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Random walk and Fibonacci matrices

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Abstract. We study a discrete random walk on a one-dimensional finite lattice, where each state has different probabilities to move one step forward, backward, staying for a moment, or being absorbed. We obtain expected number of arrivals and expected time until absorption using a new concept: Fibonacci matrices.

1. Introduction

A discrete random walk with variable absorbing probabilities is described in every state i ($i = 0, 1, \ldots, N$) by the one-step forward probability p_i, the one-step backward probability q_i, the probability to stay for a moment in the same position r_i, and s_i is the probability of absorption in state i, where $p_i + q_i + r_i + s_i = 1$ ($i = 0, 1, \ldots, N$). For this type of random walk, we use the notation $[pqrs]$. In literature (see references), there is a focus on random walks with one or two reflecting and/or absorbing barriers. In this paper, we have the freedom of *absorption/reflection in any point at any time with state dependent probabilities.* In this way, we can model more complicated situations in physics and operations research. In Section 2, we analyze a set of difference equations which is strongly related to the expected number of arrivals and expected time until absorption. Fibonacci numbers and Fibonacci matrices (a new concept) play an important role in this setting. In Sections 3 and 4, we obtain results for expected number of arrivals and expected time until absorption for a $[pqrs]$ random walk on $[0, N]$. In Section 5, we analyze two simple random walks and their relations to Fibonacci numbers.

2. Difference equations and Fibonacci matrices

In Section 3, we calculate the expected number of arrivals, and we have to solve equations (see (3))

$$x_n = p_{n-1}x_{n-1} + q_{n+1}x_{n+1} + r_n x_n + \delta(n, i_0) \quad (0 \le n \le N),$$

and in Section 4, we obtain results for expected time until absorption, where we have to deal with (see (20))

$$m_i = p_i m_{i+1} + q_i m_{i-1} + r_i m_i + 1 \quad (1 \le i \le N-1).$$

We shall see that both sets of equations can be handled by solving the next set, which will be the object of research in this section:

$$(1) \qquad x_{i+1} = \lambda_{m+i} x_i + \mu_{m+i-1} x_{i-1} \quad (i = 1, 2, \dots, N), \qquad x_0 = 1, \quad x_1 = \lambda_m \quad (m \in \mathbb{Z}).$$

We will define Fibonacci matrices which generate in a natural way a unique solution of difference equations (1). We start with a Fibonacci sequence

$$f_0 = f_1 = 1, \quad f_{n+1} = f_n + f_{n-1} \quad (n = 1, 2, \dots)$$

Definition 2.1. Fibonacci matrices: $F_0 = [1]$, $F_1 = [\lambda_m]$ $(m \in \mathbb{Z})$, where F_{n+1} $(n = 1, 2, \dots)$ with elements $\tau_{ij}^{(m)}$ $(i = 1, 2, \dots, n+1,\ j = 1, 2, \dots, f_{n+1})$ is recursively defined by Table 1.

$\tau_{ij}^{(m)}$	$1 \dots f_n$	$f_n + 1 \dots f_{n+1}$
1		
$\vdots$	F_n	F_{n-1}
n		$1 \dots 1$
$n+1$	$\lambda_{m+n} \dots \lambda_{m+n}$	$\mu_{m+n-1} \dots \mu_{m+n-1}$

TABLE 1. F_{n+1}

So we have

$$F_2 = \begin{bmatrix} \lambda_m & 1 \\ \lambda_{m+1} & \mu_m \end{bmatrix},$$

$$F_3 = \begin{bmatrix} \lambda_m & 1 & \lambda_m \\ \lambda_{m+1} & \mu_m & 1 \\ \lambda_{m+2} & \lambda_{m+2} & \mu_{m+1} \end{bmatrix},$$

$$F_4 = \begin{bmatrix} \lambda_m & 1 & \lambda_m & \lambda_m & 1 \\ \lambda_{m+1} & \mu_m & 1 & \lambda_{m+1} & \mu_m \\ \lambda_{m+2} & \lambda_{m+2} & \mu_{m+1} & 1 & 1 \\ \lambda_{m+3} & \lambda_{m+3} & \lambda_{m+3} & \mu_{m+2} & \mu_{m+2} \end{bmatrix}.$$

Lemma 2.2. $\tau_{i,f_n+j} = \tau_{i,j}$, *where* $1 \le i \le n-1$, $1 \le j \le f_{n-1}$.

Proof. The element F_n in Table 1 can be split in F_{n-1} and F_{n-2} (and some λ, μ, and 1 below). The element F_{n-1} is in the upper left corner with rows 1 until $n-1$ and columns 1 until f_{n-1}. The same element F_{n-1} can be found in rows 1 until $n-1$ and columns $f_n + 1$ until $f_n + f_{n-1}$ $(= f_{n+1})$. □

Definition 2.3. $F_0^* = 1$; $F_n^* = \sum_{j=1}^{f_n} \prod_{k=1}^{n} \tau_{kj}^{(m)}$ $(n = 1, 2, \dots, N+1)$.

Proposition 2.4. *F_n^*, where $n = 0, 1, \dots, N+1$, is a solution of* (1).

Proof.

$$\begin{aligned} F_{n+1}^* &= \sum_{j=1}^{f_{n+1}} \prod_{k=1}^{n+1} \tau_{kj}^{(m)} = \sum_{j=1}^{f_n} \prod_{k=1}^{n+1} \tau_{kj}^{(m)} + \sum_{j=f_n+1}^{f_{n+1}} \prod_{k=1}^{n+1} \tau_{kj}^{(m)} \\ &= \sum_{j=1}^{f_n} \Bigl[\prod_{k=1}^{n} \tau_{kj}^{(m)} \tau_{n+1,j}^{(m)}\Bigr] + \sum_{j=f_n+1}^{f_{n+1}} \Bigl[\prod_{k=1}^{n-1} \tau_{kj}^{(m)} \tau_{n,j}^{(m)} \tau_{n+1,j}^{(m)}\Bigr] \\ &= \lambda_{m+n} \sum_{j=1}^{f_n} \prod_{k=1}^{n} \tau_{kj}^{(m)} + 1.\mu_{m+n-1} \sum_{j=f_n+1}^{f_{n+1}} \prod_{k=1}^{n-1} \tau_{kj}^{(m)} \\ &= \lambda_{m+n} F_n^* + \mu_{m+n-1} \sum_{j=1}^{f_{n-1}} \prod_{k=1}^{n-1} \tau_{kj}^{(m)} = \lambda_{m+n} F_n^* + \mu_{m+n-1} F_{n-1}^*, \end{aligned}$$

where we used Lemma 2.2 in the penultimate step. □

Theorem 2.5. *The solution of the linear system* (1) *is*

$$x_0 = 1, \quad x_i = \sum_{j=1}^{f_n} \prod_{k=1}^{n} \tau_{kj}^{(m)} \quad (i = 1, 2, \dots, N+1), \tag{2}$$

where

- *for $j \le f_{i+1}$,*

$$\tau_{ij}^{(m)} = \begin{cases} \lambda_{m+i-1} & (j = 1, 2, \dots, f_{i-1}), \\ \mu_{m+i-2} & (j = f_{i-1}+1, \dots, f_i), \\ 1 & (j = f_i + 1, \dots, f_{i+1}); \end{cases}$$

- *for $j > f_{i+1}$, there exists $n \in \mathbb{N}$ such that $1 + f_n \le j \le f_{n+1}$. Let*

$$\begin{aligned} &j_\ell = j - (f_n + f_{n-2} + \dots + f_{n-2\ell}) \quad \Bigl(\ell = 0, 1, \dots, \Bigl\lfloor \frac{n-1}{2} \Bigr\rfloor\Bigr), \\ &k = \min\{\ell \in \mathbb{N} \mid j_\ell \le f_{i+1}\}; \end{aligned}$$

then

$$\tau_{i,j}^{(m)} = \tau_{i,j_k}^{(m)} = \begin{cases} \lambda_{m+i-1} & (j_k = 1, 2, \dots, f_{i-1}), \\ \mu_{m+i-2} & (j_k = f_{i-1}+1, \dots, f_i), \\ 1 & (j_k = f_i + 1, \dots, f_{i+1}). \end{cases}$$

Proof. We start with the first case.

Case $j \le f_{i+1}$: Substituting (2) in (1) yields

$$\sum_{j=1}^{f_{i+1}} \prod_{k=1}^{i+1} \tau_{kj}^{(m)} = \lambda_{m+i} \sum_{j=1}^{f_i} \prod_{k=1}^{i} \tau_{kj}^{(m)} + \mu_{m+i-1} \sum_{j=1}^{f_{i-1}} \prod_{k=1}^{i-1} \tau_{kj}^{(m)},$$

so

$$\sum_{j=1}^{f_{i+1}} \tau_{1j}^{(m)}\tau_{2j}^{(m)}\dots\tau_{i+1,j}^{(m)} = \sum_{j=1}^{f_i} \tau_{1j}^{(m)}\tau_{2j}^{(m)}\dots\tau_{i,j}^{(m)}\lambda_{m+i} + \sum_{j=1}^{f_{i-1}} \tau_{1j}^{(m)}\tau_{2j}^{(m)}\dots\tau_{i-1,j}^{(m)}.1.\mu_{m+i-1}.$$

Using Lemma 2.2, we get

$$\sum_{j=1}^{f_{i+1}} \tau_{1j}^{(m)}\tau_{2j}^{(m)}\dots\tau_{i+1,j}^{(m)} = \sum_{j=1}^{f_i} \tau_{1j}^{(m)}\tau_{2j}^{(m)}\dots\tau_{i,j}^{(m)}\lambda_{m+i} + \sum_{j=f_i+1}^{f_{i+1}} \tau_{1j}^{(m)}\tau_{2j}^{(m)}\dots\tau_{i-1,j}^{(m)}.1.\mu_{m+i-1}.$$

It follows

$$\tau_{i+1,j}^{(m)} = \lambda_{m+i} \qquad (j \le f_i)$$

and

$$\tau_{ij}^{(m)} = 1, \quad \tau_{i+1,j}^{(m)} = \mu_{m+i-1} \quad (f_i + 1 \le j \le f_{i+1}).$$

Case $j > f_{i+1}$: there exists $n \in \mathbb{N}$ such that $1 + f_n \le j \le f_{n+1}$, so

$$1 \le j_0 = j - f_n \le f_{n-1}.$$

Let j_ℓ ($\ell = 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor$) be defined by

$$j_\ell = j_{\ell-1} - f_{n-2\ell} \le f_{n-2\ell-1}.$$

It follows

$$j_\ell = j - (f_n + f_{n-2} + \dots + f_{n-2\ell}) \quad \Big(\ell = 0, 1, \dots, \Big\lfloor \frac{n-1}{2} \Big\rfloor\Big).$$

Using Lemma 2.2, $\tau_{i,j}^{(m)} = \tau_{i,j_0}^{(m)} = \tau_{i,j_1}^{(m)} = \dots = \tau_{i,j_k}^{(m)}$, where $i \le n-1$.

Let $k = \min\{\ell \in \mathbb{N} \mid j_\ell \le f_{i+1}\}$; then $j_k \le f_{i+1}$, so we can apply the first part of this proof. □

Notation 2.6.

$$A_i^{(m)} = A_i^{(m)}(\lambda, \mu) = \sum_{j=1}^{f_i} \prod_{k=1}^{i} \tau_{kj}^{(m)} \quad (i = 1, 2, \dots, N+1),$$

where $\lambda = (\lambda_m, \lambda_{m+1}, \dots, \lambda_{m+N})$, $\mu = (\mu_m, \mu_{m+1}, \dots, \mu_{m+N-1})$.

Definition 2.7. $A_0^{(m)} = 1$, $A_{-1}^{(m)} = 0$ ($m \in \mathbb{Z}$).

Theorem 2.8. *$A_{N+1}^{(m)} = \lambda_m A_N^{(m+1)} + \mu_m A_{N-1}^{(m+2)}$, where $m \in \mathbb{Z}$, $N = 0, 1, 2, \ldots$.*

Proof. We write the linear system

$$\begin{aligned} x_{i+1} &= \lambda_{m+i} x_i + \mu_{m+i-1} x_{i-1} \quad (i = 1, 2, \ldots, N), \\ x_0 &= 1, \quad x_1 = \lambda_m \end{aligned}$$

in matrix notation

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \lambda_m & -1 & 0 & & & & 0 \\ \mu_m & \lambda_{m+1} & -1 & & & & 0 \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{m+N-1} & \lambda_{m+N} & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The determinant of the matrix is $(-1)^{N+1}$. Using Cramer's rule, we get

$$\begin{aligned} A_{N+1}^{(m)} &= x_{N+1} \\ &= (-1)^{N+1} \det \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & 1 \\ \lambda_m & -1 & 0 & & & & 0 \\ \mu_m & \lambda_{m+1} & -1 & & & & 0 \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{m+N-1} & \lambda_{m+N} & 0 \end{bmatrix} \\ &= (-1)^{N+1}(-1)^{N+1} \det \begin{bmatrix} \lambda_m & -1 & & & \\ \mu_m & \lambda_{m+1} & -1 & & \\ & & & & \\ & & & \mu_{m+N-1} & \lambda_{m+N} \end{bmatrix} \\ &= \lambda_m \det \begin{bmatrix} \lambda_{m+1} & -1 & & \\ & & & \\ & & \mu_{m+N-1} & \lambda_{m+N} \end{bmatrix} \\ &\quad + \mu_m \det \begin{bmatrix} \lambda_{m+2} & -1 & & \\ & & & \\ & & \mu_{m+N-1} & \lambda_{m+N} \end{bmatrix} \\ &= \lambda_m A_N^{(m+1)} + \mu_m A_{N-1}^{(m+2)}. \end{aligned}$$

□

3. Expected number of arrivals

We start with two definitions.

Definition 3.1. $p_{ij}^{(k)} = P(\text{system is in state } j \text{ after } k \text{ steps} \mid \text{start in } i)$.

Definition 3.2. $x_j = x_{i,j} = \sum_{k=0}^{\infty} p_{ij}^{(k)}$, where x_j is the expected number of arrivals in j when starting in i.

We analyze a finite discrete random walk with different absorbing probabilities: in every state i $(i = 0, 1, \ldots, N)$, we have one-step forward probability p_i, one-step backward probability q_i, probability to stay for a moment in the same position r_i, and s_i is the probability of absorption in state i, where $p_i + q_i + r_i + s_i = 1$ $(i = 0, 1, \ldots, N)$ and $q_0 = p_N = 0$. The starting point of the random walk on $[0, N]$ is i_0 $(0 \le i_0 \le N)$.

Theorem 3.3.

$$x_n = p_{n-1}x_{n-1} + q_{n+1}x_{n+1} + r_n x_n + \delta(n, i_0) \quad (0 \le n \le N). \tag{3}$$

Proof. When $0 < i_0 < N$, we have

$$\begin{aligned} x_n &= \sum_{k=0}^{\infty} p_{i_0,n}^{(k)} = p_{i_0,n}^{(0)} + \sum_{k=1}^{\infty}\sum_{l} p_{i_0,l}^{(k-1)} p_{l,n} = \delta(i_0, n) + \sum_{l} p_{l,n} \sum_{k=1}^{\infty} p_{i_0,l}^{(k-1)} \\ &= \delta(i_0, n) + p_{n-1}x_{n-1} + q_{n+1}x_{n+1} + r_n x_n. \end{aligned}$$

When $i_0 = 0$ or $i_0 = N$, the prove goes along the same lines. □

Lemma 3.4. *The linear system*

$$x_{i+1} = \lambda_i x_i + \mu_{i-1} x_{i-1} \quad (i = i_0 + 1, i_0 + 2, \ldots, N) \tag{4}$$

given x_{i_0} *and* x_{i_0+1} *has the solution*

$$x_{i_0+k+1} = x_{i_0+1} A_k^{(i_0+1)}(\lambda, \mu) + \mu_{i_0} x_{i_0} A_{k-1}^{(i_0+2)}(\lambda, \mu) \quad (k = 1, 2, \ldots, N - i_0). \tag{5}$$

Proof. We use induction.

(i) Substituting $k = 1$ in (5) gives

$$x_{i_0+2} = x_{i_0+1} A_1^{(i_0+1)}(\lambda, \mu) + \mu_{i_0} x_{i_0} A_0^{(i_0+2)}(\lambda, \mu) = \lambda_{i_0+1} x_{i_0+1} + \mu_{i_0} x_{i_0}.$$

(ii) We give that (4) is correct for $i = i_0 + k - 1$ and $i = i_0 + k$. We have

$$\begin{aligned} x_{i_0+k+1} &= \lambda_{i_0+k} x_{i_0+k} + \mu_{i_0+k-1} x_{i_0+k-1} \\ &= \lambda_{i_0+k}[x_{i_0+1} A_{k-1}^{(i_0+1)} + \mu_{i_0} x_{i_0} A_{k-2}^{(i_0+2)}] \\ &\quad + \mu_{i_0+k-1}[x_{i_0+1} A_{k-2}^{(i_0+1)} + \mu_{i_0} x_{i_0} A_{k-3}^{(i_0+2)}] \\ &= [\lambda_{i_0+k} A_{k-1}^{(i_0+1)} + \mu_{i_0+k-1} A_{k-2}^{(i_0+1)}] x_{i_0+1} \\ &\quad + [\lambda_{i_0+k} A_{k-2}^{(i_0+2)} + \mu_{i_0+k-1} A_{k-3}^{(i_0+2)}] \mu_{i_0} x_{i_0} \\ &= x_{i_0+1} A_k^{(i_0+1)} + \mu_{i_0} x_{i_0} A_{k-1}^{(i_0+2)}, \end{aligned}$$

where in the last step we used that $A_i^{(m)}$ is a solution of (1); substitute $m = i_0 + 1$, $i = k - 1$, and $m = i_0 + 2$, $i = k - 2$ in (1). So (4) is also correct for $i = i_0 + k + 1$.

(iii) Apply induction. □

In this section, we use $A_i^{(m)}(\lambda, \mu)$ $(m \geq 0)$, where

$$\lambda = (\lambda_m, \lambda_{m+1}, \ldots, \lambda_N), \qquad \mu = (\mu_m, \mu_{m+1}, \ldots, \mu_{N-1}),$$
$$\lambda_i = \frac{1 - r_i}{q_{i+1}}, \qquad \mu_i = -\frac{p_i}{q_{i+2}}.$$

We also use $A_i^{(m)}(\rho, \theta)$ $(m < 0)$, where

$$\rho = (\rho_m, \rho_{m+1}, \ldots, \rho_0), \qquad \theta = (\theta_m, \theta_{m+1}, \ldots, \theta_{-1}),$$
$$\rho_{-i} = \frac{1 - r_i}{p_{i-1}}, \qquad \theta_{-i} = -\frac{q_i}{p_{i-2}}.$$

Theorem 3.5.

- *For* $0 < i_0 < N$,

$$x_{i_0} = \left[1 - r_{i_0} + p_{i_0-1}\theta_{-i_0}\frac{A_{i_0-1}^{(2-i_0)}(\rho,\theta)}{A_{i_0}^{(1-i_0)}(\rho,\theta)} + q_{i_0+1}\mu_{i_0}\frac{A_{N-i_0-1}^{(i_0+2)}(\lambda,\mu)}{A_{N-i_0}^{(i_0+1)}(\lambda,\mu)}\right]^{-1}. \tag{6}$$

For $k = 0, 1, \ldots, N - i_0$,

$$\begin{aligned} x_{i_0+k+1} = \mu_{i_0} x_{i_0} [A_{k-1}^{(i_0+2)}(\lambda,\mu) A_{N-i_0}^{(i_0+1)}(\lambda,\mu) \\ - A_{N-i_0-1}^{(i_0+2)}(\lambda,\mu) A_k^{(i_0+1)}(\lambda,\mu)][A_{N-i_0}^{(i_0+1)}(\lambda,\mu)]^{-1}, \end{aligned} \tag{7}$$

and for $k = 0, 1, \ldots, i_0 - 1$,

$$\begin{aligned} x_{i_0-(k+1)} = \theta_{-i_0} x_{i_0} [A_{k-1}^{(2-i_0)}(\rho,\theta) A_{i_0}^{(1-i_0)}(\rho,\theta) \\ - A_{i_0-1}^{(2-i_0)}(\rho,\theta) A_k^{(1-i_0)}(\rho,\theta)][A_{i_0}^{(1-i_0)}(\rho,\theta)]^{-1} \end{aligned} \tag{8}$$

- *For* $i_0 = 0$,

$$x_0 = \frac{A_N^{(1)}}{q_1 A_{N+1}^{(0)}}, \tag{9}$$

$$x_i = \frac{A_N^{(1)} A_i^{(0)} - A_{i-1}^{(1)} A_{N+1}^{(0)}}{q_1 A_{N+1}^{(0)}} \quad (i = 1, \ldots, N) \tag{10}$$

Proof. We treat the two cases separately.

Case $0 < i_0 < N$: We introduce two artificial states, $N + 1$ and -1, with specifications

$$x_{N+1} = 0, \quad x_{-1} = 0, \tag{11}$$

$$q_{N+1} > 0, \quad p_{-1} > 0. \tag{12}$$

Using (3), we get forward and backward equations

$$x_{i+1} = \frac{1-r_i}{q_{i+1}}x_i - \frac{p_{i-1}}{q_{i+1}}x_{i-1} = \lambda_i x_i + \mu_{i-1}x_{i-1} \quad (i = i_0+1, i_0+2, \dots, N), \tag{13}$$

$$x_{i-1} = \frac{1-r_i}{p_{i-1}}x_i - \frac{q_{i+1}}{p_{i-1}}x_{i+1} = \rho_{-i}x_i + \theta_{-(i+1)}x_{i+1} \quad (i = i_0-1, i_0-2, \dots, 0). \tag{14}$$

By induction, we can prove that solutions of (13) and (14) are (see Lemma 3.4 for a proof of the first linear system; the second one can be proved the same way)

$$x_{i_0+k+1} = x_{i_0+1}A_k^{(i_0+1)}(\lambda,\mu) + \mu_{i_0}x_{i_0}A_{k-1}^{(i_0+2)}(\lambda,\mu) \quad (k = 1,2,\dots,N-i_0), \tag{15}$$

$$x_{i_0-(k+1)} = x_{i_0-1}A_k^{(1-i_0)}(\rho,\theta) + \theta_{-i_0}x_{i_0}A_{k-1}^{(2-i_0)}(\rho,\theta) \quad (k = 1,2,\dots,i_0). \tag{16}$$

Using (11), (12), (15), and (16), we get

$$x_{N+1} = x_{i_0+1}A_{N-i_0}^{(i_0+1)}(\lambda,\mu) + \mu_{i_0}x_{i_0}A_{N-i_0-1}^{(i_0+2)}(\lambda,\mu) = 0, \tag{17}$$

$$x_{-1} = x_{i_0-1}A_{i_0}^{(1-i_0)}(\rho,\theta) + \theta_{-i_0}x_{i_0}A_{i_0-1}^{(2-i_0)}(\rho,\theta) = 0. \tag{18}$$

Substituting $n = i_0$ in (3) gives

$$(1-r_{i_0})x_{i_0} = 1 + p_{i_0-1}x_{i_0-1} + q_{i_0+1}x_{i_0+1}. \tag{19}$$

Using (17), (18), and (19), we get the expected number of arrivals in the starting point i_0; see (6).

For $k = 0,1,\dots,N-i_0$, we find the expected number of arrivals in i_0+k+1 (use (19), (17), and (15)); see (7).

For $k = 0,1,\dots,i_0$, we get (8) (use (19), (18), and (16)).

Case $i_0 = 0$ ($i_0 = N$ proceeds along the same lines): Instead of two artificial states, we now need one artificial state $N+1$ with $x_{N+1} = 0$, $q_{N+1} > 0$. We get (use (3))

$$x_{i+1} = \frac{1-r_i}{q_{i+1}}x_i - \frac{p_{i-1}}{q_{i+1}}x_{i-1} = \lambda_i x_i + \mu_{i-1}x_{i-1} \quad (i = 1,2,\dots,N),$$

$$x_1 = \frac{1-r_0}{q_1}x_0 - \frac{1}{q_1} = \lambda_0 x_0 - \frac{1}{q_1} \quad (i = 0),$$

with solution (proved by induction)

$$x_i = x_0 A_i^{(0)} - \frac{1}{q_1}A_{i-1}^{(1)} \quad (i = 1,2,\dots,N+1),$$

where $A_i^{(m)} = A_i^{(m)}(\lambda, \mu)$. Using the artificial state, we get

$$x_{N+1} = x_0 A_{N+1}^{(0)} - \frac{1}{q_1} A_N^{(1)} = 0,$$

resulting in (9) and (10). □

Remark. Equation (9) can also be derived from (6): use Theorem 2.8 and $i_0 = 0$, $p_{-1} = 0$.

4. Expected time until absorption

Let T_i be the time until absorption when starting in i ($i = 0, 1, \ldots, N$).

Definition 4.1. $m_i = E[T_i] = \sum_{k=1}^{\infty} kP(T_i = k)$ ($i = 0, 1, \ldots, N$), where m_i is the expected time until absorption when starting in i.

In this section, we demand $s_i > 0$ ($i = 0, 1, \ldots, N$). Let

$$s = \min(s_0, s_1, \ldots, s_N).$$

Then $P(\text{no absorption after } n \text{ steps}) \leq (1 - s)^n$, so absorption will always occur: $\sum_{k=1}^{\infty} P(T_i = k) = 1$.

Theorem 4.2.

$$\begin{aligned} m_0 &= p_0 m_1 + r_0 m_0 + 1, \\ m_i &= p_i m_{i+1} + q_i m_{i-1} + r_i m_i + 1 \quad (1 \leq i \leq N-1), \\ m_N &= q_N m_{N-1} + r_N m_N + 1. \end{aligned} \tag{20}$$

Proof. We prove (20). The rest is going along the same lines.

$$\begin{aligned} m_i = E[T_i] &= \sum_{k=1}^{\infty} kP(T_i = k) = \sum_{k=1}^{\infty} (k-1) P(T_i = k) + \sum_{k=1}^{\infty} P(T_i = k) \\ &= \sum_{k=2}^{\infty} (k-1)\{p_i P(T_{i+1} = k-1) + q_i P(T_{i-1} = k-1) \\ &\qquad\qquad + r_i P(T_i = k-1)\} + 1 \\ &= p_i m_{i+1} + q_i m_{i-1} + r_i m_i + 1. \end{aligned}$$ □

Another way to obtain (20) is by observing the next step of the random walk: with probability p_i, we move to state $i+1$, and then our expectation of time until absorption is m_{i+1}. But we did one step, so we have to deal with $1 + m_{i+1}$. The last term is about absorption in one step:

$$m_i = p_i(1 + m_{i+1}) + q_i(1 + m_{i-1}) + r_i(1 + m_i) + s_i.1 \quad (1 \leq i \leq N-1).$$

In this section, we use the abbreviations

$$\omega_i = \frac{1-r_i}{p_i} \quad (i=0,\dots,N-1), \qquad \omega_N = 1-r_N,$$
$$\phi_i = -\frac{q_{i+1}}{p_{i+1}} \quad (i=0,\dots,N-2), \quad \phi_{N-1} = -q_N,$$
$$\alpha_i = -\frac{1}{p_i} \quad (i=0,\dots,N-1), \qquad \alpha_N = -1.$$

Theorem 4.3. *For* $0 \le i \le N$*,*

$$m_i = \sum_{k=1}^{i} A_{i-k}^{(k)} \alpha_{k-1} - \frac{A_i^{(0)}[\omega_N \sum_{k=1}^{N} A_{N-k}^{(k)} \alpha_{k-1} + \phi_{N-1} \sum_{k=1}^{N-1} A_{N-1-k}^{(k)} \alpha_{k-1} + \alpha_N]}{\omega_N A_N^{(0)} + \phi_{N-1} A_{N-1}^{(0)}}. \tag{21}$$

Proof. The $N+1$ forward equations are (using Theorem 4.2)

$$m_1 = \frac{(1-r_0)}{p_0} m_0 - \frac{1}{p_0} = \omega_0 m_0 + \alpha_0,$$
$$m_{i+1} = \frac{(1-r_i)}{p_i} m_i - \frac{q_i}{p_i} m_{i-1} - \frac{1}{p_i}$$
$$= \omega_i m_i + \phi_{i-1} m_{i-1} + \alpha_i \quad (i=1,2,\dots,N-1),$$
$$0 = (1-r_N) m_N - q_N m_{N-1} - 1 = \omega_N m_N + \phi_{N-1} m_{N-1} + \alpha_N. \tag{22}$$

By induction (as in Lemma 3.4), we can prove

$$m_i = m_0 A_i^{(0)} + \sum_{k=1}^{i} A_{i-k}^{(k)} \alpha_{k-1} \quad (i=0,1,\dots,N), \tag{23}$$

where $A_i^{(m)} = A_i^{(m)}(\omega,\phi)$, with $\omega = (\omega_m,\dots,\omega_N)$ and $\phi = (\phi_m,\dots,\phi_{N-1})$. Substituting (23) in (22), we obtain m_0, and again using (23), we get (21). □

5. Random walk and Fibonacci numbers

In this section, we study two simple random walks and their relation to Fibonacci numbers.

5.1. **Homogeneous transition probabilities.** We first consider a random walk on $[0,N]$ where we have homogeneous transition probabilities and there is no option to stay for a moment in any state: $p_i = p$, $q_i = q$, $r_i = 0$, $s_i = s$ $(i=1,2,\dots,N-1)$, $p+q+s=1$, $p_0 = p$, $s_0 = 1-p$, $q_N = q$, $s_N = 1-q$. We start in 0.

Theorem 5.2.

$$x_0 = x_0^{[N]} = \frac{\sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N-k}{k} (-pq)^k}{\sum_{k=0}^{\lfloor \frac{N+1}{2} \rfloor} \binom{N+1-k}{k} (-pq)^k} \quad (p+q+s=1). \tag{24}$$

Proof. Using (9) and homogeneity (superscripts in $A_i^{(m)}$ can be omitted, and $\lambda_i = \lambda = \frac{1}{q}$, $\mu_i = \mu = -\frac{p}{q}$), we get, when $N = 1$,

$$x_0^{[1]} = \frac{A_1^{(1)}}{q_1 A_2^{(0)}} = \frac{A_1}{qA_2} = \frac{\lambda}{q(\lambda^2+\mu)} = \frac{1}{1-pq}. \tag{25}$$

Using (9) and Theorem 2.8 yields

$$x_0^{[j]} = \frac{A_j^{(1)}}{q_1 A_{j+1}^{(0)}} = \frac{A_j}{qA_{j+1}},$$

so

$$\begin{aligned} \frac{A_j}{A_{j+1}} &= qx_0^{[j]}, \\ x_0^{[j+1]} &= \frac{A_{j+1}}{qA_{j+2}} = \frac{A_{j+1}}{q\{\lambda A_{j+1} + \mu A_j\}} = \frac{A_{j+1}}{A_{j+1} - pA_j} \\ &= \frac{1}{1 - p\frac{A_j}{A_{j+1}}} = \frac{1}{1 - pqx_0^{[j]}}. \end{aligned}$$

We get

$$\begin{aligned} x_0^{[2]} &= \frac{1-pq}{(1-pq) - pq.1} = \frac{1-pq}{1-2pq}, \\ x_0^{[3]} &= \frac{1-2pq}{(1-2pq) - pq(1-pq)} = \frac{1-2pq}{1-3pq+p^2q^2} \\ &\vdots \end{aligned}$$

This leads to (24), which can be proven by using induction to N.

(i) $N = 1$ is correct; see (25).

(ii) Suppose (24) is correct up to $N+1$. We first rewrite (24) to $x_0^{[N]} = \frac{\sigma_N}{\sigma_{N+1}}$; then

$$x_0^{[N+1]} = \frac{1}{1-pqx_0^{[N]}} = \frac{\sigma_{N+1}}{\sigma_{N+1} - pq\sigma_N},$$

and for all terms except the first and last one in $\sigma_{N+1} - pq\sigma_N$,

$$\begin{aligned} \sigma_{N+1} - pq\sigma_N &= \sum \binom{N-k+1}{k} (-pq)^k - pq \sum \binom{N-k}{k} (-pq)^k \\ &= \sum \left\{ \binom{N-k+1}{k} + \binom{N-k+1}{k-1} \right\} (-pq)^k \\ &= \sum \binom{N-k+2}{k} (-pq)^k = \sigma_{N+2}. \end{aligned}$$

The first term in $\sigma_{N+1} - pq\sigma_N$ is the first term in σ_{N+1}; the last term in $\sigma_{N+1} - pq\sigma_N$ is the last term in $-pq\sigma_N$ with index $\lfloor \frac{N}{2} \rfloor + 1 = \lfloor \frac{N+2}{2} \rfloor$. So (24) is also correct for $N+2$.

(iii) Apply induction to N. □

5.3. **Partial absorbing barriers in the endpoints.** Our next random walk is more restricted. We study simple random walk with partial absorbing barriers in the endpoints. See the previous random walk, but now with $s_i = 0$ $(i = 1, 2, \dots, N-1)$, $p + q = 1$.

Theorem 5.4.

$$\sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N-k}{k} (-pq)^k = \begin{cases} \dfrac{q^{N+1} - p^{N+1}}{q - p} & (p \neq q), \\ \dfrac{N+1}{2^N} & (p = q = \frac{1}{2}). \end{cases} \tag{26}$$

Proof. The well-known probability of absorption in state 0 is

$$qx_0 = \frac{1 - (\frac{p}{q})^{N+1}}{1 - (\frac{p}{q})^{N+2}} \quad (p \neq q).$$

So we have

$$x_0 = \frac{q^{N+1} - p^{N+1}}{q^{N+2} - p^{N+2}} \quad (p \neq q).$$

Using (24), we guess the first part of (26), which can be proven by induction to N.

The second part of (26) is obtained by applying de l'Hospitals rule on the first result, or by induction to N. □

Theorem 5.5.

$$\sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N-k}{k} t^k = \frac{1}{2^N} \sum_{n=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N+1}{2n+1} (1+4t)^n.$$

Proof. Let $f_{N,t} = \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N-k}{k} t^k$ be the continuation of $\sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N-k}{k} (-pq)^k$ to $\mathbb{R}$. Taking $t = -pq$ gives $p^2 - p - t = 0$ and

$$f_{N,t} = \frac{q^{N+1} - p^{N+1}}{q - p} = \frac{(1 + \sqrt{1+4t})^{N+1} - (1 - \sqrt{1+4t})^{N+1}}{2^{N+1}\sqrt{1+4t}}$$
$$= \frac{1}{2^N} \sum_{n=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N+1}{2n+1} (1+4t)^n. \qquad \square$$

Corollary 5.6.

$$f_N = \frac{1}{2^N} \sum_{n=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N+1}{2n+1} .5^n \quad (N = 0, 1, 2, \dots).$$

By repeated differentiating of $f_{N,t}$, we get the "moments" of

$$f_N = \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N-k}{k},$$

for example,

$$(f'_{N,t})_{t=1} = \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} k \binom{N-k}{k} = \frac{1}{2^{N-2}} \sum_{n=1}^{\lfloor \frac{N}{2} \rfloor} \binom{N+1}{2n+1} .n.5^{n-1}.$$

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