrm{SE})_n$, we now prove it has $(\mathrm{Im})_n$: let $K_1, \cdots, K_n$ be small models of the theory of (K, v) – as always we are working in a monster model, thus all valuations are restriction of a given valuation on the monster – and b a singleton such that $K_1 \cdots K_n(b)/K_i$ is immediate. We let L be the henselization of the relative perfect hull of $K_1 \cdots K_n(b)$. By the properties of the henselization, L is uniquely determined by the isomorphism type of b over $K_1 \cdots K_n$.

Now we consider L', the relative tame closure of L. This is uniquely determined up to isomorphism by [KPR86, Thm. 5.1] because L is Kaplansky. By [Del82, Thm. 5.1], L' is an elementary extension of K_i (for any i).

Thus, the isomorphism type of b over $K_1, \cdots, K_n$ (that is, its qf type) uniquely determines a model containing it, so it implies the full type. Quantifier free formulas in the language of valued fields are NIP, thus in particular NIP_n; which means (K, v) has $(\mathrm{Im})_n$, and we have transfer by Theorem 3.3.9. □

Note that we did not specify the characteristic – the way we wrote it assumes the residue characteristic is p, but in equicharacteristic 0, it's even simpler, since $K_1 \cdots K_n(b) \equiv k((\Gamma)) \equiv K_i$.

In equicharacteristic, we already proved that NIP_n henselian valued fields are SAMK (or trivial), so this suffices to have the equivalence, and only the mixed characteristic case remains.

Lemma 3.4.3. *If (K, v) is henselian, of mixed characteristic and unramified, then it has (SE)$_n$.*

Proof. As before, it is an easy adaptation of the proof in the case $n = 1$, see [JS20, Lem. 3.1] and [AJ19a, Prop. 4.1], using Lemma 3.3.6. $\square$

Lemma 3.4.4. *If (K, v) is (mixed-char) unramified with NIP$_n$ residue, then it is NIP$_n$.*

Proof. Again, NIP$_1$-transfer has been proved using (SE)$_1$+(Im)$_1$ by Anscombe and Jahnke, see [AJ19a, Lem. 4.4]. We now go towards arbitrary n.

We let $K_1, \cdots, K_n$ be small models – of a given monster model, as above – and b be a singleton such that $K_1 \cdots K_n(b)/K_i$ is immediate for each i. We also assume that one of them, say K_1, is $\aleph_1$-saturated. Each of them is equipped with a valuation which is the restriction of the monster's valuation and that we denote v in each of them.

Let $L = K_1 \cdots K_n(b)$, by assumption L/K_1 is immediate, so we write Γ for the value group and k for the residue field. By unramification, $\Gamma = \Delta \oplus \mathbb{Z}$, with $\Delta = \Gamma/\mathbb{Z}$ and $v(p) = (0, 1) \in \Delta \oplus \mathbb{Z}$, and we let w be the coarsening of v corresponding to $\mathbb{Z}$. We denote the residue field of $(\cdot, w)$ by $\overline{\cdot}$.

Now $(\overline{L}, \overline{v})$ is an immediate extension of $(\overline{K_1}, \overline{v})$. But by $\aleph_1$-saturation, $(\overline{K_1}, \overline{v})$ is spherically complete, hence maximal. So, $\overline{L} = \overline{K_1}$.

Finally, we consider the henselization L^h of L. It is immediate over L – and over K_1. Decomposing it into its Δ part and its $\mathbb{Z}$ part, we have that $\overline{L^h} = \overline{L}^h = \overline{L}$, since it is equal to $\overline{K_1}$ which is henselian.

$$
\begin{array}{ccccc}
L^h & \xrightarrow{\ \Delta\ } & \overline{L^h} & \xrightarrow{\ \mathbb{Z}\ } & k \\
\Big| & & \Big\| & & \\
L & \xrightarrow{\ \Delta\ } & \overline{L} & \xrightarrow{\ \mathbb{Z}\ } & k \\
\Big| & & \Big\| & & \\
K_1 & \xrightarrow{\ \Delta\ } & \overline{K_1} & \xrightarrow{\ \mathbb{Z}\ } & k
\end{array}
$$

The $\mathbb{Z}$ part of L^h and K_1 are exactly the same, this implies that (K_1, v) is an elementary substructure of (L^h, v^h) by [AJ19b, Cor. 12.5]†.

This means that the quantifier free type of b over K_1 completely determines a model containing $K_1 \cdots K_n(b)$, that is, it implies the full type $\text{tp}(a/K_1, \cdots, K_n)$. Note that we fixed K_1 but we could have worked over any K_i instead.

$\square$

We need to go from unramified to finitely ramified, and to study compositions of valuations in the standard decomposition. The following results will be useful:

Proposition 3.4.5. *Let $\mathcal{L}$ be relational, let M be a NIP$_n$ $\mathcal{L}$-structure, let D be $\emptyset$-definable and n-hanced stably embedded. Consider an extension D' of D_{ind} to a relational language $\mathcal{L}_p$, and let M' be the corresponding extension of M to $\mathcal{L}' = \mathcal{L} \cup \mathcal{L}_p$.*

Then, D' is n-hanced stably embedded in M', and if furthermore D' is NIP$_n$, then so is M'.

†This paper by Anscombe and Jahnke is still in the preprint stage, and has changed structure and numbering many times; in fact, we refer to the second version on arXiv, which is not the most recent one.

Before proving it, let us specify how we will use it: we aim to obtain a NIP$_n$ version of [AJ19a, Prop. 3.3]. To do so, we apply the proposition above with $\mathcal{L}$ a relational version of the language of valued fields, $M = (K, w)$, $D = k_w$, and $\mathcal{L}_p$ containing a predicate for a valuation $\overline{v}$ on $D = k_w$, and we get:

Corollary 3.4.6. *Let (K, v) be a valued field and w be a coarsening of v. Assume that (K, w) and $(k_w, \overline{v})$ are both NIP$_n$ and that k_w is n-hanced stably embedded (as a pure field) in (K, w). Then (K, v) is NIP$_n$.*

Proof of Proposition 3.4.5. We may assume that D' has QE in $\mathcal{L}_p$ and M in $\mathcal{L}$; then (the proof of) [CS15, Lem. 46] implies that every $\mathcal{L}'$-formula is equivalent to a D-bounded formula, that is, a formula of the form:

$$Qy \in D \bigvee_{i<m} \varphi_i(x, y) \wedge \psi_i(x, y)$$

with Q a tuple of quantifiers, φ_i qf-$\mathcal{L}$-formulas and ψ_i qf-$\mathcal{L}_p$-formulas (with x restricted to D).

Thus, D' is n-hanced stably embedded in M', and its induced structure is exactly coming from $\mathcal{L}_p$.

We now assume D' is NIP$_n$ and we prove by induction on the number of quantifiers that every D-bounded formula is NIP$_n$. If it has no quantifier, it is NIP$_n$ by assumption. Now let $\varphi(x, y_1, \cdots, y_n) = \exists z \in D\, \psi(x, y_1, \cdots, y_n z)$, where ψ is D-bounded and NIP$_n$.

Suppose φ has IP$_n$. Then, in a sufficiently saturated model, we can find $(a_g)_{g \in G_n}$ and b such that $(a_g)_{G_n}$ is G_n-indiscernible over b and O_n-indiscernible over $\emptyset$. Fix an edge $(g_1, \cdots, g_n)$, now $\exists z \in D \psi(b, a_{g_1}, \cdots, a_{g_n} z)$ holds and we can find $c_{g_n} \in D$ witnessing it. Interpolate the sequence using Lemma 3.3.8 to get $(a'_g)_{g \in G_n}$ and $(c_g)_{g \in G_n}$ such that:

- $a'_{g_i} = a_{g_i}$,
- $\mathrm{tp}((a'_g)_{g \in G_n}/b) = \mathrm{tp}((a_g)_{g \in G_n}/b)$,
- $(a'_g c_g)_{g \in G_n}$ is O_n-indiscernible over $\emptyset$ and G_n-indiscernible over b.

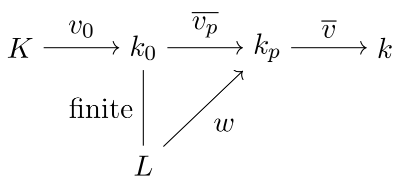
By G_n-indiscernability over b, since $\psi(b, a_{g_1}, \cdots, a_{g_n} c_{g_n})$ holds, it also holds for any edge. By assumption, $\forall z \in D \neg\psi(b, a_{g'_1}, \cdots, a_{g'_n} z)$ holds for any non-edge, thus in particular not for $z = c_{g'_n}$.

Hence there is an IP$_n$ pattern for ψ, which contradicts our induction hypothesis. □

Proposition 3.4.7. *Let (K, v) be henselian of mixed characteristic such that (K, v_p) is finitely ramified and $(k_p, \overline{v})$ is NIP$_n$; then (K, v) is NIP$_n$.*

Proof. Since v_p is finitely ramified, it is definable by the generalized Robinson formula, see Corollary 1.4.3. Thus, if we consider an $\aleph_1$-saturated extension (K^*, v^*) of (K, v), we have that (K^*, v^*_p) is also finitely ramified, and $(k^*_p, \overline{v}^*)$ is also NIP$_n$. Furthermore, (K, v) is NIP$_n$ iff (K^*, v^*) is NIP$_n$; thus we may assume that (K, v) is $\aleph_1$-saturated.

As usual, we consider the standard decomposition. By $\aleph_1$-saturation, $(k_0, \overline{v}_p)$ is complete; it is also rank-1 by definition and finitely ramified by assumption. By [War93, Thm. 22.7], there is a field L such that k_0/L is finite and such that, writing $w = \overline{v_p}|_L$, we have that (L, w) is complete, unramified, and has residue field $k_w = k_p$.



Since we know that k_p is NIP$_n$, by Lemma 3.4.4, (L, w) is NIP$_n$; we also know that k_p is n-hanced stably embedded in (L, w). We are thus in the setting of Corollary 3.4.6, so $(L, \overline{v} \circ w)$ is NIP$_n$. Since k_0 is a finite extension of L, we conclude that $(k_0, \overline{v})$ is NIP$_n$ as well.

Finally, we apply Lemma 3.4.4 once more to the fields (K, v_0) and $(k_0, \overline{v})$: because (K, v_0) is of equicharacteristic 0, k_0 is n-hanced stably embedded, and since it is NIP$_n$, we know (K, v_0) is NIP$_n$ by equicharacteristic 0 transfer. Since we just proved that $(k_0, \overline{v})$ is NIP$_n$, (K, v) itself is NIP$_n$. □

We are now finally ready to prove our main theorem.

Proof of Theorem 3.1.1. Let (K, v) be henselian. If (K, v) is NIP$_n$, then k is NIP$_n$ by interpretability. If it is of equicharacteristic, then (K, v) is SAMK or trivial, by Lemma 3.2.1. If it is of mixed characteristic, then by Proposition 3.2.3 either (K, v_p) is finitely ramified and $(k_p, \overline{v})$ is SAMK or trivial, or $(k_0, \overline{v})$ is AMK. This proves one direction.

In the other direction, assume that k is NIP$_n$. If v is trivial then (K, v) is NIP$_n$. Assume v is non-trivial. If K is of equicharacteristic and SAMK, then (K, v) is NIP$_n$ by Proposition 3.4.2. If K is of mixed characteristic, (K, v_p) finitely ramified, and $(k_p, \overline{v})$ SAMK or trivial; then (K, v) is NIP$_n$ by Proposition 3.4.7. Finally, if K is of mixed characteristic and $(k_0, \overline{v})$ is AMK, then $(k_0, \overline{v})$ is NIP$_n$ by Proposition 3.4.2 – since AMK and SAMK are the same thing for a characteristic 0 field such as k_0. Finally, we conclude that (K, v) is NIP$_n$ by applying Corollary 3.4.6: (K, v_0) is of equicharacteristic 0 so k_0 is stably embedded in it, $(k_0, \overline{v})$ is NIP$_n$, hence (K, v) is NIP$_n$. □

Note that the theorem reads: "a henselian valued field is NIP$_n$ iff its residue is NIP$_n$ and it satisfies algebraic conditions". In particular, these algebraic conditions do not depend on n, so we have the following:

Corollary 3.4.8. *Let (K, v) be a NIP$_n$ henselian valued field. If k is NIP$_m$ for some $m < n$, then (K, v) is NIP$_m$. In particular, if k is NIP, (K, v) is NIP.*

It also yields the following:

Corollary 3.4.9. *Conjecture 2.3.6 and Conjecture 2.3.7 are equivalent, that is, if no strictly NIP$_n$ pure field exists, no strictly NIP$_n$ henselian valued field exists.*

We also know some new examples of cases where these conjectures hold: there are no strictly NIP$_n$ algebraic extensions of $\mathbb{Q}_p$ or of $\mathbb{F}_p((t))$, as we will see in the next chapter.

Finally, we partly generalize the main result from [Jah19] which says that augmenting the structure of a NIP pure field by an arbitrary henselian valuation preserves NIP:

Corollary 3.4.10. *Let (K, v) be henselian of residue characteristic p. Suppose K is NIP$_n$ as a pure field, then (K, v) is NIP$_n$ as a valued field.*

Proof.

Step 1: we may assume (K, v) is SAMK. There are 3 cases. In equicharacteristic p, we are SAMK – or trivial, but in case the valuation is trivial, there is nothing to prove.

In mixed characteristic, if v_p is finitely ramified, we know that it is definable, thus we know that (K, v_p) is NIP$_n$ and in particular that k_p is NIP$_n$. Now, (k_p, v) is SAMK; assuming we can prove the result for it, we can now apply Proposition 3.4.7 and conclude.

Finally, in the other mixed characteristic case, we have that $(k_0, \overline{v})$ is AMK. Assuming we can prove that $(k_0, \overline{v})$ is NIP$_n$, since we know equicharacteristic 0 henselian valued fields have (SE)$_n$ by Proposition 3.4.1, we know (K, v_0) is NIP$_n$ and we can apply Corollary 3.4.6.

Step 2: we only need to define a henselian refinement of v. Assume w is a definable henselian refinement of v. Hence, (K, w) is NIP_n. Because $(k_v, \overline{w})$ is an equicharacteristic p non-trivially valued field (if it is trivially valued, $v = w$ and we're done), it must be SAMK by explicit Artin-Schreier lifting (up to K). Hence, since we took (K, v) SAMK, $(K, \overline{w} \circ v) = (K, w)$ is SAMK itself, its value group is stably embeded. Bearing in mind that any OAG is NIP, we can augment it by the convex subgroup Δ_v corresponding to v, preserving its NIPity by Shelah's expansion theorem; so by Proposition 3.4.5, (K, w, Δ_v) remains NIP_n, and in particular (K, v).

Step 3: well then let's define v_K. We know that the absolute Galois group of K is non-universal since K is SAMK and thus can't have Galois extension of degree divisible by p.

We can apply Corollary 1.4.8: if k_{v_K} is neither SCF nor RCF, then v_K is definable in K. In which case, by the definition of v_K, it is the finest henselian valuation on K. In particular, it is a refinement of v, and we conclude.

If k_{v_K} is SCF, then, by step 1, (K, v_K) is NIP_n. Assume first that v_K is a refinement of v. Now, (K, v) is SAMK by step 2, and $(k_v, \overline{v_K})$ is equicharacteristic p, hence also SAMK by AS-lifting. Because its residue is NIP_n, $(k_v, \overline{v_K})$ is NIP_n; in particular k_v is NIP_n. On the other hand, if v_K is a coarsening of v, then k_v is SCF, thus also NIP_n. Now because SAMK fields have NIP_n transfer, we conclude.

Finally, k_{v_K} can't be RCF; indeed, v_K is still a refinement of v; so $(k_v, \overline{v_K})$ would have RCF residue. But this is an equicharacteristic p valuation, so it can't be. □

As of now, there is no proof for the equicharacteristic 0 case as a whole; indeed, in equicharacteristic 0, there is no way to prove that the Galois group of K is non-universal – to the extent of our knowledge.

Chapter 4

Combinatorial Complexity in Algebraic Extensions of $\mathbb{Q}_p$

4.1 Definability of v_p in algebraic extensions of $\mathbb{Q}_p$

When talking about combinatorial complexity of a valued field, there is always a possibility that the pure field and the valued field structures differ in their complexity. In algebraic extensions of $\mathbb{Q}_p$ though, the p-adic valuation is always definable, as we will prove.

4.1.1 Explicit definitions

Let (K, v) be an algebraic extension of $\mathbb{Q}_p$. We have:

$$\mathbb{Z} \subset \Gamma \subset \mathbb{Q} \quad \& \quad \mathbb{F}_p \subset k \subset \mathbb{F}_p^{\text{alg}}.$$

In the following cases, v is definable in K:

- If Γ is not q-divisible for some $q \neq p$, we can use Robinson's generalized formula, see Corollary 1.4.3.
- If $k \neq \mathbb{F}_p^{\text{alg}}$, it is either finite or PAC; in both cases, we can use Fehm's method, see Corollary 1.4.6.

However, both definitions fail when Γ is divisible by all $q \neq p$ and $k = \mathbb{F}_p^{\text{alg}}$. For these fields, no explicit definition is known; yet we can still show that v_p is ring-definable – except, of course, in the algebraic closure itself – by using the canonical ℓ-henselian valuations.

4.1.2 Canonical ℓ-henselian valuations on extensions of $\mathbb{Q}_p$

Let's first look at $\mathbb{Q}_p$ in itself. Since v_p is henselian, it is in particular ℓ-henselian for any prime ℓ. It must therefore be comparable with the canonical ℓ-henselian valuation (which is non-trivial since $\mathbb{Q}_p$ is henselian and not p-closed), and we have to look at two cases:

- If $\mathcal{O}_{v_p} \subset \mathcal{O}_{v^\ell_{\mathbb{Q}_p}}$, then there must be a convex subgroup of Γ corresponding to this coarsening; but since $\Gamma = \mathbb{Z}$, the only possibility is $\mathcal{O}_{v^\ell_{\mathbb{Q}_p}} = \mathcal{O}_{v_p}$.
- If $\mathcal{O}_{v^\ell_{\mathbb{Q}_p}} \subset \mathcal{O}_{v_p}$, then $\mathcal{O}_{v^\ell_{\mathbb{Q}_p}}/\mathcal{M}_{v_p}$ is a valuation ring of $k = \mathbb{F}_b$, which has no non-trivial valuation, so again $\mathcal{O}_{v^ell_{\mathbb{Q}_p}} = \mathcal{O}_{v_p}$.

The argument works in the same manner for an algebraic extension (K, v) of $(\mathbb{Q}_p, v_p)$, since $\mathbb{Z} \subset \Gamma \subset \mathrm{Div}(\mathbb{Z}) = \mathbb{Q}$ has no non-trivial convex subgroup, and $\mathbb{F}_p \subset k \subset \mathbb{F}_p^{\mathrm{alg}}$ has no non-trivial valuation; note however that if K were to be ℓ-closed then v_K^ℓ would be trivial.

Therefore v_p is the canonical ℓ-henselian valuation on any non ℓ-closed algebraic extension of the p-adics, and is in particular definable in any such extension containing an ℓ^{th}-root of unity.

4.1.3 There and back again

In order to deal with arbitrary algebraic extensions, we will need to go "there" by, among other things, adjoining a root of unity when needed, and then "back again" by interpreting this field extension in the original one.

Let $K/\mathbb{Q}_p$ be algebraic, and $K \neq K^{\mathrm{alg}}$. Then there exists a finite algebraic extension of L of degree $n \geqslant 2$, which can be extended to a Galois extension N of degree at most $n!$. If ℓ divides $[N\colon K]$, then $\mathrm{Gal}(N/K)$ has an ℓ-Sylow subgroup S_ℓ; denote F its fixed field. Now N/F is a Galois extension of ℓ-power degree, therefore F is not ℓ-closed, and F/K is finite. Consider $M = F[\zeta_\ell]$: we are there. M is still not ℓ-closed since it is a finite extension of F (if $\ell = 2$ then we have to argue that $\mathbb{Q}_p$ is not orderable and therefore no extension of it can be euclidean), so ψ_ℓ defines v_p on M. Finally, we interpret M in L (with coefficients of minimal polynomials of generators of M as parameters), and the restriction of v_p to L is therefore definable: we are back again.

4.2 NIP$_n$-ity of algebraic extensions of $\mathbb{Q}_p$

As an immediate consequence of the definability of the valuation, we know that an algebraic extension K of $\mathbb{Q}_p$ is NIP$_n$ as a pure field iff (K, v_p) is NIP$_n$ as a valued field (except $\mathbb{Q}_p^{\mathrm{alg}}$, since in it the valuation is not definable; but we know that ACF is strongly minimal and ACVF is NIP). Hence, we can use the NIP$_n$ version of Anscombe-Jahnke's classification, Theorem 3.1.1, to understand NIP$_n$ extensions of $\mathbb{Q}_p$. As we will note later however, residue fields of these fields are either finite and thus NIP, SCF and thus NIP, or PAC not SCF and thus IP$_n$ for any n; in other words, Corollary 3.4.8 tells us that algebraic extensions of $\mathbb{Q}_p$ are NIP$_n$ iff they are NIP, so we will set $n = 1$ and phrase everything in terms of NIP for this section.

4.2.1 Applying Anscombe-Jahnke's classification

If K is an algebraic extension of $\mathbb{Q}_p$ equipped with the p-adic valuation v, then several simplifications occur in Anscombe-Jahnke's classification: we can obviously ignore the equicharacteristic case, and since we are in rank 1, the standard decomposition gives v_0 trivial and $v_p = v$, so $k_0 = K$ and $k_p = k$. Furthermore, since $\mathbb{Z} \subset \Gamma \subset \mathbb{Q}$, (K, v) is finitely ramified iff Γ is isomorphic to $\mathbb{Z}$. Parsing these properties together:

Corollary 4.2.1. *Let $K/\mathbb{Q}_p$ be algebraic and let v be the p-adic valuation on K. Then (K, v) is NIP if and only if the following holds:*

(1) k is NIP, and

(2) either (b) $\Gamma \simeq \mathbb{Z}$, or (c) (K, v) is AMK.

We will reformulate this characterization of NIP extensions of $\mathbb{Q}_p$ in somewhat more concrete terms.

A first easy case to consider is when k is finite. Then it is NIP, which takes care of (1). All finite fields have extensions of degree p, so (K, v) can't be Kaplansky and can't satisfy (2c).

We must then have $\Gamma \simeq \mathbb{Z}$ for K to check (2), so both the ramification and inertia degrees are finite, which is equivalent to having $K/\mathbb{Q}_p$ finite. Such a K immediately checks (1) and (2b); it is also obviously NIP by interpretability of finite extensions. The following lemma is therefore not very insightful:

Lemma 4.2.2. *Let $K/\mathbb{Q}_p$ be algebraic with k finite. Then K is NIP iff $K/\mathbb{Q}_p$ finite.*

This tackles the finite case. Now if k is infinite, by (1) it must be separably closed, since infinite extensions of finite fields are PAC, and PAC not SC fields have IP and IP_n, see Proposition 2.3.8. So $k = \mathbb{F}_p^{\text{alg}}$. Remains for this field to check (2), which gives two distinct cases, and we then have the following case distinction:

1. k finite & $\Gamma \simeq \mathbb{Z}$,
2. $k = \mathbb{F}_p^{\text{alg}}$ & $\Gamma \simeq \mathbb{Z}$,
3. $k = \mathbb{F}_p^{\text{alg}}$ & K AMK.

These are the three types of NIP algebraic extensions of $\mathbb{Q}_p$. We will now study cases 2 and 3 and reformulate them with the help of Galois theory.

4.2.2 Inertia and ramification groups

Let (K, v) be any valued field. Let $G = \text{Gal}(K^{\text{sep}}/K)$ be its absolute Galois group. Let's fix an extension of v to K^{sep} and denote it by v^{sep}. We will define several interesting closed subgroups of G with their corresponding extensions and list their properties without proving them. Details can be found in [EP10].

Definition 4.2.3 (Closed subgroups of G of interest)**.**

- The *decomposition subgroup* G^h and the associated field extension K^h are defined as follows:
$$G^h = \{\sigma \in G \mid \sigma(\mathcal{O}_{v^{\text{sep}}}) = \mathcal{O}_{v^{\text{sep}}}\}$$
$$K^h = \text{Fix}(G^h),\ v^h = v^{\text{sep}}|_{K^h}$$
(K^h, v^h) is called the *henselization* of K, hence the h.

- The *inertia subgroup* G^t and the associated *inertia extension* of K are defined as follows:
$$G^t = \{\sigma \in G \mid \sigma(x) - x \in \mathcal{M}_{v^{\text{sep}}}\ \forall x \in \mathcal{O}_{v^{\text{sep}}}\}$$
$$K^t = \text{Fix}(G^t),\ v^t = v^{\text{sep}}|_{K^t}$$
The t stands for "träge".

- The *ramification subgroup* and the associated *ramification extension* of K are defined as follows:
$$G^v = \{\sigma \in G \mid \sigma(x) - x \in x\mathcal{M}_{v^{\text{sep}}}\ \forall x \in K^{\text{sep}}\}$$
$$K^v = \text{Fix}(G^v),\ v^v = v^{\text{sep}}|_{K^v}$$
The v stands for "verzweigt".

Looking at the definitions, it is clear that these are ordered as follows:

$$G^v \subset G^t \subset G^h \subset G \qquad K \subset K^h \subset K^t \subset K^v \subset K^{\text{sep}}$$

Let us study them in order:

Proposition 4.2.4 (Henselization, [EP10, Thm. 5.2.2])**.** *(K^h, v^h) is a henselian valued field. It also uniquely embeds in any henselian extension of (K, v). K^h is trivial iff (K, v) is already henselian. A priori K^h depends on the choice of v^{sep}, but these choices give extensions conjugate over K. (K^h, v^h) is an immediate extension of (K, v).*

Proposition 4.2.5 (Inertia field, [EP10, Thm. 5.2.7])**.** *G^t is a normal subgroup of G^h, so K^t is a Galois extension of K^h. (K^t, v^t) is also a purely inertial extension of (K^h, v^h), in the following sense: if $K^h \subset L \subset M \subset K^t$ with M/L finite, then $[M : L] = [k_M : k_L]$.*[*] *We also have $\Gamma_{v^t} = \Gamma$ and $k_{v^t} = k^{\mathrm{sep}}$. Finally, if an extension L/K^h is such that $k^{\mathrm{sep}} \subset k_L$, then already $K^t \subset L$.*

Proposition 4.2.6 (Ramification field, [EP10, Thm. 5.3.3])**.** *G^v is a normal subgroup of G^t, so K^v is a Galois extension of K^t. (K^v, v^v) is a purely ramified extension of (K^t, v^t), in the following sense: if $K^t \subset L \subset M \subset K^v$ with M/L finite, then $[M : L] = [\Gamma_M : \Gamma_L]$. We also have $k_{v^v} = k^{\mathrm{sep}}$ and $\Gamma_{v^v} = \bigcup_{q \neq \mathrm{ch}(k)} \mathrm{Div}_q(\Gamma)$, the q-divisible hull of Γ for all q prime different form* $\mathrm{ch}(k)$. *If k is of characteristic 0 then it is the full divisible hull, and $K^v = K^{\mathrm{sep}} = K^{\mathrm{alg}}$. If k is of characteristic p then G^v is the unique p-Sylow subgroup of G^t.*

We summarize some of this information in Figure 4.1.

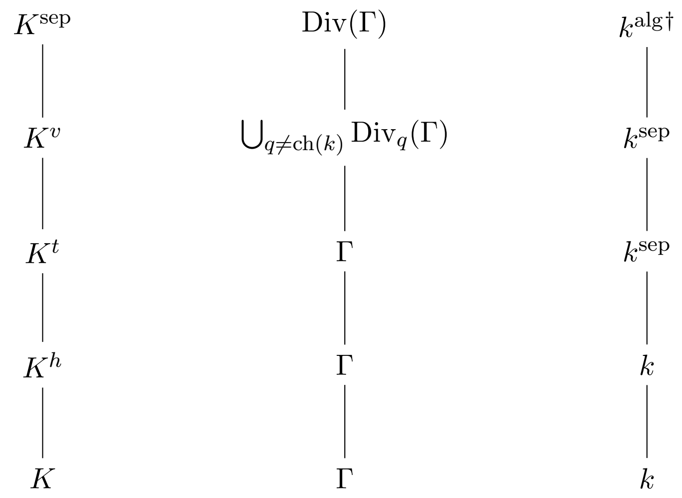


Figure 4.1: Special extensions of a valued field and their corresponding value groups and residue fields.

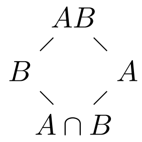
The last field extension we will define and study is the complement of the ramification group, represented in Figure 4.2. Its existence is guaranteed by the following theorem:

Theorem 4.2.7 (Kuhlmann-Pank-Roquette, [KPR86])**.** *Let (K, v) be a valued field and fix an extension of v to the separable closure. Then there exists at least one G^h-complement of G^v, that is, a closed subgroup $G^k \subset G^h$ such that $G^k G^v = G^h$ and $G^k \cap G^v = \{\mathrm{id}\}$. Denoting $K^k = \mathrm{Fix}(G^k)$, we then have $K^k K^v = K^{\mathrm{sep}}$ and $K^k \cap K^v = K^h$.*

Note that this theorem states existence of such complements, but a priori not uniqueness. A lot of these complements could exist. Complements are better understood via diagrams drawings, see Figure 4.2. In these drawings anything going up (straight or slanted) is a field extension, and there will be a lot of "diamonds":

[*] Since v^h is henselian, any extension of K^h is canonically associated with a unique valuation.

[†] or k^{sep} if v is trivial.



Anytime such a diamond appears, it will be drawn such that the bottom vertex is the intersection of the left and right vertices, and the top vertex is their compositum.

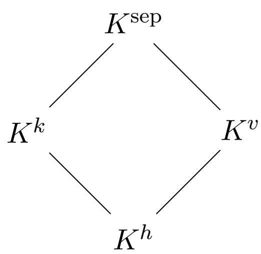


Figure 4.2: Complement of ramification group

4.2.3 Special extensions of $\mathbb{Q}_p$

We can now apply this to $\mathbb{Q}_p$. Since it is henselian, we know $G^h = G$ and $\mathbb{Q}_p^h = \mathbb{Q}_p$. We also have $\mathbb{F}_p^{\mathrm{sep}} = \mathbb{F}_p^{\mathrm{alg}}$, and $\mathbb{Q}_p^{\mathrm{sep}} = \mathbb{Q}_p^{\mathrm{alg}}$. We conclude that an algebraic extension $K/\mathbb{Q}_p$ has residue $\mathbb{F}_p^{\mathrm{alg}}$ iff $\mathbb{Q}_p^t \subset K$.

Our goal is to classify NIP infinite algebraic extensions of $\mathbb{Q}_p$. We already know that they must have residue $\mathbb{F}_p^{\mathrm{alg}}$, that is, they must contain $\mathbb{Q}_p^t$. To be NIP, such an extension still needs to check condition (2) of Corollary 4.2.1.

Lemma 4.2.8. *Let $K/\mathbb{Q}_p$ be algebraic with $k = \mathbb{F}_p^{\mathrm{alg}}$. Then $\Gamma \simeq \mathbb{Z}$ iff $K/\mathbb{Q}_p^t$ is finite.*

This tackles the case (2b) of Corollary 4.2.1. The last remaining case is (2c), when (K, v) is AMK. In our case, since k is already algebraically closed, to be Kaplansky we need only to worry about the value group. Furthermore, we need to make sure that (K, v) is algebraically maximal. Looking at $\mathbb{Q}_p^v$, we see that its value group is everything but p-divisible, and has no reason to be algebraically maximal. On the other hand, complements of $\mathbb{Q}_p^v$ are exactly in the inverse situation, so they should have p-divisible value group and no defect, which will imply algebraic maximality. More precisely, we apply Theorem 4.2.7 to $\mathbb{Q}_p^t$ and find $(\mathbb{Q}_p^t)^k$ such that $(\mathbb{Q}_p^t)^k\mathbb{Q}_p^v = \mathbb{Q}_p^{\mathrm{alg}}$ and $\mathbb{Q}_p^v \cap (\mathbb{Q}_p^t)^k = \mathbb{Q}_p^t$. We claim that any extension of any complement $(\mathbb{Q}_p^t)^k$ is AMK, and that any AMK extension of $\mathbb{Q}_p^t$ will contain a complement:

Lemma 4.2.9. *Let $K/\mathbb{Q}_p$ be algebraic with $k = \mathbb{F}_p^{\mathrm{alg}}$ – so $\mathbb{Q}_p^t \subset K$. Then the following are equivalent:*

1. *K contains some $\mathbb{Q}_p^t$-complement of $\mathbb{Q}_p^v$,*
2. *$K\mathbb{Q}_p^v = \mathbb{Q}_p^{\mathrm{alg}}$,*
3. *K is AMK.*

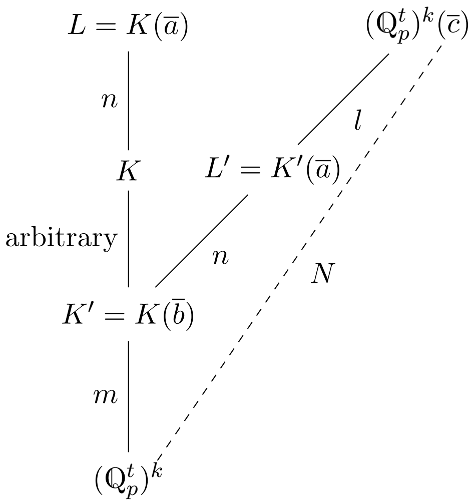
Proof.

1 ⇒ 2 This is by definition of a complement. This implication is not needed to prove the lemma since the route we're taking is 1 ⇒ 3 ⇒ 2 ⇒ 1.

1 ⇒ 3 Let $(\mathbb{Q}_p^t)^k$ be a $\mathbb{Q}_p^t$-complement of $\mathbb{Q}_p^v$ such that $(\mathbb{Q}_p^t)^k \subset K$. Let L/K be finite, and write $L = K(a)$, where $a \in \mathbb{Q}_p^{\mathrm{alg}}$. Let $\bar{b}$ be the tuple of coefficients of the minimal polynomial of a over K. Take $K' = (\mathbb{Q}_p^t)^k(\bar{b})$ and $L' = K'(a)$, then $[L' \colon K'] = [L \colon K] = n$ and $K'/(\mathbb{Q}_p^t)^k$ is finite; denote its degree by m. Finally, since a and $\bar{b}$ lie in $\mathbb{Q}_p^{\mathrm{alg}} = (\mathbb{Q}_p^t)^k\mathbb{Q}_p^v$, we might write them as elements of $(\mathbb{Q}_p^t)^k(\mathbb{Q}_p^v)$. Take $\bar{c}$ any finite tuple of $\mathbb{Q}_p^v$ containing every element of $\mathbb{Q}_p^v$ appearing in a and $\bar{b}$. Then $L' \subset (\mathbb{Q}_p^t)^k(\bar{c})$, and let N be the degree of $\bar{c}$ over $(\mathbb{Q}_p^t)^k$ and l its degree over L'. We have:

$$N = [(\mathbb{Q}_p^t)^k(\bar{c}) \colon (\mathbb{Q}_p^t)^k] = [(\mathbb{Q}_p^t)^k(\bar{c}) \colon L'][L' \colon K'][K' \colon (\mathbb{Q}_p^t)^k] = nml$$

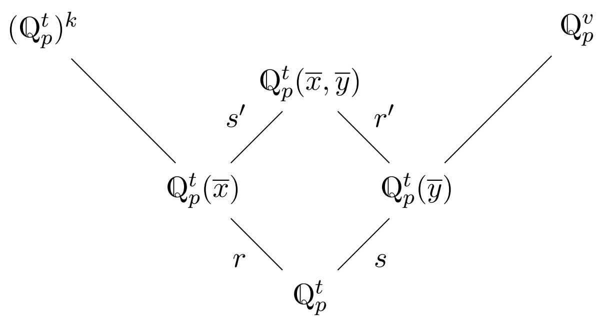
This information is compiled in the following diagram:



We will prove that N is not divisible by p, hence giving $n = [L \colon K]$ not divisible by p. We will thus have defectlessness, hence algebraic maximality, and p-divisibility of Γ; since k is already algebraically closed, this will also yield Kaplansky.

Claim. $(\mathbb{Q}_p^t)^k$ and $\mathbb{Q}_p^v$ are linearly disjoint‡over $\mathbb{Q}_p^t$.

Indeed, for any $\bar{x} \in (\mathbb{Q}_p^t)^k$ and $\bar{y} \in \mathbb{Q}_p^v$:



‡Several equivalent definitions of linear disjointness exist. Here, we say that L and M are linearly disjoint over $K \subset L \cap M$ iff any time we have $K \subset L_0 \subset L$ and $K \subset M_0 \subset M$ with $[L_0 \colon K] = l$ and $[M_0 \colon K] = m$, then $[L_0M_0 \colon M_0] = l$ and $[L_0M_0 \colon L_0] = m$.

Now we have $r = p^d$ and s coprime with p. By definition $rs' = r's$, so p^d divides $r's$, thus it divides r'. Finally, $s' \leqslant s$, giving:

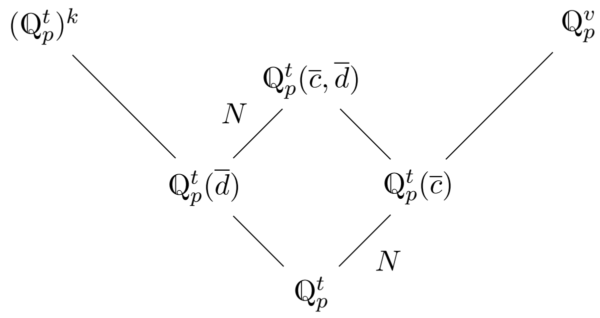
$$s' = \frac{r'}{p^d} s \leqslant s$$

Thus $s' = s$, $r' = r$, and we have linear disjointness.

Now let $\overline{d}$ be a tuple in $(\mathbb{Q}_p^t)^k$ containing coefficients of the minimal polynomials of elements in $\overline{c}$ over $(\mathbb{Q}_p^t)^k$. We know have by linear disjointness:

$$N = [(\mathbb{Q}_p^t)^k(\overline{c})\colon (\mathbb{Q}_p^t)^k] = [\mathbb{Q}_p^t(\overline{d}, \overline{c})\colon \mathbb{Q}_p^t(\overline{d})] = [\mathbb{Q}_p^t(\overline{c})\colon \mathbb{Q}_p^t]$$

So in the following diagram, we know the top N and deduce the bottom N:



Hence N correspond to an extension inside $\mathbb{Q}_p^v$. We know those extensions have degree prime to p, thus p does not divide N and K is AMK.

3 $\Rightarrow$ 2 Let K containing $\mathbb{Q}_p^t$ be AMK. Consider $L = K\mathbb{Q}_p^v$. It must have divisible value group and algebraically closed residue field, thus $\mathbb{Q}_p^{\mathrm{alg}}/L$ must be an immediate extension. Now take $a \in \mathbb{Q}_p^{\mathrm{alg}}$ and consider $L(a)/L$. It is finite and immediate, hence purely defect; so $[L(a)\colon L] = p^n$. Now consider K' which is obtained by adding to K the coefficients of the minimal polynomial of a over L. We have $[L(a)\colon L] = [K'(a)\colon K'] = p^n$, and $[K'(a)\colon K] = [K'(a)\colon K'][K'\colon K] = p^n[K'\colon K]$. But K is AMK, hence defectless; see [Kuh13, Cor. 3.11]. No finite extension of K can have degree divisible by p, thus $n = 0$ and $a \in L$; so to say, $L = \mathbb{Q}_p^{alg}$.

2 $\Rightarrow$ 1 Let K containing $\mathbb{Q}_p^t$ be big enough to have $K\mathbb{Q}_p^v = \mathbb{Q}_p^{\mathrm{alg}}$. In terms of Galois group, keeping the same notation as in section 4.2.2, we have that $H = \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{alg}}/K)$ is a closed subgroup of G^t, and the "big enough" condition on K yields $H \cap G^v = \{\mathrm{id}\}$. Recall that G^v is the unique p-Sylow of G^t. Thus H is a p'-subgroup, meaning that its order is not divisible by p.

Fact. In a prosolvable group, p'-subgroups can be extended into p'-Hall-subgroup, and the later are G-complements of p-Sylow subgroups.

This is a reformulation of known results about profinite groups, details can be found in [RZ00, sec. 2.3].

Since $G = \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{alg}}/\mathbb{Q}_p)$ is a prosolvable group[§] and G^t is a normal and closed subgroup, G^t is also prosolvable, and we can extend H into a G^t-complement of G^v; denote it by G^k and let $(\mathbb{Q}_p^t)^k = \mathrm{Fix}(G^k)$. Now $H \subset G^k$ yields $(\mathbb{Q}_p^t)^k \subset K$, and $(\mathbb{Q}_p^t)^k$ is indeed the wanted complement.[¶] □

We can now state the characterization of NIP algebraic extensions of $\mathbb{Q}_p$:

Theorem 4.2.10 (NIPity of extensions of $\mathbb{Q}_p$)**.** *The class of all NIP algebraic extensions of $\mathbb{Q}_p$ is the disjoint union of the three following classes:*

1. *finite extensions of $\mathbb{Q}_p$,*
2. *finite extensions of $\mathbb{Q}_p^t$,*
3. *algebraic extensions of $(\mathbb{Q}_p^t)^k$, where $(\mathbb{Q}_p^t)^k$ is any $\mathbb{Q}_p^t$-complement of $\mathbb{Q}_p^v$.*

Figure 4.3 shows where those NIP extensions lie compared with the usual $\mathbb{Q}_p \subset \mathbb{Q}_p^t \subset \mathbb{Q}_p^v \subset \mathbb{Q}_p^{\mathrm{alg}}$ tower. Note that this is a bit of a misdirection, since there are many possible choices of $(\mathbb{Q}_p^t)^k$, but the picture represents only 1.

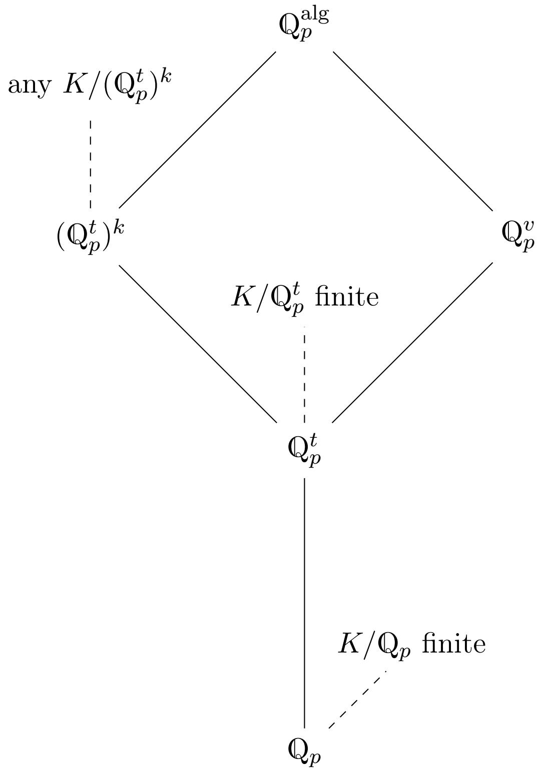


Figure 4.3: NIP algebraic extensions of $\mathbb{Q}_p$.

§This is a well-known fact, but it turns out to be quite hard to provide a good reference for it, or even to know when exactly was it first stated. An argument can be found in [HJP05, Prop. 7.2] for all p-adically closed fields, and a more elementary argument of algebraic flavour can be found in [Bos03, Cor. 3.9].

¶Most of this argument was obtained after discussing complements of p-Sylow subgroups with Tim Clausen.

4.3 State of art for some complexity classes

Remember that in any non-algebraically closed algebraic extension of $\mathbb{Q}_p$, the p-adic valuation is definable; thus its combinatorial complexity as a pure field or as a valued field is the same.

4.3.1 Complete classifications

Stable and refinements The algebraic closure $\mathbb{Q}_p^{\mathrm{alg}}$ is stable as a pure field; it is even strongly minimal. On the other hand, in any other algebraic extension of $\mathbb{Q}_p$, the valuation is definable, and the valuation witnesses the order property. Thus, there is no middle ground: it's either strongly minimal or completely unstable.

NIP and NIP$_n$ We studied extensively this case in Section 4.2, so let us summarize: an algebraic extension of $\mathbb{Q}_p$ is NIP$_n$ for some n iff it is NIP iff it is a finite extension of $\mathbb{Q}_p$, a finite extension of the inertial extension $\mathbb{Q}_p^t$, or an algebraic extension of a $\mathbb{Q}_p^t$-complement of $\mathbb{Q}_p^v$. Note that $\mathbb{Q}_p^{\mathrm{alg}}$ is strongly minimal as a pure field but NIP (and unstable) as a valued field.

dp-minimal and dp-finite We have the following theorem by Johnson, from [Joh20]:

Theorem 4.3.1. *A field is dp-finite iff it is perfect and admits a (potentially trivial) henselian valuation v which is defectless, has "almost almost divisible" value group, and has residue field either of characteristic 0 and elementarily equivalent to a local field or of characteristic p and algebraically closed, in which case we also require the value group to be roughly p-divisible.*

Furthermore, a field is dp-minimal iff all of the above holds with "almost divisible" instead of "almost almost divisible".

Here, "almost divisible" means $(\Gamma\colon n\Gamma) < \infty$ for all n, and "almost almost divisible" means:

- $(\Gamma\colon p\Gamma) < \infty$ for almost all prime p, and
- if $(\Gamma\colon p\Gamma) = \infty$ then $|\Gamma/{\sim_p}| < \infty$; where $\delta \sim_p \gamma$ means δ is p-divisible modulo Δ iff γ is also p-divisible modulo Δ, for each convex subgroup Δ.

In our case, the value group is a subgroup of $\mathrm{Div}(\mathbb{Z}) = \mathbb{Q}$, and is thus always almost divisible – and a fortiori almost almost divisible; we therefore know that an algebraic extension of $\mathbb{Q}_p$ is dp-finite iff it is dp-minimal.

Let's be more precise. Take a non-ACF algebraic extension of $\mathbb{Q}_p$: it is equipped with exactly two henselian valuations, the trivial one and the p-adic one. Consider first the trivial one: it's defectless and has residue characteristic 0, so in order to be dp-minimal, the residue field – which is the field itself – must be equivalent to a local field; that is, it must be a finite extension of $\mathbb{Q}_p$.

Now consider an infinite extension of $\mathbb{Q}_p$. Equipping it with the trivial valuation won't work, so let's consider the p-adic valuation. Its value group is always almost divisible. Its residue is of characteristic p, so it must be ACF, so the field must contain the inertial extension $\mathbb{Q}_p^t$. Its value group must be roughly p-divisible, but since it is of rank 1, it must be completely p-divisible. Finally, it must be defectless; the field is now AMK, and by the NIP classification, we know it's exactly those containing a complement of $\mathbb{Q}_p^v$, the ramified extension, over $\mathbb{Q}_p^t$.

Thus, out of the three classes of NIP extensions of $\mathbb{Q}_p$, exactly two are dp-finite and, equivalently, dp-minimal: finite extensions and extensions containing a $\mathbb{Q}_p^t$-complement of $\mathbb{Q}_p^v$.

Simple As noted in Lemma 2.4.8, a valued field is simple iff its valuation is trivial, so by definability of the p-adic valuation, no algebraic extension of $\mathbb{Q}_p$ is simple – except $\mathbb{Q}_p^{\text{alg}}$.

4.3.2 Partial results for NTP2

No complete classification of NTP2 henselian valued fields is known, but some partial results have been discussed in section 2.5.7. In particular, NTP2 transfer holds in Anscombe-Jahnke's setup. The good news is that, above $\mathbb{Q}_p$, any residue field is NTP2, since bounded PAC fields are NTP2 (even simple, see [Cha99]). Hence, we know that:

- Finitely ramified extensions of $\mathbb{Q}_p$ are NTP2,
- SAMK extensions of $\mathbb{Q}_p$ are NTP2.

Reformulating it in terms of special extensions, any field contained in $\mathbb{Q}_p^t$ is NTP2 – and so are its finite extensions. Now let $K \subset \mathbb{Q}_p^t$ be the unique algebraic extension of $\mathbb{Q}_p$ with residue $\mathbb{F}_p(p)$, the p-closure of $\mathbb{F}_p$. Consider K^k, a K-complement of $\mathbb{Q}_p^v$. Going back to the argument for the NIP case, the same proof shows that an algebraic extension of $\mathbb{Q}_p$ is SAMK iff it contains some K^k. All of these fields are thus NTP2.

The only question remaining is: did we hit all of them? Probably not; take $\mathbb{Q}_p^k$, a complement of $\mathbb{Q}_p^v$ taken directly above $\mathbb{Q}_p$. Its residue is just $\mathbb{F}_p$, whereas its value group is p-divisible; furthermore, it is defectless. It is neither SAMK nor finitely ramified, and as of yet, we do not know if it has NTP2 transfer, but it should.

On the other side, around $\mathbb{Q}_p^v$, we have a much clearer understanding. We know that if a henselian valued field is infinitely ramified, it must be roughly p-divisible; $\mathbb{Q}_p^v$ is "as ramified as possible for everthing but for p", so it is NTP2. If a field lies between $\mathbb{Q}_p^t$ and $\mathbb{Q}_p^v$, either it is a finite extension of $\mathbb{Q}_p^t$ and thus NTP2 (even dp-minimal), or it is infinitely ramified and thus TP2. An algebraic extension of $\mathbb{Q}_p^v$ will be TP2 as long as it doesn't have value group $\mathbb{Q}$; if it has, then the only thing standing between it and $\mathbb{Q}_p^{\text{alg}}$ is defect, and once more, we're uncertain.

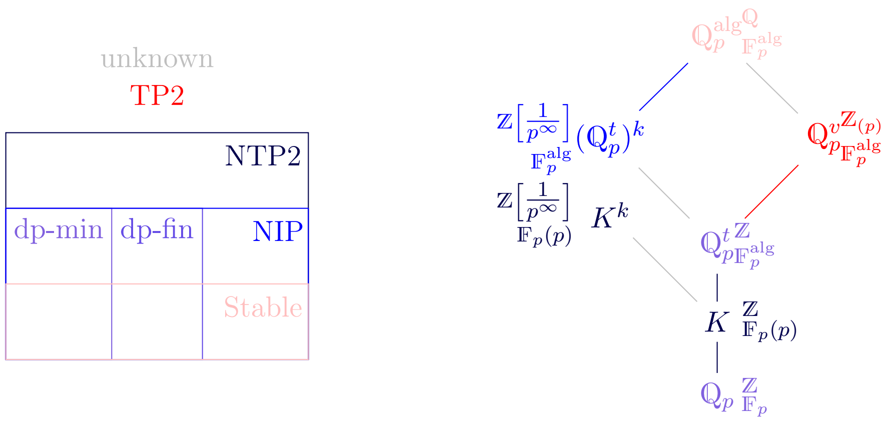


Figure 4.4: Combinatorial complexity in algebraic extensions of $\mathbb{Q}_p$.

Appendix

A Defining the canonical p-henselian valuation

In this appendix, we give an argument, as self-standing as possible, as to why the canonical p-henselian valuation is definable in some fields. This fact was first announced by Koenigsmann in [Koe95], however, his proof was incomplete. The first complete proof was given by Jahnke and Koenigsmann in [JK14]. However, their argument does rely on previous results coming from Koenigsmann's original paper, and another gap has been fixed by Chatzidakis and Perera. Because the proof is strewn about the literature, we decided to restate the argument here.

A.1 The canonical p-henselian valuation

Definition A.1. A valuation v on a field K is called *p-henselian* if it extends uniquely to the p-closure $K(p)$, which is the compositum of all p-power degree extensions of K. We also say that a field is p-henselian if it admits a non-trivial p-henselian valuation.

Similar to the henselian case, we have a number of properties equivalent to p-henselianity.

Proposition A.2 (p-Hensel's lemma)**.** *For a valued field (K, v), the following are equivalent:*

1. *v is p-henselian;*
2. *For all $P \in \mathcal{O}_v[X]$ splitting in $K(p)$, if $\overline{P}$ has a simple zero $\alpha \in k$, then P has a zero $a \in \mathcal{O}_v$ with $\overline{a} = \alpha$;*
3. *For all $P \in \mathcal{O}_v[X]$ splitting in $K(p)$, if there is $a \in \mathcal{O}_v$ such that $v(P(a)) > 2v(P'(a))$, then there is a unique $b \in \mathcal{O}_v$ with $P(b) = 0$ and $v(b - a) > v(P'(a))$;*
4. *If K^h is a henselization* of K, then $K = K^h \cap K(p)$.*

Let us now state properties derived from the study of $K(p)$:

Proposition A.3. *Let (K, v) be a valued field.*

1. *v is p-henselian iff it extends uniquely to every Galois extension of degree p;*
2. *If $K = K(p)$ (and v is non trivial), then $k = k(p)^{\mathrm{perf}}$;*
3. *If v is the composition of two valuations v_0 and v_p, then v is p-henselian iff v_0 and v_p are p-henselian.*

Now we can apply the same reasoning than for the henselian case to obtain a canonical valuation.

*See Proposition 4.2.4.

Proposition A.4. *If a field K admits two independent p-henselian valuation rings, then K is p-closed.*

Note that by Proposition A.3 any coarsening of a p-henselian valuation is still p-henselian. Now split the set H^p of all p-henselian valuation rings of K in two:

$$H_1^p = \{\mathcal{O}_v \mid v \text{ is } p\text{-henselian and } k \neq k(p)\}$$
$$H_2^p = \{\mathcal{O}_v \mid v \text{ is } p\text{-henselian and } k = k(p)\}$$

Since K itself is a p-henselian valuation ring H^p is never empty.

Proposition A.5. *Let K be a field, then H_1^p is linearly ordered by inclusion; furthermore, for any $\mathcal{O}_1 \in H_1^p$ and $\mathcal{O}_2 \in H_2^p$, we have $\mathcal{O}_2 \subset \mathcal{O}_1$.*

We can now draw the same picture as in the henselian case, see Figure 1.1. Note that henselian valuations are p-henselian, so this new tree contains the previous one.

Definition A.6. The canonical p-henselian valuation of a field K, denoted by v_K^p, is the coarsest valuation of H_2^p if H_2^p is non-empty, and the finest valuation of H_1^p if H_2^p is empty.

Proposition A.7. *It follows from the definition:*

1. *Every p-henselian valuation is comparable with v_K^p and with every coarsening of it;*
2. *v_K^p is non-trivial iff $K \neq K(p)$ and K is p-henselian;*
3. *No proper coarsening of v_K^p has p-closed residue field;*
4. *All proper refinements of v_K^p have p-closed residue field.*

Because v_K is in particular p-henselian, v_K and v_K^p are comparable, but depending on cases, their order varies.

A.2 p-henselianity is a first order property

The first step in our quest to ring-define the canonical p-henselian valuation is to show how p-henselianity can be described by first-order valued-field formulas.

Most of the results of this section were obtained by Koenigsmann in [Koe95].

Recall that by Proposition A.3 1 we only care about Galois extensions of degree p. In general, those extensions can be quite wild; but when the field is of characteristic p they are exactly of the form $K(\alpha)$, where α is a root of an Artin-Schreier polynomial $X^p - X - a$. In other characteristics, if K contains a primitive p^{th}-root of unity – which we will denote by ζ_p from now on – then all Galois extensions of degree p are of the form $K(\alpha)$ for a root of $X^p - a$. This leads to a first description of p-henselianity:

Lemma A.8. *Let (K, v) be a valued field such that $\text{ch}(k) \neq p$ and $\zeta_p \in K$, then:*

$$v \textit{ is } p\textit{-henselian} \Leftrightarrow 1 + \mathcal{M}_v \subset (K^\times)^p.$$

Proof. If v is p-henselian, take $m \in \mathcal{M}_v$ and consider $X^p - (1+m)$; it has a root by Proposition A.2 2, so $1 + \mathcal{M}_v \subset (K^\times)^p$.

Conversely, suppose $1+\mathcal{M}_v \subset (K^\times)^p$, let L/K be a Galois extension of degree p such that $L \subset K^h$, and take $w = v^h|_L$. Since $\zeta_p \in K$, $L = K(\sqrt[p]{a})$ for some $a \in K \setminus (K^\times)^p$. Now since K^h is an immediate extension, $L \subset K^h$ is also immediate, so $\Gamma_w = \Gamma_v$ and $w(\sqrt[p]{a}) = v(b)$ for some $b \in K$; we may therefore replace a by $b^{-p}a$ in order to assume $a \in \mathcal{O}_v^\times$. On the other hand $k_v = k_w$, thus $\overline{\sqrt[p]{a}} = \bar{c}$ for some $c \in K$; now we may replace a by $c^{-p}a$ and assume $a \in 1+\mathcal{M}_v \subset (K^\times)^p$, which contradicts $[L\colon K] = p$. Therefore there is no extension of degree p inside K^h, which means $K(p) \cap K^h = K$; therefore v is p-henselian. □

Lemma A.9. *Let* (K, v) *be a valued field such that* $\mathrm{ch}(K) = p$*, then:*

$$v \text{ is } p\text{-henselian} \Leftrightarrow \mathcal{M}_v \subset \wp(K)$$

where $\wp(K) = \{y \in K \mid \exists x \in K(y = x^p - x)\}$.

Proof. If v is p-henselian, take $m \in \mathcal{M}_v$ and consider $X^p - X - m$; it has a root by Proposition A.2 2, so $\mathcal{M}_v \subset \wp(K)$.

The proof of the converse direction is due to Chatzidakis and Perera in [CP17]: suppose $\mathcal{M}_v \subset \wp(K)$, let L/K be an immediate Galois extension of degree p, and take w any extension of v to L. Since $\mathrm{ch}(K) = p$, $L = K(\alpha)$ with $\alpha^p - \alpha = a \notin \wp(K)$.

Step 1: we may assume $(L, w)/(K, v)$ immediate. Since L/K is Galois, the fundamental equality Theorem 1.1.11 gives $p = [L\colon K] = ndef$, where n is the number of extensions of v to L. If $(L, w)/(K, v)$ is not immediate, then either e or f is bigger than 1, hence equal to p, thus $n = 1$ and we have p-henselianity.

Step 2: the set $C = v(K^{(p)} - a)$ admits 0 as a (strict) upper bound but has no max element. Suppose $v(x^p - x - a) > 0$ for some $x \in K$. Then, since $\mathcal{M}_v \subset \wp(K)$, we have $x^p - x - a = y^p - y$ for some $y \in K$, and thus $a = (x - y)^p - (x - y) \in \wp(K)$, which contradicts our choice of α. So 0 is a (large) upper bound of C.

Now let $b \in K$ and $\gamma = v(b^p - b - a)$. We have $w(b - \alpha) \in \Gamma_w = \Gamma_v$, so there exists $x \in K$ such that $w(b - \alpha) = v(x)$. Now $w(\frac{b-\alpha}{x}) = 0$, thus $\overline{\frac{b-\alpha}{x}} \in k_w = k_v$ and there exists $y \in K$ such that $\overline{\frac{b-\alpha}{x}} = \overline{y}$. This yields $w(\frac{b-\alpha}{x} - y) > 0$, or $w(b - xy - \alpha) > v(x) = w(b - \alpha)$. Write $c = -xy$.

We claim that $w(b - \alpha) < 0$. Indeed, if $w(b - \alpha) \geqslant 0$, then $w(b + c - \alpha) > 0$. But $w((b + c - \alpha)^p - (b + c - \alpha)) = v((b + c)^p - (b + c) - a) \in C$ can't be positive as shown before. Note: since there is nothing special about b, the same argument would work for any $z \in K$, in particular for $b + c$: $w(b + c - \alpha) < 0$.

Now $\gamma = v(b^p - b - a) = w((b - \alpha)^p - (b - \alpha)) = pw(b - \alpha)$, and $v((b + c)^p - (b + c) - a) = w((b + c - \alpha)^p - (b + c - \alpha)) = pw(b + c - \alpha) > pw(b - \alpha) = \gamma$. Thus, C can't have a max element; in particular 0 is a strict upper bound.

Step 3: we define a "good" sequence in K. Our purpose is to apply the following, which is a reformulation of [Kap42, Thm. 3]:

Fact. Let $(x_\lambda)_{\lambda<\kappa}$ be a pseudo-Cauchy sequence without pseudo-limit in K such that $(v(f(x_\lambda)))_{\lambda<\kappa}$ is stritcly increasing for some $f \in K[X]$. Let $P(X) \in K[X]$ non-constant be of minimal degree such that $(P(x_\lambda))_{\lambda<\kappa}$ admits 0 as a pseudo-limit. Then there exist an immediate extension $(K(x_\infty), \tilde{v})$ of (K, v), which checks and is uniquely determined by the conditions $P(x_\infty) = 0$ and x_∞ is a pseudo-limit of $(x_\lambda)_{\lambda<\kappa}$.

So we aim to find such a sequence, with α as a pseudo-limit. Let $(c_\lambda)_{\lambda<\kappa}$ increasingly enumerate C, and choose x_λ for each $\lambda < \kappa$ such that $c_\lambda = v(x_\lambda^p - x_\lambda - a)$. By what was done before, we know $c_\lambda = v(x_\lambda^p - x_\lambda - a) = pw(x_\lambda - \alpha)$. For all $\lambda < \mu < \kappa$, we know $c_\lambda < c_\mu$, hence:

$$v(x_\lambda - x_\mu) = w(x_\lambda - \alpha - x_\mu + \alpha) = \frac{1}{p} c_\lambda$$

So $(x_\lambda)_{\lambda<\kappa}$ is pseudo-Cauchy, and we note $\gamma_\lambda = v(x_\lambda - x_\mu) = \frac{1}{p} c_\lambda$. Furthermore, if $f(X) = X^p - X - a$, we have $v(f(x_\lambda)) = c_\lambda$ strictly increasing.

Finally, if $l \in K$ is a pseudo-limit of $(x_\lambda)_{\lambda<\kappa}$, then $v(l^p - l - a) \geqslant c_\lambda$ for all $\lambda < \kappa$. But then it is a max element of C, which can't be. So $(x_\lambda)_{\lambda<\kappa}$ does not have a pseudo-limit in K, and thus we can apply the previous fact to it. We would like to apply it while taking $P(X) = X^p - X - a$. For that, we need to show that $P(x_\lambda)_{\lambda<\kappa}$ admits 0 as pseudo-limit, and that no polynomial of smaller degree does.

Step 4: if $Q \in K_{p-1}[X]$, then $v(Q(x_\lambda))_{\lambda<\kappa}$ is eventually constant. Clearly, this is true for polynomials of degree 0 or 1. Let $1 < n < p$, suppose it is true for all polynomials of degree smaller than n, and take Q of degree n. Suppose $v(Q(x_\lambda))_{\lambda<\kappa}$ is not eventually constant. For $\lambda < \kappa$, we then have eventually:

$$v(Q(x_\lambda)) = v(Q'(\lambda)) + \gamma_\lambda = \delta' + \gamma_\lambda$$

This is a consequence of [Kap42, Lem. 8], and recall that Q' is of degree $< n$ so $\delta' = v(Q'(\lambda))$ does not depend on λ. We then write:

$$P(X) = X^p - X - a = \sum_{i=1}^{p-n} R_i(X)Q(X)^i$$

with $R_i \in K_{n-1}[X]$. Thus, $v(R_i(x_\lambda)) = \delta_i$ is eventually constant, and $v(R_i(x_\lambda)Q(x_\lambda)^i) = \delta_i + i(\delta' + \gamma_\lambda)$ eventually. Thus, eventually:

$$v(P(x_\lambda)) = v(\sum_{i=1}^{p-n} R_i(x_\lambda)Q(x_\lambda)^i) = \delta_{i_0} + i_0(\delta' + \gamma_\lambda)$$

For some $1 \leqslant i_0 \leqslant p - n$ (see [Kap42, Lem. 4]). But $v(P(x_\lambda)) = c_\lambda = p\gamma_\lambda$, so eventually $(p - i_0)\gamma_\lambda = \delta_{i_0}$, which is impossible since γ_λ is strictly increasing.

Hence, if $Q \in K_{p-1}[X]$, then $v(Q(x_\lambda))_{\lambda<\kappa}$ is eventually constant, so it can't have 0 as a pseudo-limit. On the other hand, $v(P(x_\lambda))_{\lambda<\kappa}$ is strictly increasing, thus admits 0 as a pseudo-limit; we can then apply the fact to $(x_\lambda)_{\lambda<\kappa}$ with this P. It is clear that $x_\infty = \alpha$ will work. We thus get an immediate extension $(K(a), \tilde{v})$ of (K, v). Since (L, w) checks the conditions uniquely determining $(K(a), \tilde{v})$, they must be the same. Now any other extension w' of v to L also checks those properties, hence $w' = w$ and v is p-henselian. □

Lemma A.10. *Let (K, v) be a valued field such that $\mathrm{ch}(K) = 0$, $\mathrm{ch}(k_v) = p$, $\zeta_p \in K$ and v is of rank 1, then:*

$$v \text{ is } p\text{-henselian} \Leftrightarrow 1 + p^2\mathcal{M}_v \subset (K^\times)^p.$$

Proof. If v is p-henselian, take $m \in \mathcal{M}_v$ and consider $f = X^p - (1 + p^2m)$; now $v(f(1)) = v(-p^2m) > 2v(p) = 2v(f'(1))$, so it has a root by Proposition A.2 3. Note that this also works when v is not of rank 1.

Conversely, suppose $1 + p^2\mathcal{M}_v \subset (K^\times)^p$, and take $L = K(\sqrt[p]{a})$ as before; we may assume $a \in 1 + \mathcal{M}_v$. Now consider the Cauchy completion $(\hat{K}, \hat{v})$ of (K, v) which exists since v is of rank 1. The completion is always henselian, thus (K^h, v^h) embeds uniquely in $(\hat{K}, \hat{v})$; we may therefore assume $L \subset \hat{K}$.

By density of K in $\hat{K}$, we can take $b \in K$ such that $\hat{v}(b - \sqrt[p]{a}) > v(p^2)$, so to say $b \in \sqrt[p]{a} + p^2\mathcal{M}_{\hat{v}}$. Then $b^p \in a + p^2\mathcal{M}_{\hat{v}}$, and since b and a are in K, $b^p \in a + p^2\mathcal{M}_v = a(1+p^2\mathcal{M}_v) \subset a(K^\times)^p$. This means $a \in (K^\times)^p$, and therefore $L = K$ and v is p-henselian. □

We will then combine the three cases in order to have a criterion for any (K, v) of characteristic p or containing ζ_p. The most troublesome case will be when (K, v) is of mixed characteristic $(0, p)$ with valuation of rank bigger than 1; in which case we perform the standard decomposition around p, as defined in Definition 1.1.18 and the remark following it.

Proposition A.11. *Let (K, v) be a valued field of characteristic p or containing ζ_p, then v is p-henselian iff $1 + p^2\mathcal{M}_v \subset (K^\times)^p$ and $\mathcal{M}_v \subset \wp(K) + p\mathcal{M}_v$.*

Proof. If (K, v) is not of mixed characteristic $(0, p)$, then it is an immediate consequence of the previous lemmas: when $\mathrm{ch}(k) \neq p$, $v(p) = 0$ so $p\mathcal{M}_v = \mathcal{M}_v$ and we conclude by Lemma A.8; and when $\mathrm{ch}(k) = p$, $p = 0$ so $p\mathcal{M}_v = \{0\}$ and we conclude by Lemma A.9.

If (K, v) is of mixed characteristic $(0, p)$, then we construct v_0 and v_p as above (note that $\overline{v}$ and v_p may be trivial if v is already of rank 1). Now, by composition, v is p-henselian iff $\overline{v}$, $\overline{v_0}$ and v_p are p-henselian, and by the three previous lemmas:

$$v \text{ is } p\text{-henselian } \Leftrightarrow \begin{cases} 1 + \mathcal{M}_{v_p} \subset (K^\times)^p \\ 1 + p^2\mathcal{M}_{\overline{v_0}} \subset (k_p^\times)^p \\ \mathcal{M}_{\overline{v}} \subset k_0^{(p)} \end{cases}$$

We know that if v is p-henselian, then $1 + p^2\mathcal{M}_v \subset (K^\times)^p$ by the proof of Lemma A.10. Now, since $\mathcal{M}_{v_p} \subset \mathcal{M}_{v_0} \subset \mathcal{M}_v$ and $v_p(p) = 0$, we have that:

$$1 + p^2\mathcal{M}_v \subset (K^\times)^p \Rightarrow \begin{cases} 1 + \mathcal{M}_{v_p} \subset (K^\times)^p \\ 1 + p^2\mathcal{M}_{\overline{v_0}} \subset (k_p^\times)^p \end{cases}$$

Furthermore, lifting on one way and projecting to residues on the other, we see that:

$$\mathcal{M}_{\overline{v}} \subset k_0^{(p)} \Leftrightarrow \mathcal{M}_v \subset \wp(K) + \mathcal{M}_{v_0}.$$

We thus have:

$$v \text{ is } p\text{-henselian } \Leftrightarrow \begin{cases} 1 + p^2\mathcal{M}_v \subset (K^\times)^p \\ \mathcal{M}_v \subset \wp(K) + \mathcal{M}_{v_0} \end{cases}$$

We now use a completion method to establish that $\mathcal{M}_v \subset \wp(K) + \mathcal{M}_{v_0} \Leftrightarrow \mathcal{M}_v \subset \wp(K) + p\mathcal{M}_v$: suppose $\mathcal{M}_v \subset \wp(K) + \mathcal{M}_{v_0}$ and take $a \in \mathcal{M}_v$. Let $f = X^p - X - a \in K[X]$ and let f_1, f_2 be the residues of f in k_0 and k_p. Since $a = x^p - x + m$ for some $x \in K$ and $m \in \mathcal{M}_{v_0}$, we have that f_1 has a root, and since $(k_p, \overline{v_0})$ is of rank 1, f_2 will have a root α in the completion $(\hat{k_p}, \hat{\overline{v_0}})$. We can approximate α by some $b \in k_p$ such that $\hat{\overline{v_0}}(b - \alpha) > p$. Now $b = \alpha + pm'$ for some $m' \in \mathcal{M}_{\hat{\overline{v_0}}}$, therefore:

$$\begin{aligned} b^p - b &= (\alpha - pm')^p - (\alpha - pm') \\ &= (\alpha^p - p\alpha^{p-1}pm' + \cdots + (-pm')^p) - \alpha + pm' \\ &= \alpha^p - \alpha + p(-\alpha^{p-1}pm' + \cdots + (-m')^p p^{p-1} + m') \\ &= \overline{a} + pm'' \end{aligned}$$

where $\hat{\overline{v_0}}(m'') > 0$, and since $b, \overline{a} \in k_p$, also $m'' \in k_p$. So $\overline{a} = b^p - b - pm'' \in k_p^{(p)} + p\mathcal{M}_{\overline{v_0}}$, and lifting it we have $a \in \wp(K) + p\mathcal{M}_{v_0}$. Finally $p\mathcal{M}_{v_0} \subset p\mathcal{M}_v \subset \mathcal{M}_{v_0}$, and we conclude. □

A.3 p-henselianity of a field

Definition A.12. The p-topology of a field K, denoted τ_p, is defined in the following way:

1. If $\zeta_p \in K$, τ_p is the coarsest topology for which $(K^\times)^p$ is open and all linear transformations are continuous; a subbase for τ_p is given by sets $a(K^\times)^p + b$ for $a \in (K^\times)^p$ and $b \in K$;

2. If $\mathrm{ch}(K) = p$, τ_p is the coarsest topology for which $\wp(K)$ is open and all Möbius transformations are continuous; a subbase for τ_p is given by sets $\left\{ \frac{ax+b}{cx+d} \;\middle|\; x \in \wp(K),\, x \neq -\frac{d}{c} \right\}$ for $a, b, c, d \in K$ with $ad \neq bc$.

Theorem A.13. *Let v be a non-trivial valuation on K inducing the topology τ_v, then $\tau_p = \tau_v$ iff some non-trivial coarsening w of v is p-henselian.*

In this case, τ_p admits a nice base: when $\mathrm{ch}(K) \neq p$, this base is formed by all the sets $(a(K^\times)^p + b) \cap (c(K^\times)^p + d)$ with $c, d \neq 0$; when $\mathrm{ch}(K) = p$, it is formed by all the sets $\left\{ \frac{ax+b}{cx+d} \;\middle|\; x \in \wp(K),\, x \neq -\frac{d}{c} \right\}$ with $ad \neq bc$.

Proof. Suppose $\tau_v = \tau_p$. In the case $\mathrm{ch}(K) \neq p$, then $(K^\times)^p$ must be open for τ_v, so there is a τ_v-open neighbourhood of 1 included in $(K^\times)^p$; so there is an $\mathcal{O}_v$-ideal non-trivial $\mathcal{A}$ such that $1 + \mathcal{A} \subset (K^\times)^p$, let's suppose it maximal for this property. We start with a preliminary statement:

$$b^2 \in \mathcal{A} \Rightarrow pb \in \mathcal{A}$$

Since $\mathcal{A}$ is an $\mathcal{O}_v$-ideal, $a \in \mathcal{A}$ implies $a\mathcal{O}_v \subset \mathcal{A}$, in particular any $c \in K$ such that $v(c) \geqslant v(b)$ verify $c^2 \in \mathcal{A}$, and since $1 + (-1) = 0 \notin (K^\times)^p$, we know that $\mathcal{A} \subset \mathcal{M}_v$, so $v(c) \geqslant v(b) > 0$. Now:

$$(1+c)^p = 1 + pc + \binom{p}{2} c^2 + \cdots + c^p$$

Since $v(1 + pc) = 0$, $(1 + pc)\mathcal{A} = \mathcal{A}$. Now $\binom{p}{2} c^2 + \cdots + c^p \in \mathcal{A}$, so

$$(1+c)^p \in (1+pc) + \mathcal{A} = (1+pc)(1+\mathcal{A}) \subset (1+pc)(K^\times)^p$$

Therefore $1 + pc \in (K^\times)^p$ for each $c \in b\mathcal{O}_v$, therefore $1 + pb\mathcal{O}_v \subset (K^\times)^p$, and by maximality $pb\mathcal{O}_v \subset \mathcal{A}$; this proves the statement.

If $p\mathcal{A} = \mathcal{A}$, then $\mathcal{A}$ is stable by square roots; so $\mathcal{A}$ is a radical ideal, therefore prime: if $ab \in \mathcal{A}$, suppose with $v(a) \geqslant v(b)$, then $ab^{-1} \in \mathcal{O}_v$ so $abab^{-1} = a^2 \in \mathcal{A}$, thus $a \in \mathcal{A}$. Take w the coarsening of v such that $\mathcal{M}_w = \mathcal{A}$, now $\mathrm{ch}(Kw) \neq p$: if $w(p) > 0$ then $\inf(w(\mathcal{A})) = \inf(w(p\mathcal{A})) > \inf(w(\mathcal{A}))$. This coarsening is p-henselian since $1 + \mathcal{M}_w \subset (K^\times)^p$ (see Lemma A.8).

If $p\mathcal{A} \subsetneq \mathcal{A}$, we must have $v(p) > 0$, so to say $\mathrm{ch}(k_v) = p$. Consider the coarsening w of v with $\mathcal{M}_w = \sqrt{p\mathcal{A}}$. Then $p^2\mathcal{M}_w \subset \mathcal{A}$. Indeed, take $m \in \mathcal{M}_w$, we have $m^2 \in p\mathcal{A} \subsetneq \mathcal{A}$, thus $pm \in \mathcal{A}$ and $p^2 m \in p\mathcal{A} \subsetneq \mathcal{A}$. Now using the same technique than in the proof of Proposition A.11, we can reduce to the case where w is of rank 1, and then w is p-henselian by Lemma A.10, and we are done with the case $\mathrm{ch}(K) \neq p$.

In the case $\mathrm{ch}(K) = p$, then $\wp(K)$ must be open for τ_v, so there is a τ_v-open neighbourhood of 0 included in $\wp(K)$; so there is an $\mathcal{O}_v$-ideal non-trivial $\mathcal{A}$ such that $\mathcal{A} \subset \wp(K)$, let's suppose it maximal for this property.

Now if $b^p \in \mathcal{A}$ then any c with $v(c) \geqslant b$ checks $c = c^p - (c^p - c) \in \mathcal{A} + \wp(K) = \wp(K)$. So $b\mathcal{O}_v \in \wp(K)$ and thus $b\mathcal{O}_v \in \mathcal{A}$ by maximality. So $\mathcal{A}$ is a radical ideal, hence prime, and taking $\mathcal{M}_w = \mathcal{A}$ yields w p-henselian.

For the converse, suppose w is a non-trivial p-henselian coarsening of v. Then $\tau_v = \tau_w$, so we may as well take $v = w$. Then in case $\mathrm{ch}(K) = p$, $\mathcal{M}_v \subset \wp(K) = \bigcup_{x \in \wp(K)} (x + \mathcal{M}_v)$, and in case $\mathrm{ch}(K) \neq p$, $1 + p^2\mathcal{M}_v \subset (K^\times)^p = \bigcup_{x \in (K^\times)^p} x(1 + p^2\mathcal{M}_v)$. So we have $\tau_p \subset \tau_v$.

To see that $\tau_p \subset \tau_v$, it suffices to check $\mathcal{M}_v \subset \tau_p$; and for this it suffices to find an open τ_p-neighbourhood of 0 $U \subset \mathcal{M}_v$, since then $\mathcal{M}_v = \bigcup_{x \in \mathcal{M}_v} (x + U)$.

In case $\mathrm{ch}(K) \neq p$, then we choose $a \in p^2\mathcal{M}_v \setminus K^p$, and $U = a(1-(K^\times)^p) \cap a^2(1-(K^\times)^p)$ works.

In case $\mathrm{ch}(K) = p$, then we choose $a \in K \setminus (\mathcal{M}_v \cup \wp(K))$, and $U = \left\{ \frac{a^2 x}{x+a^{-1}} \;\middle|\; x \in \wp(K) \right\}$ works.

All the details and (long) calculations can be found in the original paper [Koe95] by Koenigsmann. □

Let T_p be the theory of fields together with a sentence which says "the characteristic is p or there is a p-th root of unity".

Corollary A.14. *When $p \neq 2$, there is a first-order ring-sentence expressing the fact that a field $K \neq K(p)$, $K \vDash T_p$ is p-henselian; namely, this sentence reads "τ_p is a V-topology", which is first-order by Lemma 1.3.7.*

When $p = 2$, the sentence "τ_p is a V-topology" might not work when K is euclidean, but it still expresses p-henselianity for fields $K \neq K(2)$, $K \vDash T_2$ which are non-euclidean.

Proof. V-topologies are exactly the topologies induced by valuations or archimedean absolute values. Since we threw the euclidean case out of the window, no archimedean absolute value can exist, therefore τ_p is a V-topology iff $\tau_p = \tau_v$ for some valuation v, and by Theorem A.13 K is p-henselian.

We still have to check that this is a first-order-ring-sentence, but once again Theorem A.13 gives us a nice base for τ_p, and being a V-topology is expressible just in term of the base. All the claims above and more information on V-topologies can be found in [EP10, App. B]. □

A.4 Ring-defining v_K^p

Overview of the proof Following Jahnke and Koenigsmann in [JK14], the final step in our quest will be to exhibit a valued-field-sentence characterizing v_K^p, and apply afterwards Beth's definability theorem:

Theorem A.15 (Beth, [Hod93, Thm. 6.6.4])**.** *Let $\mathcal{L}$ be a language and T an $\mathcal{L}$-theory. Let $\mathcal{L}_P = \mathcal{L} \cup \{P\}$, where P is a new unary predicate symbol, and let $T_P \supseteq T$ be an $\mathcal{L}_P$-theory.*

If every model $\mathcal{M}$ of T can be extended uniquely to a model $\mathcal{M}_P = (\mathcal{M}, P)$ of T_P, then P is already $\mathcal{L}$-definable modulo T: there is an $\mathcal{L}$-formula φ such that if $\mathcal{M} \vDash T$, then $\varphi(M) = P^{\mathcal{M}_P}$.

Taking $\mathcal{L}_{\text{ring}}$ for $\mathcal{L}$, T_p for T and adding a new predicate symbol $\mathcal{O}_v$, we want to axiomatize the property $\mathcal{O}_v = \mathcal{O}_{v_K^p}$; we claim that this is done in the case $p \neq 2$ by the following parameter-free sentence ψ_p:

1. If $K = K(p)$ then $\mathcal{O}_v = K$, and
2. if $K \neq K(p)$ then:
   (a) $\mathcal{O}_v$ is a valuation ring of K, and
   (b) v is p-henselian, and
   (c) if $k \neq k(p)$, then k is not p-henselian, and
   (d) if $k = k(p)$, then:
      i. Γ has no non-trivial p-divisible convex subgroup, or
      ii. it has one and:
         A. $\mathrm{ch}(K) = p$ and $\forall x \in \mathcal{M}_v \setminus \{0\}$, $x^{-1}\mathcal{O}_v \not\subseteq \wp(K)$, or

B. (K, v) is of mixed characteristic and k is not perfect, or

C. (K, v) is of mixed characteristic, k is perfect and $\forall x \in \mathcal{M}_v \setminus \{0\}$, $1 + x^{-1}(\zeta_p - 1)^p \mathcal{O}_v \not\subset (K^\times)^p$.

We can already check that this is of first-order: $K = K(p)$ can be expressed by saying that $K = K^p$ or $K = \wp(K)$, p-henselianity of a valuation and a field are of first-order as seen before[†], and ring-properties of the residue field as well as ordered-group-properties of the value group can be expressed by interpretability of those structures in the valued field.

ψ_p characterizes v_K^p The next 5 lemmas will be a long series of calculations, grouping them together will yield the result.

Lemma A.16. *Let (K, v) be a valued field, let $K \vDash T_p$, and suppose:*

1. *Γ_v has no non-trivial convex p-divisible subgroup, or*

2. *$\mathrm{ch}(k_v) = p$ and k_v is not perfect.*

Then for any non-trivial proper coarsening w of v, we have $k_w \neq k_w(p)$.

Proof. Let w be a proper coarsening of v and let $\Delta < \Gamma_v$ be the corresponding non-trivial convex subgroup of Γ_v, so we have $\Gamma_w = \Gamma_v/\Delta$, and $\overline{v}\colon k_w \to \Delta$ is a valuation with residue field $k_{\overline{v}} = k_v$. We aim to find a Galois extension of k_w of degree p.

In case 1, we have $\Delta \neq p\Delta$. Thus there is $x \in k_w$ with $\overline{v}(x) \notin p\Delta$. Now if $\mathrm{ch}(k_w) \neq p$, then $\overline{\zeta_p} \neq 1$ is a p^{th}-root of unity in k_w, so $k_w[\sqrt[p]{x}]$ is a Galois extension of k_w of degree p.

On the other hand if $\mathrm{ch}(k_w) = p$, then we may assume $\overline{v}(x) < 0$ by possibly replacing x by x^{-1}. Consider the polynomial $X^p - X - x$, the roots of which can not be in k_w: if $\alpha^p - \alpha = x$, then $\overline{v}(\alpha) < 0$; therefore $\overline{v}(\alpha^p - \alpha) = p\overline{v}(\alpha) = \overline{v}(x)$. Now $k_w[\alpha]$ is a Galois extension of k_w of degree p.

In case 2, $(k_w)\overline{v} = k_v$ is not perfect. Thus we can choose some $\overline{a} \notin (k_v)^p$ and any corresponding $a \in \mathcal{O}_{\overline{v}}^\times$ is also not in $(k_w)^p$. If $\mathrm{ch}(k_w) \neq p$, then as before $k_w[\sqrt[p]{a}]$ is a Galois extension of k_w of degree p. If $\mathrm{ch}(k_w) = p$, take any $x \in \mathcal{M}_{\overline{v}}$ and consider the polynomial $X^p - X - ax^{-p}$, a root of which in k_w would satisfy $\overline{v}(\alpha) = -\overline{v}(x)$, which yields $(\alpha x)^p - a = \alpha x^p \in \mathcal{M}_{\overline{v}}$. In the residue field, we would then have $(\overline{\alpha x})^p + \overline{a}$, contradicting our choice of $\overline{a}$. Therefore any root α of the polynomial generates a Galois extension of degree p. □

Corollary A.17. *Let (K, v) be a p-henselian valued field containing ζ_p. Suppose $K \neq K(p)$, $\mathrm{ch}(k) \neq p$ and $k = k(p)$ hold, then:*

$$v = v_K^p \Leftrightarrow \Gamma \textit{ has no non-trivial } p\textit{-divisible subgroup.}$$

Proof. Right-to-left follows immediately from Lemma A.16 1: if Γ has no non-trivial p-divisible convex subgroup, then v has no proper coarsening with p-closed residue field, so to say v is the coarsest valuation with p-closed residue field; by definition, this means $v = v_K^p$.

Conversely if Γ has a non-trivial p-divisible subgroup Δ, then the corresponding coarsening w of v has p-closed residue field: take $a \in k_w$. If $\overline{v}(a) \leqslant 0$, then replace it by a^{-1}. Now if $\overline{v}(a) > 0$, then by p-divisibility of Δ, $\overline{v}(a) = p\overline{v}(b)$ for some $b \in k_w$. So replacing a by ab^{-p} if necessary, we can assume $\overline{v}(a) = 0$. But then $X^p - \overline{a}$ has a (simple) root in $k_{\overline{v}} = k$ since it is p-closed by assumption, and applying p-Hensel's lemma A.2, we get a root of $X^p - a$ in k_w. Since $\zeta_p \in K$, any Galois extension of degree p is generated by p^{th}-roots, so k_w is p-closed; therefore v is not the coarsest p-henselian valuation with p-closed residue field. □

[†]This is true only when $p \neq 2$, we will see what can be done for $p = 2$ later.

Lemma A.18. *Let (K, v) be a p-henselian valued field of equicharacteristic p with p-closed residue field. Then:*

$$v = v_K^p \Leftrightarrow \forall x \in \mathcal{M}_v \setminus \{0\}\,,\, x^{-1}\mathcal{O}_v \not\subset \wp(K).$$

Proof. If K is p-closed, then v_K^p is trivial and $\wp(K) = K$. Therefore, if $v = v_K^p$ then v is trivial and $\mathcal{M}_v = \{0\}$, so the statement on the right reads "$\forall x \in \emptyset, \ldots$" and trivially holds; now for the converse take $x \in K \setminus \{0\}$, then obviously $x^{-1}\mathcal{O}_v \subset \wp(K) = K$, so for the statement on the right side to hold, the only possibility is $\mathcal{M}_v = \{0\}$. Thus we can assume from now on $K \neq K(p)$.

From p-henselianity we can deduce that under the assumptions of the lemma, $\mathcal{O}_v \subset \wp(K)$: take $a \in \mathcal{O}_v$ and consider $X^p - X - a$, which has a root in k_v. What the statement expresses if that $v = v_K^p$ iff no proper coarsening w of v satisfy $\mathcal{O}_w \subset \wp(K)$.

First we show "$\Rightarrow$" by contradiction: suppose $\exists x \in \mathcal{M}_v \setminus \{0\}$ such that $x^{-1}\mathcal{O}_v \subset \wp(K)$. $x^{-1}\mathcal{O}_v$ is an $\mathcal{O}_v$-fractional-ideal – an $\mathcal{O}_v$-submodule I of the fraction field of $\mathcal{O}_v$ (here K) such that there exists an $a \in \mathcal{O}_v$ with $aI \subset \mathcal{O}_v$. Furthermore, if $v(y) \leqslant v(x)$, then $y^{-1}\mathcal{O}_v \subset x^{-1}\mathcal{O}_v$.

Now let $\mathcal{N} = \bigcup_{x \in A} x^{-1}\mathcal{O}_v$, where $A = \left\{x \in \mathcal{M}_v \;\middle|\; x^{-1}\mathcal{O}_v \subset \wp(K)\right\}$. We claim that $\exists a \in K$ such that $v(a) > v(A)$: if not, then $\forall x \in K$, $\exists y \in A$ such that $v(y) \geqslant v(x)$; therefore $x^{-1}\mathcal{O}_v \subset y^{-1}\mathcal{O}_v \subset \wp(K)$ and $K = \wp(K)$, so $K = K(p)$.

$\mathcal{N}$ is an $\mathcal{O}_v$-fractional-ideal since $a\mathcal{N} \subset \mathcal{O}_v$, better still, it is the maximal one such that $\mathcal{O}_v \subsetneq \mathcal{N} \subset \wp(K)$: for any $z \in \wp(K) \setminus \mathcal{N}$, take $\mathcal{Z}$ any $\mathcal{O}_v$-fractional-ideal containing z. It must contain $z\mathcal{O}_v$, which is not contained in $\wp(K)$ since $z \notin \mathcal{N}$, so $\mathcal{Z} \not\subset \wp(K)$.

Let Δ be the convex hull of the subgroup of Γ generated by $v(\mathcal{N} \setminus \mathcal{O}_v)$:

- Δ is non-trivial by assumption.
- Any $\gamma \in v(\mathcal{N} \setminus \mathcal{O}_v)$ is p-divisible in Γ: take $x \in \mathcal{N}$ such that $v(x) = \gamma$, now since $\mathcal{N} \subset \wp(K)$, there is a $y \in K$ such that $y^p - y = x$. We have $v(x) = v(y^p - y) < 0$, so $pv(y) = v(x) = \gamma$.
- Δ is p-divisible: let $\delta \in \Delta$, assume $\delta < 0$. By definition there are a finite number of $n_i \in \mathbb{Z}$, $\alpha_i \in v(\mathcal{N} \setminus \mathcal{O}_v)$ such that:

  $$\sum n_i \alpha_i \leqslant \delta < 0$$

  Take $\alpha = \min(\alpha_i)$ and $n = \sum n_i$; now $n\alpha \leqslant \delta < 0$. δ lies in exactly one interval of the form $[(k+1)\alpha, k\alpha[$, therefore for some k, $\beta = \delta - k\alpha \in [\alpha, 0[$. Now since $\alpha \in v(\mathcal{N})$ and $\alpha \leqslant \beta$, also $\beta \in v(\mathcal{N})$. By what we've seen, both α and β are p-divisible in Γ, and $\delta = \beta + k\alpha$ as well. Since Δ is convex, $\frac{\delta}{p} \in \Delta$.

Now we assume $v = v_K^p$ and aim towards a contradiction.

Since any coarsening of v has non p-closed residue field, $\mathcal{N}$ does not contain any coarsening of $\mathcal{O}_v$: if $\mathcal{O}_w \subset \mathcal{N} \subset \wp(K)$, then $X^p - X - a$ has a root in K for any $a \in \mathcal{O}_w$, so it has a root in $\mathcal{O}_w$ since valuation rings are integrally closed, and therefore $X^p - X - \overline{a}$ has a root in k_w; thus $k_w(p) = k_w$ and w cannot be a proper coarsening of v.

We claim that Δ is of rank 1: take any $\{0\} < \mathcal{E} < \Delta$ convex, and let w be the associated proper coarsening of v. We know that $\mathcal{O}_w \not\subset \mathcal{N}$, so there is $z \in \mathcal{O}_w \setminus \mathcal{N}$, in particular, $z \notin \mathcal{O}_v$. Suppose there exists $x \in \mathcal{M}_w$ such that $0 < v(x) \leqslant v(z^{-1})$, then $z^{-1}x^{-1} \in \mathcal{O}_v \subset \mathcal{O}_w$. But now $x^{-1} \in z\mathcal{O}_w \subset \mathcal{O}_w$, which contradicts the choice of $x \in \mathcal{M}_w$. This means that $v(z) \in \mathcal{E} = \{\gamma \in \Gamma \mid 0 \leqslant \pm\gamma < v(x)\, \forall x \in \mathcal{M}_w\}$. Since $z \notin \mathcal{N}$, we know that $v(z) < v(y) < 0$ for any $y \in \mathcal{N} \setminus \mathcal{O}_v$, so $v(\mathcal{N} \setminus \mathcal{O}_v) \subset \mathcal{E}$; and by definition of Δ, we have $\Delta \subset \mathcal{E}$, which contradicts our choice of $\mathcal{E}$.

Thus, Δ is of rank 1, which means it must embed in $\mathbb{R}$, and we can fix $\alpha \in \mathbb{R}$ and consider the following set:

$$\mathcal{N}_\alpha = \{x \in K \mid v(x) \geqslant \alpha v(y) \text{ for some } y \in \mathcal{N}\}$$

It is an $\mathcal{O}_v$-fractional ideal which strictly contains $\mathcal{N}$ if $\alpha > 1$:

- Let $x \in \mathcal{N}$: if $v(x) \geqslant 0$ then $v(x) \geqslant \alpha v(1)$, and if $v(x) < 0$ then $v(x) \geqslant \alpha v(x)$.
- Since $v(\mathcal{N} \setminus \mathcal{O}_v) \subset \Delta$, we know that $\gamma = \inf(v(\mathcal{N}))$ exists in $\mathbb{R}$. We also know that Δ is a p-divisible subgroup of $\mathbb{R}$, therefore it must be dense. This means that the interval $]\alpha\gamma, \gamma[$ contains an element of Δ, which is of the form $v(x)$. Now $x \in \mathcal{N}_\alpha$ but $x \notin \mathcal{N}$.
- $\mathcal{N}_\alpha$ is clearly an $\mathcal{O}_v$-module, and any $b \in \mathcal{O}_v$ such that $v(b) > -\alpha\gamma$ will verify $b\mathcal{N} \subset \mathcal{O}_v$. Such a b exists since Δ is dense.

Recall that by construction $\mathcal{N}$ is the maximal $\mathcal{O}_v$-fractional ideal such that $\mathcal{N} \subset \wp(K)$. To get a contradiction, we take $\alpha \in]1, 2 - \frac{1}{p}[$ and prove that $\mathcal{N}_\alpha \subset \wp(K)$:

Let $z \in \mathcal{N}_\alpha \setminus \mathcal{N}$, so there is $y \in \mathcal{N}$ such that $v(y) > v(z) \geqslant \alpha v(y)$. Note that $\alpha v(y) < 0$, so $v(y) < 0$. Now $0 > v(zy^{-1}) \geqslant (\alpha - 1)v(y) > v(y)$ since $\alpha < 2$. This means $zy^{-1} \in \mathcal{N} \setminus \mathcal{O}_v$, so $v(zy^{-1}) \in \Delta$ is p-divisible: $v(zy^{-1}) = v(a^p)$, thus $v(za^{-p}) = v(y)$, which means $za^{-p} \in \mathcal{N}$. Finally, since $\mathcal{N} \subset \wp(K)$, there is $b \in K$ such that $b^p - b = za^{-p}$, and we can write $z = (ab)^p - a^p b$, and we have:

$$\begin{aligned} v(a^p b) &= v(a^p) + v(b) \\ &= v(zy^{-1}) + \tfrac{1}{p} v(b^p) \\ &= v(z) - v(y) + \tfrac{1}{p} v(za^{-p}) \\ &= v(z) - v(y) + \tfrac{1}{p} v(y) \\ &\geqslant (\alpha - 1 + \tfrac{1}{p}) v(y) \\ &\geqslant v(y) \in \mathcal{N} \end{aligned}$$

Therefore $a^p b \in \mathcal{N} \subset \wp(K)$, now since $v(ab) > v(a^p b)$ also $ab \in \mathcal{N} \subset \wp(K)$, and $z = (ab)^p - ab + ab - a^p b$ is a sum of elements of $\wp(K)$ which is stable by addition: $z \in \wp(K)$, so $\mathcal{N}$ can't be maximal.

Lastly, we prove "$\Leftarrow$" by contraposition: suppose $v \neq v_K^p$, then by definition v_K^p is a proper coarsening of v with p-closed residue field. As done before for v, $\mathcal{O}_{v_K^p} \subset \wp(K)$. Now take any $x \in \mathcal{M}_v \setminus \mathcal{M}_{v_K^p}$, in particular $x \in \mathcal{O}_{v_K^p}^\times$ and:

$$x^{-1}\mathcal{O}_v \subset x^{-1}\mathcal{O}_{v_K^p} = \mathcal{O}_{v_K^p} \subset \wp(K)$$

which means $\exists x \in \mathcal{M}_v \setminus \{0\}, x^{-1}\mathcal{O}_v \subset \wp(K)$. □

The cases $\mathrm{ch}(k) \neq p$ and equicharacteristic p have been taken care in the previous lemmas, but the most tedious case of mixed characteristic $(0, p)$ is yet to be dealt with; this will require two lemmas.

Lemma A.19 (Koenigsmann, [Koe99, lemma 3.2])**.** *Let (K, v) be a p-henselian valued field of mixed characteristic $(0, p)$ containing ζ_p. Then for any $a \in \mathcal{O}_v$ we have:*

$$1 + (1 - \zeta_p)^p a \in (K^\times)^p \Leftrightarrow \exists x \in k,\ x^p - x - \overline{a} = \overline{0}$$

Proof. The proof relies on a good choice of polynomial: if we have $f(X)$ such that $1+(1-\zeta_p)^p a \in (K^\times)^p$ iff f has a zero in K and such that $\overline{f}(X) = X^p - X - \overline{a}$, the lemma holds by p-hensel's lemma.

We claim that the following polynomial is a good choice:

$$f(X) = \left(X + \frac{1}{1-\zeta_p}\right)^p - \left(\frac{1}{(1-\zeta_p)^p} + a\right)$$

Now $f(\alpha) = 0 \Leftrightarrow (1-\zeta_p)^p f(\alpha) = 0 \Leftrightarrow (\alpha(1-\zeta_p)+1)^p = 1+(1-\zeta_p)^p a$. In order to obtain $\overline{f}$, we need to calculate coefficients of f:

$$\begin{aligned} f(X) &= \sum_{k=0}^{p} \left[\binom{p}{k} X^k \frac{1}{(1-\zeta_p)^{p-k}}\right] - \frac{1}{(1-\zeta_p)^p} - a \\ &= X^p + \sum_{k=2}^{p-1} \left[\frac{(p-1)!}{(p-k)!k!} X^k \frac{p}{(1-\zeta_p)^{p-1}} (1-\zeta_p)^{k-1}\right] + \frac{p}{(1-\zeta_p)^{p-1}} X - a \end{aligned}$$

It is still unclear what the residue of f is but believe it or not, we are almost here. First note that $\overline{\zeta_p} = \overline{1}$ since 1 is the only root of unity in characteristic p. Let $g(X)$ be the minimal polynomial of ζ_p over $\mathbb{Q}$: $g(X) = X^{p-1} + \cdots + 1 = \prod_{k=1}^{p-1}(1-\zeta_p^k)$. Now:

$$\begin{aligned} p = g(1) &= (1-\zeta_p)(1-\zeta_p^2)\cdots(1-\zeta_p^{p-1}) \\ \frac{p}{(1-\zeta_p)^{p-1}} &= \frac{1-\zeta_p}{1-\zeta_p} \times \frac{1-\zeta_p^2}{1-\zeta_p} \times \cdots \times \frac{1-\zeta_p^{p-1}}{1-\zeta_p} \\ &= (1+\zeta_p)(1+\zeta_p+\zeta_p^2)\cdots(1+\zeta_p+\zeta_p^2+\cdots+\zeta_p^{p-2}) \end{aligned}$$

Therefore $\frac{p}{(1-\zeta_p)^{p-1}}$ has residue $2\times 3\times\cdots\times p-1 = (p-1)!$, but since $\mathrm{ch}(k) = p$, $(p-1)! = -1$ in k. This implies $v(p) = (p-1)v(1-\zeta_p) > 0$, and we can look again at the coefficients of f:

- for $2 \leqslant k < p$, $v(\frac{(p-1)!}{(p-k)!k!}) \geqslant 0$ since it is an integer;
- $v(\frac{p}{(1-\zeta_p)^{p-1}}) = 0$;
- for $k \geqslant 2$, $(k-1)v(1-\zeta_p) = \frac{k-1}{p-1}v(p) > 0$.

Therefore coefficients in front of X^2, X^3, ..., X^{p-1} all have positive valuation and are consequently null in the residue field. Since the coefficient in front of X is $\frac{p}{(1-\zeta_p)^{p-1}}$ which has residue -1, we have $\overline{f}(X) = X^p - X - \overline{a}$ and we conclude. □

We now prove a final lemma very similar to Lemma A.18 but for the mixed characteristic case:

Lemma A.20. *Let (K, v) be a p-henselian valued field of mixed characteristic $(0, p)$ containing ζ_p with residue field perfect and p-closed, and with no non-trivial convex p-divisible subgroup of its value group. Then:*

$$v = v_K^p \Leftrightarrow \forall x \in \mathcal{M}_v \setminus \{0\}\,,\, 1 + x^{-1}(\zeta_p - 1)^p \mathcal{O}_v \not\subset (K^\times)^p$$

Proof. Once again, if $K = K(p)$ then v_K^p is trivial, and if $v = v_k^p$ then $\mathcal{M}_v = \{0\}$ and the statement on the right side reads "$\forall x \in \emptyset, \ldots$" and holds. Conversely, if v is non-trivial, then $p \in \mathcal{M}_v \setminus \{0\}$ and $v(p) < \frac{p}{p-1}v(p) = v((\zeta_p - 1)^p)$; so to say $p^{-1}(\zeta_p-1)^p\mathcal{O}_v \subset \mathcal{M}_v \subset K \setminus \{-1\}$, and $1 + p^{-1}(\zeta_p - 1)^p \mathcal{O}_v \subset K^\times = (K^\times)^p$. We can thus assume $K \neq K(p)$.

"⇒": We assume $v = v^p_K$. Consider first the case where there exists a proper coarsening w of v such that $\mathrm{ch}(k_w) = p$. Since $v = v^p_K$, any proper coarsening $\overline{u}$ of $\overline{v}$ lifts to a proper coarsening u of v, and $k_{\overline{u}} = k_u$ is not p-closed by definition of v^p_K. Likewise, any proper refinement of $\overline{v}$ has p-closed residue field. We then have (and this holds for any field K and valuations $\mathcal{O}_v \subset \mathcal{O}_w$):

$$v = v^p_K \Rightarrow \overline{v} = v^p_{k_w}$$

In our case, $(k_w, \overline{v})$ is a valued field of equicharacteristic p with p-closed residue field, so we can apply Lemma A.18 to it:

$$\overline{v} = v^p_{k_w} \Rightarrow \forall \overline{x} \in \mathcal{M}_{\overline{v}} \setminus \{0\}\,,\ \overline{x}^{-1}\mathcal{O}_{\overline{v}} \not\subset k_w w^{(p)}$$

Given a $x \in \mathcal{M}_v \setminus \mathcal{M}_w$, we then know that for some $\overline{a} \in \mathcal{O}_{\overline{v}}$, $\overline{x}^{-1}a \notin k_w^{(p)}$, and by Lemma A.19 we have $1 + x^{-1}(1-\zeta_p)^p a \notin (K^\times)^p$. Doing this with $x = 1$ gives us an $a \in \mathcal{O}_v$ such that $1 + (1-\zeta_p)^p a \notin (K^\times)^p$. Finally, for $x \in \mathcal{M}_w \setminus \{0\}$, we have $1 + x^{-1}(\zeta_p - 1)^p xa \notin (K^\times)^p$, with $xa \in \mathcal{O}_v$; parsing everything together, we have:

$$v = v^p_K \Rightarrow \forall x \in \mathcal{M}_v \setminus \{0\}\,,\ 1 + x^{-1}(\zeta_p - 1)^p \mathcal{O}_v \not\subset (K^\times)^p$$

Thus in the case where a coarsening of v has residue characterisitc p, the proof of left-to-right is done.

Assume now that all coarsenings of v have residue characteristic 0. Then, we claim that $1 + \mathcal{M}_v \not\subset (K^\times)^p$: consider the coarsening w of v corresponding to the maximal convex p-divisible subgroup of Γ, which is non-trivial by assumption. We know that $\mathrm{ch}(k_w) = 0$, $\overline{v}$ is p-henselian, has p-divisible value group and perfect residue field. If $1 + \mathcal{M}_v \subset (K^\times)^p$, then also $1 + \mathcal{M}_{\overline{v}} \subset (k_w^\times)^p$. Note also that $\overline{\zeta_p} \neq \overline{1} \in k_w$, since otherwise the calculation of $w(\zeta_p - 1)$ in the proof of Lemma A.19 would yield $w(\zeta_p - 1) = \frac{w(p)}{p-1} = 0$, contradicting $\zeta_p - 1 \in \mathcal{M}_w$. Take now $a \in k_w$, by p-divisibility of $\Gamma_{\overline{v}}$ we can find $b \in k_w$ such that $\overline{v}(ab^{-p}) = 0$. Since $k_{\overline{v}}$ is perfect, $\overline{ab^{-p}} = \overline{c}^p$ for some $\overline{c} \in k_v$, and lifting it we have $ab^{-p} \in c^p(1 + \mathcal{M}_{\overline{v}})$ and thus $a \in k_w^p$. This means that w is a proper coarsening of v with p-closed residue field, contradicting $v = v^p_K$.

Now, we assume the following and aim for a contradiction:

$$\exists x \in \mathcal{M}_v,\ 1 + x^{-1}(\zeta_p - 1)^p \mathcal{O}_v \subset (K^\times)^p$$

As before, $v(y) \leqslant v(x)$ implies $1 + y^{-1}(\zeta_p - 1)^p \mathcal{O}_v \subset (K^\times)^p$, and we can define $\mathcal{N} = \bigcup_{x \in A} x^{-1}\mathcal{O}_v$, where $A = \left\{x \in \mathcal{M}_v \;\middle|\; 1 + x^{-1}(\zeta_p - 1)^p \mathcal{O}_v \subset (K^\times)^p\right\}$.

Let $a = -(\zeta_p - 1)^p$. Now $1 + a^{-1}(\zeta_p - 1)^p = 0 \notin (K^\times)^p$, therefore $a \notin A$ and any $x \in K$ with $v(x) \geqslant v(a)$ is also not in A. Thus any $y \in \mathcal{N}$ has value $v(y) > v(a^{-1})$, so $a\mathcal{N} \subset \mathcal{M}_v \subset \mathcal{O}_v$ and $\mathcal{N}$ is an $\mathcal{O}_v$-fractional-ideal.

Furthermore, $\mathcal{N}$ is the maximal $\mathcal{O}_v$-fractional-ideal such that $1 + (\zeta_p - 1)^p \mathcal{N} \subset (K^\times)^p$: for any $z \in (K^\times)^p \setminus \mathcal{N}$, take any $\mathcal{O}_v$-fractional-ideal $\mathcal{Z}$ containing it. $\mathcal{Z}$ must contain $z\mathcal{O}_v$, but $1 + (\zeta_p - 1)^p z\mathcal{O}_v \not\subset (K^\times)^p$ since $z^{-1} \notin A$, so $1 + (\zeta_p - 1)^p \mathcal{Z} \not\subset (K^\times)^p$.

Note also that since $1 + \mathcal{M}_v \not\subset (K^\times)^p$, we have $\mathcal{N} \subsetneq a^{-1}\mathcal{M}_v = (\zeta_p - 1)^{-p}\mathcal{M}_v$.

Let Δ be the convex hull of the subgroup of Γ generated by $v(\mathcal{N} \setminus \mathcal{O}_v)$:

- Δ is non-trivial by assumption.
- Any $\gamma \in v(\mathcal{N} \setminus \mathcal{O}_v)$ is p-divisible: take $x \in \mathcal{N}$ such that $v(x) = \gamma < 0$, then since $1 + (\zeta_p - 1)^p \mathcal{N} \subset (K^\times)^p$ we have $1 + (\zeta_p - 1)^p x = a^p$ for some $a \in K$, and since

$\mathcal{N} \subsetneq (\zeta_p - 1)^{-p}\mathcal{M}_v$ we have $\overline{a}^p = 1 + \overline{(\zeta_p - 1)^p x} = 1 \in k$ and thus $\overline{a} = 1$ since $\mathrm{ch}(k) = p$. Hence, for some $b \in \mathcal{M}_v$:

$$1 + (\zeta_p - 1)^p x = (1+b)^p = \sum_{i=0}^{p} \binom{i}{p} b^k$$

Recall that $v(\zeta_p - 1) = \frac{v(p)}{p-1}$, therefore:

$$\begin{aligned}
\min_{i=1,p}\left(\binom{p}{i} i v(b)\right) \leqslant v\left(\sum_{i=1}^{p}\binom{p}{i} b^i\right) &= v\left(x(\zeta_p - 1)^p\right) \\
&= v(x) + p\frac{v(p)}{p-1} < p\frac{v(p)}{p-1}
\end{aligned}$$

$$\begin{aligned}
\min_{i=1,p}\left(\frac{p!}{i!(p-i)!} i v(b)\right) &< p\frac{v(p)}{p-1} \\
\min_{i=1,p}\left(\frac{(p-1)!}{(i-1)!(p-i)!} v(b)\right) &< \frac{v(p)}{p-1} \\
v(b) = \min_{i=1,p}\left(\binom{p-1}{i-1} v(b)\right) &< \frac{v(p)}{p-1}
\end{aligned}$$

This then yields $v(b^p) < \frac{p}{p-1}v(p) < v(p)$, therefore $v(b^p) < v(\binom{p}{i}v(b^i))$ since p divides the cofficient. Thus $v(x(\zeta_p - 1)^p) = v(b^p)$, which means γ is p-divisible.

- Δ is p-divisible: the argument in the proof of Lemma A.18 actually shows that any convex hull of a subgroup generated by a set is p-divisible as soon as the set of generators is p-divisible.

$\mathcal{N}$ does not contain any proper coarsening of v: suppose $\mathcal{O}_v \subsetneq \mathcal{O}_w \subset \mathcal{N}$. We know that $\mathrm{ch}(k_w) = 0$, therefore $w(\zeta_p - 1) = 0$, and:

$$1 + \mathcal{M}_v \subset 1 + \mathcal{O}_w = 1 + (\zeta_p - 1)^p \mathcal{O}_w \subset 1 + (\zeta_p - 1)^p \mathcal{N} \subset (K^\times)^p$$

Which as seen before contradicts $v = v_K^p$.

Following the proof of Lemma A.18, we have $\Delta \leqslant \mathbb{R}$ and for $1 < \alpha \in \mathbb{R}$, the following is an $\mathcal{O}_v$-fractional ideal stricly containing $\mathcal{N}$:

$$\mathcal{N}_\alpha = \{x \in K \mid v(x) \geqslant \alpha v(y) \text{ for some } y \in \mathcal{N}\}$$

Recall that $(\zeta_p - 1)^p \mathcal{N} \subsetneq \mathcal{M}_v$. If $v((\zeta_p - 1)^{-p}) = \inf(v(\mathcal{N}))$, then any $x \in K$ with $v(x) > v((\zeta_p - 1)^{-p})$ would be in $\mathcal{N}$, and therefore $(\zeta_p - 1)^p \mathcal{N}$ would equal $\mathcal{M}_v$. Therefore we take α such that $\alpha > 1$, $\alpha < 2 - \frac{1}{p}$ and $\alpha < \frac{v((\zeta_p - 1)^{-p})}{\inf(v(\mathcal{N}))}$, the later being strictly bigger than 1. This yields $\mathcal{N}_\alpha \subset (\zeta_p - 1)^{-p}\mathcal{M}_v$, and we aim to contradict the maximality of $\mathcal{N}$ by proving $1 + (\zeta_p - 1)^p \mathcal{N}_\alpha \subset (K^\times)^p$.

Let $z \in \mathcal{N}_\alpha \setminus \mathcal{N}$, so there is some $y \in \mathcal{N}$ with $0 > v(y) > v(z) \geqslant \alpha v(y)$. Then $0 > v(zy^{-1}) \geqslant (\alpha - 1)v(y) > v(y)$, thus $zy^{-1} \in \mathcal{N} \setminus \mathcal{O}_v$. Therefore by p-divisibility of Γ there is $a \in K \setminus \mathcal{O}_v$ such that $v(zy^{-1}) = v(a^p)$, which gives $v(za^{-p}) = v(y)$, so $za^{-p} \in \mathcal{N} \setminus \mathcal{O}_v$, and there is $b \in \mathcal{M}_v$ such that:

$$\begin{aligned}
1 + za^{-p}(\zeta_p - 1)^p &= (1+b)^b \\
z(\zeta_p - 1)^p &= a^p(b^p + \cdots + pb) \\
1 + z(\zeta_p - 1)^p &= 1 + (ab)^p + \cdots + pa^p b
\end{aligned}$$

and as before $v(b^p) = v(za^{-p}(\zeta_p - 1)^p)$. Note also that $z(\zeta_p - 1)^p \in \mathcal{M}_v$ thanks to our choice of α, so $(ab)^p \in \mathcal{M}_v$ and ab also. We will first finish the proof modulo the following claim:

Claim.

$$pa^pb \in (\zeta_p - 1)^p \mathcal{N}.$$

This implies the following:

$$\begin{aligned}
1 + z(\zeta_p - 1)^p &= 1 + (ab)^p + \cdots + pa^pb \\
&= (1+ab)^p \underbrace{-pab - \cdots - p(ab)^{p-1}}_{\in \mathcal{M}_v} + \underbrace{pa^pb^{p-1} + \cdots + pa^pb}_{\in pa^pb\mathcal{O}_v} \\
&\in (1+ab)^p + \mathcal{M}_v + pa^pb\mathcal{M}_v \subset (1+ab)^p + pa^pb\mathcal{M}_v \\
&\qquad \subset (1+ab)^p + (\zeta_p - 1)^p \mathcal{N}\mathcal{M}_v \\
&\qquad \subset (1+ab)^p + (\zeta_p - 1)^p \mathcal{N} \\
&\qquad \subset (1+ab)^p (1 + (\zeta_p - 1)^p \mathcal{N}) \\
&\qquad \subset (K^\times)^p
\end{aligned}$$

Since this holds for any $z \in \mathcal{N}_\alpha$, we have $1+(\zeta_p-1)^p\mathcal{N}_\alpha \subset (K^\times)^p$, contradicting the minimality of $\mathcal{N}$.

Remains the claim to prove. Recall that we have:

$$v(b^p) = v(za^{-p}(\zeta_p - 1)^p)$$

We first aim to prove $a^pb \in (\zeta_p - 1)\mathcal{N}$. Since $y \in \mathcal{N}$, we just need to show $v(a^pb(\zeta_p - 1)^{-1}) > v(y)$:

$$\begin{aligned}
v(a^pb(\zeta_p - 1)^{-1}) &= pv(a) + v(b) - v(\zeta_p - 1) \\
&= pv(a) + \frac{1}{p}v(z) - v(a) \\
&= (p-1)v(a) + \frac{1}{p}v(z)
\end{aligned}$$

We chose y, z and α such that $0 > v(y) > v(z) \geqslant \alpha v(y) > (2 - \frac{1}{p})v(y)$. This yields $v(z^p) > v(y^{2p-1})$, and from that:

$$\begin{aligned}
v((zy^{-1})^{p-1}z) = v(y^{1-p}) + v(z^p) &> v(y^p) \\
v((a^p)^{p-1}z) &> v(y^p) \\
(p-1)v(a^p) + v(z) &> pv(y) \\
(p-1)v(a) + \frac{1}{p}v(z) &> v(y)
\end{aligned}$$

Hence we have $a^pb \in (\zeta_p - 1)\mathcal{N}$. Finally, we write $a^pb = (\zeta_p - 1)x$ for some $x \in \mathcal{N}$. Now:

$$\begin{aligned}
pa^pb &= p(\zeta^p - 1)^p(\zeta^p - 1)^{p-1}x \\
v(pa^pb(\zeta^p - 1)^{-p}) &= v(p) - (p-1)v(\zeta^p - 1) + v(x) \\
&= v(p) - (p-1)\frac{v(p)p-1}{+}v(x) \\
&= v(x)
\end{aligned}$$

Thus $pa^pb(\zeta^p - 1)^{-p} \in \mathcal{N}$ and the claim is proven. $\square_\Rightarrow$

"$\Leftarrow$": We assume $\forall x \in \mathcal{M}_v \setminus \{0\}\,,\, 1 + x^{-1}(\zeta_p - 1)^p\mathcal{O}_v \not\subset (K^\times)^p$, so $\forall x \in \mathcal{M}_v$ there is $a \in \mathcal{O}_v$ such that $1 + (\zeta_p)x^{-1}a \notin (K^\times)^p$. We first suppose that there exists a proper coarsening w of v with residue characteristic p. Then, Lemma A.19 tells us that $\forall x \in \mathcal{M}_v \setminus \mathcal{M}_w$, $1 + (\zeta_p)x^{-1}a \notin (K^\times)^p \Rightarrow \overline{x}^{-1}\overline{a} \notin k_w^{(p)}$, so we have:

$$\forall \overline{x} \in \mathcal{M}_w \setminus \{0\}\,,\, \overline{x}^{-1}\mathcal{O}_{\overline{v}} \not\subset k_w^{(p)}$$

Now by Lemma A.18 applied to $(k_w, \overline{v})$, we have $\overline{v} = v^p_{k_w}$, which then yields $v = v^p_K$.

Therefore we can assume that all proper coarsenings of v have residue characteristic 0. Let w be a proper coarsening of v. Since $p \in \mathcal{M}_v$, there is $a \in \mathcal{O}_v$ such that $1+\frac{1}{p}(\zeta_p-1)^p a \notin (K^\times)^p$. But $p \notin \mathcal{M}_w$ since w has residue characteristic 0, so $\mathcal{O}_v[\frac{1}{p}] \subset \mathcal{O}_w$, so $\frac{1}{p}(\zeta_p - 1)^p a \in \mathcal{O}_w$. But by p-hensel's lemma A.2, we have $1 + \mathcal{M}_w \subset (K^\times)^p$, and therefore $\frac{1}{p}(\zeta_p - 1)^p a \notin \mathcal{M}_w$. Taking the residue, $\overline{\frac{1}{p}(\zeta_p - 1)^p a} \neq 0 \in k_w$, and it cannot have a p^{root}, otherwise we would lift it to K. This means $k_w \neq k_w(p)$, so v is the coarsest valuation with p-closed residue field: $v = v^p_K$. $\square$

We now have to parse every result together to understand why ψ_p characterizes v^p_K, meaning that if $K \vDash T_p$, then $(K, v) \vDash \psi_p$ iff $v = v^p_K$. Let us go through the sentence step by step:

1. If $K = K(p)$ then $\mathcal{O}_v = K$

In the case where K is p-closed, $(K, v) \vDash \psi_p$ iff v is trivial iff $v = v^p_K$.

2. And if $K \neq K(p)$ then:

   (a) $\mathcal{O}_v$ is a valuation ring of K, and

   (b) v is p-henselian, and

   (c) if $k \neq k(p)$, then k is not p-henselian

In the case where K and k are not p-closed, then v^p_K is a refinement of v, and in k, $\overline{v^p_K} = v^p_k$, thus $v = v^p_K$ iff $\overline{v^p_K}$ is trivial iff k is not p-henselian (since it is not p-closed).

   (d) And if $k = k(p)$, then:

      i. Γ has no non-trivial p-divisible convex subgroup

By Corollary A.17, in the case where $\mathrm{ch}(k) \neq p$ we are done, so we can restrict ourselves to the case $\mathrm{ch}(k) = p$ in the next statement (note that this is a disjunction).

      ii. Or it has one and:

         A. $\mathrm{ch}(K) = p$ and $\forall x \in \mathcal{M}_v \setminus \{0\}\,,\, x^{-1}\mathcal{O}_v \not\subset \wp(K)$

This is the equicharacteristic p case, handled in Lemma A.18.

         B. Or (K, v) is of mixed characteristic and k is not perfect

This is thanks to Lemma A.16 2. This is not an equivalence: if k is not perfect (and all the previous assumptions) then $v = v^p_K$. Now if $v = v^p_K$, then either k is not perfect, or it is perfect and we fall in the next (and last) clause:

         C. Or (K, v) is of mixed characteristic, k is perfect and $\forall x \in \mathcal{M}_v \setminus \{0\}\,,\, 1 + x^{-1}(\zeta_p - 1)^p\mathcal{O}_v \not\subset (K^\times)^p$.

This is the mixed characteristic case, handled in Lemma A.20.

A.5 When $p = 2$

Until there we conveniently dodged the case $p = 2$, for a very good reason: Corollary A.14 gives us a first-order-ring-sentence saying that a field is p-henselian, but it need not work when $p = 2$ for euclidean fields. The sentence ψ_p that we wrote previously would still characterize v_K^p even for $p = 2$, but now "k is 2-henselian" is not writable in first-order. And there's no hope to be as general as for $p \neq 2$: $R((\mathbb{Q}))$ contains a primitive square root of unity, its canonical 2-henselian valuation is the t-adic valuation. But it is real closed; hence no non-trivial valuation is definable.

However, we can still characterize a 2-henselian valuation; not quite v_K^2 but something close enough:

Definition A.21. On a field K, if $k_{v_K^2}$ is not euclidean then we let $v_K^{2*} = v_K^2$; and if $k_{v_K^2}$ is euclidean then we let v_K^{2*} be the coarsest 2-henselian valuation with euclidean residue field.

This is well defined, since if a valuation has euclidean residue field then any refinement does too; therefore v_K^{2*} is always a coarsening of v_K^2. We can now use a tweaked version of ψ_2 to characterize v_K^{2*} for all $K \vDash T_2$; let ψ_2^* be the following valued-field-sentence:

- If k is not euclidean, then ψ_2, and
- If k is euclidean, then $\mathcal{O}_v$ is a 2-henselian valuation ring and no non-trivial convex subgroup of Γ is 2-divisible.

We claim that if $K \vDash T_2$, then $(K, v) \vDash \psi_2^*$ iff $v = v_*^2$: consider first the case $k_{v_K^2}$ non-euclidean, then $v = v_K^{2*}$ iff $v = v_K^2$ iff k is non euclidean and $(K, v) \vDash \psi_2$, the later being truly first-order.

Now in the case $k_{v_K^2}$ euclidean, $v = v_K^{2*}$ iff k is euclidean, v is a 2-henselian valuation and no coarsening of v have euclidean residue field. It remains to check that this is equivalent to the property of Γ given above:

Lemma A.22. *Let (K, v) be a 2-henselian valued field and suppose k euclidean, then:*

$$v = v_K^{2*} \Leftrightarrow \Gamma \textit{ has no non-trivial convex 2-divisible subgroup.}$$

Proof. Let $\Delta \leqslant \Gamma$ be a convex subgroup, and denote w the corresponding coarsening, so that $\overline{v} \colon k_w \to \Delta$. We want to show that k_w is euclidean iff Δ is 2-divisible:

- Suppose k_w euclidean and let $\delta \in \Delta$. Take $x \in k_w$ such that $\overline{v}(x) = \delta$. Since k_w is euclidean, either x or $-x$ admits a square root, the image of which by $\overline{v}$ is $\frac{\delta}{2}$.
- Suppose Δ is 2-divisible, and let $x \in k_w$. Since $\overline{v}(x) \in \Delta$, there is $y \in k_w$ such that $\overline{v}(y^2) = \overline{v}(x)$, so $a = xy^{-2} \in \mathcal{O}_{\overline{v}^\times}$ and $\overline{a} \neq 0 \in k$. Since k is euclidean, either $\overline{a}$ or $-\overline{a}$ has a square root in k, so to say one polynomial $X^2 \pm \overline{a}$ has a simple root in k (since euclidean implies characteristic 0), and by 2-henselianity we lift it to a square root of $\pm a = \pm xy^{-2}$. Finally if -1 was a square in k_w, by taking the residue we would have a square root of -1 in k as well; so k_w is indeed euclidean.

Since by definition v_K^{2*} is the only valuation having no coarsening with euclidean residue field, we have the equivalence. $\square$

We can now apply Beth's definability theorem, and grouping everything together we have the following:

Theorem A.23 (Jahnke-Koenigsmann, [JK14]). *For any prime number $p \neq 2$, there is a $\emptyset$-ring-formula φ_p such that if $K \vDash T_p$ then $\varphi_p(K) = \mathcal{O}_{v_K^p}$; and for $p = 2$ there is a $\emptyset$-ring-formula φ_2 such that if $K \vDash T_2$, then $\varphi_2(K) = \mathcal{O}_{v_K^2}$ when $k_{v_K^2}$ is not euclidean and $\varphi_2(K) = \mathcal{O}_{v_K^{2*}}$ when $k_{v_K^2}$ is euclidean.*

B Shelah's expansion and the random graph

B.1 Shelah's expansion theorem

We first recall the definition of the Shelah expansion and, without proof, Shelah's expansion theorem. They are adapted from [Sim15, Def. 3.6 & 3.9, Prop. 3.23, Cor. 3.24].

Definition B.1. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $\varphi(x, y)$ an $\mathcal{L}$-formula, $\mathcal{N} \succcurlyeq \mathcal{M}$ be $|\mathcal{M}|^+$-saturated, and $b \in \mathcal{N}$. We consider a new relation symbol $S^{y \leftarrow b}_{\varphi(x,y)}(x)$, and we extend $\mathcal{M}$ to $\mathcal{M}'$ by interpreting this new symbol as as $\mathcal{M}' \vDash S^{y \leftarrow b}_{\varphi(x,y)}(a)$ iff $\mathcal{N} \vDash \varphi(a, b)$. We say that $X \subset \mathcal{M}^k$ is externally definable if there is φ and b such that $X = \left\{a \in \mathcal{M} \;\middle|\; \mathcal{M}' \vDash S^{y \leftarrow b}_{\varphi(x,y)}(a)\right\}$, or equivalently $X = \{a \in \mathcal{M} \mid \mathcal{N} \vDash \varphi(a, b)\}$.

If we do this for all formulas and all external parameters, we obtain the language $\mathcal{L}^{Sh} = \mathcal{L} \cup \left\{S^{y \leftarrow b}_{\varphi(x,y)} \;\middle|\; \varphi(x, y) \text{ an } \mathcal{L}\text{-formula}, b \in \mathcal{N}\right\}$, and the structure $\mathcal{M}^{Sh}$ in this language, which we call *the Shelah expansion.*

Note that the structure $\mathcal{M}^{Sh}$ depends on the choice of $\mathcal{N}$; however, the definable sets of $\mathcal{M}^{Sh}$, that is, the externally definable sets of $\mathcal{M}$, are exactly the same, regardless of the choice of $\mathcal{N}$ – as long as it is $|\mathcal{M}|^+$-saturated.

Theorem B.2 (Shelah's expansion theorem). *Let $\mathcal{M}$ be NIP, then $\mathcal{M}^{Sh}$ has quantifier elimination in $\mathcal{L}^{Sh}$. It follows that ${\mathcal{M}^{Sh}}^{Sh}$ and $\mathcal{M}^{Sh}$ have the same definable sets, and that $\mathcal{M}^{Sh}$ is NIP.*

It is folklore that Shelah's expansion theorem fails for NTP2 or NIP_n structures, and notably, for the random graph. We now expose this lore and explain how this failure happens.

B.2 The R^{Sh} lore

Let R be (a model of the theory of) the random graph, in the language containing only one binary relation symbol E which stands for the edges of the graph. R is the prototypical example of a structure having IP. It is nonetheless simple and NIP_2, but as we will see, its Shelah expansion is as wild as it gets. The following is a folklore result:

Lemma B.3. *Any subset of R^{Sh} is definable, that is, any subset of R is externally definable.*

Proof. Let $A \subset R$. Consider the type $\{E(x, a) \mid a \in A\} \cup \{\neg E(x, b) \mid b \notin A\}$. This type is finitely consistent, thus it is realized in a saturated model by some b. Now, $x \in R$ is connected to b iff $x \in A$, and thus A is externally definable: $A = S^{y \leftarrow b}_{E(x,y)}(R)$. □

It is often quoted as a reason why R^{Sh} should be very wild, namely, TP2 and/or IP_n; however, this is not per se enough to ensure so:

Example B.4. Let $\mathcal{L} = \{P_J \mid J \subset \omega\}$ and consider the $\mathcal{L}$-structure $\mathcal{N}$ having base set ω and such that $\mathcal{N} \vDash P_J(i)$ iff $i \in J$. Any subset of $\mathcal{N}$ is definable. Nonetheless, $\mathcal{N}$ has QE and is stable.

Of course, this example is very different from R^{Sh}. Most notably, it is true that any subset is definable in $\mathcal{N}$, but not in a sufficiently saturated extension of $\mathcal{N}$.

But bear in mind that subsets of R^{Sh} have an induced structure which can be very wild. In fact, reformulating slightly the previous folklore, we get:

Proposition B.5. *Any structure in a finite relational language is interpretable in R^{Sh}.*

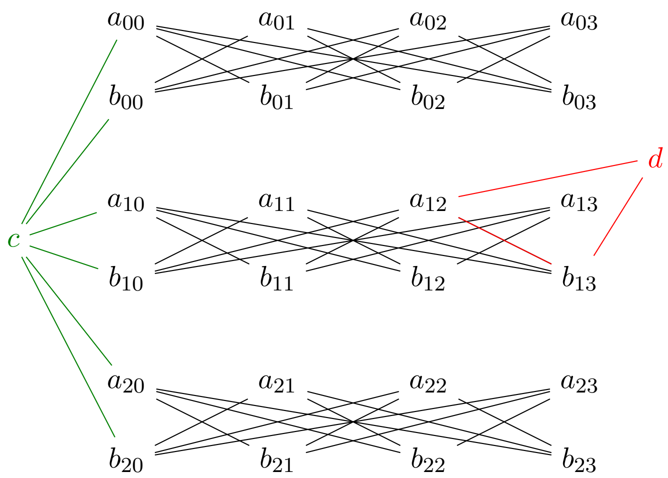
Proof. Any structure in a finite relational language is bi-interpretable with a graph. This fact is well known, see for example [Hod93, Thm. 5.5.1]. Any graph can be embedded in (a sufficiently saturated model of the theory of) the random graph R. The base set of the image of this embedding is definable in R^{Sh}, and its induced structure is exactly the structure of the graph we wanted to embed. Therefore, the original structure is interpretable in R^{Sh}. □

Corollary B.6. *R^{Sh} has IP_n and TP2.*

Proof. We just need to find a structure in a finite relational language that has TP2 or IP_n. Of course, Peano's Arithmetic or even ZFC are examples, but we prefer to use this proof as an excuse to discuss how random graphs can provide such examples.

The random n-hypergraph has IP_{n-1}: indeed, an IP_{n-1} pattern of finite length can be seen as a finite n-hypergraph. In the Fraïssé limit, an infinite IP_{n-1} pattern exists.

The triangle-free random graph has TP2, as noted in [Che14, ex. 3.13]. We repeat the argument here. Consider the formula $\varphi(x; yz)$: $E(x,y) \wedge E(x,z)$. Let $(a_{ij}, b_{ij})_{i,j\in\omega^2}$ be such that a_{ij} and $b_{k\ell}$ are connected iff $i = k$ but $j \neq \ell$. This array then witnesses TP2 for φ, as one can see on this drawing:



Any vertical path is consistent, as illustrated in green with c, and any row is 2-inconsistent, as illustrated in red with d. □

This means that Shelah's expansion theorem fails wildly outside of NIP theories: there are structures which are NIP_n and/or simple, the Shelah expansion of which has IP_n, TP2, etc. Even worse: adding one externally definable set is enough, at least in the random graph, to break Shelah's expansion theorem.

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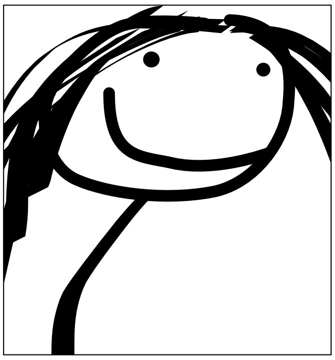
The text presented here is a work of pure mathematics. Any similarity to real life, whether useful or not, is entirely coincidental.

The author of this thesis would like to specify that under no circumstance should one push Anscombe, Jahnke, or anyone else, up or down any ladder, stairs, shaft, chasm, hole, cliff, or any other height.

The content of this thesis has been rated R for explicit formulas.

About the author



Before being a mathematician, Blaise was nothing but an annoying brat; now Blaise is an annoying mathematician – and an annoying brat.

"*A playfull thesis*" – Artem Chernikov
"*Truly iconic*" – Simone Ramello
"*That is brilliant*" – Sylvy Anscombe
"*You should maybe remove some jokes*" – my non-fun advisor
"*It will be 65.54*" – the copy shop
"*How much?????*" – my spouse